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NUMERICAL SOLUTIONS TO BEAN'S CRITICAL-STATE MODEL FOR TYPE-II SUPERCONDUCTORS

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Abstract. In this paper we study the numerical solution for an $p-$Laplacian type of evolution system $\mathbf{H}_t + \nabla \times \left[|\nabla \times \mathbf{H}|^{p-2}\nabla \times \mathbf{H}\right] = \mathbf{F}(x,t), p > 2$ in two space dimensions. For large p this system is an approximation of Bean's critical-state model for type-II superconductors. By introducing suitable transformation, the system is equivalent to a nonlinear parabolic equation. For the nonlinear parabolic problem we obtain the numerical solution by combining approximation schemes for the linear equation and the nonlinear semigroup. The convergence and stability of the scheme are proved. Finally, a numerical experiment is presented.

Key Words. Approximation of Bean's Critical-State model, Numerical solutions.

1. Introduction

Bean's critical-state model for type-II superconductors describes the evolution of a magnetic field in an alloy-type of metal material under the external force ([4, 7]). The electric field $\mathbf{E}$ and the current density $\mathbf{J} = \sigma\mathbf{E}$ by Ohm's law are characterized as follows: there exists a critical current, denoted by J_c, such that $|\mathbf{J}| \leq J_c$ and

$$|\mathbf{E}| = \begin{cases} 0, & \text{if } |\mathbf{J}| < J_c, \\ [0,\infty), & \text{if } |\mathbf{J}| = J_c, \\ \emptyset, & \text{if } |\mathbf{J}| > J_c. \end{cases}$$

Here and thereafter, a bold letter represents a vector or vector function in R^3.

If one scales the value of critical current by assuming $J_c = 1$ without loss of generality, then the graph of $|\mathbf{E}|$ and $|\mathbf{J}|$ can be obtained formally from Ampere's law:

$$\mathbf{E} = |\nabla \times \mathbf{H}|^{p-2}\nabla \times \mathbf{H}$$

as $p \to \infty$, where $\mathbf{H}$ represents the magnetic field and the resistivity $\rho = \frac{1}{\sigma}$ is equal to $|\nabla \times \mathbf{H}|^{p-2}$.

This leads us to consider the following problem:

$$\mathbf{H}_t + \nabla \times \left[|\nabla \times \mathbf{H}|^{p-2}\nabla \times \mathbf{H}\right] = \mathbf{F}(x,t), \qquad (x,t) \in Q_T, \tag{1.1}$$

$$\nabla \cdot \mathbf{H} = 0, \qquad (x,t) \in Q_T, \tag{1.2}$$

$$\mathbf{n} \times \mathbf{H}(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,T], \tag{1.3}$$

$$\mathbf{H}(x,0) = \mathbf{H}_0(x), \qquad x \in \Omega, \tag{1.4}$$

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where Ω is a bounded simply-connected domain in R^3 and $Q_T = \Omega \times (0, T]$, $p \geq 2$ is fixed, $\mathbf{n}$ is the outward unit normal on $\partial\Omega$ and $x = (x_1, x_2, x_3)$, $\mathbf{F}(x,t)$ represents the applied magnetic current.

For large p the electric resistivity $\rho = |\nabla \times \mathbf{H}|^{p-2}$ is small in the region $S_\varepsilon = \{(x,t) : |\nabla \times \mathbf{H}| \leq 1 - \varepsilon\}$ while it is very large in $\{(x,t) : |\nabla \times \mathbf{H}| \geq 1 + \varepsilon\}$, where ε is a small constant. Thus, the resistivity ρ in S_ε becomes smaller and smaller as p increases and eventually S_ε becomes the superconductor region as $\varepsilon \to 0$ (no resistivity). The region $\{(x,t) : 1 - \varepsilon < |\nabla \times \mathbf{H}| < 1 + \varepsilon\}$ is the transition zone and formally becomes a sharp interface between the normal and superconductor regions as $\varepsilon \to 0$.

