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L^2-NORM ERROR BOUNDS OF CHARACTERISTICS COLLOCATION METHOD FOR COMPRESSIBLE MISCIBLE DISPLACEMENT IN POROUS MEDIA

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Abstract. A nonlinear parabolic system is derived to describe compressible miscible displacement in a porous medium in non-periodic space. The concentration is treated by a characteristics collocation method, while the pressure is treated by a finite element collocation method. Optimal order estimates in L^2 is derived.

Key Words. compressible miscible displacement; characteristics line; collocation scheme; error estimate.

1. Introduction

The mathematical controlling model for compressible flow in porous media is given by

$$(1)\quad \begin{aligned} &(a)\ \ d(c)\frac{\partial p}{\partial t} + \nabla\cdot u = d(c)\frac{\partial p}{\partial t} - \nabla\cdot(a(c)\nabla p) = q, \qquad (x,y)\in\Omega,\ t\in(0,T] \\ &(b)\ \ \phi\frac{\partial c}{\partial t} + b(c)\frac{\partial p}{\partial t} + u\cdot\nabla c - \nabla\cdot(D\nabla c) = (\bar{c}-c)q, \quad (x,y)\in\Omega,\ t\in(0,T] \end{aligned}$$

where $\qquad c = c_1 = 1 - c_2, \quad a(c) = a(x,y,c) = k(x,y)/\mu(c),$

$$b(c) = b(x,y,c) = \phi(x,y)c_1\{z_1 - \sum_{j=1}^{2} z_j c_j\}, \quad d(c) = d(x,y,c) = \phi(x,y)\sum_{j=1}^{2} z_j c_j.$$

c_i denote the concentration of the ith component of the fluid mixture, and z_i is the "constant compressibility" factor [1] for the ith component. The model is a nonlinear coupled system of two partial differential equations. Let $\Omega = (0,1)\times(0,1)$ with the boundary $\partial\Omega$, $p(x,y,t)$ is the pressure in the mixture,u is the Darcy velocity of the fluid, and $c(x,y,t)$ is the relative concentration of the injected fluid. $k(x,y)$ and $\phi(x,y)$ are the permeability and the porosity of porous media, $\mu(c)$ is the viscosity of fluid, $D(x,y)$ is molecular dissipation coefficient, q and $\bar{c}(t)$ etc. are just like the definition of [1,2].

We shall assume that no flow occurs across the boundary

$$(2)\quad \begin{aligned} &(a)\ \ u\cdot\nu = 0 \quad on\ \partial\Omega, \\ &(b)\ \ D\nabla c\cdot\nu = 0 \quad on\ \partial\Omega, \end{aligned}$$

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where ν is the outer normal to $\partial\Omega$, and the initial conditions

$$(3) \qquad \begin{aligned} &(a)\ \ p(x,y,0) = p_0(x,y), \quad (x,y) \in \Omega, \\ &(b)\ \ c(x,y,0) = c_0(x,y), \quad (x,y) \in \Omega. \end{aligned}$$

The collocation methods are widely used for solving practice problems in engineering due to its easiness of implementation and high-order accuracy. But the most parts of mathematical theory focused on one-dimensional or two-dimensional constant coefficient problems [3-6]. In 1990's the collocation method of two-dimensional variable coefficients elliptic problems is given in [7].

The mathematical controlling model for compressible flow in porous media is strongly nonlinear coupling system of partial differential equations of two different types. Nonlinear terms introduce many difficulties for convergence analysis of algorithms. In the present article, we use different collocation technique to treat equations of different types, usual collocation method to solve the equation for pressure and characteristic collocation scheme to approximate the equation for concentration. We develop some technique to analyze convergence of collocation algorithm for this strongly nonlinear system and prove the optimal order L^2 error estimate. And we shall assume the coefficients $a(c), D(x,y), \phi(x,y), d(c), b(c)$ to be bounded above and below by positive constants independently of c as well as being smooth.

The organization of the rest of the paper is as follows. In Section 2, we will present the formulation of the characteristic collocation scheme for nonlinear system (1). In section 3, we will analyze convergent rate of the scheme defined in section 2. Throughout, the symbols K and ε will denote, respectively, a generic constant and a generic small positive constant.

2. Fully Discrete Characteristic Collocation Scheme

In this section, we will give some basic notations and definition for collocation methods, which will be used in this article. Then we will present the fully discrete characteristic collocation scheme for nonlinear system (1).

2.1. Notations and definition for collocation methods.

We make the partition of the domain Ω, which is quasi-uniform and equally spaced rectangular grid. The grid points are (x_i, y_j), $i = 0, 1 \cdots N_x; j = 0, 1 \cdots N_y$. Let

$$\delta_x : 0 = x_0 < x_1 < \cdots < x_{N_x} = 1, \qquad \delta_y : 0 = y_0 < y_1 < \cdots < y_{N_y} = 1$$

be the grid points along x-direction and y-direction respectively, and

$$h_x = x_i - x_{i-1}, \quad h_y = y_j - y_{j-1}, \quad h = \max\{h_x, h_y\}$$

be grid size along x-direction and y-direction and maximum size of partition respectively. Introduce the following notations:

$$\Omega_{ij} = (x_{i-1}, x_i) \times (y_{j-1}, y_j), \qquad I = [0,1]$$

$$I_x^i = [x_{i-1}, x_i], \quad I_y^j = [y_{j-1}, y_j],$$

for $i = 1, 2 \cdots N_x$ and $j = 1, 2 \cdots N_y$. Define function spaces as follows:

$$\mathcal{M}_1(3, \delta_x) = \{v \in C^1(I)|\ v \in P_3(I_x^i),\ i = 1 \cdots N_x\},$$

$$\mathcal{M}_1(3, \delta_y) = \{v \in C^1(I)|\ v \in P_3(I_y^j),\ j = 1 \cdots N_y\},$$

where P_3 denotes the set of polynomials of degree ≤ 3, and

$$\mathcal{M}_{1,P}(3, \delta_x) = \{v \in \mathcal{M}_1(3, \delta_x) :\ v(0) = v(1) = 0\},$$

$$\mathcal{M}_{1,P}(3, \delta_y) = \{v \in \mathcal{M}_1(3, \delta_y) :\ v(0) = v(1) = 0\},$$

then let $m_1(3,\delta)$ and $m_{1,P}(3,\delta)$ be the spaces of piecewise Hermite bicubics defined by

$$\mathcal{M}_1(3,\delta)=\mathcal{M}_1(3,\delta_x)\bigotimes\mathcal{M}_1(3,\delta_y),$$

and

$$\mathcal{M}_{1,P}(3,\delta)=\mathcal{M}_{1,P}(3,\delta_x)\bigotimes\mathcal{M}_{1,p}(3,\delta_y).$$

Next, we take four Gauss points as collocation points in Ω_{ij}: (ξ^x_{ik},ξ^y_{jl}), $k,l=1,2$,

$$\xi^x_{ik}=x_{i-1}+h_x\xi_k,\quad \xi^y_{jl}=y_{j-1}+h_y\xi_l,$$

where

$$\xi_1=(3-\sqrt{3})/6,\quad \xi_2=(3+\sqrt{3})/6.$$

Let T_{3,δ_x} and T_{3,δ_y} be the interpolation operators of piecewise Hermite bicubics of $\mathcal{M}_1(3,\delta_x)$ in x and $\mathcal{M}_1(3,\delta_y)$ in y, respectively, and $T_{3,\delta}$ be the interpolation operator of piecewise Hermite bicubics in $m_1(3,\delta)$ on Ω, which may be defined by

$$T_{3,\delta}v=T_{3,\delta_x}T_{3,\delta_y}v=T_{3,\delta_y}T_{3,\delta_x}v,$$

for sufficiently smooth function v.

