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# ON TWO ITERATION METHODS FOR THE QUADRATIC MATRIX EQUATIONS

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**Abstract.** By simply transforming the quadratic matrix equation into an equivalent fixed-point equation, we construct a successive approximation method and a Newton's method based on this fixed-point equation. Under suitable conditions, we prove the local convergence of these two methods, as well as the linear convergence speed of the successive approximation method and the quadratic convergence speed of the Newton's method. Numerical results show that these new methods are accurate and effective when they are used to solve the quadratic matrix equation.

**Key Words.** Quadratic matrix equation, iteration method, convergence property.

## 1. Introduction

The *quadratic matrix equation* (**QME)**

(1) $$\mathcal{Q}(X) \equiv X^2 - BX - C = 0, \quad B, C \in \mathbb{C}^{n\times n}$$

occurs in a variety of applications. For example, it may arise in the quadratic eigenvalue problem[3, 4, 6, 8, 12, 13]

$$\mathcal{Q}(\lambda)x \equiv \lambda^2 x - \lambda Bx - Cx = 0, \quad B, C \in \mathbb{C}^{n\times n},$$

or the noisy Wiener-Hopf problems for Markov chains[5, 7, 10, 11]. Evidently, some Riccati equations are QMEs, and vice versa, and theory of Riccati equations and numerical methods for their solution are well developed[2, 9]; however, these two classes of equations require different techniques for analysis and solution in general. See also [1].

Recently, Higham and Kim[6] studied Newton's methods with and without exact line searches for solving the QME(1). In the Newton's method, the quadratic matrix function $\mathcal{Q}(X)$ is successively linearized at each of the current iterate $X^{(k)}$ which is required to be located in a neighborhood of a solution $X_\star$ of the QME(1), and the next iterate $X^{(k+1)}$ is obtained by solving the corresponding Newton equation which is a special case of the generalized Sylvester equation. And in the Newton's method with line search, the current Newton direction $E^{(k)}$ is used as a search direction and the next iterate

$$X^{(k+1)} = X^{(k)} + t^{(k)} E^{(k)}$$

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is defined by exactly minimizing the objective function

$$p(t) = \|\mathcal{Q}(X^{(k)} + t^{(k)}E^{(k)})\|_F^2$$

along this direction, i.e.,

$$t^{(k)} = \text{argmin}_{0<t<2}p(t),$$

where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix. It was proved in [6] that the latter has global convergence property.

In particular, when $B$ is a diagonal matrix and $C$ is an $M$-matrix, Guo[5] studied the existence and uniqueness of $M$-matrix solutions and iterative method for finding the desired $M$-matrix solution of the QME(1) by transforming it into a special nonsymmetric *algebraic Riccati equation* **(ARE)**, and proved the monotone convergence of the obtained iterative methods.

In this paper, for general matrices $B, C \in \mathbb{C}^{n\times n}$, we first simply transform the QME(1) into an equivalent fixed-point equation, and then based on it we construct a successive approximation method and a Newton's method for solving the quadratic matrix equation (1). Under suitable conditions, we prove the local convergence of these two methods, as well as the linear convergence speed of the successive approximation method and the quadratic convergence speed of the Newton's method. Numerical results show that these new methods are more accurate and effective than the known ones in [6, 5].

Without loss of generality, throughout this paper we will assume that the constant matrix term $C \in \mathbb{C}^{n\times n}$ in the QME(1) is nonsingular. In the case that the matrix $C$ is singular, we can shift the variable and make the constant matrix term in the equivalently transformed quadratic matrix equation be nonsingular. More specifically, by letting $Y = \sigma I - X$ we can rewrite the QME(1) as

$$Y^2 - (2\sigma I - B)Y + (\sigma^2 I - \sigma B - C) = 0,$$

where $\sigma$ is a real constant. We can now choose the parameter $\sigma$ such that the matrix $(\sigma^2 I - \sigma B - C)$ is nonsingular. See [5].

