INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 2, Supp , Pages 83–99

ROBIN TRANSMISSION CONDITIONS FOR OVERLAPPING ADDITIVE SCHWARZ METHOD APPLIED TO LINEAR ELLIPTIC PROBLEMS

HONGWEI LI AND JIACHANG SUN

Abstract. We consider overlapping Additive Schwarz Method(ASM) with Robin conditions as the transmission conditions(interior boundary conditions). The main difficulty left in this field is how to select the parameters for Robin conditions – these parameters have strong effect on the convergence rate of ASM. In this paper, we proposed the parameters for linear elliptic problems which seemed to be near optimal.

Key Words. domain decomposition, additive Schwarz methods, Robin transmission conditions.

1. Introduction

Classical additive Schwarz method(ASM) converges very slow for general problems. So, in most circumstances, this method can only be used as a preconditioner. On the other hand, ASM has high parallelism and is very suitable for coarse grain parallel computing. Many recent papers contribut to accelerating ASM. The technique is to replace the Dirichlet transmission conditions posed on the interfaces with some more general or exact conditions such as absorbing conditions, open conditions etc. The essence of these conditions is that they are more *exact* on the interfaces so that the corresponding ASM should converge faster. However, these conditions are always global coupled. So, in actual applications, these conditions should be localized by some kind of approximations. Taylor expansion was first used, and some other approximations were also introduced[6]. But it seems that these approximations hold only for simple problems that Fourier analysis can apply.

In this paper, the Dirichlet transmission conditions of the classical overlapping additive Schwarz method are replaced by Robin conditions directly. We hope that by selecting proper parameters for the Robin conditions, the corresponding ASM would converge more rapidly.

Robin transmission conditions were first introduced into domain decomposition by P.L.Lions in [9, 10, 11]. Since then, many papers followed.

Generalized Schwarz splitting method with Robin transmission conditions was proposed by Tang [12], which gave the initial impetus to our work in this field. Optimized Schwarz methods, proposed by M.J. Gander, L.Halpern and F.Nataf, try to get the optimal Robin parameters by Fourier analysis [6]. This idea was further utilized in [1, 8, 5, 7, 4].

Received by the editors January 1, 2004 and, in revised form, March 22, 2004.
2000 *Mathematics Subject Classification.* 65Mxx, 65Nxx, 65Yxx, 68Wxx.
This research was supported by the Basis Research Foundation (CXK25073) of Institute of Software, Chinese Academy of Sciences.

Absorbing conditions for domain decomposition methods have been analyzed by Zhao[2]. In that paper, Robin transmission conditions were analyzed by Taylor expansion.

Though many authors and papers have talked about Robin transmission conditions for additive Schwarz methods, the main difficulty – lacking of a simple and uniform way to choose good Robin parameters, is still remaining, even if for simple problems like Laplace equation.

This paper is motivated by generalized Schwarz splittings proposed by W.P. Tang and optimized Schwarz methods proposed by M.J. Gander. And we try to determine the optimal (or near optimal) Robin parameters for general linear elliptic problems.

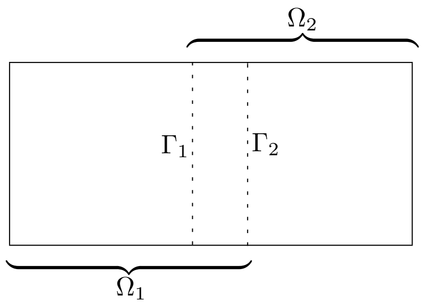
The key model problem for this paper is

$$-\Delta u + qu = f \quad (\Omega) \tag{1}$$

$$u = g \quad (\partial\Omega) \tag{2}$$

where $\Omega = (0,1)^d$, $d = 2, 3$, $q > 0$.

Suppose domain Ω is partitioned into two overlapping subdomains Ω_1 and Ω_2



Our aim is to derive the optimal(or near optimal) Robin parameters λ for the following additive Schwarz method (two subdomain case)

For any given initial values u^0, v^0, solve the following problems iteratively until convergence

$$-\Delta u^n + qu^n = f, \quad (\Omega_1) \tag{3}$$

$$\frac{\partial u^n}{\partial n} + \lambda u^n = \frac{\partial v^{n-1}}{\partial n} + \lambda v^{n-1} \quad (\Gamma_2) \tag{4}$$

$$-\Delta v^n + qv^n = f, \quad (\Omega_2) \tag{5}$$

$$\frac{\partial v^n}{\partial n} + \lambda v^n = \frac{\partial u^{n-1}}{\partial n} + \lambda u^{n-1} \quad (\Gamma_1) \tag{6}$$

where n denotes the outward normal direction of the subdomain under consideration. We will call above method as RASM(λ), so that it can be distinguished from ASM.

The main result of this paper is that for high dimensional model problems, the optimal Robin parameters can be determined as $\lambda_{opt} = \sqrt{q + (d-1)\pi^2}$, $d = 2, 3$.

2. Analysis for one dimensional Laplace equation

Suppose the domain $\Omega = (0,1)$, $0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < ... < \alpha_{ns-1} < \beta_{ns-1} < 1$. $\Omega_1 = (0, \beta_1)$, $\Omega_2 = (\alpha_1, \beta_2)$, ..., $\Omega_{ns-1} = (\alpha_{ns-2}, \beta_{ns-1})$, $\Omega_{ns} = (\alpha_{ns-1}, 1)$.

FIGURE 1. Domain Ω is decomposed into 4 subdomains

The model problem for this section is

$$-\frac{d^2u}{d^2x} = f(x), \quad x \in \Omega, \quad u(0) = u(1) = 0 \tag{7}$$

We know that the exact transmission conditions can be expressed as Steklov-Poincaré operators which depend on the interior boundaries. So the transmission conditions should be different on different interior boundaries. Therefore, when being applied to multi-subdomains, RASM(λ) should take the following form

$$\begin{aligned} -\frac{d^2u_1^{n+1}}{d^2x} &= f(x), \quad x \in \Omega_1, \\ u_1^{n+1}(0) &= 0 \\ \frac{du_1^{n+1}(\beta_1)}{dx} + \tilde{\lambda}_1 u_1^{n+1}(\beta_1) &= \frac{du_2^n(\beta_1)}{dx} + \tilde{\lambda}_1 u_2^n(\beta_1) \end{aligned}$$