Unlike Ginzburg-Landau's model for superconductors (see [7, 8]), the model problem (1.1)-(1.4) describes the macro-motion of magnetic currents and is often used by experimental physicists in searching of Type-II superconductor materials. For two-space dimensions, by a variational argument Prigozhin [13] proved the existence of a unique weak solution to Bean's model. Well-posedness of the problem (1.1)-(1.4) were established in [17] for R^3 and for a bounded simply-connected domain in R^3 in [18]. Regularity of the weak solution as well as the limit solution as $p \to \infty$ were also investigated in these papers (see [17, 18]). Particularly, the authors of [18] established the existence of a unique weak solution to Bean's model in three-space dimensions. Several researchers have investigated numerical solutions for Bean's model. Bossavit [1] and Prigozhin [14] studied the numerical solutions for the case where $\mathbf{H}$ has only one non-zero component. More recently, the authors of [10] studied the numerical solution of Bean's model via a finite element method and derived error estimate. For the problem (1.1)-(1.4), Barrett and Prigozhin in [3] discussed the finite element solution by assuming that $\mathbf{H}$ has one non-zero component. In the present paper, we study the numerical solution for the problem (1.1)-(1.4) in two space dimensions. By using a suitable transformation, we convert the system (1.1) to a nonlinear parabolic equation with possible degeneracy in the leading term. Based on the nonlinear semigroup theory, an algorithm is presented by a finite difference method. We calculate the numerical solution by solving a linear heat equation and using time-marching iteration technique. The method is quite easy to implement. We also show that this numerical method is convergence and stable in L^1 if we choose parameter properly. Moreover, the numerical scheme is unconditionally stable. Finally, a numerical experiment is given to verify our result.

The paper is organized as follows. In § 2, we transform the problem into a fully nonlinear parabolic equation. In § 3, we first present a numerical method to solve the nonlinear parabolic problem and then prove the convergence and stability of the numerical solution. In § 4 a numerical experiment is presented.

2. A New Formulation of the Problem

Consider the problem (1.1)-(1.4) in two space dimensions. Assume that $\mathbf{H}$ and $\mathbf{F}$ depend on (x_1, x_2) and component in z-direction is zero, i. e., $\mathbf{H}(x,t) = \{h_1(x,t), h_2(x,t), 0\}$, $\mathbf{F} = \{f_1(x,t), f_2(x,t), 0\}$,$\mathbf{H}_0 \in C^{2+\alpha}(R^2)$, $\mathbf{F} \in C^{2+\alpha}(0, T; R^2)$, $\mathbf{H}_0$ and $\mathbf{F}$ have compact support.

By the Theorem 2.2 of [17], we know that the problem (1.1)-(1.4) has a unique weak solution $\mathbf{H}(x,t)$ in $Q_T = \mathbf{R}^2 \times (0, T]$. Moreover $\mathbf{H}_t \in L^2(\mathbf{R}^2 \times (0, T])$, $\mathbf{H} \in L^\infty(0, T; B^d(\mathbf{R}^2))$, where $B^d(\mathbf{R}^2) = \{\mathbf{G}(x) \in W^{1,p}(\mathbf{R}^2) : \nabla \cdot \mathbf{G} = 0$, a. e. $x \in \mathbf{R}^2\}$. Moreover, $\mathbf{H}(x,t)$ has compact support for each $t \in [0, T]$. Therefore we can assume that Ω is sufficiently large such that the weak solution $\mathbf{H}$ satisfies *supp*$(\mathbf{H}(\cdot,t)) \subset \Omega$ for all $t \in (0,T]$. That is, $\mathbf{H}(x,t)|_{\partial\Omega} = 0$, $\frac{\partial \mathbf{H}(x,t)}{\partial \mathbf{n}}|_{\partial\Omega} = 0$, where $\mathbf{n}$ is the outward normal on $\partial\Omega$. For simplicity, we set $\Omega = [-a,a] \times [-a,a]$ for some sufficient large constant $a > 0$. Then

$$\nabla \times \mathbf{H} = (h_{2x_1}(x,t) - h_{1x_2}(x,t))\mathbf{k}$$

where $\mathbf{k}$ is the unit vector in z-direction. It follows that

$$\begin{aligned} &\nabla \times (|\nabla \times \mathbf{H}|^{p-2}\nabla \times \mathbf{H}) = \nabla \times \{0, 0, |h_{2x_1} - h_{1x_2}|^{p-2}(h_{2x_1} - h_{1x_2})\} \\ = \quad &\{[|h_{2x_1} - h_{1x_2}|^{p-2}(h_{2x_1} - h_{1x_2})]_{x_2}, -[|h_{2x_1} - h_{1x_2}|^{p-2}(h_{2x_1} - h_{1x_2})]_{x_1}, 0\} \end{aligned}$$