## 2. Two iteration methods

If $X_\star \in \mathbb{C}^{n\times n}$ is a solution of the QME(1), i.e.,

$$\mathcal{Q}(X_\star) = X_\star^2 - BX_\star - C = 0,$$

then we have

$$(X_\star - B)X_\star = C.$$

It then follows that both $X_\star$ and $(X_\star - B)$ are nonsingular matrices, provided $C$ is a nonsingular matrix. In this case, we can construct the following fixed-point equation for the QME(1):

$$X = \mathcal{F}(X), \quad \text{where} \quad \mathcal{F}(X) = (X - B)^{-1}C. \tag{2}$$

Therefore, $X_\star \in \mathbb{C}^{n\times n}$ is a solution of the QME(1) if and only if it is a fixed-point of the matrix operator $\mathcal{F}(X)$, or equivalently, a zero point of the matrix equation

$$X - \mathcal{F}(X) = 0.$$

Furthermore, by denoting

$$\mathcal{G}(X) = X - \mathcal{F}(X)$$

and using the first-order approximation to $\mathcal{G}(X)$, we have

$$\mathcal{G}(X + E) = \mathcal{G}(X) + \mathcal{J}(X, E) + \mathcal{O}(E^2),$$

where

$$\mathcal{J}(X,E) = E + (X-B)^{-1}E(X-B)^{-1}C.$$

This straightforwardly results in the following fixed-point equation for the QME(1):

$$X = \mathcal{N}(X) \quad \text{with} \quad \mathcal{N}(X) = X + E, \tag{3}$$

where $E$ satisfies

$$\mathcal{J}(X,E) = -\mathcal{G}(X). \tag{4}$$

We call $\mathcal{N}(X)$ the Newton operator and (4) the Newton equation of the nonlinear matrix function $\mathcal{G}(X)$. Evidently, we also have the fact that $X_\star \in \mathbb{C}^{n\times n}$ is a solution of the QME(1) if and only if it is a fixed-point of the matrix operator $\mathcal{N}(X)$.

Based on (2) and (3)-(4), we can immediately define the following two iteration methods, called as the *successive approximation method* and the *Newton's method*, respectively, for solving the QME(1) when the matrix $C \in \mathbb{C}^{n\times n}$ is nonsingular.

**Method 2.1.** (The Successive Approximation Method).
Given an initial guess $X^{(0)} \in \mathbb{C}^{n\times n}$, for $k = 0, 1, 2, \ldots$ until $\{X^{(k)}\}$ convergence, compute

$$X^{(k+1)} = (X^{(k)} - B)^{-1}C.$$

**Method 2.2.** (The Newton's Method).
Given an initial guess $X^{(0)} \in \mathbb{C}^{n\times n}$, for $k = 0, 1, 2, \ldots$ until $\{X^{(k)}\}$ convergence, compute

$$X^{(k+1)} = X^{(k)} + E^{(k)},$$

where $E^{(k)}$ is a solution of the ARE

$$(X^{(k)} - B)E^{(k)} + E^{(k)}N^{(k)} = (X^{(k)} - B)(N^{(k)} - X^{(k)}), \tag{5}$$

with

$$N^{(k)} = (X^{(k)} - B)^{-1}C. \tag{6}$$

These two methods, each has its own advantages and disadvantages. The successive approximation method is very simple and economical because at each iteration step it only needs to solve the systems of linear equations

$$(X^{(k)} - B)N^{(k)} = C$$

with respect to $N^{(k)}$; however, it only has linear convergence speed. And the Newton's method has quadratic convergence speed, however, it is comparatively complicated and costly because at each iteration step it needs to solve a nonlinear ARE(5), besides computing $N^{(k)}$ according to (6).

## 3. Local convergence theorems

In this section, we will establish local convergence theorems for both successive approximation method and Newton's method for solving the quadratic matrix equation (1). We first prove the local convergence of the successive approximation method.

**Theorem 3.1.** *Let $C \in \mathbb{C}^{n\times n}$ be a nonsingular matrix and $X_\star \in \mathbb{C}^{n\times n}$ be a solution of the QME(1) such that*

$$\|C\| \le c \quad \textit{and} \quad \|(X_\star - B)^{-1}\| \le \beta,$$

*where $c$ and $\beta$ are two positive constants. Assume that $X^{(0)} \in \mathbb{C}^{n\times n}$ and there exists a $\delta > 0$ such that $\|X^{(0)} - X_\star\| \leq \delta$. Then, if*

$$0 < \beta < \frac{\sqrt{\delta^2 + 4c} - \delta}{2c},$$

*the iterative sequence $\{X^{(k)}\}$ generated by the successive approximation method with $X^{(0)}$ as the initial guess satisfies*

$$\|X^{(k+1)} - X_\star\| \leq \gamma \|X^{(k)} - X_\star\|, \qquad k = 0, 1, 2, \ldots,$$

*where*

$$\gamma = \frac{\beta^2 c}{1 - \beta\delta} \in (0, 1).$$

*Proof.* From the definition of the sequence $\{X^{(k)}\}$ we obtain

$$\begin{aligned} X^{(k+1)} - X_\star &= (X^{(k)} - B)^{-1}C - (X_\star - B)^{-1}C \\ &= -(X^{(k)} - B)^{-1}(X^{(k)} - X_\star)(X_\star - B)^{-1}C. \end{aligned} \tag{7}$$

In addition, we easily have the equality

$$(X^{(k)} - B) - (X_\star - B) = X^{(k)} - X_\star. \tag{8}$$

Now, the proof can be proceeded by induction.

