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TAYLOR EXPANSION ALGORITHM FOR THE BRANCHING SOLUTION OF THE NAVIER-STOKES EQUATIONS

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Abstract. The aim of this paper is to present a general algorithm for the branching solution of nonlinear operator equations in a Hilbert space, namely the k-order Taylor expansion algorithm, $k \geq 1$. The standard Galerkin method can be viewed as the 1-order Taylor expansion algorithm; while the optimum nonlinear Galerkin method can be viewed as the 2-order Taylor expansion algorithm. The general algorithm is then applied to the study of the numerical approximations for the steady Navier–Stokes equations. Finally, the theoretical analysis and numerical experiments show that, in some situations, the optimum nonlinear Galerkin method provides higher convergence rate than the standard Galerkin method and the nonlinear Galerkin method.

Key Words. Nonlinear operator equation, the Navier-Stokes equations, Taylor expansion algorithm, Optimum nonlinear Galerkin method.

1. Introduction

Many integral equations and differential equations in mathematical physics can be reduced to the operator equations. The operator equations and their numerical approximation are very important in the areas of theoretical mathematics and computational mathematics(see[1]). The main feature of the approximate theory of the operator equations is to apply the functional analytic method to the study of the numerical approximation of the operator equations, which will provide new ideas and new algorithms for the computational mathematics.

This paper is devoted to present the k-order Taylor expansion algorithms for the branching Solution of the nonlinear operator equations. The standard Galerkin (SG) method and the optimum nonlinear Galerkin (ONG) method can be viewed as specific Taylor expansion algorithms.As the important application of the algorithms, we consider the numerical approximations of the 2–D steady Navier-Stokes equations and estimate the convergence rates of the corresponding algorithms.Moreover, we also recall the convergence rate of the nonlinear Galerkin (NG) methods presented in[4–9]. Our theoretical analysis and numerical experiments show that the ONG method is of the higher convergence rate than the NG method and the SG method.

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2. Operator Equation and Taylor Expansion Algorithms

We are given a Hilbert space H with a scalar product$(\cdot,\cdot)$and a norm $|\cdot|$. The abstract operator equation that we will study has the form

$$F(u) = f. \tag{2.1}$$

Here $F : D(F) \subset H \to H$ is a nonlinear operator, $D(F) = v \in H; F(v) =\in H$is the domain of operator F, $f \in H$is given and $u \in D(f)$ is a unknown function(or vector function)defined in a bounded domain Ω of R^2or R^3.

We now recall the following Taylor expansion (see[1]).

Theorem 2.1. *Assume that $F : D(F) \to H$ is the continuous Fréchet differentiable of the k-order. Then for each $p \in D(F)$,$q \in H$, $p+q \in D(F)$ there holds the Taylor expansion with the integral remainder, namely*

$$\begin{aligned} F(p+q) &= F(p) + \tfrac{1}{1!}DF(p)q + \cdots + \tfrac{1}{(k-1)!}D^{k-1}F(p)q^{k-1} \\ &+\tfrac{1}{(k-1)!}\int_0^1 (1-t)^{k-1} D^k F(p+tq)q^k \,\mathrm{d}t. \end{aligned} \tag{2.2}$$

For each $n > 0$, we let H_n be a n–dimensional subspace of H and $P_n : H \to H_n$ be an orthogonal projection operator. To introduce the Taylor expansion algorithm, we select a large n and rewrite the solution u of (2.1) as

$$u = p + q, p = P_n u \in H_n, q = (I - P_n)u \in H\backslash H_n,$$

such that p represents the large eddies of the flow and q represents the small eddies of the flow, namely $|q| \to 0$(as $n \to \infty$) (refer to [2-3]). Hence, we apply respectively P_n and $Q_n = I - P_n$ to (2.1):

$$P_n F(p+q) = P_n f, \tag{2.3}$$

$$Q_n F(p+q) = Q_n f. \tag{2.4}$$

Assume that $F(u) = F(p+q)$ can be rewritten as the Taylor expansion (2.2). Thanks to q being the small eddies of the flow, it is then reasonable to neglect some small terms as $DF(p)q, \frac{1}{2!}DF(p)q^2, \cdots \frac{1}{(k-1)!}\int_0^1 (1-t)^{k-1}D^kF(p+tq)q^k \,\mathrm{d}t$, in (2.2). Then we obtain the following Taylor expansion algorithms:

the 1-order Algorithm : Find $u_{app} = y \in H_n$ such that

$$P_n F(y) = P_n f; \tag{2.5}$$

the 2-order Algorithm : find $u_{app} = y + z \in H, y \in H_n, z \in H\backslash H_n$, such that

$$P_n F(y+z) = P_n f, \tag{2.6}$$

$$Q_n(F(y) + DF(y)z) = Q_n f; \tag{2.7}$$

the k-order Algorithm : find $u_{app} = y + z \in H, y \in H_n, z \in H\backslash H_n$, such that

$$P_n F(y+z) = P_n f, \tag{2.8}$$

$$Q_n(F(y) + DF(y)z + \cdots \frac{1}{(k-1)!}D^{k-1}F(y)z^{k-1}) = Q_n f. \tag{2.9}$$

Notice that (2.7),(2.9) are the infinite dimensional system. From the computational point of view we have to replace $H\backslash H_n$ and Q_n by $H_N\backslash H_n$ and $Q_n^N = P_N - P_n$ in (2.7),(2.9), where $N > n$ will be chosen according to some convergence analysis.

In particular, we notice that the 1-order Taylor expansion Algorithm is then the standard Galerkin (SG) method. Moreover, the 2-order Taylor expansion Algorithm is called the optimum nonlinear Galerkin (ONG) method which id different from the nonlinear Galerkin methods presented in the papers[4-9].

3. Numerical Approximations of the Navier-Stokes Equations

The 2-D steady Navier-Stokes equations in the primitive variable formulation are written as

$$-\nu\Delta u + (u \cdot \bigtriangledown)u + \bigtriangledown\pi = g, \tag{3.1}$$

$$\text{div} u = 0, \tag{3.2}$$

where Ω is an open bounded set in R^2 with sufficient smooth boundary, $\nu > 0$ is the kinematic viscosity and $g = g(x)$ represents the external body force. The unknowns are the vector function u (velocity) and the scalar function π (pressure).