$$\begin{aligned} -\frac{d^2u_i^{n+1}}{d^2x} &= f(x), \quad x \in \Omega_i, \\ \frac{du_i^{n+1}(\alpha_{i-1})}{dx} + \lambda_{i-1}u_i^{n+1}(\alpha_{i-1}) &= -\frac{du_{i-1}^n(\alpha_{i-1})}{dx} + \lambda_{i-1}u_{i-1}^n(\alpha_{i-1}) \\ \frac{du_i^{n+1}(\beta_i)}{dx} + \tilde{\lambda}_i u_i^{n+1}(\beta_i) &= \frac{du_{i+1}^n(\beta_i)}{dx} + \tilde{\lambda}_i u_{i+1}^n(\beta_i) \\ i &= 2, 3, \dots, ns-1. \end{aligned}$$

$$\begin{aligned} -\frac{d^2u_{ns}^{n+1}}{d^2x} &= f(x), \quad x \in \Omega_{ns}, \\ u_{ns}^{n+1}(1) &= 0 \\ -\frac{du_{ns}^{n+1}(\alpha_{ns-1})}{dx} + \lambda_{ns-1}u_{ns}^{n+1}(\alpha_{ns-1}) &= -\frac{du_{ns-1}^n(\alpha_{ns-1})}{dx} + \lambda_{ns-1}u_{ns-1}^n(\alpha_{ns-1}) \end{aligned}$$

Notice that the Robin parameters can be different on different interior boundaries.

Theorem 2.1. *Let* $\lambda_i = \frac{1}{\alpha_i}$, $\tilde{\lambda}_i = \frac{1}{1-\beta_i}$, *i=1,2,..., ns-1. then the above method converges in ns iterations.*

Proof. It suffices to give the proof in the case of $f(x) \equiv 0$. So we can suppose

$$u_i^{n+1} = C_i^{n+1}x + d_i^{n+1}, \quad i = 1, 2, \dots, ns.$$

Using the transmission conditions and the corresponding parameters $\lambda_i = \frac{1}{\alpha_i}, \quad \widetilde{\lambda}_i = \frac{1}{1-\beta_i}$, $i = 1, 2, \ldots, ns$, , we have

$$\begin{cases} d_1^{n+1} = 0 \\ c_1^{n+1} + d_1^{n+1} = c_2^n + d_2^n \end{cases}$$
$$\begin{cases} d_2^{n+1} = d_1^n \\ c_2^{n+1} + d_2^{n+1} = c_3^n + d_3^n \end{cases}$$
$$\begin{cases} d_3^{n+1} = d_2^n \\ c_3^{n+1} + d_3^{n+1} = c_4^n + d_4^n \end{cases}$$
$$\vdots$$
$$\begin{cases} d_{ns}^{n+1} = d_{ns-1}^n \\ c_{ns}^{n+1} + d_{ns}^{n+1} = 0 \end{cases}$$

Now we need only to verify that for any given initial values c_i^0, d_i^0, we will have $c_i^{ns} = 0$, $d_i^{ns} = 0$, $i = 1, 2, \ldots, ns$. According to above formulas, for any i, we have

$$d_i^n = d_{i-1}^{n-1} = d_{i-2}^{n-2} = \cdots = d_1^{n-i+1}$$

if $n \geq i$, then $d_1^{n-i+1} = 0$, so $d_i^n = 0$. Obviously, ns is the minimum number that meets $n \geq i$ for any i. So

$$d_i^{ns} = 0, \quad i = 1, 2, \ldots, ns.$$

Secondly, let $e_i^n = c_i^n + d_i^n$, $i = 1, 2, \ldots, ns$. It's easy to verify that

$$e_i^{ns} = 0, \quad i = 1, 2, \ldots, ns.$$

Therefore, because of $d_i^{ns} = 0$, we have

$$c_i^{ns} = 0, \quad i = 1, 2, \ldots, ns$$

□

Suppose domain $\Omega = (0, 1)$, $\Omega_1 = (0, \beta_1)$, $\Omega_i = (\alpha_{i-1}, \beta_i), i = 2, \ldots, ns-1$, $\Omega_{ns} = (\alpha_{ns-1}, 1)$. $n = ns \times m$, $h = 1/(n+2)$, $\alpha_i = i \times mh$, $\beta_i = i \times (m+2)h$, let $\gamma_i = i \times (m+1)h$, i=1,ns-1.

For one dimensional problems, if no specification, the domain Ω will always take this kind of decomposition in this paper.

Numerical experiments 2.1. *Initial zero interior boundary values, central difference scheme. Right-hand term: $f(x) = 2x(1-x)$, subdomain solver: CG. Convergence criterion: $\|r^n\|/\|r^0\| \leq 10^{-5}$. The optimal Robin parameters are determined by Theorem (2.1). The results are showed in Table 1 and Table 2*

	Iter. time(s)		Iter. num.	
m	ASM	RASM(λ_{opt})	ASM	RASM(λ_{opt})
9	0.09	0.	217	4
19	0.64	0.	450	4
99	63.67	0.11	2405	4

TABLE 1. One dimensional, 4 subdomains. Comparison of iteration time and iteration number

It should be pointed out that, in *Theorem* 2.1, ns is the minimum iterations for the method to converge. The iteration number depends only on the domain size,

	Iter. time(s)		Iter. num	
m	ASM	RASM(λ_{opt})	ASM	RASM(λ_{opt})
9	0.62	0.01	803	8
19	4.84	0.02	1702	8
99	491.4	0.44	9240	8

TABLE 2. One dimensional, 8 subdomains. Comparison of iteration time and iteration number

e.g. the number of subdomains. Furthermore, the optimal parameters λ_i, $\widetilde{\lambda}_i$ are not unique. In fact, we can confine ourself to take same Robin parameters on every interior boundary pair $\{\alpha_i, \beta_i\}$. In this case, we still can find a group of parameters which satisfy that RASM(λ) converges in ns iterations.