Hence the equation (1.1) is equivalent to the following systems

$$h_{1t} + \frac{\partial}{\partial x_2}[|h_{2x_1} - h_{1x_2}|^{p-2}(h_{2x_1} - h_{1x_2})] = f_1(x_1, x_2, t), \tag{2.1}$$

$$h_{2t} + \frac{\partial}{\partial x_1}[|h_{2x_1} - h_{1x_2}|^{p-2}(h_{2x_1} - h_{1x_2})] = f_2(x_1, x_2, t), \tag{2.2}$$

Since $\mathbf{H} = 0$ on $x_2 = -a$, for all $-a \le x_1 \le a$ and $t \in (0,T]$. Then from Eq.(1.2), we see that

$$h_2(x_1, x_2, t) = -\int_{-a}^{x_2} h_{1x_1}(x_1, s, t)ds = -(\widehat{H}_1)_{x_1}(x_1, x_2, t),$$

where

$$\widehat{H}_1(x_1, x_2, t) = \int_{-a}^{x_2} h_1(x_1, s, t)ds.$$

It follows that

$$\widehat{H}_{1x_2x_2} = h_{1x_2} \qquad \text{and} \qquad \widehat{H}_{1x_1x_1} = -h_{2x_1}, \qquad (x_1, x_2, t) \in Q_T.$$

We use the above observation in Eq. (2.2) to obtain

$$\frac{\partial}{\partial x_1}\widehat{H}_{1t} - \frac{\partial}{\partial x_1}[|\triangle \widehat{H}_1|^{p-2}\triangle \widehat{H}_1] = -f_2(x,t), \qquad (x,t) \in Q_T.$$

Integrating x_1 over $(-a, x_1)$ yields

$$\widehat{H}_{1t} - |\triangle \widehat{H}_1|^{p-2}\triangle \widehat{H}_1 = -\int_{-a}^{x_1} f_2(s, x_2, t)ds + M_0(x_2, t).$$

where $M_0(x_2, t)$ is an unknown function of x_2 and t.

Since $\mathbf{H}(x,t)$ has compact support, we see

$$M(x_2, t) = \widehat{H}_{1t} - |\triangle \widehat{H}_1|^{p-2}\triangle \widehat{H}_1 \; |_{x_1=-a} = 0.$$

It follows that Eq.(2.2) becomes

$$\widehat{H}_{1t} - |\triangle \widehat{H}_1|^{p-2}\triangle \widehat{H}_1 = \widehat{f}_1(x,t)$$

where

$$\widehat{f}_1(x,t) = -\int_{-a}^{x_1} f_2(s, x_2, t)ds$$

Similarly, we can eliminate $h_2(x,t)$ to derive the following equation from (2.1)

$$\widehat{H}_{2t} - |\Delta \widehat{H}_2|^{p-2}\Delta \widehat{H}_2 = \widehat{f}_2(x,t), \qquad (x,t) \in Q_T,$$

where

$$\widehat{H}_2(x,t) = \int_{-a}^{x_1} h_2(s, x_2, t)ds, \qquad \text{and } \widehat{f}_2(x,t) = -\int_{-a}^{x_2} f_1(x_2, s, t)ds.$$

Hence the problem (1.1)-(1.4) is transformed to the following fully nonlinear parabolic equations subject to initial and boundary conditions:

$$\begin{cases} \widehat{H}_{it} - |\Delta \widehat{H}_i|^{p-2}\Delta \widehat{H}_i = \widehat{f}_i, & (x,t) \in Q_T, \\ \widehat{H}_i(x_1,-a,t) = \widehat{H}_i(x_1,a,t) = 0, & -a \le x_1 \le a, t \in [0,T], \\ \widehat{H}_i(-a,x_2,t) = \widehat{H}_i(a,x_2,t) = 0, & -a \le x_2 \le a, t \in [0,T], \\ \widehat{H}_i(x_1,x_2,0) = \widehat{h}_{0i}(x_1,x_2), & i = 1,2 \end{cases} \tag{2.3}$$

where

$$\widehat{h}_{01}(x_1,x_2) = \int_{-a}^{x_2} h_{01}(x_1,s)ds, \qquad \text{and} \qquad \widehat{h}_{02}(x_1,x_2) = \int_{-a}^{x_1} h_{02}(s,x_2)ds.$$