We will consider either the homogeneous Dirichlet boundary conditions, for which we denote:

$$V = \{v \in (H_0^1(\Omega)^2;\ \text{div} u = 0\},$$

or the periodic boundary conditions for which

$$V = \{v \in (H_p^1(\Omega)^2;\ \text{div} u = 0, \int_\Omega v(x)\,\mathrm{d}x = 0\}.$$

In both cases,we set

$$H = closure \quad of \quad V \quad in \quad (L^2(\Omega))^2.$$

Let$P$ be the orthonormal projection of $(L^2(\Omega))^2$ onto H we define the Stoke operator

$$Au = -P\triangle u, \qquad \forall u \in D(A) = V \cap (H^2(\Omega))^2,$$

and the bilinear operator

$$B(u, v) = P[(u \cdot \nabla)v], \forall u, v \in V.$$

The Stokes operator A is an unbounded positive self-adjoint closed operator in H with domain $D(A)$ and its inverse A^{-1} is compact in H. Consequently, there exists an orthonormal basis of H consisting of the eigenvectors w_j of A:

$$Aw_j = \lambda_j w_j, 0 < \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lambda_j \to \infty (asj \to \infty), \tag{3.3}$$

(see[2]). We denote the scalar products and norms of H and V by

$$|u| = (\int_\Omega |u(x)|^2\,\mathrm{d}x)^{1/2} \quad and \quad ||u|| = (\int_\Omega |\nabla u(x)|^2\,\mathrm{d}x)^{1/2}$$

The corresponding scalar products are denoted by $(\cdot, \cdot)$ and $((\cdot, \cdot))$ respectively.

We define a trilinear form on $V \times V \times V$ by

$$B(u, v, w) = < B(u, v), w >_{V', V}, \qquad \forall u, v, w \in V$$

It is easy to verify that b satisfies the following important property

$$b(u, v, w) = -b(u, w, v), \qquad \forall u, v, w \in V. \tag{3.4}$$

We recall some continuity properties satisfied by B and b (see[2,4-7]):

$$|b(u, v, w)| \leq c_0 |u|^{1/2} ||u||^{1/2} \|v\| \|w\|^{1/2} \|w\|^{1/2}, \tag{3.5}$$

$$|b(u, v, w)| \leq c_0 |u| ||v|| |w|^{1/2} |Aw|^{1/2}, \tag{3.6}$$

$$|b(u, v, w)| \leq c_0 |u|^{1/2} |Au|^{1/2} \|v\| |w|,\ |B(u, v)| \leq c_0 |u|^{1/2} |Au|^{1/2} \|v\|, \tag{3.7}$$

$$|B(u, v) \leq c_0 |u|^{1/2} \|u\|^{1/2} \|v\|^{1/2} |Av|^{1/2}. \tag{3.8}$$

Under the above notations,the system (3.1)-(3.2) is equivalent to the following abstract equation

$$\mathcal{F}(u,\nu) \equiv \nu Au + B(u,u) - f = 0, \tag{3.9}$$

where $f = Pg$. The following results are well known(see for instance [11,12]).

Theorem 3.1. *Let $f \in V'$, Ω be a bounded domain of R^2 with a Lipschitz continuous boundary Γ. Then the equation (3.9) admits at least a weak solution $u \in V$ such that*

$$\begin{cases} |A^{1/2}| \le \nu^{-1}||A^{-1/2}f|| \equiv M_1, \\ |A^{3/4}u| \le \nu^{-1}|A^{-1/4}f| + c\nu^{-3}|A^{-1/2}f|. \end{cases} \tag{3.10}$$

Moreover, if $f \in H$ then $u \in D(A)$ satisfies

$$|Au| \le \nu^{-1}|f| + c\nu^{-5}|A^{1/2}f|^3 + c\nu^{-2}|A^{-1/4}f||A^{-1/2}f| \equiv M_2. \tag{3.11}$$

If ν and f satisfy

$$c_0\nu^{-2}|f|_{V'} < 1, \tag{3.12}$$

then the solution of (3.9) is unique.

The proof can be found in [10,11,12].

Hereafter, we denote

$$H_m = span\{w_1, \cdots, w_m\},\ Q_m = I - P_m,$$

$P_m : H(or \quad V \quad or \quad D(A)) \to H_m$ is the orthogonal projector, where $m = n, N$. According to the definitions of H_m and P_m, there hold the following properties:

$$\begin{cases} P_mA = AP_m, \qquad Q_mA = AQ_m, \\ \|u\| \ge \lambda_1^{1/2}|u|, \qquad \forall u \in V, \\ \|q\| \ge \lambda_{m+1}^{1/2}|q|, \qquad \forall q \in V\backslash H_m, \\ \|p\| \le \lambda_m^{1/2}|p|, \qquad \forall p \in H_m, \\ |A^sp|^2 + |A^sq|^2 = |A^s(p+q)|^2, \quad \forall p \in H_m, \quad q \in D(A^s)\backslash H_m, \end{cases} \tag{3.13}$$

where $s = 0, \frac{1}{2}, \quad |A^0 \cdot| = |\cdot|, \quad |A^{1/2}\cdot| = \|\cdot\|$.

In order to apply the Taylor expansion algorithm to the abstract equation (3.9), we write

$$F(u) = \nu Au + B(u,u) - f = 0,$$

where $D(F) = D(A)$.We set $u = p + q \in D(A)$ be the solution of (3.9), where $p = P_nu \in H_n, q = Q_nu \in D(A)\backslash H_n$.

Thanks to Theorem 3.1 and (3.13), we have

$$\|q\|^2 = ((q,q)) = (q, Aq) \le |q||Aq| \le \lambda_{n+1}^{-1/2}\|q\||Aq|,$$

$$|q| \le \lambda_{n+1}^{-1/2}\|q\| \le \lambda_{n+1}^{-1}|Aq| \le \lambda_{n+1}^{-1}|Au| \le \lambda_{n+1}^{-1}M_2. \tag{3.14}$$

Hence, the Taylor expansion algorithm can be applied to the numerical approximation of problem (3.9). For this, we need the following Fréchet differentials of the nonlinear operator F at $p \in D(F)$:

$$DF(p)q = \nu Aq + B(p,q) + B(q,p),$$

$$D^2F(p)q^2 = 2B(q,q),\ D^3F(p)q^3 = 0.$$

Now, we can apply the r-order Taylor expansion algorithm with $k = 2, r = 1, 2$. Then we obtain the following numerical methods for solving (3.9):

The SG method: Find $u_G = y \in H_n$ such that

$$\nu Ay + P_nB(y,y) = P_nf; \tag{3.15}$$

the ONG method: Find $u_O = y + z, y \in H_n, z \in H_N \backslash H_n$ such that

$$\nu Ay + P_n B(y + z, y + z) = P_n f, \tag{3.16}$$

$$\nu Az + Q_n^N [B(y + z, y) + B(y, z)] = Q_n^N f, \tag{3.17}$$

where $Q_n^N = P_N - p_n = Q_n - Q_N, P_n A(y + z) = Ay, Q_n A(y + z) = Az$.