Theorem 2.2. *Let* $\lambda_i = \widetilde{\lambda}_i = \begin{cases} \dfrac{1}{\alpha_i}, i \leq ns/2+1 \\ \dfrac{1}{1-\beta_i}, i \geq ns/2+1 \end{cases}$ *RASM(λ) converges in ns iterations.*

3. Coercive Laplace equation

In order to analyze high dimensional problems, we need to study the following Coercive Laplace equation first

$$\begin{array}{l} -u'' + qu = f(x), \quad x \in \Omega = (0,1), \quad q > 0 \\ u(0) = 0, \quad u(1) = 0 \end{array} \tag{8}$$

For simplicity and concision, the two subdomain case will be taken for instance. Suppose ns=2, $\Omega_1 = (0, \beta_1)$, $\Omega_2 = (\alpha_1, 1)$. Applying RASM(λ) to problem (8), we have

$$-\frac{d^2 v^{n+1}}{d^2 x} + q\ v^{n+1} = f(x), \quad x \in \Omega_1, \tag{9}$$

$$v^{n+1}(0) = 0, \quad \frac{dv^{n+1}(\beta_1)}{dx} + \lambda v^{n+1}(\beta_1) = \frac{dw^n(\beta_1)}{dx} + \lambda w^n(\beta_1) \tag{10}$$

$$-\frac{d^2 w^{n+1}}{d^2 x} + q w^{n+1} = f(x), \quad x \in \Omega_2, \tag{11}$$

$$w^{n+1}(1) = 0, \quad -\frac{dw^{n+1}(\alpha_1)}{dx} + \lambda w^{n+1}(\alpha_1) = -\frac{dv^n(\alpha_1)}{dx} + \lambda v^n(\alpha_1) \tag{12}$$

In this section, the above method will be analyzed in discrete form, and the main means is matrix analysis. The continuous problems are approximated by their discrete forms. Then the optimal Robin parameters will be determined in discrete form.

For continuous problems, the optimal Robin parameters depend on two factors: the problem itself and the pattern of domain decomposition. Therefore, if the corresponding discrete scheme approximates the continuous problem quite exactly, then the optimal parameters derived from matrix analysis should be good approximations to those for continuous case. In this way, the optimal Robin parameters for certain discrete scheme obtained by matrix computations can be applied to other discrete methods. Here, the central difference scheme is used to discrete the second order term in (9)–(12).

The discrete form of problem (9)–(12) can be thought of as block Jacobi iteration method for the following linear algebraic equations

$$\widetilde{A}u = g \tag{13}$$

$$\widetilde{A} = \begin{pmatrix} A_1 & -E \\ -F & A_2 \end{pmatrix} \quad u = \begin{pmatrix} v \\ w \end{pmatrix} \quad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

$$g_i = h^2 f|_{\Omega_i}, i = 1, 2; v = u|_{\Omega_1}, w = u|_{\Omega_2}$$

$$A_1 = \begin{pmatrix} 2+\beta & -1 & & \\ -1 & 2+\beta & -1 & \\ & \ddots & & \\ & -1 & 2+\beta-\sigma & \end{pmatrix}_{m+1,m+1} \qquad E = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ -\sigma & 1 & 0 & \cdots & 0 \end{pmatrix}_{m+1,m+1}$$

$$A_2 = \begin{pmatrix} 2+\beta-\sigma & -1 & & \\ -1 & 2+\beta & -1 & \\ & \ddots & & \\ & -1 & 2+\beta & \end{pmatrix}_{m+1,m+1} \qquad F = \begin{pmatrix} 0 & \cdots & 0 & 1 & -\sigma \\ 0 & \cdots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}_{m+1,m+1}$$

where $\beta = 2 + qh^2$. And by simple calculations, we have

$$\sigma = \frac{1}{1 + \lambda h} \tag{14}$$

Hereafter, the above method will be called DRASM(σ), which is the discrete counterpart of RASM(λ). Notice that, DRASM(0) corresponds to the classical additive Schwarz method, which takes Dirichlet conditions as the transmission conditions.

Now the problem is how to select the parameter σ, so that DRASM(σ) converges as fast as possible. It is well known that, for any iteration method, the convergence rate is determined by the spectrum radius of the iteration matrix, more small the spectrum radius, more rapid the convergence speed.

when DRASM(σ) is applied to problem (13), The iteration matrix is

$$J = \begin{pmatrix} A_1^{-1} & \\ & A_2^{-1} \end{pmatrix} \begin{pmatrix} & E \\ F & \end{pmatrix} = \begin{pmatrix} & A_1^{-1}E \\ A_2^{-1}F & \end{pmatrix}$$

Now we need to calculate the spectrum radius of matrix J, e.g. the maximum absolute eigenvalue of J

Define $T_1 = A_1^{-1}E$, $T_2 = A_2^{-1}F$. Suppose

$$A_1^{-1} = (t_{ij})_{m+1,m+1} = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1,m+1} \\ t_{21} & t_{22} & \cdots & t_{2,m+1} \\ \vdots & \vdots & & \vdots \\ t_{m+1,1} & t_{m+1,2} & \cdots & t_{m+1,m+1} \end{pmatrix}$$

then by the property of algebraic complement and Laplace expansion, we have

$$A_2^{-1} = \begin{pmatrix} t_{m+1,m+1} & \star & \cdots & \star \\ t_{m,m+1} & \star & \cdots & \star \\ \vdots & \vdots & & \vdots \\ t_{m+1,1} & \star & \cdots & \star \end{pmatrix}$$

By some simple calculations, we have

$$T_1 = \begin{pmatrix} -\sigma t_{1,m+1} & t_{1,m+1} & 0 & \cdots & 0 \\ -\sigma t_{2,m+1} & t_{2,m+1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -\sigma t_{m+1,m+1} & t_{m+1,m+1} & 0 & \cdots & 0 \end{pmatrix}_{m+1,m+1}$$

$$T_2 = \begin{pmatrix} 0 & \cdots & 0 & t_{m+1,m+1} & -\sigma t_{m+1,m+1} \\ 0 & \cdots & 0 & t_{m,m+1} & -\sigma t_{m,m+1} \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & t_{1,m+1} & -\sigma t_{1,m+1} \end{pmatrix}_{m+1,m+1}$$

By some simple matrix transformations of J, we see that the nonzero eigenvalues of J are included in the eigenvalues of the following matrix

$$G = \begin{pmatrix} 0 & 0 & -\sigma t_{m,m+1} & t_{m,m+1} \\ 0 & 0 & -\sigma t_{m+1,m+1} & t_{m+1,m+1} \\ t_{m+1,m+1} & -\sigma t_{m+1,m+1} & 0 & 0 \\ t_{m,m+1} & -\sigma t_{m,m+1} & 0 & 0 \end{pmatrix}$$