Define $u_i(x,t) = \Delta \widehat{H}_i(x,t)$ and $g_i(x,t) = \Delta \widehat{f}_i(x,t), (x,t) \in Q_T$, for $i = 1,2$. Then u_i $(i = 1,2)$ will satisfy the following equation:

$$\begin{cases} u_{it} - \Delta\psi(u_i) = g_i(x,t), (x,t) \in Q_T \\ u_i(x,0) = u_{0i}(x), \qquad x \in \Omega \\ u(x,t) \mid_{\partial\Omega} = 0, \qquad t \in (0,T), \end{cases} \tag{2.4}$$

for $i = 1,2$, where $u_{01}(x) = \Delta \int_{-a}^{x_2} h_{01}(x_1,s)ds$, $u_{02}(x) = \Delta \int_{-a}^{x_1} h_{02}(s,x_2)ds$, and the function ψ is defined by $\psi(s) = |s|^{p-2}s$.

Note that $\mathbf{F} \in C^{2+\alpha}(Q_T)$ and $H_0 \in C^{2+\alpha}(R^2)$, it follows that $g_i(x,t)$ is continuous in Q_T and u_{0i} is continuous in Ω. By the result of [9], there exists a unique solution $u_i(x,t) \in L^\infty(Q_T) \cap C^\alpha(Q_T)$ for some $\alpha > 0$. Moreover, the solution is Hölder continuous in Q_T.

3. Numerical Method, Convergence and Stability

Let us consider the numerical method to solve the following problem:

$$\begin{cases} u_t - \Delta\psi(u) = g, & \text{in } Q_T, \\ u(x,t) = 0, & x \in \partial\Omega, t \in (0,T), \\ u(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{3.1}$$

where a function by $\psi(s) = |s|^{p-2}s$ and $u_0 \in L^1(\Omega)$. For simplicity, we assume that g is independent t and continuous in Ω.

We define an operator $Lu = -\Delta u$ with $D(L) = \{u \in W_0^{1,1}(\Omega); Lu \in L^1(\Omega)\}$. Then L generates a contractive semigroup $\{T(t), t \ge 0\}$ in $L^1(\Omega)$ and $\overline{D(L)} = L^1(\Omega)$. Let N be a large integer $\tau = T/N$ be the time step, $t_n = n\tau, u^n(\cdot) = u(\cdot, t_n)$. Suppose $\sigma : (0,+\infty) \to (0,+\infty)$ be a function such that $\lim_{\tau\to 0} \sigma_\tau = 0$. The algorithm is given as formula

$$\frac{u^{k+1} - u^k}{\tau} + [\frac{I - T(\sigma_\tau)}{\sigma_\tau}]\psi(u^k) = g, \qquad x \in \Omega \tag{3.2}$$

where I is the identity operator. That is, u^{k+1} is determined from u^k by

$$u^{k+1} = F(\tau)u^k, \qquad x \in \Omega$$

where

$$F(\tau)\varphi := \varphi + \frac{\tau}{\sigma_\tau}[T(\sigma_\tau)\psi(\varphi) - \psi(\varphi)] + \tau g \tag{3.3}$$

for $\varphi \in L^1(\Omega)$.