Introduce the following summation notation:

$$< u,v >=\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}< u,v >_{ij}=\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{1}{4}h_xh_y\sum_{k,l=1}^{2}(uv)(\xi^x_{ik},\xi^y_{jl}),$$
$$< u,v >_x=\sum_{i=1}^{N_x}< u,v >_{ix}=\sum_{i=1}^{N_x}\frac{h_x}{2}\sum_{k=1}^{2}(uv)(\xi^x_{ik}),$$
$$< u,v >_y=\sum_{j=1}^{N_y}< u,v >_{jy}=\sum_{j=1}^{N_y}\frac{h_y}{2}\sum_{l=1}^{2}(uv)(\xi^y_{jl}),$$

$$< u,v >=<< u,v >_x,1 >_y=<< u,v >_y,1 >_x,\quad < u,u >=|||u|||^2,$$

and discrete norms

$$|||u|||^2_{H^1_0(\Omega)}=\int_0^1< Du_x,u_x >_y\,dx+\int_0^1< Du_y,u_y >_x\,dy,\quad \forall u\in\mathcal{M}_1(3,\delta),$$

and

$$|||u|||^2_E=\int_0^1< u_x,u_x >_y\,dx+\int_0^1< u_y,u_y >_x\,dy,\quad \forall u\in\mathcal{M}_1(3,\delta).$$

2.2. Fully discrete CCS.

At first time can be discretized $0=t^0<t^1<\cdots<t^n=T$, $\triangle t=t^n-t^{n-1}$. We consider the concentration equation, let $\psi=[\phi^2+u_1^2+u_2^2]^{\frac{1}{2}}$, and the characteristic direction associated with the operator $\phi c_t+u\cdot\nabla c$ is denoted by $\tau(x,y)$, hence

$$\psi\frac{\partial c}{\partial\tau}=\phi\frac{\partial c}{\partial t}+u\cdot\nabla c.$$

The equation $(1)(b)$ can be put in the form

$$\psi\frac{\partial c}{\partial\tau}+b(c)\frac{\partial p}{\partial t}-\nabla\cdot(D\nabla c)=(\bar{c}-c)q,\quad (x,y)\in\Omega,\quad t\in(0,T]. \tag{4}$$

For (4), we use a backward difference quotient for $\partial c/\partial\tau$ along the characteristic line

$$\psi\frac{\partial c^n}{\partial\tau}\approx\psi\frac{c^n(x,y)-c^{n-1}(\check{x},\check{y})}{\triangle t[1+|u|^2/\phi^2]^{\frac{1}{2}}}=\phi\frac{c^n-\check{c}^{n-1}}{\triangle t}, \tag{5}$$

where

$$\breve{f}^n = f(\breve{x}^n, \breve{y}^n, t^n), \quad f^n = f(t^n),$$

with

$$\breve{x}^{n-1} = x - \frac{u_1^n}{\phi}\triangle t, \quad \breve{y}^{n-1} = y - \frac{u_2^n}{\phi}\triangle t.$$

Then, we have the following discrete equation

$$(6) \qquad \phi\frac{c_h^n - \breve{c}_h^{n-1}}{\triangle t} + b(c_h^{n-1})\frac{P^n - P^{n-1}}{\triangle t} - \nabla\cdot(D\nabla c_h^n) - (\bar{c}^{n-1} - c_h^{n-1})q = 0, \quad n = 1, 2\cdots.$$

Now by using the interpolation operator $T_{3,\delta}$ and the Gauss points $\{ (\xi_{ik}^x, \xi_{jl}^y)$, $1 \le i \le N_x;\ 1 \le j \le N_y; k, l = 1, 2\}$, we give the fully discrete characteristic collocation scheme:

Characteristic Collocation Scheme: If (C^{n-1}, P^{n-1}) has been known at $t = t^{n-1}$, at $t = t^n$ the (C^n, P^n) should be

$$(7) \qquad \begin{aligned} &(a)\ \ C^0 = T_{3,\delta}c_0(x,y), \quad P^0 = T_{3,\delta}p_0(x,y),\\ &(b)\ \ \{\ d(C^{n-1})\frac{P^n - P^{n-1}}{\triangle t} - \nabla\cdot(a(C^{n-1})\nabla P^n) - q\ \}(\xi_{ik}^x, \xi_{jl}^y) = 0,\\ &(c)\ \ \{\ \phi\frac{C^n - \hat{C}^{n-1}}{\triangle t} + b(C^{n-1})\frac{P^n - P^{n-1}}{\triangle t} - \nabla\cdot(D\nabla C^n)\\ &\qquad\qquad - (\bar{C}^{n-1} - C^{n-1})q\ \}(\xi_{ik}^x, \xi_{jl}^y) = 0,\\ &(d)\ \ \frac{\partial C^n}{\partial \nu}\bigg|_{\partial\Omega} = 0 \end{aligned}$$

where

$$\hat{f}^n = f(\hat{x}^n, \hat{y}^n, t^n), \quad f^n = f(t^n)$$

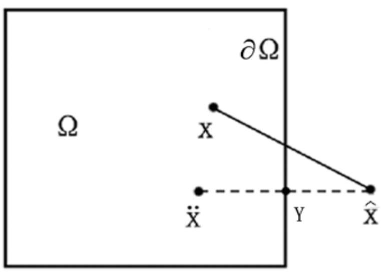
and

$$(8) \qquad U^{n-1} = -a(C^{n-1})\nabla P^{n-1}$$

with

$$\hat{x}^{n-1} = x - \frac{U_1^n}{\phi}\triangle t, \quad \hat{y}^{n-1} = y - \frac{U_2^n}{\phi}\triangle t,$$

for $1 \le i \le N_x$, $1 \le j \le N_y$, $k, l = 1, 2$ and $n, m \ge 0$, computed in the order: at first P^n can been computed from (7)(b), then from (8) and (7)(c) we can obtain C^n.



When $\hat{x}$ is through the boundary $\partial\Omega$, we will do continuation according to specular reflection method, namely when $\hat{x}$ is outside Ω, we do the normal from $\hat{x}$ to $\partial\Omega$, and the normal intersects $\partial\Omega$ at Y. Then we do inner normal at Y, and we choose point $\ddot{x}$ so as to $|\hat{x}Y| = |\ddot{x}Y|$, and the value of $c(\ddot{x})$ replaces the one of $c(\hat{x})$, in this way c and C etc. functions are certain meaning. Because c satisfies (2)(b), the continuation is right[10].

In next section, we will analyze existence and convergence of the solution of the characteristic collocation scheme.