However the nonzero eigenvalues of G can be easily derived as

$$\lambda_{1,2} = \pm(t_{m,m+1} - \sigma t_{m+1,m+1})$$

So, the spectrum radius of J is

$$\rho(J) = |t_{m,m+1} - \sigma t_{m+1,m+1}| \tag{15}$$

Apparently, the optimal Robin parameter σ is

$$\sigma = \frac{t_{m,m+1}}{t_{m+1,m+1}} \tag{16}$$

In order to figure out $t_{m,m+1}$ and $t_{m+1,m+1}$, the following Lemma 3.1 and Lemma 3.2 are needed

Lemma 3.1. [3] *Let* $\beta \geq 2$, *and*

$$T_n = \begin{pmatrix} \beta & -1 & & \\ -1 & \beta & -1 & \\ & \ddots & & \\ & -1 & \beta & \end{pmatrix}$$

$D_n(\beta) = det(T_n)$. *Then*

$$D_n(\beta) = \begin{cases} \sinh(n+1)\theta/\sinh\theta, & \beta > 2, 2\cosh\theta = \beta \\ n+1, & \beta = 2 \end{cases} \tag{17}$$

Moreover, if let $t_n^{-1} = (t_{ij})_{n\times n}$, *then*

$$t_{ij} = \begin{cases} D_{j-1}(\beta)D_{n-i}(\beta)/D_n(\beta), & i \geq j \\ D_{i-1}(\beta)D_{n-j}(\beta)/D_n(\beta), & i < j \end{cases} \tag{18}$$

Lemma 3.2. *Let*

$$A = \begin{pmatrix} \beta & -1 & & \\ -1 & \beta & -1 & \\ & \ddots & & \\ & -1 & \beta & \end{pmatrix}_{n\times n}, \quad B = \begin{pmatrix} \beta & -1 & & \\ -1 & \beta & -1 & \\ & \ddots & & \\ & -1 & \beta-\sigma & \end{pmatrix}_{n\times n}$$

and let $A^{-1} = (t_{ij})_{n\times n}$, $B^{-1} = (f_{ij})_{n\times n}$, $D_0(\beta) = 1$, $D_n(\beta) = det(A)$, $F_n = det(B)$. *Then*

$$F_n = D_n(\beta) - \sigma D_{n-1}(\beta) \tag{19}$$

$$f_{in} = \frac{D_{i-1}(\beta)}{D_n(\beta) - \sigma D_{n-1}(\beta)} \tag{20}$$

Proof. By the theorem of Laplace expansion, expand $D_n(\beta)$ and F_n according to their last rows

$$D_n(\beta) = \alpha + 2D_{n-1}(\beta)$$
$$F_n = \alpha + (2-\sigma)D_{n-1}(\beta)$$

where α is some certain algebraic complement. So

$$F_n - D_n(\beta) = -\sigma D_{n-1}(\beta)$$

Therefore

$$F_n = D_n(\beta) - \sigma D_{n-1}(\beta)$$

Besides, if $A^\star$, $B^\star$ are the adjoint matrixes of A and B respectively, then by the property of adjoint matrix, we have

$$AA^\star = det(A)I, \quad BB^\star = det(B)I$$

And by the definitions of adjoint matrix and the matrix A and B, the last columns of $A^\star$ and $B^\star$ should have no difference at all.

Because of $A^{-1} = \frac{1}{det(A)}A^\star$, and A^{-1} can be determined by Lemma 3.1

$$t_{ij} = \begin{cases} D_{j-1}(\beta)D_{n-i}(\beta)/D_n(\beta), & i \geq j \\ D_{i-1}(\beta)D_{n-j}(\beta)/D_n(\beta), & i < j \end{cases}$$

Especially, let $j = n$, we have

$$t_{in} = \frac{D_{i-1}(\beta)}{D_n(\beta)}$$

Therefore

$$\begin{aligned} & det(A)t_{in} = det(B)f_{in}, \quad i = 1, 2, \ldots, n. \\ & \Rightarrow D_n(\beta)t_{in} = F_n f_{in} \\ & \Rightarrow f_{in} = \frac{D_n(\beta)t_{in}}{F_n} \\ & \Rightarrow f_{in} = \frac{D_{i-1}(\beta)}{D_n(\beta) - \sigma D_{n-1}(\beta)} \end{aligned}$$

□

Theorem 3.1. *The optimal Robin parameter for our model problem is*

$$\sigma = \sinh(m\theta)/\sinh(m+1)\theta, \tag{21}$$

where θ *satisfies*

$$2\cosh\theta = \beta, \quad \beta = 2 + qh^2. \tag{22}$$

and $DRASM(\sigma)$ *converges in two iterations.*

Proof. By (16), the optimal Robin parameter $\sigma = t_{m,m+1}/t_{m+1,m+1}$. Moreover, by (15), the spectrum radius of the iteration matrix corresponding to $DRASM(\sigma)$ equals zero in this case. So $DRASM(\sigma)$ converges in two iterations. And then we need only to verify the following formula

$$\frac{t_{m,m+1}}{t_{m+1,m+1}} = \frac{\sinh(m\theta)}{\sinh(m+1)\theta}$$

By *Lemma* 3.2

$$t_{m,m+1} = \frac{D_{m-1}(\beta)}{D_{m+1}(\beta) - \sigma D_m(\beta)} \tag{23}$$

$$t_{m+1,m+1} = \frac{D_m(\beta)}{D_{m+1}(\beta) - \sigma D_m(\beta)} \tag{24}$$

Therefore

$$\sigma = t_{m,m+1}/t_{m+1,m+1} = D_{m-1}(\beta)/D_m(\beta)$$

By *Lemma* 3.1, we have

$$\frac{D_m(\beta)}{D_{m+1}(\beta)} = \frac{\sinh(m\theta)}{\sinh(m+1)\theta}$$

□

We can express the optimal Robin parameter more directly. By (22)

$$\begin{aligned} &2\cosh\theta = \beta \\ &\Rightarrow e^{\theta} + e^{-\theta} = \beta \\ &\Rightarrow e^{\theta} = \frac{\beta + \sqrt{\beta^2 - 4}}{2} \end{aligned}$$

On the other hand,

$$\sigma = \frac{\sinh(m\theta)}{\sinh(m+1)\theta} = \frac{e^{m\theta} - e^{-m\theta}}{e^{(m+1)\theta} - e^{-(m+1)\theta}} = e^{\theta}\frac{(e^{\theta})^{2m} - 1}{(e^{\theta})^{2(m+1)} - 1} \tag{25}$$