Remark 3.1. *The SG method consists in solving a nonlinear problem in a small space H_n; while the ONG method consists in solving a similar nonlinear subproblem in a small space H_n and solving a linear subproblem in $H_N \backslash H_n$.*

Finally, applying the nonlinear Galerkin (NG) method presented in [4-9] to the abstract equation (3.9), we derive the following numerical method:

the NG method: find $u_N = y + z, y \in H_n, z \in H_N \backslash H_n$ such that

$$\nu Ay + P_n[B(y + z, y) + B(y, z)] = P_n f, \tag{3.18}$$

$$\nu Az + Q_n^N B(y, y) = Q_n^N f. \tag{3.19}$$

Remark 3.2. *The NG method can not be viewed as the particular Taylor expansion algorithm. Also, the NG method consists in solving a nonlinear subproblem in small space H_n and solving a linear subproblem in the space $H_N \backslash H_n$.*

4. Existence and Uniqueness of the Numerical Solutions

In this section, we aim to prove the existence, uniqueness and regularity of the numerical solutions u_G, u_O and u_N.

First, we provide the existence uniqueness and regularity of the numerical solution $u_G \in H_n$.

Theorem 4.1. *Assume that $f \in V'$. Then the approximate problem (3.15) admits at least a weak solution $u_G \in H_n$ such that*

$$||u_G|| \leq \nu^{-2} ||f||_{V'}. \tag{4.1}$$

Moreover, if $f \in H$ then

$$|Au_G| \leq M_2. \tag{4.2}$$

If ν and f satisfy the unique condition (3.12) then the solution of (3.15) is unique.

This proof is classical, it can be omitted.

To prove the existence and uniqueness of the numerical solution $u_0 \in H_N$; we need the following lemma. In order to completing we give the proof (see[10], [11],[12]).

Lemma 4.2. *Let X be a finite dimensional Hilbert space with scalar product $[\cdot, \cdot]$ and norm $[\cdot]$. Let G be a continuous mapping from X into itself such that*

$$[G(\xi), \xi] \geq 0, \quad \forall \xi \in X \quad \textit{with} \quad [\xi] = \mu > 0. \tag{4.3}$$

Then there exists an element $\xi \in X$, such that

$$G(\xi) = 0, |\xi| \leq \mu. \tag{4.4}$$

Proof. The proof proceeds by contradiction. Suppose $G(\xi) \neq 0$ in the closed sphere $S = \{\xi \in X; |\xi| \leq \mu\}$. Then the mapping $\xi \to -\mu G(\xi)/|G(\xi)|$ is continuous from S into S. As the dimension of X id finite and sine the set S is obvious non-void, convex and compact, we may apply classical fixed point theorem due to Brouwer, we can conclude that there exists and $\xi \in S$ such that

$$\xi = -\mu G(\xi)/|G(\xi)|.$$

Thus, we have exhibited and $\xi \in X$ such that $|\xi| = \mu$ and

$$(G(\xi), \xi) = -\mu |G(\xi)| < 0.$$

This contradicts (4.3). The proof is completed. □

Theorem 4.3. *Assume that $f \in V'$ and Ω be a bounded domain with Lipschitz boundary. Then, the approximate problem (3.16)-(3.17) admits at least one solution $u_O = y + z$ such that*

$$\|u_O\| \leq \nu^{-1} \|f\|_{V'}. \tag{4.5}$$

Moreover, if $f \in H$, then

$$|Au_O| \leq M_2. \tag{4.6}$$

Proof. We rewrite (3.16)-(3.17) as the equivalent problem

$$\nu A u_O + P_N B(u_O, u_O) - Q_n^N B(Q_n^N u_O, Q_n^N u_O) = P_N f. \tag{4.7}$$

Let $X = H_N$ with the scalar product $((\cdot,\cdot))$ and norm $||\cdot||$, we define the mapping $G(\xi)$ by

$$((G(\xi), \eta)) = \nu((\xi, \eta)) + b(\xi, \xi, \eta) - b(Q_n^N \xi, Q_n^N \xi, Q_n^N \eta) - (f, \eta), \qquad \forall \xi, \eta \in H_N.$$

The continuity of the mapping G is obvious; Thanks to (3.4),

$$\begin{aligned} (G(\xi), \eta) &= \nu \|\xi\|^2 + b(\xi, \xi, \xi) - b(Q_n^N \xi, Q_n^N \xi, Q_n^N \xi) - (f, \xi) \\ &= \nu \|\xi\|^2 - (f, \xi) \geq \|\xi\| (\nu \|\xi\| - \|f\|_{V'}). \end{aligned}$$

Hence, by choosing $\mu = \frac{1}{\nu} \|f\|_{V'}$, we have

$$(G(\xi), \xi) \geq 0, \qquad \forall \xi \in H_N, \ \|\xi\| = \nu.$$

Therefore we can apply lemma 4.2, there exist at least one solution of (4.7) and

$$\|u_O\| \leq \mu = \frac{1}{\nu} \|f\|_{V'}.$$

This yields (4.5).

In order to prove (4.6) we recall that

$$|B(u_O, u_O)| \leq c_0 |u_O|^{1/2} \|Au_O\|^{1/2} \|u_O\| \leq \frac{\nu}{4} |Au_O| + c_0^2 \nu^{-1} |u_O| \|u_O\|^2, \tag{4.8}$$

$$|B(Q_n^N u_O, Q_n^N u_O)| \leq \frac{\nu}{4} |Au_O| + \nu^{-1} c_0^2 |u_O| \|u_O\|^2. \tag{4.9}$$

Combining (4.7)-(4.8) with (4.9), we have

$$\begin{aligned} \nu |Au_O| &\leq 2|f| + c_0^2 \nu^{-1} |u_O| \|u_O\|^2 \\ &\leq 2|f| + c_0^2 \nu^{-1} \lambda_1^{-1/2} M_1^3. \end{aligned}$$

This yields (4.6). The proof ends. □

By a similar manner we can derive the following results.