3. Convergence Analysis

In this section, we first analyze the existence of the solution of the characteristic collocation scheme, and then analyze convergence. We assume that

(R) $$c \in L^\infty(H^6)\bigcap L^\infty(W^2_\infty)\bigcap H^1(W^2_\infty)\bigcap H^2(H^1)$$
$$p \in L^\infty(H^6)\bigcap H^1(H^6)\bigcap L^\infty(W^1_\infty)\bigcap H^2(H^1).$$

3.1. Preliminary results.

We list some basic results in [3,8].

Lemma 3.1 . *Let $e = v - T_{3,\delta_x}v$, then there exists constant $K > 0$ such that*

$$(1)\quad < e^{(l)}, e^{(l)} >_x \le K h_x^{2(4-l)} \cdot \sum_{i=1}^{N_x}\int_{x_{i-1}}^{x_i}\sum_{\alpha\le 4}(\frac{\partial^\alpha v}{\partial x^\alpha})^2 dx, \quad l = 0, 1$$

$$(2)\quad < e_{xx}, e_{xx} >_x \le K h_x^6 \cdot \sum_{i=1}^{N_x}\int_{x_{i-1}}^{x_i}\sum_{\alpha\le 5}(\frac{\partial^\alpha v}{\partial x^\alpha})^2 dx$$

$$(3)\quad | < e_x, 1 >_x |^2 \le K h_x^9 \cdot \sum_{i=1}^{N_x}\int_{x_{i-1}}^{x_i}\sum_{\alpha\le 5}(\frac{\partial^\alpha v}{\partial x^\alpha})^2 dx$$

$$(4)\quad | < e_{xx}, 1 >_x |^2 \le K h_x^9 \cdot \sum_{i=1}^{N_x}\int_{x_{i-1}}^{x_i}\sum_{\alpha\le 6}(\frac{\partial^\alpha v}{\partial x^\alpha})^2 dx.$$

There is the same conclusions in y direction.

Lemma 3.2 *There exists constant $K \ge 0$ such that for sufficiently smooth function v*

$$\|v - T_{3,\delta}v\|_{L^2(\Omega)} \le K h^4 (\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\|v^{(4)}\|_{L^2(\Omega_{ij})})^{\frac{1}{2}},$$

$$\|v_t - T_{3,\delta}v_t\|_{L^2(\Omega)} \le K h^4 (\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\|v_t^{(4)}\|_{L^2(\Omega_{ij})})^{\frac{1}{2}}.$$

The following conclusions are proved in [3,5].

Lemma 3.3 *For any $v \in \mathcal{M}_1(3,\delta)$, if we have*

$$v(\xi^x_{ik}, 0) = v(\xi^x_{ik}, 1) = v(0, \xi^y_{jl}) = v(1, \xi^y_{jl}) = v(0,0) = v(0,1) = v(1,0)$$
$$= v(1,1) = v(\xi^x_{ik}, \xi^y_{jl}) = 0,$$

for $1 \le i \le N_x$, $1 \le j \le N_y$ and $k, l = 1, 2$, then $v = 0$.

Lemma 3.4 *For any $v \in \mathcal{M}_{1,P}(3,\delta)$,there exists constant $K > 0$ such that*

$$|||v|||_E^2 \le \; - < \triangle v, v > \le K|||v|||_E^2.$$

Lemma 3.5 *Assume that the inverse supposition for $m_1(3,\delta)$ holds* [9]*, then exists constant $K > 0$ such that for any $v \in \mathcal{M}_1(3,\delta)$*

$$\|v\|^2_{H^1(\Omega)} \le K\{ < v, v > + |||v|||^2_{H^1_0(\Omega)} \}.$$

Lemma 3.6 *Assume that $v \in \mathcal{M}_1(3,\delta)$ holds, there exists constant $K_1 \ge 0$ and $K_2 \ge 0$ such that*

$$\|v\|_{L^2(\Omega)} \le |||v||| \le K_1\|v\|_{L^2(\Omega)}, \quad \|v\|_{L^\infty(\Omega)} \le K_2 h^{-1}\|v\|_{L^2(\Omega)}.$$

Proof. We may see 2.2 and 2.4 in [4].

Lemma 3.7. *Assume that $D(x,y)$ is sufficiently smooth. There exists constants $0 < K_* \le K^*$ such that for each $v \in \mathcal{M}_{1,P}(3,\delta)$*

$$K_* < -\triangle v, v > \le - < \nabla\cdot(D\nabla v), v > \le K^* < -\triangle v, v > .$$

Proof. The Peano representation of the remainder in the two-point Gauss-Legendre quadrature and Leibnitz's formula, (see Theorem 4.2 in [7]), reads

$$< -\frac{\partial}{\partial x}(D\frac{\partial v}{\partial x})(\cdot,\eta_{jl}), v(\cdot,\eta_{jl}) >_x = I_1(D,v,\eta_{jl}) + I_2(D,v,\eta_{jl}),$$

where

$$\begin{aligned} I_1(D,v,\eta_{jl}) &= \int_0^1 [D(\frac{\partial v}{\partial x})^2](x,\eta_{jl})dx \\ &+ 4\sum_{k=1}^{N_x}(h_x)^4 \int_{I_k^x}[D(\frac{\partial^3 v}{\partial x^3})^2](x,\eta_{jl})\mathcal{K}(\frac{x-x_{k-1}}{h_x})dx \\ &= I_3(D,v,\eta_{jl}) + I_4(D,v,\eta_{jl}), \end{aligned}$$

and

$$I_2(D,v,\eta_{jl}) = \sum_{l=1}^{5}\sum_{\substack{i+j=6-l\\ 0\le i,j\le 3}} \alpha_{i,j}^l \sum_{k=1}^{N_x}(h_x)^4 \times \int_{I_k^x}[\frac{\partial^l D}{\partial x^l}\frac{\partial^i v}{\partial x^i}\frac{\partial^j v}{\partial x^j}](x,\eta_{jl})\mathcal{K}(\frac{x-x_{k-1}}{h_x})dx,$$

the constant α_{ij}^l are independent of h and symmetrical $\alpha_{ij}^l = \alpha_{ji}^l$, and

$$0 \le \mathcal{K}(\beta) = \frac{1}{24}\{\ (1-\beta)^4 - 2[(\xi_1-\beta)_+^3 + (\xi_2-\beta)_+^3]\ \} \le K, \qquad \beta\in[0,1].$$

Since $I_2(1,v,\eta_{jl}) = 0$, we see that

$$D_* < \ -\frac{\partial^2 v}{\partial x^2}(\cdot,\eta_{jl}), v(\cdot,\eta_{jl})\ >_x \ \le \ I_1(D,v,\eta_{jl}), \qquad D_* \in \min_{(x,y)\in\bar\Omega} D(x,y).$$

On the other hand, the Cauchy-Schwarz inequality in $L^2(I_k^x)$ give

$$|I_2(D,v,\eta_{jl})| \le KK_1^x \sum_{l=1}^{5}\sum_{\substack{i+j=6-l\\ 0\le i,j\le 3}}\sum_{k=1}^{N_x}(h_x)^4 \|\frac{\partial^i v}{\partial x^i}(\cdot,\eta_{jl})\|_{L^2(I_k^x)}\|\frac{\partial^j v}{\partial x^j}(\cdot,\eta_{jl})\|_{L^2(I_k^x)},$$

where

$$K_1^x = \max_{1\le l\le 5}\max_{(x,y)\in\bar\Omega}|\frac{\partial^l D}{\partial x^l}(x,y)|.$$