As $\beta > 2$, $e^{\theta} > 1$. So, if m is a relative large natural number, then $(e^{\theta})^{2m} \gg 1$. Therefore

$$\sigma \approx e^{-\theta} = \frac{2}{\beta + \sqrt{\beta^2 - 4}} = \frac{1}{1 + \frac{qh + \sqrt{q^2h^2 + 4q}}{2}h} \tag{26}$$

However, by (14), $\sigma = \dfrac{1}{1 + \lambda h}$, so

$$\lambda = \frac{qh + \sqrt{4q + q^2h^2}}{2} \tag{27}$$

If $h \to 0$, the discrete scheme should approximate the corresponding continuous problem better and better, so λ should approximate the optimal Robin parameter for the continuous problem better and better. Therefore, we have reason to think that the optimal Robin parameter for the continuous problem should be

$$\lambda_{opt} = \lim_{h \to 0} \frac{qh + \sqrt{4q + q^2h^2}}{2} = \sqrt{q} \tag{28}$$

and the optimal Robin parameter for the corresponding discrete problem should be

$$\sigma_{opt} = \frac{1}{1 + \lambda_{opt}h} \tag{29}$$

Numerical experiments 3.1. *Initial zero interior boundary values, central difference scheme. Right-hand term $f(x) = 2x(1-x)$, q=10, subdomain solver: CG. Convergence criterion: $\|r^n\|/\|r^0\| \le 10^{-5}$. The optimal Robin parameter is determined by (28) and (29). Table 3 shows the results.*

	Iter.time(s)		Iter.num	
m	ASM	DRASM(σ_{opt})	ASM	DRASM(σ_{opt})
9	0.01	0.	38	6
19	0.05	0.	76	6
99	4.53	0.06	401	6

TABLE 3. One dimensional, two subdomains. Comparison of iteration time and iteration steps. q = 10

Notation 3.1. *By (26), if $q > 0$ and the subdomain size m is relatively large, then the optimal Robin parameter σ_{opt} or λ_{opt} can be thought of as no coupling with the relative position of the interior boundaries. Based on this observation, for multi-subdomain problems, we can take the same Robin parameters on all the interior boundaries.*

Numerical experiments 3.2. *Initial zero interior boundary values, central difference scheme. Right-hand term: $f(x) = 2x(1-x)$, q=100, subdomain solver: CG. Convergence criterion: $\|r^n\|/\|r^0\| \leq 10^{-5}$. The optimal Robin parameter is determined by (28) and (29). Table 4 shows the results.*

	Iter.time(s)		Iter.num	
m	ASM	DRASM(σ_{opt})	ASM	DRASM(σ_{opt})
9	0.05	0.01	81	10
19	0.39	0.03	171	11
99	42.16	0.5	927	11

TABLE 4. One dimensional, 8 subdomains. Comparison of iteration time and iteration steps, q = 100

Notation 3.2. *We will analyze high dimensional problems in the following sections, and the optimal Robin parameters for high dimensional problems will be reduced to a series of one dimensional problems just like the model problem in this section, which has zero order term. So, for high dimensional problems, if no specification, when reduced to one dimensional multi-subdomain problems, we always take the same Robin parameters on all the interior boundaries.*

In order to quantify the effects of q on the spectrum radius of the iteration matrix, (15) needs to be analyzed further. By (23), (24) and (15) (substituting $f_{m,m+1}$ and $f_{m+1,m+1}$ for $t_{m,m+1}$ and $t_{m+1,m+1}$ respectively), we have

$$\rho(J) = \left| \frac{D_{m-1}(\beta) - \sigma D_m(\beta)}{D_{m+1}(\beta) - \sigma D_m(\beta)} \right| \tag{30}$$

By (3.1)

$$\begin{aligned} \rho(J) &= \left| \frac{\sinh(m\theta) - \sigma \sinh(m+1)\theta}{\sinh(m+2)\theta - \sigma \sinh(m+1)\theta} \right| \\ &= \left| \frac{\dfrac{\sinh(m\theta)}{\sinh(m+1)\theta} - \sigma}{\dfrac{\sinh(m\theta)}{\sinh(m+1)\theta} - \sigma + \dfrac{\sinh(m+2)\theta - \sinh(m\theta)}{\sinh(m+1)\theta}} \right| = \left| \frac{\eta_\sigma}{\eta_\sigma + \tau} \right| \end{aligned}$$

where θ satisfies $2\cosh\theta = \beta$, $\beta = 2 + qh^2$, and $\eta_\sigma = \dfrac{\sinh(m\theta)}{\sinh(m+1)\theta} - \sigma$, $\tau = \dfrac{2\sinh\theta\cosh(m+1)\theta}{\sinh(m+1)\theta} = 2\sinh\theta\coth(m+1)\theta > 2\sinh\theta > 0$

It's clear that, if q gets larger, then β and θ get larger, so τ larger. That's to say that, the sensitivity of $\rho(J)$ on σ will decrease as q gets larger.

Notation 3.3. *High dimensional problems can be reduced to a series of one dimensional problems just like this section, which have zero order terms. So, for high dimensional problems, we can consider only the reduced one dimensional problem which has the minimum coefficient for the zero order term.*

4. Two dimensional problem

We borrow the idea in [12] to reduce high dimensional problems to lower ones. Consider the model problem

$$\begin{aligned} &-\Delta u(x,y) + qu(x,y) = f(x,y), \quad (x,y) \in \Omega = (0,1)\times(0,1) \\ &u(x,y)|_{\partial\Omega} = g(x,y) \end{aligned} \tag{31}$$

where $q \geq 0$.

We take the following pattern of domain decomposition and grid partition.

$\Omega = (0,1)\times(0,1)$, $\Omega_1 = (0,\beta_1)\times(0,1)$, $\Omega_i = (\alpha_{i-1},\beta_i)\times(0,1), i = 2,\dots,ns-1$, $\Omega_{ns} = (\alpha_{ns-1},1)\times(0,1)$. $n = ns\times m$, $h = 1/(n+2)$, $\alpha_i = i\times mh$, $\beta_i = i\times(m+2)h$. Let $\gamma_i = i\times(m+1)h$, $i = 1, ns-1$. For two dimensional model problems, we always take this kind of domain decomposition and grid partition if no specification.