Theorem 4.4. *Assume that $f \in V'$. Then the proximate problem (3.18)-(3.19) admits at least one solution $u_N \in H_N$ such that*

$$\|u_N\| \leq \nu^{-1} \|f\|_{V'}. \tag{4.10}$$

Moreover, if $f \in H$ then

$$|Au_N| \leq M_2. \tag{4.11}$$

It is obvious that the uniqueness of approximates solution u_O depends upon parameter ν and infinite dimensional solution u. Consequently, we have to consider the nonsingular solution u of (3.9). In the neighborhood of singular solution u of (3.9), my be to occurs bifurcation at (ν, u), we will study in other paper.

We say that the solution of (3.9) is an nonsingular solution at ν if only if the Fréchet derivative operator $D_u\mathcal{F}(u,\nu)$ at (u,ν) is an isomorphism from V onto V. If u is not an nonsingular solution we say that u is a singular solution.

Owing to the continuity of $D_u\mathcal{F}(u,\nu)$ for any nonsingular solution u, by the Banach theorem, we assert that there exists $D_u\mathcal{F}(u,\nu)^{-1}$, and $D_u\mathcal{F}(u,\nu)^{-1}$ is linear and continuous. In view of the bounded inverse theorem, $D_u\mathcal{F}(u,\nu)$ is bounded below

$$||D_u\mathcal{F}(u,\nu)v|| \geq \alpha\|v\|, \quad \forall v \in V, \tag{4.12}$$

where $\alpha > 0$ (see Dunford and Schwartz [13]).

Let us introduce the functional

$$\rho(w) = \inf_{v\in V} \frac{b(v,w,v)}{\|v\|^2}, \quad \forall w \in V. \tag{4.13}$$

It is clear that $\forall w \in V$,

$$|b(v,w,v)| \leq \rho(w)||v||^2, \quad \forall v \in V. \tag{4.14}$$

By virtue of (3.10), for any solution u of (3.9)

$$|b(v,u,v)| \leq c||u||||v||^2 \leq c\nu^{-1}||f||_{V'}||v||^2.$$

Consequently, for any solution u of (3.9,)

$$\rho(u) \leq c_0\nu^{-1}||f||_{V'}. \tag{4.15}$$

Recall that

$$(D_u\mathcal{F}(u,\nu)v,w) = \nu((v,w)) + b(u,v,w) + b(v,u,w),$$

we define bilinear form $V \times V \to R$:

$$C_u(w,v) \equiv (D_u\mathcal{F}(u,\nu)w,v) = \nu((w,v)) + b(u,w,v) + b(w,u,v). \tag{4.16}$$

Then,existence of $D_u\mathcal{F}(u,\nu)^{-1}$ is equivalent to that elliptic problem:
find $w \in V$ such that for any $g \in V'$

$$C_u(w,v) = (g,v), \quad \forall v \in V, \tag{4.17}$$

admits a unique solution $w \in W$.

According to generalized Lax-Milgram theorem ([10], [11]) if $C_u(w,v)$ is continuous and satisfying weak coerciveness:
(i) There exist $\delta > 0$ such that

$$\inf_{\|w\|=1} \sup_{\substack{\|v\|\leq 1\\ v\neq 0}} |C_u(w,v)| \geq \delta > 0, \tag{4.18}$$

(ii)

$$\sup_{\substack{w\in V\\ w\neq 0}} |C_u(w,v)| > 0, \qquad \forall v \in V, v \neq 0, \tag{4.19}$$

then, $\forall g \in V'$ there exist a unique solution w of (4.17) with the following estimate

$$\|w\| \leq \delta^{-1}\|g\|_{V'}. \tag{4.20}$$

This can derive that if $C_u(\cdot,\cdot)$ is weak coercive then there exists $D_u\mathcal{F}(u,\nu)^{-1}$ and $\alpha = \delta$, where α is a constant defined by (4.12). Therefore,

$$\left\{ \begin{array}{l} \|D_u\mathcal{F}(u,\nu)v\| \geq \alpha\|v\| = \delta\|v\|, \\ \|D_u\mathcal{F}(u,\nu)^{-1}\|_{\mathcal{L}(V,V)} \leq \delta^{-1}. \end{array} \right. \tag{4.21}$$

Lemma 4.5. *Assume that the solution u of (3.9) satisfies*

$$\nu - \rho(u) \geq \delta > 0. \tag{4.22}$$

Then u is a nonsingular solution.

Proof. It is sufficient to prove that bilinear $C_u(\cdot,\cdot)$ defined by (4.16) is weak coercive. In fact,

$$\begin{array}{rcl} \inf\limits_{||w||=1} \sup\limits_{\substack{||v||\leq 1 \\ ||v||\neq 0}} C_u(w,v) & = & \inf\limits_{||w||=1} \sup\limits_{\substack{||v||\leq 1 \\ ||v||\neq 0}} ||\nu((w,v)) + b(w,u,v)|| \\ & \geq & \inf\limits_{||w||=1} \{\nu||w||^2 + b(w,u,w)\} \geq \nu - \rho(u) \equiv \delta > 0 \end{array}$$

On the other hand,

$$\begin{array}{rcl} \sup|C_u(w,v)| & \geq & \nu||v||^2 + b(v,u,v) \\ & \geq & (\nu - \rho(u))||v||^2, \qquad \forall v \in V. \end{array}$$

Therefore, (4.22) yields that $C_u(\cdot,\cdot)$ satisfies weak coercive condition. □

Lemma 4.6. *Assume that ν and f satisfy*

$$\nu - c_0\nu^{-1}\|f\|_{V'} > 0. \tag{4.23}$$

Then the solution u of (3.9) is a nonsingular solution.

Proof. It is clear that (4.23) and (4.23) are equivalent, therefore, solution of (3.9) is unique in view of theorem 3.1.