Hence, by using the inverse inequality

$$\|u^{(i)}\|_{L^2(I_k^x)} \le \ Kh_x^{l-i}\|u^{(l)}\|_{L^2(I_k^x)}, \quad 0\le l\le i\le 3 \ \ , \ \ u\in P_3,$$

with $l = 1, 2 \le i \le 3$, the Cauchy-Schwarz inequality in R^{N_x}, and the Poinc*á*re inequality $\|u\|_{L^2(0,1)} \ \le \ K\|u'\|_{L^2(0,1)}$, for $u \in m_{1,P}(3,\delta_x)$, we get

$$|I_2(D,v,\eta_{jl})| \le KK_1^x h_x \|\frac{\partial v}{\partial x}(\cdot,\eta_{jl})\|_{L^2(0,1)}^2$$

and

$$|I_4(D,v,\eta_{jl})| \le KD^*\|\frac{\partial v}{\partial x}(\cdot,\eta_{jl})\|_{L^2(0,1)}^2, \quad D^* = \max_{(x,y)\in\bar\Omega} D(x,y).$$

Further, lemma 3.3 of [3] implies that

$$|I_2| \le \ KK_1^x h_x < \ -\frac{\partial^2 v}{\partial x^2}(\cdot,\eta_{jl}), v(\cdot,\eta_{jl})\ >_x$$

and

$$|I_4| \le \ KD^* < \ -\frac{\partial^2 v}{\partial x^2}(\cdot,\eta_{jl}), v(\cdot,\eta_{jl})\ >_x .$$

Putting above estimates together, we have

$$(D_* - KK_1^x h_x) < -\triangle v, v > \leq < -\frac{\partial}{\partial x}(D\frac{\partial v}{\partial x}), v >$$
$$\leq (D^* + KK_1^x h_x + KD^*) < -\triangle v, v > .$$

For $< -\frac{\partial}{\partial y}(D\frac{\partial v}{\partial y}), v \ >$ has the similar estimate. Let

$$K_1 = \max_{1\leq l\leq 5} \max_{(x,y)\in\bar\Omega}\{ \ |\frac{\partial^l D}{\partial x^l}(x,y)|, \ \ |\frac{\partial^l D}{\partial y^l}(x,y)| \ \ \}$$

and

$$K_* = 2(D_* - KK_1 h) \qquad\qquad K^* = 2(D^* + KK_1 h + KD^*).$$

For sufficient small h, K_* and K^* are positive. The lemma is proved.

Lemma 3.8 *Under the same conditions as in lemma 3.7, there exists constant* $0 < C_* \leq C^*$ *such that*

$$C_* |||v|||^2_{H_0^1(\Omega)} \ \leq < -\nabla\cdot(D\nabla v), v \ > \leq C^* \ |||v|||^2_{H_0^1(\Omega)}, \quad \forall v \in \mathcal{M}_{1,P}(3,\delta).$$

Proof. Since 2.1 section and the condition of $D(x,y)$ satisfied, we obtain

$$D_*|||v|||^2_E \ \leq \ |||v|||^2_{H_0^1(\Omega)} \leq \ D^*|||v|||^2_E, \quad v \in \mathcal{M}_{1,P}(3,\delta)$$

Since lemma 3.4 and lemma 3.7, we have

$$\frac{K_*}{D^*}|||v|||^2_{H_0^1(\Omega)} \leq \ K_*|||v|||^2_E \leq \ K_* < -\triangle v, v >$$
$$\leq \ - < \nabla\cdot(D\nabla v), v > \leq \ K^* < -\triangle v, v >$$
$$\leq \ K^*K|||v|||^2_E \leq \ \frac{K^*K}{D_*}|||v|||^2_{H_0^1(\Omega)}, \qquad v \in \mathcal{M}_{1,P}(3,\delta)$$

Let $C_* = \frac{K_*}{D^*}$, $C^* = \frac{K^*K}{D_*}$, the proof is completed.

3.2. Existence of the solution of CCS.

In this section we consider the existence and uniqueness of the numerical solution. $(7)(b)(c)$ can be rewritten as the discrete Galerkin method given by

$$(9)\quad \begin{aligned} &(a) \quad < d(C^{n-1})\frac{P^n - P^{n-1}}{\triangle t} - \nabla\cdot(a(C^{n-1})\nabla P^n) - q, \chi >= 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall \chi \in \mathcal{M}_{1,P}(3,\delta) \\ &(b) \quad < \phi\frac{C^n - \hat{C}^{n-1}}{\triangle t} + b(C^{n-1})\frac{P^n - P^{n-1}}{\triangle t} - \nabla\cdot(D\nabla C^n) \\ &\qquad - (\bar{C}^{n-1} - C^{n-1})q, Z >= 0, \ \ \forall Z \in \mathcal{M}_{1,P}(3,\delta). \end{aligned}$$

We only discuss the pressure equation, and the concentration equation is similar. It is clear that any solution of $(7)(b)$ is a solution of $(9)(a)$. Thus, it is sufficient to prove existence for $(7)(b)$ and uniqueness for $(9)(a)$ (lemma 4.1 of [3]). For sufficiently small $\triangle t$, existence for $(7)(b)$ follows from lemma 3.3, since it implies that matrix generated by the time derivative term is nonsingular for any choice of the basis for $m_{1,P}(3,\delta)$, and uniqueness for solutions of $(9)(a)$ also is implies by lemma 3.3, since the matrix generated by time-derivative term in $(9)(a)$ must be nonsingular since $d(c)$ is bounded below by a positive constant.

So CCS(7) and the discrete Galerkin method (9) each possess a unique solution for $0 < t \leq T$; moreover, these solutions are identical if the processes are started from the same initial values.

3.3. Error estimate.

In this section, we will obtain the optimal L^2-norm error estimate.

Theorem 3.1. Suppose (R) and $r=3$ hold, and $\triangle t = o(h)$, then there exists a constant $K = K(\Omega, a_*, b_*, d_*, \phi_*, D_*, \cdots, K^*, K_1, K_2)$ such that, for h sufficiently small,

$$\max_{0\le n\le[\frac{T}{\triangle t}]} \| c^n - C^n \|^2 + \sum_{n=0}^{T/\triangle t} \| p^n - P^n \|^2 \triangle t \le K(\triangle t^2 + h^8).$$

Proof. Let

$$\tilde{c} = T_{3,\delta} c, \quad \zeta = c - \tilde{c}, \quad \xi = \tilde{c} - C, \quad \tilde{p} = T_{3,\delta} p, \quad \eta = p - \tilde{p}, \quad \pi = \tilde{p} - P.$$