When $k = 0$, by (8) and the perturbation lemma in matrix analysis we can obtain

$$\|(X^{(0)} - B)^{-1}\| \leq \frac{\|(X_\star - B)^{-1}\|}{1 - \|(X_\star - B)^{-1}\|\|X^{(0)} - X_\star\|} \leq \frac{\beta}{1 - \beta\delta}.$$

It then follows from (7) that

$$\begin{aligned} \|X^{(1)} - X_\star\| &\leq \|(X^{(0)} - B)^{-1}\|\|(X_\star - B)^{-1}\|\|C\|\|X^{(0)} - X_\star\| \\ &\leq \frac{\beta^2 c}{1 - \beta\delta}\|X^{(0)} - X_\star\| \\ &:= \gamma\|X^{(0)} - X_\star\|. \end{aligned}$$

That is to say, the conclusion holds for $k = 0$. Moreover, the above estimate immediately yields that

$$\|X^{(1)} - X_\star\| \leq \delta.$$

Now, assume that

$$\|X^{(k)} - X_\star\| \leq \gamma\|X^{(k-1)} - X_\star\|.$$

Then it holds that

$$\|X^{(k)} - X_\star\| \leq \delta.$$

For $k$, again by (8) and the perturbation lemma in matrix analysis we can obtain

$$\|(X^{(k)} - B)^{-1}\| \leq \frac{\|(X_\star - B)^{-1}\|}{1 - \|(X_\star - B)^{-1}\|\|X^{(k)} - X_\star\|} \leq \frac{\beta}{1 - \beta\delta}.$$

It then follows from (7) again that

$$\begin{aligned} \|X^{(k+1)} - X_\star\| &\leq \|(X^{(k)} - B)^{-1}\|\|(X_\star - B)^{-1}\|\|C\|\|X^{(k)} - X_\star\| \\ &\leq \frac{\beta^2 c}{1 - \beta\delta}\|X^{(k)} - X_\star\| \\ &:= \gamma\|X^{(k)} - X_\star\|. \end{aligned}$$

That is to say, the conclusion holds for $k$, too. Moreover, the above estimate immediately yields that

$$\|X^{(k+1)} - X_\star\| \le \delta.$$

Therefore, by the induction principle, we have proved the conclusion. ∎

Theorem 3.1 shows that the iterative sequence $\{X^{(k)}\}$ generated by the successive approximation method converges linearly to a solution $X_\star$ of the QME(1), provided the initial guess $\{X^{(0)}\}$ is sufficiently close to $X_\star$.

We now turn to demonstrate the local convergence of the Newton's method for solving the QME(1). To this end, we first prove the following properties of the mappings $\mathcal{G}(X)$ with respect to $X$ and $\mathcal{J}(X, E)$ with respect to $E$.

**Lemma 3.1.** *Let $X_\star \in \mathbb{C}^{n\times n}$ be a solution of the QME(1) and $X$ be in a neighborhood of $X_\star$. The following properties hold for the mappings $\mathcal{G}(X)$ and $\mathcal{J}(X, E)$:*

(i) *$\mathcal{J}(X, E)$ is a linear mapping with respect to $E$;*
(ii) *$\mathcal{G}(X)$ is a smooth mapping and it holds that*

$$\|\mathcal{G}(X + E) - \mathcal{G}(X) - \mathcal{J}(X, E)\| \le \frac{1}{2}\left(1 + \|(X - B)^{-1}\|^2\|C\|\right)\|E\|^2.$$