Moreover, by virtue of (4.15) and (4.23), we have

$$\nu - \rho(u) \geq \nu - c_0\nu^{-1}||f||_{V'} > 0$$

Therefore, by lemma 4.5 it is obvious that u is an nonsingular solution. □

Theorem 4.7. *Assume that the solution u of (3.9) satisfies (4.22), and n is sufficiently large such that*

$$\delta - c_0\nu^{-1}\lambda_{n+1}^{-1/2}||f||_{V'} > 0. \tag{4.24}$$

Then the solutions u_O and u_N of (3.16)–(3.17) and (3.18)–(3.19) are unique, respectively.

Proof. It is enough to prove the uniqueness of u_O. Assume that u_1 and u_2 are two solutions of (3.16)-(3.17). Let $w = u_1 - u_2$. Then it follows from (3.16), (3.17) that

$$\nu Aw + P_N(B(w,u_1) + B(u_2,w)) - Q_n^N[B(Q_n^N w, Q_n^N u_1) + B(Q_n^N u_2, Q_n^N w)] = 0. \tag{4.25}$$

Taking the scalar product in H of (4.25) with w and using (3.5), we derive

$$\begin{array}{rcl} (\nu - \rho(u_1))||w||^2 & \leq & |b(Q_n^N w, Q_n^N u_1, Q_n^N w)| \leq c_0|Q_n^N w|\|Q_n^N w\|\|Q_n^N u_1\| \\ & \leq & c_0\lambda_{n+1}^{-1/2}\nu^{-1}\|f\|_{V'}\|w\|^2. \end{array}$$

By applying (4.24) it follows that $||w|| = 0$. □

5. Convergence Analysis

In this section, we aim to derive the convergence rates of the numerical solutions u_G, u_O and u_N to the nonsingular solution u of the abstract equation (3.9).

Theorem 5.1. *Assume that $f \in H$ and u is a nonsingular solution of (3.9). Then u_G satisfies the following convergence rate:*

$$||u - u_G|| \leq M_2(c_0\alpha^{-1}\lambda_1^{-1/2}M_2\lambda_{n+1}^{-1/2} + 1)\lambda_{n+1}^{-1/2} \tag{5.1}$$

Remark 5.1. *If n is chosen such that*

$$c_0\alpha^{-1}\lambda_1^{-1/2}M_2\lambda_{n+1}^{-1/2} \leq 1, \tag{5.2}$$

then

$$\|u - u_G\| \leq 2M_2\lambda_{n+1}^{-1/2}. \tag{5.3}$$

This proof is classical, it can be omitted.

Theorem 5.2. *Assume that $f \in H$ and u is a nonsingular solution of (3.9). Then the following estimate holds:*

$$\|u - u_O\| \leq \alpha^{-1}[c_0M_2^2\lambda_{n+1}^{-3/2} + (|f| + c_0\lambda_1^{-1/2}M_1^{3/2}M_2^{1/2})\lambda_{N+1}^{-1/2}], \tag{5.4}$$

where α is a constant defined by (4.20).

Proof. Setting $E = u - u_O$, we derive from (3.9) and (4.7) that

$$\nu AE + P_N(B(E,u) + B(u_O,E)) + Q_NB(u,u) + Q_n^NB(z,z) = Q_Nf,$$

where $z = Q_n^Nu_0$. Taking the scalar product in H of the above equality with $v \in V$, we derive

$$\begin{aligned} \nu(AE,v) \quad &+b(E,u,P_Nv) + b(u_O,E,P_Nv) + b(u,u,Q_Nv) \\ &+B(z,z,Q_n^Nv) = (f,Q_Nv). \end{aligned} \tag{5.5}$$

Noting that

$$b(u_O,E,P_Nv) = b(u,E,P_Nv) - b(E,E,P_Nv),$$

we obtain

$$\nu(AE,v) + b(E,u,v) + b(u,E,v) = (g,v), \tag{5.6}$$

where

$$\begin{aligned} (g,v) \quad &\equiv b(E,E,P_Nv) + b(E,u,Q_Nv) + b(u,E,Q_Nv) \\ &-b(u,u,Q_n^Nv) - b(z,z,Q_n^Nv) + (f,Q_Nv). \end{aligned} \tag{5.7}$$

(5.7) is equivalent to the following

$$(D_u\mathcal{F}(u,\nu)E,v) = (g,v). \tag{5.8}$$

Therefore, E can be looked as a solution of (5.9). On the other and

$$\|D_uF(u,\nu)E\| = \sup_{v\in V}\frac{(D_uF(u,\nu)E,v)}{\|v\|} = \sup_{v\in V}\frac{(g,v)}{\|v\|} \geq \frac{(g,E)}{\|E\|}. \tag{5.9}$$

Simplifying calculation shows that

$$\begin{aligned}(g,E) &= b(E,E,P_NE)+b(E,u,Q_NE)+b(u,E,Q_NE)\\ &\quad -b(u,uE,Q_NE)-b(z,z,Q_n^NE)+(f,Q_NE)\\ &= -b(E,E,Q_NE)+b(E,u,Q_NE)-b(u,u_O,Q_NE)\\ &\quad -b(z,z,Q_n^NE)+(f,Q_NE)\\ &= b(E,u_O,Q_NE)-b(u,u_O,Q_NE)\\ &\quad -b(z,z,Q_n^NE)+(f,Q_NE)\\ &= -b(u_O,u_O,Q_NE)-b(z,z,Q_n^NE)+(f,Q_NE).\end{aligned}$$

Comparing (5.6) with (4.12), we assert that

$$\begin{aligned}\alpha\|E\|^2 &\le (g,E)\le c_0|u_O|^{1/2}|Au_O|^{1/2}\|u_O\||Q_NE|+c_0|z|\|z\|\|Q_n^NE\|\\ &\le c_0\lambda_1^{-1/2}M_1^{3/2}M_2^{1/2}\lambda_{N+1}^{-1/2}\|E\|+c_0\lambda_{n+1}^{-3/2}|Au_O|\|E\|\\ &+ |f|\lambda_{N+1}^{-1/2}\|E\|.\end{aligned}$$

Therefore

$$\|E\|\le\alpha^{-1}((c_0\lambda_1^{-1/2}M_1^{3/2}M_2^{1/2}+|f|)\lambda_{N+1}^{-1/2}+c_0M_2^2\lambda_{n+1}^{-3/2}).$$

This yields (5.4). □

Remark 5.2. *If n and N are chosen such that*

$$\lambda_{N+1}^{-1}=O(\lambda_{n+1}^{-3}), \tag{5.10}$$

then

$$\|u-u_N\|=O(\lambda_{n+1}^{-3/2}). \tag{5.11}$$

Recalling (5.3), we obtain

$$\|u-u_O\|=O(\|u-u_G\|^3). \tag{5.12}$$

Theorem 5.3. *Assume that $f\in H$ and u is a nonsingular solution of (3.9). Then the following estimate holds*

$$\|u-u_N\|\le\alpha^{-1}[(|f|+c_0\lambda_1^{-1/2}M_1^{3/2}M_2^{1/2})\lambda_{N+1}^{-1/2}+2c_0M_2^2\lambda_{n+1}^{-1}], \tag{5.13}$$

where α is a constant defined by (4.12).