We first consider the pressure equation. Subtracting (9)(a) from the Galerkin method of (1)(a), we obtain

$$\begin{aligned}
&< d(C^{n-1}) d_t \pi^n, \chi > - < \nabla \cdot (a(C^{n-1}) \nabla \pi^n), \chi > \\
&\quad =< [\ d(C^{n-1}) - d(c^n)\ ] d_t \tilde{p}^n, \chi > - < d(c^n) d_t \eta^n, \chi > \\
&\quad + < d(c^n)(d_t p^n - \frac{\partial p^n}{\partial t}), \chi > + < \nabla \cdot (a(c^n) \nabla \eta^n), \chi > \\
&\quad + < \nabla \cdot [\ (a(c^n) - a(C^{n-1})) \nabla \tilde{p}^n\ ], \chi >, \qquad \forall \chi \in \mathcal{M}_{1,P}(3,\delta)
\end{aligned} \tag{10}$$

where $d_t f^n = \dfrac{f^n - f^{n-1}}{\triangle t}$, and choosing the test function $\chi = \pi^n$ in (10), and the right terms can be denoted by $T_i', i = 1, 2 \cdots 5$ in turn. Then by lemma 3.1, lemma 3.2 and lemma 3.6, we have

$$\begin{aligned}
|T_1'| &=< [\ d(C^{n-1}) - d(c^{n-1}) + d(c^{n-1}) - d(c^n)\ ] d_t \tilde{p}^n, \pi^n > \\
&=< [\ \frac{\partial d}{\partial c}(c^1)(C^{n-1} - c^{n-1}) + \frac{\partial d}{\partial c}(c^2)(c^{n-1} - c^n)\ ] d_t \tilde{p}^n, \pi^n > \\
&\le K(|||\zeta^{n-1}||| + |||\xi^{n-1}||| + |||c^{n-1} - c^n|||) \sup_n |d_t \tilde{p}^n| \cdot |||\pi^n||| \\
&\le K(h^8 + \triangle t^2 + ||\xi^{n-1}||^2) + \varepsilon ||\pi^n||^2.
\end{aligned} \tag{11}$$

And

$$\begin{aligned}
|T_2'| &\le |< d(c^n) \frac{\eta^n - \eta^{n-1}}{\triangle t}, \pi^n > | \\
&\le K|||\eta_t|||^2 + \varepsilon |||\pi^n||| \le K h^8 ||p_t||^2_{H^4} + \varepsilon ||\pi^n||^2,
\end{aligned} \tag{12}$$

where using lemma 3.1, lemma 3.2, lemma 3.6.

For T_3', we can get from the standard backward-difference error equation or Taylor expansion[10]

$$|T_3'| \le |< d(c^n)(\frac{p^n - p^{n-1}}{\triangle t} - \frac{\partial p^n}{\partial t}), \pi^n > \le K(\triangle t)^2 + \varepsilon ||\pi^n||^2. \tag{13}$$

To obtain T_4', we have the following conclusion. ξ^n, ζ^n are defined as the above, such that for ε sufficiently small

$$\begin{aligned}
&< (D\zeta_x^n)_x, \xi^n >_x | \le \varepsilon \{ (\xi_x^n, \xi_x^n)_x + < \xi^n, \xi^n >_x \} \\
&\qquad + K\ h_x^8 \sum_{i=1}^{N_x} \int_{x_{i-1}}^{x_i} \sum_{\alpha \le 6} (\frac{\partial^\alpha c^n}{\partial x^\alpha})^2 dx.
\end{aligned} \tag{14}$$

Because we let $\check{\xi}_i^n = h_x^{-1} < \xi^n, 1 >_i$, by the definition of section 2.1 we obtain

$$\begin{aligned} < \xi^n, 1 >_i^2 &= \frac{h_x^2}{4}\{\xi^n(\xi_{i1}^x) + \xi^n(\xi_{i2}^x)\}^2 \le K\frac{h_x^2}{4}\{(\xi^n(\xi_{i1}^x))^2 + (\xi^n(\xi_{i2}^x))^2\} \\ &= K\frac{h_x}{2}\{\frac{h_x}{2}[(\xi^n(\xi_{i1}^x))^2 + (\xi^n(\xi_{i2}^x))^2]\} \\ &\le Kh_x < \xi^n, \xi^n >_i = Kh_x |||\xi^n|||_i^2. \end{aligned}$$

Thus

$$|\check{\xi}_i^n| \le K h_x^{-\frac{1}{2}} |||\xi^n|||_i \tag{15}$$

And

$$< (D\zeta_x^n)_x, \xi^n >_x = < D_x\zeta_x^n, \xi^n >_x + < D\zeta_{xx}^n, \xi^n >_x . \tag{16}$$

We estimate the first term of the right-side of (16)

$$\begin{aligned} | < D_x\zeta_x^n, \xi^n >_i | &\le | < D_x\zeta_x^n, \xi^n - \check{\xi}_i^n >_i | + | < D_x\zeta_x^n, \check{\xi}_i^n >_i | \\ &= S_1 + S_2. \end{aligned}$$

By lemma 3.1 , Poincáre inequality [3], we obtain

$$\begin{aligned} |S_1| &\le K \max\{|D_x(\xi_{i1}^x)|, |D_x(\xi_{i2}^x)|\} \ \ |||\zeta_x^n|||_i \cdot |||\xi^n - \check{\xi}_i^n|||_i \\ &\le K \max\{|D_x(\xi_{i1}^x)|, |D_x(\xi_{i2}^x)|\} h_x^4 \ (\int_{x_{i-1}}^{x_i} \sum_{\alpha \le 4} (\frac{\partial^\alpha c^n}{\partial x^\alpha})^2 dx)^{\frac{1}{2}} \cdot \|\xi_x^n\|_{L^2(I_i)} \\ &\le \varepsilon(\xi_x^n, \xi_x^n)_i + K \cdot h_x^8 \cdot \int_{x_{i-1}}^{x_i} \sum_{\alpha \le 4} (\frac{\partial^\alpha c^n}{\partial x^\alpha})^2 dx. \end{aligned} \tag{17}$$

By lemma 3.1 and (15) , we obtain

$$\begin{aligned} |S_2| &\le K \max\{|D_x(\xi_{i1}^x)|, |D_x(\xi_{i2}^x)|\} \ \ | < \zeta_x^n, 1 >_i | \cdot |\check{\xi}_i^n| \\ &\le K \max\{|D_x(\xi_{i1}^x)|, |D_x(\xi_{i2}^x)|\} h_x^4 \ (\int_{x_{i-1}}^{x_i} \sum_{\alpha \le 5} (\frac{\partial^\alpha c^n}{\partial x^\alpha})^2 dx)^{\frac{1}{2}} \cdot |||\xi^n|||_i \\ &\le \varepsilon < \xi^n, \xi^n >_i + K \cdot h_x^8 \cdot \int_{x_{i-1}}^{x_i} \sum_{\alpha \le 5} (\frac{\partial^\alpha c^n}{\partial x^\alpha})^2 dx. \end{aligned} \tag{18}$$

Next we estimate the second term of (16)

$$\begin{aligned} | < D\zeta_{xx}^n, \xi^n >_i | &\le | < D\zeta_{xx}^n, \xi^n - \check{\xi}_i^n >_i | + | < D\zeta_{xx}^n, \check{\xi}_i^n >_i | \\ &= S_1' + S_2'. \end{aligned}$$