We define the approximation $u_n(x,t)$ to $u(x,t)$ by

$$u_n(x,t) = u^n = [F(\frac{t}{n})]^n u_0(x). \tag{3.4}$$

Then we have:

Theorem 1 *Assume $u_0 \in L^\infty(\Omega)$, set $M = \|u_0\|_{L^\infty(\Omega)}$ and $\mu = (p-1)M^{p-2}$. If the following stability condition holds: If*

$$\mu\tau/\sigma_\tau \le 1 \qquad \text{for every } \tau > 0. \tag{3.5}$$

Then

$$\|F(\tau)\eta - F(\tau)\xi\|_{L^1(\Omega)} \le \|\eta - \xi\|_{L^1(\Omega)} \qquad \text{for } \eta, \xi \in L^1(\Omega), \tag{3.6}$$

and $\lim_{n\to+\infty} u_n(\cdot,t) = u(\cdot,t)$ in $L^1(\Omega)$. Moreover, the convergence is uniform for t in any given bounded interval, where $u(x,t)$ is the weak solution of (3.1).

Proof: At first, we claim that for every $k \ge 0$, $-M \le u^k \le M$. It is shown by induction. Assume $-M \le u^k \le M$. Since $\psi(r) = |r|^{p-2}r$ is nondecreasing and $T(\sigma_\tau)$ is contraction, it follows that $\psi(-M) \le \psi(u^k) \le \psi(M)$ and

$$\psi(-M) \le T(\sigma_\tau)\psi(u^k) \le \psi(M). \tag{3.7}$$

By the fact:

$$|\psi(r) - \psi(s)| \le (p-1)M^{p-2}|r - s|$$

for all $s, r \in R$ with $|s| \le M, |r| \le M$ (see, Ch26, [19]) and (3.5), we obtain

$$-M - \tau\psi(-M)\sigma_\tau \le u^k - \tau\psi(u^k)\sigma_\tau \le M - \tau\psi(M)\sigma_\tau. \tag{3.8}$$

Combining (3.7) and (3.8), one obtains $-M \le u^{k+1} \le M$. So in the following, we can assume $\psi(s)$ defined in domain $[-M, M]$. Then the function $r - \tau\psi(r)/\sigma$ is nondecreasing in r. Hence

$$\frac{\tau}{\sigma_\tau}|\psi(r) - \psi(s)| + |r - s - \frac{\tau}{\sigma_\tau}(\psi(r) - \psi(s))| = |r - s| \tag{3.9}$$

It follows from the contraction of $\{T(t)\}$ that

$$\begin{aligned} &\|F(\tau)\eta - F(\tau)\xi\|_{L^1(\Omega)} \\ \le\ & \frac{\tau}{\sigma_\tau}\|\psi(\eta) - \psi(\xi)\|_{L^1(\Omega)} + \|(\eta - \xi) - \frac{\tau}{\sigma_\tau}(\psi(\eta) - \psi(\xi))\|_{L^1(\Omega)}. \end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10), we obtain the estimate (3.6).

Let A be an operator defined by $Au = L\psi(u)$ in $L^1(\Omega)$ with domain $D(A) = \{u \in L^1(\Omega); \psi(u) \in D(L)\}$. It is known ([15], lemma 2.6.2) that A is m-accretive and $-A$ generates a contraction semigroup$\{S(t), t \ge 0\}$ on $\overline{D(A)} = L^1(\Omega)$.

By applying the nonlinear version of famous Chernoff theorem (see theorem 2.2 of [2]), we remain to verify that for every $\zeta \in L^1(\Omega)$ and every $\lambda > 0$:

$$[I + \frac{\lambda}{\tau}(I - F(\tau))]^{-1}\zeta \to [I + \lambda A]^{-1}\zeta \qquad \text{as } \tau \to 0.$$

That is, we need show that the solution θ_τ of the following equation

$$\theta_\tau + \frac{\lambda}{\tau}(I - F(\tau))\theta_\tau = \zeta \tag{3.11}$$

tends θ in $L^1(\Omega)$ as $\tau \to 0$, where θ is a solution of the equation

$$\theta + \lambda A\theta = \zeta. \tag{3.12}$$

In fact, $\theta \in D(A)$ implies $\psi(\theta) \in D(L)$. Then since $T(t)$ is a contraction C_0-semigroup of L, we know that

$$\frac{I - T(\sigma_\tau)}{\sigma_\tau}\psi(\theta) \longrightarrow L\psi(\theta) \equiv A\theta \qquad \text{in } L^1(\Omega) \text{ as } \tau \to 0.$$

By the definition of $F(\tau)$, (3.10), and (3.11), one can see

$$(1+\frac{\lambda}{\tau})(\theta_\tau-\theta)=\frac{\lambda}{\tau}(F(\tau)\theta_\tau-F(\tau)\theta)-\lambda\frac{I-T(\sigma_\tau)}{\sigma_\tau}\psi(\theta)+\lambda A\theta$$

So, by the inequality (3.6), we obtain

$$\|\theta_\tau-\theta\|_{L^1(\Omega)}\leq|\lambda|\|\frac{I-T(\sigma_\tau)}{\sigma_\tau}\psi(\theta)+A\theta\|_{L^1(\Omega)}.$$

The proof is complete.