Proof. Set $E_N=u-u_N$. Then (3.18) and (3.19) can be rewritten as

$$\nu Au_N+P_NB(u_N,u_N)-P_NB(z,z)-Q_n^N(B(y,z)+B(z,y))=P_Nf. \tag{5.14}$$

Subtracting (5.14) from (3.9) we derive

$$\begin{aligned}\nu AE_N &+P_N(B(u,u)-B(u_N,u_N))+Q_NB(u,u)+Q_n^NB(z,z)\\ &+Q_n^N(B(y,z)+B(z,y))=Q_Nf.\end{aligned} \tag{5.15}$$

Noticing that

$$B(u,u)-B(u_N,u_N)=B(E_N,u)+B(u_N,E_N)=B(E_N,z)+B(u,E_N)-B(E_N,E_N),$$

and taking scalar product in H of (5.15) with $v\in V$, we have

$$\begin{aligned}\nu(AE_N,v) &+ b(E_N,u,P_Nv)+b(u,E_N,P_Nv)-b(E_N,E_N,P_Nv)\\ &+ b(u,u,Q_nv)+b(z,z,P_Nv)+b(y,z,Q_n^Nv)\\ &+ b(z,y,Q_n^Nv)=(f,Q_Nv).\end{aligned}$$

It is equivalent to the following formulation

$$(D_u\mathcal{F}(u,\nu)E_N,v)=(g,v),\quad\forall v\in V, \tag{5.16}$$

where

$$\begin{aligned}(g,v) \quad = \quad & b(E_N,u,Q_Nv)+b(u,E_N,Q_Nv)+b(E_N,E_N,P_Nv)\\ & -b(u,u,Q_Nv)-b(z,z,P_Nv)-b(y,z,Q_n^Nv)-b(z,y,Q_n^Nv)+(f,Q_Nv).\end{aligned}$$

By similar manner as in proof of theorem 5.1 we obtain

(5.17)

$$\begin{aligned}\alpha\|E_N\|^2 \quad & \le (g,E_N)=b(E_N,u,Q_NE_N)+b(u,E_N,Q_NE_N)-b(E_N,E_N,Q_NE_N)\\ & -b(u,u,Q_NE_N)-b(z,z,P_NE_N)-b(y,z,Q_NE_N)\\ & -b(z,y,Q_n^NE_N)+(f,Q_NE_N),\end{aligned}$$

$$\begin{aligned}|b(u_N,u_N,Q_NE_N)| \quad & \le c_0|u_N|^{1/2}|Au_N|^{1/2}||u_N|||Q_NE_N\|\\ & \le c_0\lambda_1^{-1/2}M_1^{3/2}M_2^{1/2}\lambda_{N+1}^{-1/2}\|E_N\|,\end{aligned} \tag{5.18}$$

$$\begin{aligned}|b(z,z,P_NE_N)| \quad & \le c_0|z|||z||||P_NE_N|| \le c_0\lambda_{n+1}^{-3/2}|Au_N|^2\|E_N\|\\ & \le c_0M_2^2\lambda_{n+1}^{-3/2}\|E_N\|,\end{aligned} \tag{5.19}$$

$$\begin{aligned}|b(z,y,Q_n^NE_N) \quad & +b(y,z,Q_n^NE_N)| \le 2c_0|y|^{1/2}|Ay|^{1/2}\|Q_n^NE_N\|||z|\\ & \le 2c_0\lambda^{-1/2}M_2\|E_N\|\lambda_{N+1}^{-1/2}|Au_N|\\ & \le 2c_0\lambda_1^{-1/2}M_2^2\lambda_{n+1}^{-1}\|E_N\|,\end{aligned} \tag{5.20}$$

$$|(f,Q_NE_N)| \le |f|\lambda_{N+1}^{-1/2}\|E_N\|, \tag{5.21}$$

we derive from (5.17) that

$$\begin{aligned}\|E_N\| \quad \le \quad & \alpha^{-1}[(|f|+c_0\lambda_1^{-1/2}M_1^{3/2}M_2^{1/2})\lambda_{N+1}^{-1/2}\\ & +c_0M_2^2\lambda_{n+1}^{-3/2}+2c_0\lambda_{N+1}^{-1/2}M_2^2\lambda_{n+1}^{-1}].\end{aligned}$$

This yields (5.13). The proof ends. □

Remark 5.3 *If n and N are chosen such that*

$$\lambda_{N+1}^{-1}=O(\lambda_{n+1}^{-2}), \tag{5.22}$$

then

$$\|u-u_N||=O(\lambda_{N+1}^{-1}), \tag{5.23}$$

Recalling (5.3) we obtain

$$\|u-u_N\|=O(\|u-u_G\|^2). \tag{5.24}$$

Remark 5.4 *According to Remarks 5.1-5.3, we conclude that the ONG method provides the higher convergence than the SG method:*

$$\|u-u_O\|=O(\|u-u_G\|^3), \|u-u_N\|=O(\|u-u_G\|^2),$$

provide

$$\lambda_{N+1}^{-1}=O(\lambda_{n+1}^{-3}) \quad \textit{for the ONG method,}$$

$$\lambda_{N+1}^{-1}=O(\lambda_{n+1}^{-2}) \quad \textit{for the NG method.}$$

However, the extra cost of the ONG method and the NG method consist of solving a linear subproblem in space $H_N\backslash H_n$. Hence the ONG Method is superior to the NG Method and the NG Method is superior to the SG Method.