Similar to (17)

$$\begin{aligned} |S_1'| &\le K \max\{|D(\xi_{i1}^x)|, |D(\xi_{i2}^x)|\} \ \ |||\zeta_{xx}^n|||_i \cdot |||\xi^n - \check{\xi}_i^n|||_i \\ &\le K \max\{|D(\xi_{i1}^x)|, |D(\xi_{i2}^x)|\} h_x^4 \ (\int_{x_{i-1}}^{x_i} \sum_{\alpha \le 5} (\frac{\partial^\alpha c^n}{\partial x^\alpha})^2 dx)^{\frac{1}{2}} \cdot \|\xi_x^n\|_{L^2(I_i)} \\ &\le \varepsilon(\xi_x^n, \xi_x^n)_i + K \cdot h_x^8 \cdot \int_{x_{i-1}}^{x_i} \sum_{\alpha \le 5} (\frac{\partial^\alpha c^n}{\partial x^\alpha})^2 dx. \end{aligned} \tag{19}$$

Similar to (18)

$$
\begin{aligned}
|S_2'| &\le K\max\{|D(\xi_{i1}^x)|,|D(\xi_{i2}^x)|\} \quad |<\zeta_{xx}^n,1>_i| \cdot |\check{\xi}_i^n| \\
&\le K\max\{|D(\xi_{i1}^x)|,|D(\xi_{i2}^x)|\} h_x^4 \ (\int_{x_{i-1}}^{x_i} \sum_{\alpha\le 6} (\frac{\partial^\alpha c^n}{\partial x^\alpha})^2 dx)^{\frac{1}{2}} \cdot |||\xi^n|||_i \\
&\le \varepsilon <\xi^n,\xi^n>_i + K\cdot h_x^8 \cdot \int_{x_{i-1}}^{x_i} \sum_{\alpha\le 6} (\frac{\partial^\alpha c^n}{\partial x^\alpha})^2 dx.
\end{aligned}
\tag{20}
$$

By summing over i , it follows that

$$
\begin{aligned}
&|<(D\zeta_x^n)_x,\xi^n>_x| = |\sum_{i=1}^{N_x} <(D\zeta_x^n)_x,\xi^n>_i| \\
&= |\sum_{i=1}^{N_x} [<D_x\zeta_x^n,\xi^n>_i + <D\zeta_{xx}^n,\xi^n>_i]| \\
&\le \varepsilon\{(\xi_x^n,\xi_x^n)_x + <\xi^n,\xi^n>_x\} + K\ h_x^8 \sum_{i=1}^{N_x} \int_{x_{i-1}}^{x_i} \sum_{\alpha\le 6} (\frac{\partial^\alpha c^n}{\partial x^\alpha})^2 dx.
\end{aligned}
$$

And there is the same conclusion in y direction, in this time let $\check{\xi}_j^n = h_y^{-1} <\xi^n,1>_j$, the (14) is right. And because of

$$
|<(D\zeta_x^n)_x,\xi^n>| = |\sum_{j=1}^{N_y} \frac{h_y}{2} [<(D\zeta_x^n)_x,\xi^n>_x (\xi_{j1}^y) + <(D\zeta_x^n)_x,\xi^n>_x (\xi_{j2}^y)]|
$$

we have the following conclusion.

$$
\begin{aligned}
|<(D\zeta_x^n)_x,\xi^n>| &\le \varepsilon\{(\xi_x^n,\xi_x^n) + <\xi^n,\xi^n>\} \\
&\quad + K\ h^8 \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} \int_{\Omega_{ij}} \sum_{\alpha\le 6} (\frac{\partial^\alpha c^n}{\partial x^\alpha})^2 d\Omega,
\end{aligned}
\tag{21}
$$

where α is a two-fold index, and there is the same conclusion in y direction.

Then for T_4' similar to (17)-(20), lemma 3.6 and lemma 3.7, we obtain

$$
\begin{aligned}
|T_4'| &= |<\nabla\cdot(a(c^n)\nabla\eta^n),\pi^n>| \\
&\le |<a(c^n)\triangle\eta^n,\pi^n>| \\
&+ |<a(c^n)_x(\eta^n)_x,\pi^n>| + |<a(c^n)_y(\eta^n)_y,\pi^n>|. \\
&\le Kh^8 + \varepsilon(||\pi^n||^2 + ||\nabla\pi^n||^2)
\end{aligned}
\tag{22}
$$

For T_5', we shall need an induction hypothesis. We assume that

$$
||C^n||_{W_\infty^1} \le K, \qquad 0\le n\le l-1. \tag{23}
$$

We start this induction by seeing that

$$
||C^0||_{W_\infty^1} \le ||\tilde{c}^0||_{W_\infty^1} + ||\xi^0||_{W_\infty^1} \le ||\tilde{c}^0||_{W_\infty^1} \le K,
$$

for h sufficiently small. We shall check that if $n=l$, (23) is right at the end of the proof. Similar to the proof of T_1' and T_4' and using lemma 3.1, lemma 3.2, lemma 3.6 and (23), we can get

$$
\begin{aligned}
|T_5'| &\le |<[\ a(c^n) - a(C^{n-1})\ ]\Delta\tilde{p}^n,\pi^n>| \\
&+ |<\nabla[\ a(c^n) - a(C^{n-1})\ ]\cdot\nabla\tilde{p}^n),\pi^n>| \\
&\le K(||\xi^{n-1}||_1^2 + h^8 + \triangle t^2) + \varepsilon(||\pi^n||^2 + ||\nabla\pi^n||^2).
\end{aligned}
\tag{24}
$$

Next using the inequality $a(a-b) \geq \frac{1}{2}(a^2-b^2)$, we see that the first left-hand side term of (10),

$$\begin{aligned} &< d(C^{n-1})d_t\pi^n, \pi^n > \\ &\qquad \geq \frac{1}{2\triangle t}\{ < d(C^{n-1})\pi^n, \pi^n > - < d(C^{n-1})\pi^{n-1}, \pi^{n-1} > \} \end{aligned} \tag{25}$$

Similar to the proof of lemma 3.7 and (23), the second left-hand side term of (10) get

$$- < \nabla \cdot (a(C^{n-1})\nabla\pi^n), \pi^n > \geq (a_* - KK_2h) \parallel \nabla\pi^n \parallel^2, \tag{26}$$

then for sufficiently small h there exists constant $C > 0$, we have $a_* - KK_2h \geq C > 0$.