Remark 3.1: If g depends on t, we can get similar convergence theorem by using the formula

$$u(t)=S(t)u_0+\int_0^t S(t-\tau)g(\tau)d\tau$$

since $u_n(x,t)\to u(x,t)$ is uniform for t.

Let Σ denote a regular triangulation of Ω with vertices $\{x_j\}$, edges $\{s_{ij}\}$, and triangles $\{\tau_{ijk}\}$. The interior of the polygon associated with all the edges connecting the vertex x_j is denoted by Ω_j. If a vertex x_i is on the boundary of Ω, the region Ω_i is modified to include only the portion that is inside Ω. The area of Ω is denoted by $|\Omega|$. Let J be the number of grid point. Suppose one has approximate solution values U_j^n at time $t^n=n\tau$ and grid points $\{x_j\}\subset\overline{\Omega}$. For example, $U_j^0=\frac{1}{|\Omega_j|}\int_{\Omega_j}u_0(x)dx, j=1,\cdots J$. To obtain the approximate solution values U_j^{n+1} at the next time level $t^{n+1}=t^n+\tau$, we can using the following algorithm:

Set

$$Q_j^n=\frac{\psi(U_j^n)}{\alpha},\qquad j=1,2,\cdots,J$$

where α can be chosen by any positive number satisfying $\mu/\alpha\leq 1$. Solve the following linear heat equation by using any appropriate numerical method

$$\begin{cases} Q_t=\alpha\Delta Q & \text{in } \Omega\\ Q(x,t)=0 & \text{on } \partial\Omega\\ Q(x_j,t^n)=Q_j^n, & \quad\text{for } j=1,\cdots,J\end{cases}\tag{3.13}$$

We can obtain values $\{Q_j^{n+1}\}$ at the underlying grid point $\{(x_j)\}_{j=1}^J$ at $t=t^n+\tau$, then

$$U_j^{n+1}=U_j^n+Q_j^{n+1}-\frac{\psi(U_j^n)}{\alpha}+G_j^n\tag{3.14}$$

for j such that $(x_j)\in\Omega$ and

$$U_j^{n+1}\equiv 0\qquad\text{when } (x_j)\in\partial\Omega.$$

where $G_j^n=\frac{1}{|\Omega_j|}\int_{\Omega_j}g(x,t^n)dx$.

If $U^n=U_j^n$ and $\overline{U^n}$ be two different starting values in (3.13) and (3.14), it follows from (3.6) that

$$\|U^{n+1}-\overline{U^{n+1}}\|_{L^1(\Omega)}\leq\|U^n-\overline{U^n}\|_{L^1(\Omega)}.$$

Theorem 2 *(Stability) The numerical method (3.13) and (3.14) is L^1 stable if $\alpha\geq\mu$ and the numerical method used in linear heat equation (3.13) is L^1 stable, for example, by standard Crank-Nicolson or implicit scheme.*

The numerical solution of $\widehat{H}$ can be obtained by solving the following Laplace equation using finite element method or finite difference method for every $t^n = n\tau$.

$$\begin{cases} \Delta\widehat{H}(x,t^n) = U^n, & x \in \Omega, \\ \widehat{H}(x,t^n) = 0, & x \in \partial\Omega. \end{cases} \tag{3.15}$$

where $U^n = I_h U_j^n$ and $I_h g$ stands for the piecewise linear interplant of g, for any $n = 1, 2, \cdots, N$.