Definition 4.1.

$$T_n(\beta) \triangleq Tridiagonal\{-1,\beta,-1\}_{n\times n}, \quad (\beta \geq 2)$$

and Denote $T_n(x_1,x_2,x_3)$ as $T_n(x_2)$ except the first diagonal element is x_1, and the last is x_3.

Consider the two-subdomain case. when DRASM(σ) is applied to (31), the coefficient matrix is

$$\widetilde{A} = \begin{pmatrix} A_1 & -E_1 \\ -F_1 & A_2 \end{pmatrix}$$

where

$$A_1 = T_1 \otimes I_n + I_m \otimes T_n(2)$$

$$A_2 = T_2 \otimes I_n + I_m \otimes T_n(2)$$

$$\begin{aligned} E_1 &= E \otimes I_n \\ F_1 &= F \otimes I_n \end{aligned}$$

and

$$\begin{aligned} T_1 &= T_{m+1}(\beta,\beta,\beta-\sigma) \\ T_2 &= T_{m+1}(\beta-\sigma,\beta,\beta) \end{aligned}$$

E, F defined as before. $\beta = 2 + qh^2$.

The iteration matrix for DRASM(σ) is

$$J = \begin{pmatrix} A_1^{-1} & \\ & A_2^{-1} \end{pmatrix} \begin{pmatrix} & E_1 \\ F_1 & \end{pmatrix} \triangleq M^{-1}N$$

It's well known that $T_n(2)$ has the following spectrum decomposition

$$X_n T_n(2) X_n^T = D_n = diag\{d_i\}, \quad d_i = 4\sin^2 \frac{i\pi}{2(n+1)}, \quad i = 1, \ldots, n. \tag{32}$$

Let

$$U = \begin{pmatrix} I_m \otimes X_n & \\ & I_m \otimes X_n \end{pmatrix}$$

then

$$J' = UJU^T = (UMU^T)^{-1}N = \widetilde{M}^{-1}N$$

where

$$\widetilde{M} = \begin{pmatrix} \widetilde{A_1} & \\ & \widetilde{A_2} \end{pmatrix} \quad \widetilde{A_i} = (I_m \otimes X_n)A_i(I_m \otimes X_n)^T = T_i \otimes I_n + I_m \otimes D_n$$

Let P denotes the permutation matrix that makes the rows $(k-1)n+i$ and rows $2(i-1)(m+1)+k$ permute their positions for each other, $k = 1, \ldots, 2(m+1), i = 1, \ldots, n$. Then

$$J_1 = PJ'P^T = \begin{pmatrix} J(d_1) & & & \\ & J(d_2) & & \\ & & \ddots & \\ & & & J(d_n) \end{pmatrix}$$

where $J(d_i)$ are the iteration matrices when DRASM(σ) is applied to the following one dimensional problems

$$\begin{array}{l} -u'' + (q + d_i h^{-2})u = f, \quad (0,1) \\ u(0) = u(1) = 0 \end{array}$$

However, According to *Note* (3.3), we need only consider the one dimensional problem with minimum zero order coefficient. So we need only analyze the optimal Robin parameter for $J(d_1)$. That's to say,

$$\lambda_{opt} = \sqrt{q + d_1 h^{-2}}, \quad \sigma_{opt} = \frac{1}{1 + \lambda_{opt} h} \tag{33}$$

On the other hand,

$$d_1 = 4\sin^2 \frac{\pi}{2(n+1)} = 4\sin^2 \frac{\pi h}{2} \approx \pi^2 h^2$$

so

$$\lambda_{opt} = \sqrt{q + \pi^2}, \quad \sigma_{opt} = \frac{1}{1 + \lambda_{opt} h} \tag{34}$$

This λ_{opt} will be taken as the optimal Robin parameter for the two dimensional model problem. And numerical experiments showed that λ_{opt} is near optimal.

Numerical experiments 4.1. *Initial zero interior boundary values, central difference scheme. Right-hand term: $f(x,y) = x(1-x) + y(1-y)$, $q = 0$, subdomain solver: CG. Convergence criterion: $\|r^n\|/\|r^0\| \le 10^{-5}$. The Robin parameter is determined by (34). The results are showed in Table 5 and Table 6.*

The above strip domain decomposition pattern can be generalized to multi-direction decomposition. The optimal Robin parameter can be generalized to this situation in a simple and straightforward way, e.g. ignoring the coupling among the different directions, and the optimal Robin parameter is still

$$\lambda_{opt} = \sqrt{q + \pi^2}, \quad \sigma_{opt} = \frac{1}{1 + \lambda_{opt} h} \tag{35}$$

	Iter.time(s)		Iter.num	
m	ASM	DRASM(σ_{opt})	ASM	DRASM(σ_{opt})
9	0.19	0.05	38	9
19	3.35	0.7	76	12
49	95.92	9.29	196	19

Table 5. Two dimensional, two subdomains, comparison of iteration time and iteration numbers. Strip domain decomposition

	Iter.time(s)		Iter.num	
m	ASM	DRASM(σ_{opt})	ASM	DRASM(σ_{opt})
9	2.55	0.31	113	13
19	45.22	3.54	234	15
29	237.96	13.79	358	23

Table 6. Two dimensional, 4 subdomains, comparison of iteration time and iteration numbers. Strip domain decomposition

Numerical experiments 4.2. *Initial zero interior boundary values, central difference scheme. Right-hand term* $f(x, y) = x(1-x) + y(1-y)$, $q = 0$, *subdomain solver: CG. Convergence criterion:* $\|r^n\|/\|r^0\| \leq 10^{-5}$. *The Robin parameter is determined by (34). The results are showed in Table 7 and Table 8*

	Iter.time(s)		Iter.num	
m	ASM	DRASM(σ_{opt})	ASM	DRASM(σ_{opt})
9	0.68	0.17	63	14
19	9,98	1.54	129	18
29	39.07	3.8	195	21

Table 7. Two dimensional, 4 subdomains. Comparison of iteration time and iteration numbers. Domain decomposition: 2×2

	Iter.time(s)		Iter.num	
m	ASM	DRASM(σ_{opt})	ASM	DRASM(σ_{opt})
9	3.44	0.62	126	18
19	47.75	5.08	258	23
29	196.35	12.68	393	27

Table 8. Two dimensional, 9 subdomains. Comparison of iteration time and iteration numbers. Domain decomposition: 3×3

5. Three dimensional problem

Consider the following model problem

$$\begin{aligned} &-\Delta u(x,y,z) + qu(x,y,z) = f(x,y,z),\ (x,y,z) \in (0,1)^3 = \Omega \\ &u(x,y,z)|_{\partial\Omega} = g(x,y,z) \end{aligned} \tag{36}$$

where $q > 0$.