By (11)-(26), we multiplied by $2\triangle t$ and sum in time n, for ε sufficiently small,

$$\begin{aligned} &\sum_{n=1}^{m}(< d(C^{n-1})\pi^n, \pi^n > - < d(C^{n-1})\pi^{n-1}, \pi^{n-1} >) + C\sum_{n=1}^{m}||\nabla\pi^n||^2\triangle t \\ &\quad \leq K(h^8 + \triangle t^2 + \sum_{n=1}^{m-1}||\xi^n||_1^2\triangle t) + \varepsilon\sum_{n=1}^{m}(||\pi^n||^2 + ||\nabla\pi^n||^2)\triangle t, \end{aligned}$$

and

$$\begin{aligned} &d'_*\sum_{n=1}^{m-1}||\pi^n||^2\triangle t + d_*||\pi^m||^2 + \sum_{n=1}^{m}||\nabla\pi^n||^2\triangle t \\ &\qquad\qquad \leq K(h^8 + \triangle t^2 + \sum_{n=1}^{m-1}||\xi^n||_1^2\triangle t). \end{aligned} \tag{27}$$

We can turn to the derivation of a corresponding evolution inequality for ξ^n. Subtracting (9)(b) from the discrete Galerkin scheme of (1)(b), we obtain

$$\begin{aligned} &< \phi\frac{\xi^n - \xi^{n-1}}{\triangle t}, Z > - < \nabla \cdot (D\nabla\xi^n), Z > \\ &\quad = - < \phi\frac{\partial c^n}{\partial t} + u^n \cdot \nabla c^n - \phi\frac{c^n - \check{c}^{n-1}}{\triangle t}, Z > \\ &\quad + < \phi\frac{\check{c}^{n-1} - \hat{c}^{n-1}}{\triangle t}, Z > - < \phi\frac{\xi^{n-1} - \hat{\xi}^{n-1}}{\triangle t}, Z > \\ &\quad - < \phi\frac{\zeta^n - \hat{\zeta}^{n-1}}{\triangle t}, Z > + < \nabla \cdot (D\nabla\zeta^n), Z > \\ &\quad + < [-(\xi^{n-1} + \zeta^{n-1}) + (c^{n-1} - c^n)\ ]\ q, Z > \\ &\quad + < b(C^{n-1})\frac{P^n - P^{n-1}}{\triangle t} - b(c^n)\frac{\partial p^n}{\partial t}, Z > \quad \forall Z \in \mathcal{M}_{1,P}(3,\delta). \end{aligned} \tag{28}$$

To obtain L^2 estimate for ξ, we choose $Z = \xi^n$ as test function in (28), and we denote the resulting right-hand side terms by $T_1, T_2, \cdots, T_7$. First we shall discuss the right-hand side of (28).

For T_1, similar to the discussion in [2,10], so that

$$\psi\frac{\partial c^n}{\partial\tau} = \phi\frac{\partial c^n}{\partial t} + u^n \cdot \nabla c^n,$$

The standard backward-difference error equation is given by

$$\frac{\partial c^n}{\partial t} - \frac{c^n - c^{n-1}}{\triangle t} = \frac{1}{\triangle t}\int_{t^{n-1}}^{t^n}(t - t^{n-1})\frac{\partial^2 c}{\partial t^2}dt,$$

analogously, along the tangent to the characteristic

$$\begin{aligned}\psi\frac{\partial c^n}{\partial \tau} &- \phi\frac{c^n - \breve{c}^{n-1}}{\triangle t}\\ &= \frac{\phi}{\triangle t}\int_{(\breve{x},\breve{y},t^{n-1})}^{(x,y,t^n)}\sqrt{(x(\tau)-\breve{x})^2 + (y(\tau)-\breve{y})^2 + (t(\tau)-t^{n-1})^2}\frac{\partial^2 c}{\partial \tau^2}d\tau\end{aligned} \tag{29}$$

So by the definition of section 2.1, we obtain

$$\begin{aligned}&< \psi\frac{\partial c^n}{\partial \tau} - \phi\frac{c^n - \breve{c}^{n-1}}{\triangle t}, \psi\frac{\partial c^n}{\partial \tau} - \phi\frac{c^n - \breve{c}^{n-1}}{\triangle t} >\\ &= \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{1}{4}h_x h_y\sum_{k,l=1}^{2}\cdot\\ &\{(\frac{\phi}{\triangle t}\int_{(\breve{x},\breve{y},t^{n-1})}^{(x,y,t^n)}\sqrt{(x-\breve{x})^2 + (y-\breve{y})^2 + (t-t^{n-1})^2}\frac{\partial^2 c}{\partial \tau^2}d\tau)(\xi_{ik}^x, \xi_{jl}^y)\}^2.\end{aligned}$$

Let E_{ij} be the plane from $(\breve{\xi}_{ik}^x, \breve{\xi}_{jl}^y, t^{n-1})$ to $(\xi_{ik}^x, \xi_{jl}^y, t^n)$ along the characteristic direction, then

$$\begin{aligned}&|||\psi\frac{\partial c^n}{\partial \tau} - \phi\frac{c^n - \breve{c}^{n-1}}{\triangle t}|||^2\\ &\quad\le Ch^2\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\sum_{k,l=1}^{2}\max_{(x,y)\in E_{ij}}|\frac{\partial^2 c}{\partial \tau^2}|^2\{\frac{\phi}{\triangle t}\cdot(\frac{\psi\triangle t}{\phi})\int_{(\breve{\xi}_{ik},\breve{\xi}_{jl},t^{n-1})}^{(\xi_{ik},\xi_{jl},t^n)}d\tau\}^2\\ &\quad\le K\triangle t^2 h^2\max_{(x,y)\in E}|\frac{\partial^2 c}{\partial \tau^2}|^2\end{aligned}$$

Thus, we can obtain the estimate of T_1

$$\begin{aligned}|T_1| &\le K|||\psi\frac{\partial c^n}{\partial \tau} - \phi\frac{c^n - \breve{c}^{n-1}}{\triangle t}|||\cdot|||\xi^n|||\\ &\le K\triangle t^2 h^2\max_{(x,y)\in E}|\frac{\partial^2 c}{\partial \tau^2}|^2 + \varepsilon|||\xi^n|||^2.\end{aligned} \tag{30}$$

By (8), we get

$$\begin{aligned}|T_2| &= |< \phi\frac{\breve{c}^{n-1} - \hat{c}^{n-1}}{\triangle t}, \xi^n >| = |< \nabla\bar{c}\cdot(u^n - U^n), \xi^n >|\\ &= |< \nabla\bar{c}\cdot[a(c^n)\nabla\eta^n + a(C^n)\nabla\pi^n + (a(c^n) - a(C^n))\nabla\tilde{p}^n], \xi^n >|.\end{aligned}$$

Similar to the estimation of T_4' in the pressure equation, (23) and lemma 3.1, we can get

$$|T_2| \le K(h^8 + \triangle t^2 + ||\xi^n||^2) + \varepsilon(||\xi^n||_1^2 + ||\nabla\pi^n||^2). \tag{31}$$

To handle T_3, we shall need another induction hypothesis. We assume that

$$||\nabla P^n||_{L^\infty} \le K, \qquad 0 \le n \le l-1. \tag{32}$$

If $l = 1$, we can start the induction by (27) to get

$$||\nabla P^0||_{L^\infty} \le ||\nabla\tilde{p}^0||_{L^\infty} + ||\nabla\pi^0||_{L^\infty} \le K + Kh^{-1}(h^4 + \triangle t) \le K,$$

for h sufficiently small and $\triangle t = o(h)$. We shall check that if $n = l$ (32) is right at the end of the proof. Then for T_3, we can obtain by lemma 3.6, [2,10], the induction hypotheses (23) and (32),

$$\begin{aligned}|T_3| &\le K|||\frac{\xi^{n-1}-\hat{\xi}^{n-1}}{\triangle t}|||\cdot|||\xi^n||| \le \varepsilon||\frac{\xi^{n-1}-\hat{\xi}^{n-1}}{\triangle t}||^2 + K||\xi^n||^2 \\ &\le \varepsilon||\nabla\xi^{n-1}||^2 + K||\xi^n||^2.\end{aligned} \tag{33}$$