*Proof.* The linearity of the mapping $\mathcal{J}(X, E)$ with respect to $E$ is evident. We now verify the validity of (ii). Obviously, $\mathcal{G}(X)$ is a smooth mapping. By making use of the mean-value theorem we obtain

$$\mathcal{G}(X + E) - \mathcal{G}(X) = \int_0^1 \mathcal{J}(X, tE)\, dt.$$

It then follows that

$$\begin{aligned}
\|\mathcal{G}(X + E) - \mathcal{G}(X) - \mathcal{J}(X, E)\| &= \left\|\int_0^1 \mathcal{J}(X, tE)\, dt - \mathcal{J}(X, E)\right\| \\
&\le \int_0^1 \|\mathcal{J}(X, tE) - \mathcal{J}(X, E)\|\, dt \\
&= \int_0^1 \|\mathcal{J}(X, (1 - t)E)\|\, dt \\
&= \int_0^1 \|tE + (X - B)^{-1} \cdot tE \cdot (X - B)^{-1}C\|\, dt \\
&\le \frac{1}{2}\left(1 + \|(X - B)^{-1}\|^2\|C\|\right)\|E\|^2,
\end{aligned}$$

here we have used the linearity of the mapping $\mathcal{J}(X, E)$ with respect to $E$. ∎

Now, we are ready to establish the local convergence theorem of the Newton's method for the QME(1).

**Theorem 3.2.** *Let $C \in \mathbb{C}^{n\times n}$ be a nonsingular matrix and $X_\star \in \mathbb{C}^{n\times n}$ be a solution of the QME(1) such that*

$$\|C\| \le c \quad \textit{and} \quad \|(X_\star - B)^{-1}\| \le \beta,$$

*where $c$ and $\beta$ are two positive constants. Assume that $X^{(0)} \in \mathbb{C}^{n\times n}$ and there exists a $\delta > 0$ such that $\|X^{(0)} - X_\star\| \le \delta$. Then, if*

$$\beta\delta < 1 \quad \textit{and} \quad \left(1 + \frac{(1 - \beta\delta)^2}{\beta^2 c}\right)\delta < 2,$$

*the iterative sequence* $\{X^{(k)}\}$ *generated by the Newton's method with* $X^{(0)}$ *as the initial guess satisfies*

$$\|X^{(k+1)} - X_\star\| \le \gamma \|X^{(k)} - X_\star\|^2, \qquad k = 0, 1, 2, \dots,$$

*where*

$$\gamma = \frac{1}{2}\left(1 + \frac{(1-\beta\delta)^2}{\beta^2 c}\right).$$

*Proof.* For the Newton sequence $\{X^{(k)}\}$ we have

$$X^{(k+1)} - X_\star = X^{(k)} - X_\star + E^{(k)}.$$

By the linearity of the mapping $\mathcal{J}(X, E)$ with respect to $E$ and the definition of the Newton sequence $\{X^{(k)}\}$, we can obtain

$$\begin{aligned}
\mathcal{J}(X^{(k)}, X^{(k+1)} - X_\star) &= \mathcal{J}(X^{(k)}, X^{(k)} - X_\star) + \mathcal{J}(X^{(k)}, E^{(k)}) \\
&= \mathcal{J}(X^{(k)}, X^{(k)} - X_\star) - \mathcal{G}(X^{(k)}) \\
&= \mathcal{G}(X_\star) - \mathcal{G}(X^{(k)}) - \mathcal{J}(X^{(k)}, X_\star - X^{(k)}).
\end{aligned}$$

It then follows from the estimate

$$\begin{aligned}
&\|\mathcal{J}(X^{(k)}, X^{(k+1)} - X_\star)\| \\
&= \left\|(X^{(k+1)} - X_\star) + (X^{(k)} - B)^{-1}(X^{(k+1)} - X_\star)(X^{(k)} - B)^{-1}C\right\| \\
&\ge \left|\|X^{(k+1)} - X_\star\| - \|(X^{(k)} - B)^{-1}(X^{(k+1)} - X_\star)(X^{(k)} - B)^{-1}C\|\right| \\
&\ge \left|1 - \|(X^{(k)} - B)^{-1}\|^2\|C\|\right| \|X^{(k+1)} - X_\star\|
\end{aligned}$$

and Lemma 3.1 (ii) we straightforwardly get

$$\|X^{(k+1)} - X_\star\| \le \frac{1}{2}\frac{1 + \|(X^{(k)} - B)^{-1}\|^2\|C\|}{|1 - \|(X^{(k)} - B)^{-1}\|^2\|C\||}\|X^{(k)} - X_\star\|^2. \tag{9}$$

Analogously to the proof of Theorem 3.1 we have

$$\|(X^{(k)} - B)^{-1}\| \le \frac{\|(X_\star - B)^{-1}\|}{1 - \|(X_\star - B)^{-1}\|\|X^{(k)} - X_\star\|}. \tag{10}$$