By the formula $h_1 = \widehat{H}_{1x_2}$ and $h_2 = \widehat{H}_{2x_1}$, we can using any numerical differentiation method for the first derivative to get h_1 and h_2, for example

$$h_{1ij}^n = \frac{\widehat{H}_{i,j+2}^n + 4\widehat{H}_{i,j+1}^n - 3\widehat{H}_{i,j}^n}{2h} \tag{3.15}$$

for all $(x_{1i}, x_{2j}) \in \Omega$ and $n = 1, 2, \cdots, N$. For the boundary, because of support H is compact, it is always zero.

Remark 3.2: All the results in this section hold for n-space dimensions.

4. Numerical Examples

We study the performance of the method when applied to test problem

$$\begin{cases} u_t - \Delta|u|^{p-2}u = g, & (x,t) \in Q_T, \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) = \cos(\frac{\pi x}{2a})\cos(\frac{\pi y}{2a})/\pi & x \in \Omega. \end{cases} \tag{4.2}$$

where the right-hand side g is chosen such that

$$u(x,y,t) = (1-t)\cos(\frac{\pi x}{2a})\cos(\frac{\pi y}{2a})/\pi$$

is the exact solution.

We take $\alpha = p$ and use standard Crank-Nicolson difference method by taking $J = 40 \times 40$ grid points on $[-10,10] \times [-10,10]$ and using a variable number N of time steps to reach time $T = 8$. The numerical solution graphs and exact result graphs are showed in Figure 1,2,3 by taking $p = 3, 6, 9$ respectively.

Suppose the discrete L^1 error E at time T was calculated by

$$E_1 = \frac{20 \times 20}{J-1} \sum_{i,j=1}^{40} |U^N(x_i,y_j) - u(x_i,y_j,T)|$$

to get Table 1 from differential $\triangle t = \frac{T}{N}$ at time $T = 8$. Suppose J is large enough so that the spatial discretization error is relatively negligible, and assume that E_1 is given by $E_1 = C_1(\triangle t)^\gamma$, where C_1 is a constant independent of $\triangle t$. The numerical rate of convergence γ computed by

$$\gamma = (\ln(E_1) - \ln(\ \overline{E_1}\ )/\ln(2)$$

where E_1 and $\overline{E}_1$ are errors corresponding to N and $\overline{N} = 2N$. Table 1 is obtained by taking $p = 3, p = 6$ and $p = 9$ with different $\triangle t$. It shows that our method is effective.

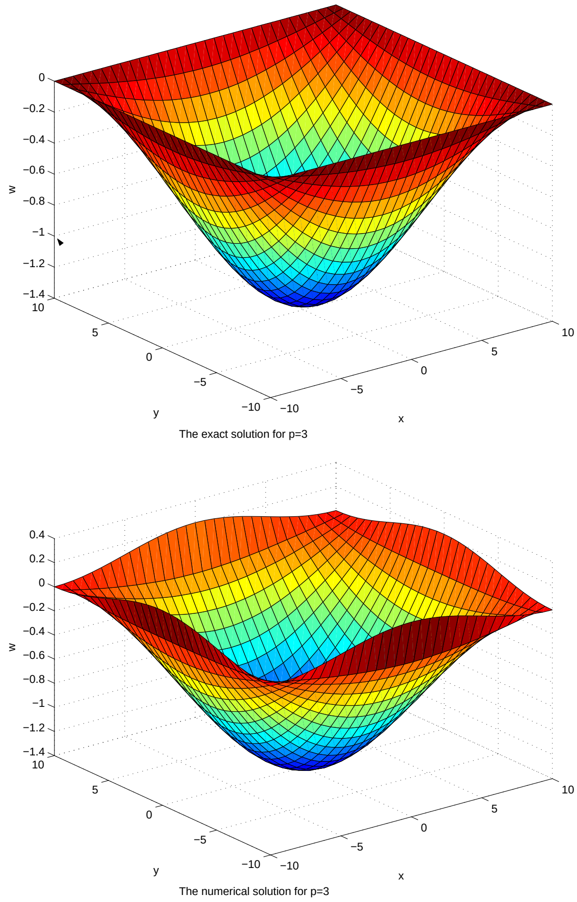

FIGURE 1. Exact and numerical solutions for $p = 3$

	$p=3$		$p=6$		$p=9$	
N	Error E_1	Rate γ	Error E_1	Rate γ	Error E_1	Rate γ
8	1.8504		2.0764		2.6477	
		0.7916		0.9732		0.8848
16	1.0690		1.0577		1.4339	
		0.6574		0.9529		1.0761
32	0.6706		0.5464		0.6801	
		0.4250		0.9189		0.9880
64	0.4995		0.2890		0.3429	
		0.2344		0.8877		0.9277
128	0.4246		0.1562		0.1932	