6. Approximation of Branches of Nonsingular Solution

We rewrite the Navier-Stokes Equations by the following

$$\text{(NSE)} \qquad \mathcal{F}(u,\nu) \equiv u + \nu A^{-1}(B(u,u) - f) = 0$$

where A is the abstract stokes operator and A^{-1} is its inverse which is compact from H into H. By a similar manner, (3.18)-(3.19) can be rewritten by

$$\text{(ANSE)} \quad \mathcal{F}_N(u_O,\nu) \equiv u_O + \nu^{-1}A^{-1}(B(u_O,u_O) - Q_n^N B(Q_n^N u_O, Q_n^N u_O) - f) = 0.$$

Let

$$G(u,u) = \nu^{-1}(B(u,u) - f), \tag{6.1}$$

$$G_N(u,u) = \nu^{-1}(B(u,u), Q_n^N B(Q_n^N u, Q_n^N u) - f). \tag{6.2}$$

Then

$$D_u\mathcal{F}(u,\nu)v = v + A^{-1}D_uG(u,u)v, \tag{6.3}$$

$$D_u\mathcal{F}_N(u_O,\nu)v = v + A^{-1}D_uG_N(u_O,u_O)v, \tag{6.4}$$

where

$$D_uG(u,u)v = \nu^{-1}(B(u,v) + B(v,u)), \quad \forall v \in H, \tag{6.5}$$

$$\begin{aligned} D_uG_N(u_O,u_O)v = \quad & \nu^{-1}[(B(u_O,v) + B(v,u_O) \\ & -Q_n^N(B(Q_n^N u_O, v) + B(v, Q_n^N u_O))], \quad \forall v \in H_N. \end{aligned} \tag{6.6}$$

Let us assume that $\{(\nu, u(\nu)); \nu \in I \subset R\}$ is a branch of nonsingular solutions of (NES). We want to find some sufficient conditions ensuring the existence and uniqueness of a branch $\{(\nu, u(\nu)); \nu \in I\}$ of the solutions of (ANSE) in a suitable neighborhood of the branch $\{(\nu, u(\nu)); \nu \in I\}$ of the solutions of (NSE).

In first stage we fix ν in I. We introduce

$$\gamma(u,\nu) = \|D_u\mathcal{F}(u,\nu)^{-1}\|_{\mathcal{L}(V)},$$

$$\mu_N(\nu,\widetilde{u}_0) = \|D_u\mathcal{F}(u,\nu) - D_u\mathcal{F}_N(\widetilde{u}_0,\nu)\|_{\mathcal{L}(V)},$$

where $\widetilde{u}_0$ is an arbitrary element of H_N.

Lemma 6.1. *Under the condition*

$$\gamma(u,\nu)\mu_N(\nu,\widetilde{u}_0) < 1, \tag{6.7}$$

the mapping $D_u\mathcal{F}_N(\widetilde{u}_0,\nu)$ is an isomorphism from H_N onto H_N. For the proof, the reader can refer to lemma 3.3 in chapter V of [11].

Corollary 6.2. *Under the condition*

$$\mu_N(\nu,\widetilde{u}_0)\alpha^{-1} < 1, \tag{6.8}$$

where α is a constant defined by (4.12), the mapping $D_u\mathcal{F}_N(\widetilde{u}_0,\nu)$ is an isomorphism from H_N onto H_N.

Proof. (4.12) shows, $\gamma(\nu) \leq \alpha^{-1}$. Taking (6.7) and (6.8) into account, we obtain the conclusion of Corollary 6.2. $\square$

Next,we calculate $\mu_N(\nu, u_0)$

$$\mu_N(\nu, u_0) = \sup_{v,w\in V} \frac{(((D_u\mathcal{F}(u,\nu) - D_u\mathcal{F}_N(u_0,\nu))v, w))}{\|v\|\|w\|}. \tag{6.9}$$

In view of (6.3)-(6.6), we have

$$\begin{aligned}
I &= (((D_u\mathcal{F}(u,\nu) - D_u\mathcal{F}_N(u_0,\nu))v, w)) \\
&= \nu^{-1}((A^{-1}(B(u-u_0,v) + B(v,u-u_0)) + Q_n^N(B(Q_n^N u_0, v) + B(v, Q_n^N u_0)), w)) \\
&= \nu^{-1}((A^{-1/2}(B(u-u_0,v) + B(v,u-u_0)) \\
&+ Q_n^N(B(Q_n^N u_0, v) + B(v, Q_n^N u_0)), A^{-1/2}w)).
\end{aligned}$$

Let $A^{-1/2}g = \xi, A^{-1/2}h = \eta, i.e.\quad g = A^{1/2}\xi, \eta = A^{1/2}\eta$. This yields

$$((A^{-1/2}g, A^{-1/2}h)) = ((\xi,\eta)) = (A^{1/2}, A^{1/2}\eta) = (g,h). \tag{6.10}$$

Therefore,

$$I = \nu^{-1}(b(u-u_0,v,w) + b(v,u-u_0,w) + b(Q_n^N u_0, v, Q_n^N w) + b(v, Q_n^N u_0, Q_n^N w)),$$

or

$$\begin{aligned}
|I| &\le \nu^{-1}[c_0\|u_0\|\|v\|\|w\| + c_0\|v\||Q_n^N u_0|^{1/2}\|Q_n^N u_0\|^{1/2}|Q_n^N|w|^{1/2}\|Q_n^N w\|^{1/2} \\
&\quad + c_0|v|^{1/2}\|v\|^{1/2}\|Q_n^N u_0\|\|Q_n^N w\|] \\
&\le \nu^{-1}[c_0\|u_0\|\|v\|\|w\| + c_0(1+\lambda_1^{-1/2})\lambda_{n+1}^{-1/2}\|v\||u_0\|\|w\|],
\end{aligned}$$

$$\mu_N(\nu,u_0) \le \nu^{-1}c(\lambda_{n+1}^{-1}\|u_0\| + \|u-u_0\|).$$

By using (5.4),

$$\begin{aligned}
\mu_N(\nu,u_0) &\le \nu^{-1}c(M_1\lambda_{n+1}^{-1/2} + (|f| + c_0\lambda_1^{-1/2}M_1^{3/2}M_0^{1/2})\lambda_{N+1}^{-1/2} + M_2^2\lambda_{N+1}^{-3/2}) \\
&\le \overline{c}\lambda_{n+1}^{-1/2}.
\end{aligned}$$