We take the following pattern of domain decomposition and grid partition.

$\Omega = (0,1) \times (0,1) \times (0,1)$, $\Omega_1 = \times(0,1) \times (0,\beta_1)$, $\Omega_i = (0,1) \times (0,1) \times (\alpha_{i-1}, \beta_i), i = 2, \ldots, ns-1$, $\Omega_{ns} = (0,1) \times (0,1) \times (\alpha_{ns-1}, 1)$. $n = ns \times m$, $h =$

$1/(n+2)$, $\alpha_i = i \times mh$, $\beta_i = i \times (m+2)h$. let $\gamma_i = i \times (m+1)h$, $i = 1, ns-1$. For three dimensional problems, we always take this kind of domain decomposition and grid partition if no specification.

Consider the two-subdomain case. when DRASM(σ) is applied to (36), the coefficient matrix can be expressed as

$$\widetilde{A} = \begin{pmatrix} A_1 & -E_1 \\ -F_1 & A_2 \end{pmatrix}$$

where

$$A_1 = T_1 \otimes I_n \otimes I_n + I_m \otimes T_n(2) \otimes I_n + I_m \otimes I_n \otimes T_n(2)$$
$$A_2 = T_2 \otimes I_n \otimes I_n + I_m \otimes T_n(2) \otimes I_n + I_m \otimes I_n \otimes T_n(2)$$

$$E_1 = E \otimes I_n \otimes I_n$$
$$F_1 = F \otimes I_n \otimes I_n$$

where

$$T_1 = T_{m+1}(\beta, \beta, \beta - \sigma)$$
$$T_2 = T_{m+1}(\beta - \sigma, \beta, \beta)$$

$E,\ F$ are defined as before. $\beta = 2 + qh^2$. The iteration matrix is

$$J = \begin{pmatrix} A_1^{-1} & \\ & A_2^{-1} \end{pmatrix} \begin{pmatrix} & E_1 \\ F_1 & \end{pmatrix} \triangleq M^{-1}N$$

According to (32) , there is a matrix X_n satisfies

$$X_n T_n(2) X_n^T = D_n = diag\{d_i\}, \quad d_i = 4\sin^2 \frac{i\pi}{2(n+1)}, \quad i = 1, \ldots, n.$$

Let

$$U = \begin{pmatrix} I_m \otimes I_n \otimes X_n & \\ & I_m \otimes I_n \otimes X_n \end{pmatrix}$$

we have

$$\begin{aligned} & (I_m \otimes I_n \otimes X_n)(T_i \otimes I_n \otimes I_n)(I_m \otimes I_n \otimes X_n)^T \\ = \ & (T_i \otimes I_n \otimes X_n)(I_m \otimes I_n \otimes X_n) \\ = \ & T_i \otimes I_n \otimes I_n \end{aligned}$$

$$\begin{aligned} & (I_m \otimes I_n \otimes X_n)(I_m \otimes T_n \otimes I_n)(I_m \otimes I_n \otimes X_n)^T \\ = \ & (I_m \otimes T_n \otimes X_n)(I_m \otimes I_n \otimes X_n^T) \\ = \ & I_m \otimes T_n \otimes I_n \end{aligned}$$

$$\begin{aligned} & (I_m \otimes I_n \otimes X_n)(I_m \otimes I_n \otimes T_n(2))(I_m \otimes I_n \otimes X_n)^T \\ = \ & [I_m \otimes I_n \otimes (X_n T_n(2))][I_m \otimes I_n \otimes X_n^T] \\ = \ & I_m \otimes I_n \otimes (X_n T_n(2) X_n^T) \\ = \ & I_m \otimes I_n \otimes D_n \end{aligned}$$

So

$$J' = UJU^T = (UMU^T)^{-1}N = \widetilde{M}^{-1}N$$

where

$$\widetilde{M} = \begin{pmatrix} \widetilde{A_1} & \\ & \widetilde{A_2} \end{pmatrix}$$

and

$$\widetilde{A_i} = T_i \otimes I_n \otimes I_n + I_m \otimes T_n(2) \otimes I_n + I_m \otimes I_n \otimes D_n \quad i = 1, 2.$$

Let $B_i = T_i \otimes I_n + I_m \otimes T_n(2)$, $I_{mn} = I_m \otimes I_n$, then the above formula can be written as

$$\widetilde{A_i} = B_i \otimes I_n + I_{mn} \otimes D_n \tag{37}$$

From (37), we can see that, B_i are the coefficient matrices when DRASM(σ) is applied to two dimensional model problem (31). Using the same method as in above section to reduce two dimensional problem to one dimensional problems, we can also reduce our three dimensional problem to two dimensional problems. That's to say that the optimal Robin parameter for our three dimensional problem can be approximated by the following two dimensional problems

$$\begin{aligned} -\Delta u(x,y) + (q + d_i h^{-2})u(x,y) = f(x,y),\ (x,y) \in \Omega = (0,1)^2 \\ u(x,y)|_{\partial\Omega} = g(x,y) \end{aligned}$$

According to *Note* (3.3), we need only consider the minimum eigenvalue d_1. So by (34), the optimal Robin parameter λ for the three dimensional model problem can be written as

$$\lambda_{opt} = \sqrt{q + d_1 h^{-2} + d_1 h^{-2}} = \sqrt{q + 2\pi^2}, \quad \sigma_{opt} = \frac{1}{1 + \lambda_{opt} h} \tag{38}$$

It's obvious that when DRASM(σ) is applied to three dimensional problems, its sensitivity to optimal Robin parameter gets decreased compared to two dimensional problems.