Next we estimate T_4,

$$|T_4| \le K(\ |<\phi\frac{\zeta^n-\zeta^{n-1}}{\triangle t},\xi^n>|+|<\phi\frac{\zeta^{n-1}-\hat{\zeta}^{n-1}}{\triangle t},\xi^n>|\ ),$$

by the Taylor expansion, Cauchy inequality and lemma 3.1, lemma 3.6, we obtain

$$|<\phi\frac{\zeta^n-\zeta^{n-1}}{\triangle t},\xi^n>| \le K|||\zeta_t^n|||^2+\varepsilon|||\xi^n|||^2 \le Kh^8||c_t^n||_{H^4}^2+\varepsilon||\xi^n||^2,$$

and by two dimensional Taylor expansion and (32), similar to (17) and (18), it follows that

$$\begin{aligned}&|<\phi\frac{\zeta^{n-1}-\hat{\zeta}^{n-1}}{\triangle t},\xi^n>| \\ &\le K(\ |<U_1^n\zeta_x^{n-1},\xi^n>|+|<U_2^n\zeta_y^{n-1},\xi^n>|\ )+K\triangle t|||\xi^n||| \\ &\le K\ (h^8\|c^{n-1}\|_{H^5(\Omega)}^2+\triangle t^2)+\varepsilon(||\xi^n||^2+||\nabla\xi^n||^2),\end{aligned}$$

so we can get

$$|T_4| \le K\ h^8(\|c^{n-1}\|_{H^5}^2+\|c_t^n\|_{H^4}^2)+K\triangle t^2+\varepsilon(||\xi^n||^2+||\nabla\xi^n||^2). \tag{34}$$

Then, similar to T_4, by (14) and (21), we have

$$\begin{aligned}|T_5| &= |<\nabla\cdot(D\nabla\zeta^n),\xi^n>| \\ &\le |<(D\zeta_x^n)_x,\xi^n>|+|<(D\zeta_y^n)_y,\xi^n>| \\ &\le K\ h^8\|c^n\|_{H^6}^2+\varepsilon(||\xi^n||^2+||\nabla\xi^n||^2).\end{aligned} \tag{35}$$

And using lemma 3.1, lemma 3.2, lemma 3.6, we shall get

$$|T_6| \le K(\ h^8+\triangle t^2+||\xi^{n-1}||^2)+\varepsilon||\xi^n||^2. \tag{36}$$

Similar to the pressure equation estimate (10), T_7 can be written as

$$\begin{aligned}|T_7| &\le |<d(C^{n-1})d_t\pi^n,\xi^n>|+|<[\ d(C^{n-1})-d(c^n)\ ]d_t\tilde{p}^n,\xi^n>| \\ &+|<d(c^n)d_t\eta^n,\xi^n>|+|<d(c^n)(d_tp^n-\frac{\partial p^n}{\partial t}),\xi^n>| \\ &\le K(h^8+\triangle t^2+||\xi^{n-1}||^2)+\varepsilon||\xi^n||^2 \\ &+|<d(C^{n-1})\frac{\pi^n-\pi^{n-1}}{\triangle t},\xi^n>|.\end{aligned} \tag{37}$$

Thus we obtain the estimate of the right-side of (28) by the preceding, next for the left-hand side of (28) we use the inequality $\frac{1}{2}(a^2-b^2) \le a(a-b)$ and lemma 3.8, such that

$$\begin{aligned}&\frac{1}{2\triangle t}\{<\phi\xi^n,\xi^n>-<\phi\xi^{n-1},\xi^{n-1}>\}+C_*|||\xi^n|||_{H_0^1(\Omega)}^2 \\ &\qquad\le <\phi\frac{\xi^n-\xi^{n-1}}{\triangle t},\xi^n>-<\nabla\cdot(D\nabla\xi^n),\xi^n>.\end{aligned} \tag{38}$$

So by (30)-(38), we now have

$$\begin{aligned}
&\frac{1}{2\triangle t}\{< \phi\xi^n, \xi^n > - < \phi\xi^{n-1}, \xi^{n-1} >\} + C_*|||\xi^n|||^2_{H_0^1(\Omega)} \\
&\qquad \le K(\triangle t^2 + \triangle t^2 h^2 + h^8 + ||\xi^{n-1}||^2 + ||\xi^n||^2) \qquad (39)\\
&\qquad + \varepsilon(||\xi^n||_1^2 + ||\nabla\pi^n||^2) + |< d(C^{n-1})\frac{\pi^n - \pi^{n-1}}{\triangle t}, \xi^n >|.
\end{aligned}$$

If (39) is multiplied by $2\triangle t$ and summed in time n ($\xi^0 = 0, \triangle t = o(h)$), then it follows that

$$\begin{aligned}
&< \phi\xi^m, \xi^m > + C_*\sum_{n=1}^{m}|||\xi^n|||^2_{H_0^1(\Omega)}\triangle t \\
&\qquad \le K\ (\triangle t^2 + h^8 + \sum_{n=1}^{m}||\xi^n||^2\triangle t\ ) + \varepsilon\sum_{n=1}^{m}(||\xi^n||_1^2 + ||\nabla\pi^n||^2)\triangle t \qquad (40)\\
&\qquad + 2\sum_{n=1}^{m}|< d(C^{n-1})(\pi^n - \pi^{n-1}), \xi^n >|,
\end{aligned}$$

where the right-hand side last term of (40) can be written as

$$\begin{aligned}
&\sum_{n=1}^{m}|< d(C^{n-1})(\pi^n - \pi^{n-1}), \xi^n >| \qquad (41)\\
&\qquad \le d'^*\sum_{n=1}^{m-1}||\pi^n||^2\triangle t + d^*||\pi^m||^2 + \varepsilon\sum_{n=1}^{m}||\xi^n||^2\triangle t.
\end{aligned}$$

So the relations (40) and (41) can be combined with (27) and the Gronwall lemma for sufficiently small ε to show that

$$\max_{1\le n\le m}\|\xi^n\|^2 + C_*\sum_{n=1}^{m}|||\xi^n|||^2_{H_0^1(\Omega)}\triangle t \le K\{\triangle t^2 + h^8\}, \qquad (42)$$

then lemma 3.5 and (42) can be combined with (27) to show that

$$\sum_{n=1}^{m}||\nabla\pi^n||^2\triangle t \le\ K\{\ \triangle t^2 + h^8\ \}, \qquad (43)$$

At last we shall check the induction hypotheses (32) and (23)

$$\begin{aligned}
||\nabla P^l||_{L^\infty} &\le ||\nabla\tilde{p}^l||_{L^\infty} + ||\nabla\pi^l||_{L^\infty} \le K + Kh^{-1}||\nabla\pi^l|| \\
&\le K + Kh^{-2}(\triangle t + h^4) \le K, \\
||C^l||_{W^1_\infty} &\le ||\tilde{c}^l||_{W^1_\infty} + ||\xi^l||_{W^1_\infty} \le K + Kh^{-2}||\xi^l|| \\
&\le K + Kh^{-2}(\triangle t + h^4) \le K,
\end{aligned}$$

for h sufficiently small , and the proof is complete.

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