TABLE 1. Discrete L^1 errors and numerical convergence rates

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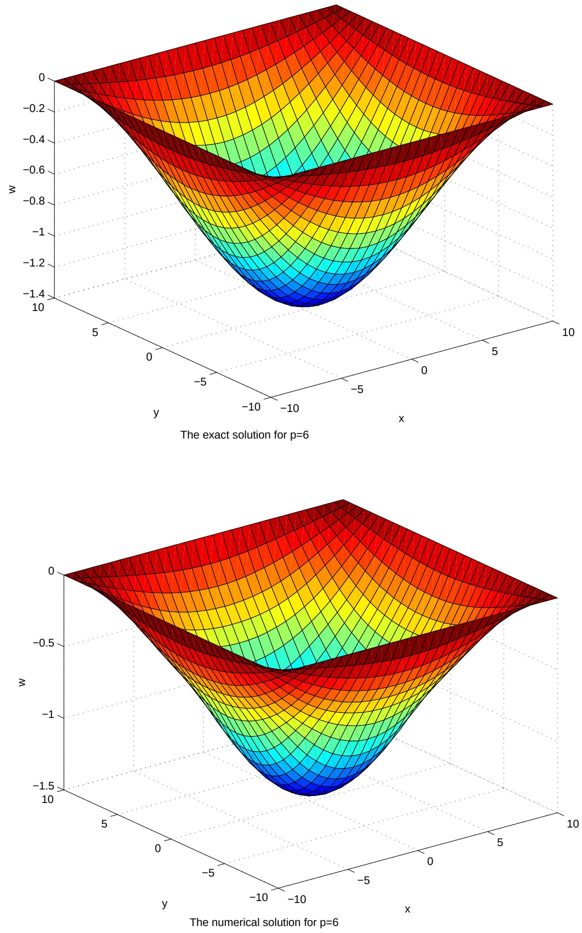

FIGURE 2. Eexact and numerical solutions for p=6

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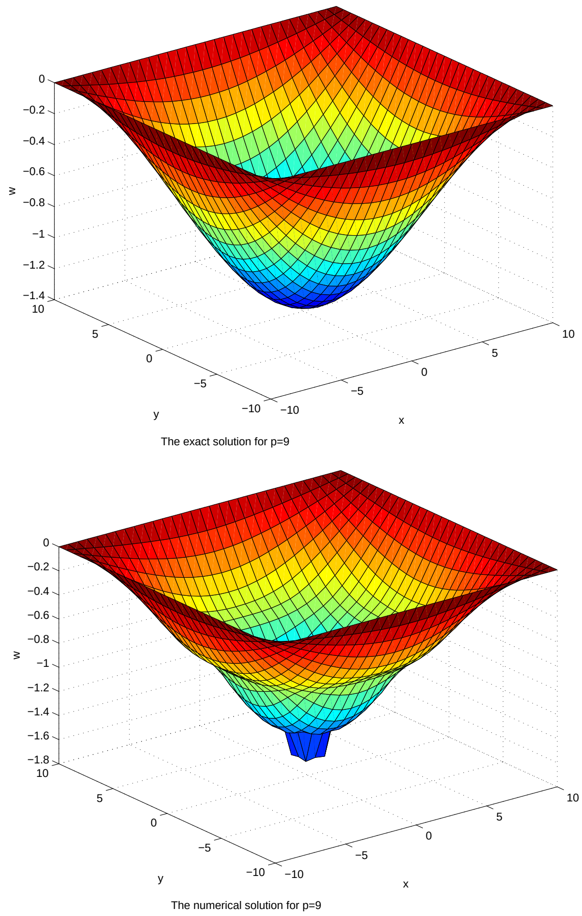

FIGURE 3. Exact and numerical solutions for $p = 9$

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