Numerical experiments 5.1. *Initial zero interior boundary values, central difference scheme. Right-hand term $f(x,y,z) = x(1-x) + y(1-y) + z(1-z)$, q=0, subdomain solver CG. Convergence criterion: $\|r^n\|/\|r^0\| \leq 10^{-5}$. The Robin parameter is determined by (38). The results are showed in Table 9 and Table 10.*

	Iter.time(s)		Iter.num	
m	ASM	DRASM(σ_{opt})	ASM	DRASM(σ_{opt})
9	3.87	1.54	28	8
19	77.08	25.28	57	10
29	622.46	97.57	87	12

TABLE 9. Three dimensional, two subdomains. Comparison of iteration time and iteration numbers, strip domain decomposition

	Iter.time(s)		Iter.num	
m	ASM	DRASM(σ_{opt})	ASM	DRASM(σ_{opt})
9	18.90	5.68	79	10
19	718.22	99.56	163	11

TABLE 10. Three dimensional, 4 subdomains. Comparison of iteration time and iteration numbers, strip domain decomposition

The above strip domain decomposition pattern can also be generalized to multi-direction decomposition in a simple and straightforward way, e.g. ignoring the coupling among the different directions, and the optimal Robin parameter is still taken as the same.

Numerical experiments 5.2. *Initial zero interior boundary values, central difference scheme. Right-hand term* $f(x,y,z) = x(1-x) + y(1-y) + z(1-z)$*,* $q = 0$*, subdomain solver: CG. Convergence criterion:* $\|r^n\|/\|r^0\| \leq 10^{-5}$*. The Robin parameter is determined by (38). The results are showed in Table 11.*

	Iter.time(s)		Iter.num	
m	ASM	DRASM(σ_{opt})	ASM	DRASM(σ_{opt})
9	1.63	0.46	61	13
19	53.51	7.98	124	17
29	484.45	46.94	187	19

TABLE 11. Three dimensional, 8 subdomains. Comparison of iteration time and iteration numbers. Domain decomposition pattern: $2 \times 2 \times 2$

6. Conclusions and Remarks

The main point of this paper is that for model problems, the optimal (near optimal) Robin parameters have been determined as

$$\lambda_{opt} = \sqrt{q + (d-1)\pi^2}, \quad d = 2, 3 \tag{39}$$

we started out with one dimensional problems, derived out the optimal Robin parameters by discrete scheme and matrix analysis, then after some reductions and approximations, the near optimal Robin parameters for continuous problems are obtained. For large scale model problems, the optimal Robin parameter can accelerate classical additive Schwarz method by tens of magnitude.

The optimal Robin parameters are *near optimal*, and DRASM(σ_{opt}) has a weak dependence on the grid size h. For high dimensional problems, the convergence rate is less sensitive to the optimal Robin parameters. So *near optimal* and *optimal* have little difference in practice. And this is also the main reason that we can take the same Robin parameter on all the different interior boundaries. Indeed, we can take the following Robin parameter for our two or three dimensional model problems

$$\lambda_{opt} = \sqrt{q + 3\pi^2} \tag{40}$$

It comes from the considerations not only the minimum eigenvalue $d_1 \approx \pi^2$, but also the second minimum eigenvalue $d_2 \approx 4\pi^2$. Numerical experiments showed that this parameter may work somewhat better than (39) in some cases, though the advantage is negligible.

The key idea different from other papers is that we gave up the efforts to seek for the real optimal Robin parameters. Instead, we just try to find the "near optimal" or good enough Robin parameters. In some cases, the Robin parameters determined by our approach may be far away from the optimal in some sense, but the Robin parameters may still work perfect in reality. Because in these situations, the problem may converge fast for a relative large scope of Robin parameters. It's no need to look for the real optimal one.

The optimal Robin parameter is just for our model problems, which is of constant coefficients and rectangle domain. Note that the optimal Robin parameter λ_{opt} can be thought of as a constant for continuous problems, so in practice it can be applied to other discrete methods.

We would like to point out that, for variable coefficient problems, our Robin parameters are still near optimal and work well. We will analyze the convection-diffusion problems and general variable coefficient problems in other papers.

References

[1] V. Dolean, S. Lanteri, and F. Nataf. Optimized interface conditions for domain decomposition methods in fluid dynamic. *Int. J. Numer. Meth. Fluids*, 40:1539–1550, 2002.

[2] B. Engquist and H.K. Zhao. Absorbing boundary conditions for domain decomposition. *Applied Numerical Mathematics*, 27:341–365, 1998.

[3] F. Fischer and R. A. Usmani. Properties of some tridiagonal matrices and their application to boundary value problems. *SIAM J. Numer. Anal.*, (6):127–142, 1969.

[4] M.J. Gander. Optimized schwarz methods for helmholtz problems. *Proceedings of the 13th International Conference on Domain Decomposition*, pages 245–252, 2001.

[5] M.J. Gander and G.H. Golub. A non-overlapping optimized schwarz method which converges with arbitrarily weak dependence on h. *Proceedings of the 14th International Conference on Domain Decomposition*, 2002.

[6] M.J. Gander, L. Halpern, and F. Nataf. Optimized schwarz methods. *Proceedings of the 12th International Conference on Domain Decomposition*, pages 15–27, 2000.

[7] M.J. Gander, F. Magoules, and F. Nataf. Optimized schwarz methods without overlap for the helmholtz equation. *SIAM Journal on Scientific Computing*, 24(1):38–60, 2002.

[8] C. Japhet, F. Nataf, and F. Rogier. The optimized order 2 method application to convection-diffusion problems. *Future Generation Computer Systems*, 18(1), 2000.

[9] P.L.Lions. On the schwarz alternating method I. In R.Glowinski, G.H.Golub, G.A.Meurant, and J.Periaux, editors, *First International Symposium on Domain Decomposition Methods for Partial Differential Equations*, pages 1–42. SIAM Publications, 1988.

[10] P.L.Lions. On the schwarz alternating method II. In T.F.Chan, R.Glowinski, J.Periaux, and O.B.Widlund, editors, *Domain Decomposition Methods*, pages 47–70. SIAM Publications, 1989.

[11] P.L.Lions. On the schwarz alternating method III:a variant for non-overlapping subdomains. pages 202–223, 1990.

[12] Wei Pai Tang. Generalized schwarz splitting. *SIAM J. Sci. Stat. Comp.*, 13(2):573–595, 1992.

Laboratory of Parallel Computing, Institute of Software, Chinese Academy of Sciences. P.O. BOX 8718, BEIJING, 100080 P. R. China

E-mail: lhw@mail.rdcps.ac.cn and sun@mail.rdcps.ac.cn

URL: http://www.rdcps.ac.cn/ lhw and http://www.rdcps.ac.cn/shouye/sjc.html