Under the assumptions of the theorem, by making use of the estimates (10) and (9) we know that

$$\|(X^{(0)} - B)^{-1}\| \le \frac{\beta}{1 - \beta\delta}$$

and

$$\begin{aligned}
\|X^{(1)} - X_\star\| &\le \frac{1}{2}\frac{1 + \|(X^{(0)} - B)^{-1}\|^2\|C\|}{|1 - \|(X^{(0)} - B)^{-1}\|^2\|C\||}\|X^{(0)} - X_\star\|^2 \\
&\le \frac{1}{2}\frac{1 + \beta^2 c/(1-\beta\delta)^2}{\beta^2 c/(1-\beta\delta)^2}\|X^{(0)} - X_\star\|^2 \\
&= \frac{1}{2}\left(1 + \frac{(1-\beta\delta)^2}{\beta^2 c}\right)\|X^{(0)} - X_\star\|^2 \\
&= \gamma\|X^{(0)} - X_\star\|^2.
\end{aligned}$$

That is to say, the conclusion what we are proving holds for $k = 0$.

Assume this conclusion be true for some positive integer $k - 1$. Then we have

$$\begin{aligned}
\|X^{(k)} - X_\star\| &\le \gamma\|X^{(k-1)} - X_\star\|^2 \le \gamma\delta\|X^{(k-1)} - X_\star\| \le \|X^{(k-1)} - X_\star\| \\
&\le \dots \le \|X^{(0)} - X_\star\| \le \delta.
\end{aligned}$$

By making use of the estimates (10) and (9) again we can obtain

$$\|(X^{(k)}-B)^{-1}\| \le \frac{\beta}{1-\beta\delta}$$

and

$$\begin{aligned}\|X^{(k+1)}-X_\star\| &\le \frac{1}{2}\frac{1+\|(X^{(k)}-B)^{-1}\|^2\|C\|}{|1-\|(X^{(k)}-B)^{-1}\|^2\|C\||}\|X^{(k)}-X_\star\|^2\\ &\le \frac{1}{2}\frac{1+\beta^2 c/(1-\beta\delta)^2}{\beta^2 c/(1-\beta\delta)^2}\|X^{(k)}-X_\star\|^2\\ &= \frac{1}{2}\left(1+\frac{(1-\beta\delta)^2}{\beta^2 c}\right)\|X^{(k)}-X_\star\|^2\\ &= \gamma\|X^{(k)}-X_\star\|^2.\end{aligned}$$

That is to say, the conclusion what we are proving holds for $k$, too. By induction principle, we have completed our proof. ∎

## 4. Numerical results

In the study of noisy Wiener-Hopf problems for Markov chain, we need to find, for a given diagonal matrix $V$ and a given positive number $\epsilon$, specific $Q$-matrices [1] $\Gamma_\pm$ satisfying

$$\frac{1}{2}\epsilon^2 Z^2 \mp VZ + Q = 0, \tag{11}$$

respectively. Here, $V$ has positive *and* negative diagonal elements[2] and $\epsilon$ is the level of noise from Brownian motion independent of the Markov chain. The solutions $\Gamma_\pm$ will be generators of two Markov chains. See [7, 10, 11] for more details. From the discussion in [5] we know that *one* of the equations in (11) does not necessarily have a unique $Q$-matrix solution, and $\Gamma_+$ (resp. $\Gamma_-$) is the unique singular $Q$-matrix solution when the "−" equation (resp. "+" equation) in (11) has no nonsingular $Q$-matrix solutions. Moreover, $\Gamma_+$ (resp. $\Gamma_-$) is the unique nonsingular $Q$-matrix solution when the "−" equation (resp. "+" equation) in (11) has singular and nonsingular $Q$-matrix solutions. If a Markov chain has a singular (nonsingular) $Q$-matrix as a generator, then the chain will live forever (die out).

We will apply our new successive approximation method and Newton's method to find the matrices $\Gamma_\pm$. As in [5] we will also limit our attention to the more difficult case that $Q$ is an irreducible *singular* $Q$-matrix. This is the case of primary interest in the study of noisy Wiener-Hopf problems. It means that the original Markov chain will live forever.

In the quadratic matrix equations in (11) we may assume $\epsilon = \sqrt{2}$ as we can always divide the equations in (11) by $\frac{\epsilon^2}{2}$. Thus, we only need to consider the quadratic matrix equations

$$Z^2 - VZ + Q = 0 \tag{12}$$

and

$$Z^2 + VZ + Q = 0. \tag{13}$$

To find the solution $\Gamma_+$ of (12), we let $X := Z$, $B := V$ and $C := -Q$. The solution $\Gamma_-$ of (13) can be found by taking $X := Z$, $B := -V$ and $C := -Q$.

[1] A $Q$-matrix has nonnegative off-diagonal elements and nonpositive row sums; $Q$ is the generator of an irreducible continuous-time finite Markov chain.

[2] This is essentially where the name Wiener-Hopf comes from.

**Example 4.1.** [5] *We consider the quadratic matrix equations (12) and (13) with*

$$V = \begin{pmatrix} aI_{10} & 0 \\ 0 & bI_{10} \end{pmatrix}, \qquad Q = \begin{pmatrix} -1 & 1 & & & \\ & -1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & -1 \end{pmatrix} \in \mathbb{R}^{20\times 20},$$

*where $a$ and $b$ are parameters to be specified. We consider four cases:*

(a) *$a = 1$, $b = -1$, so $\Gamma_\pm$ are both singular Q-matrices;*
(b) *$a = 2$, $b = -1$, so $\Gamma_+$ ($\Gamma_-$) is a singular (nonsingular) Q-matrices;*
(c) *$a = 2$, $b = -0.1$, so $\Gamma_+$ ($\Gamma_-$) is a singular (nonsingular) Q-matrices;*
(d) *$a = 1$, $b = -3$, so $\Gamma_+$ ($\Gamma_-$) is a nonsingular (singular) Q-matrices.*

For each case, the approximations $\widetilde{\Gamma}_\pm$ for $\Gamma_\pm$ are found by the successive approximation method (SA) and the Newton's method (NM) presented in this paper, the fixed-point iteration (FP) and the Newton's method (NM0) presented in [5]. We list the numerical results in Tables 1 and 2.

All results are obtained by using MATLAB 6.5 on a personal computer (Pentium IV/2.4G), with machine precision $2.2\times 10^{-16}$. In the tables, we use "IT" to denote the number of iteration steps, "RES" the errors defined by

$$\text{RES} := \|(\tilde{\Gamma}_\pm)^2 \mp V\tilde{\Gamma}_\pm + Q\|_\infty.$$

The stopping criterion for each iteration method is $\|X^{(k)} - X^{(k-1)}\|_\infty < 10^{-5}$, where $X^{(k)}$ is the current, say the $k$-th, iteration value.

From Tables 1 and 2, we see that the successive approximation method is better than the fixed-point iteration, and the Newton's method in this paper outperforms the Newton's method in [5], in the sense of iteration step and approximation accuracy. Therefore, our new methods are more accurate and effective than the known ones in [6, 5], correspondingly.

Table 1. Numerical Results for the Quadratic Matrix Equation (12)

| Method | IT | | | | RES | | | |
|---|---|---|---|---|---|---|---|---|
| | (a) | (b) | (c) | (d) | (a) | (b) | (c) | (d) |
| SA | 43 | 57 | 26 | 104 | 1.3E-5 | 1.1E-5 | 5.0E-6 | 2.6E-5 |
| FP | 62 | 72 | 40 | 116 | 1.4E-5 | 1.6E-5 | 1.1E-5 | 3.2E-5 |
| NW | 5 | 5 | 4 | 5 | 1.1E-15 | 1.6E-15 | 3.7E-13 | 5.6E-15 |
| NW0 | 6 | 6 | 5 | 6 | 2.6E-14 | 3.5E-14 | 1.0E-13 | 5.3E-14 |

Table 2. Numerical Results for the Quadratic Matrix Equation (13)

| Method | IT | | | | RES | | | |
|---|---|---|---|---|---|---|---|---|
| | (a) | (b) | (c) | (d) | (a) | (b) | (c) | (d) |
| SA | 43 | 67 | 43 | 66 | 1.3E-5 | 1.9E-5 | 1.3E-5 | 1.4E-5 |
| FP | 62 | 84 | 54 | 74 | 1.4E-5 | 2.3E-5 | 2.1E-5 | 1.5E-5 |
| NW | 5 | 5 | 5 | 4 | 1.7E-15 | 3.5E-15 | 3.1E-15 | 1.2E-13 |
| NW0 | 6 | 6 | 6 | 5 | 2.0E-14 | 3.3E-14 | 2.6E-14 | 8.0E-14 |

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