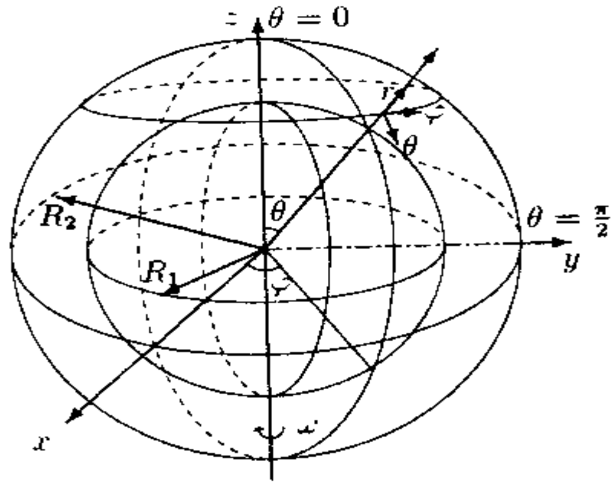
Combining this inequality with (4.12) yields the following result.
Lemma 6.3. *Assume that n is chosen enough large such that*

$$\overline{c}\lambda_{n+1}^{-1/2}\alpha^{-1} < 1,$$

where $\overline{c}$ is defined by (6.11). Then the ONG approximate solution u_O of (3.16)-(3.17) is a nonsingular solution.

7. Numerical Example

Here we study numerically the Navier-Stokes equations between two concentric rotating spheres by using the nonlinear Galerkin method and the optimal nonlinear Galerkin method, respectively.



(1)

We introduce the following notations:
(r, ϕ, θ) spherical polar coordinates,
R_1, R_2 radii of inner and outer spheres, respectively,
ω_1, ω_2 angular velocity of inner and outer spheres,
$\omega_2 = 0$ outer sphere is held fixed,
$\omega_2 > 0$ corotating case,
$\omega_2 < 2$ counter rotating case,
$\omega = \omega_2/\omega_1, \epsilon = 1 - \omega = \frac{\omega_1 - \omega_2}{\omega_1}, \eta = R_2/R_1, \eta = 1 + \sigma, \sigma = (R_2 - R_1)/R_1,$
$Re = \omega_1 R_1^2/\nu$ Reynolds Number,$\lambda = Re^{-1}$,
$u = (u_r, u_\phi, u_\theta), p$ physical components of velocity of the fluids and pressure,
u^i, u_i contravariant components and covariant components of velocity,
$u_r = u^1 = u_1, u_\phi = u^2 r \sin\theta = u_2/r \sin\theta, u_\theta = ru^3 = u_3/r,$
ψ, ξ Stream Function and Vorticity Function, respectively.

The Navier-Stokes equations and boundary conditions of a constant-density fluid between two concentric rotating at R_1 and R_2 with different angular velocity ω_1, ω_2 respective are given

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p - \nu \Delta u = 0, \tag{7.1}$$

$$\nabla \cdot u = 0, \tag{7.2}$$

$$u|_{r=1} = \sin\theta \overrightarrow{e}_\phi, u|_{r=\eta} = \omega\eta \sin\theta \overrightarrow{e}_\phi, \tag{7.3}$$

where we use dimensionless,and $(\overrightarrow{e_r}, \overrightarrow{e_\phi}, \overrightarrow{e_\theta})$ is a local coordinate frame of spherical coordinate.

Later on,we shall refer to the ϕ–component of the velocity u_ϕ as the agimuthal flow and to the remaining components of the velocity as the meridional flow $u_m = u_r \overrightarrow{e}_r + u_\theta \overrightarrow{e}_\theta$. The Stokes flow u^* is the time-independent solution to (7.1)-(7.3) in the limit $Re \to 0 (\nu \to \infty)$:

$$u^* = (\eta^3\omega - 1)/(\eta^3 - 1), \beta = \eta^3(1 - \omega)/(\eta^3 - 1), \tag{7.4}$$

where

$$\alpha = (\eta^3\omega - 1)/(\eta^3 - 1), \beta = \eta^3(1 - \omega)/(\eta^3 - 1). \tag{7.5}$$

Although the Stokes solution is exclusively agimuthal,it is a function of θ. Note that the angular velocity $u^*/r\sin\theta$ is not a function of θ, so that each radial shell moves with a constant velocity. In the spherical couette flow a small meridional velocity is generated from the nonlinear interaction of the Stokes solution with itself via the adventive terms in the Navier-Stokes equation. This caused the true flow to deviate from the Stokes solution at all finite Reynolds numbers. At very small Re, the flow is still mostly agimuthal and does not depart greatly from stokes flow.The meridional motion which driven by Ekman pumping, expele fluids out from the pales along the surface of the rotating inner sphere. The streamline resulting from the superposition of the agimuthal and the weaker meridional motion are helices. Despite being three-dimensional, the flow remains axis-symmetric. Set

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(r\sin\theta\psi), u_\theta = \frac{-1}{r\sin\theta}\frac{\partial}{\partial r}(r\sin\theta\psi), \tag{7.6}$$
$$\xi = (\xi_r, \xi_\phi, \xi_\theta) = \nabla\times u,$$

we obtain vorticity-stream function formulation for the Navier-Stokes equations with axis-symmetry

$$\frac{\partial u_\phi}{\partial t} + \frac{1}{r^3\sin^2\theta}\frac{\partial(r\sin\theta u_\phi, r\sin\theta\psi)}{\partial(r,\theta)} - \lambda L^2 u_\phi = 0 \tag{7.7}$$

$$\frac{\partial\xi_\phi}{\partial t} + \frac{1}{r^2\sin\theta}\frac{\partial(\xi_\phi, r\sin\theta\psi)}{\partial(r,\theta)} + (-2u_\phi N u_\phi + \xi_\phi N\psi) - \lambda L^2\xi_\phi = 0, \tag{7.8}$$

$$L^2\psi = \xi_\phi, \tag{7.9}$$

where

$$N = \frac{\cot\theta}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial}{\partial\theta},$$

$$\begin{aligned} L^2 &= \tfrac{\partial^2}{\partial r^2} + \tfrac{2}{r}\tfrac{2}{\partial r} + \tfrac{\cot\theta}{r^2}\tfrac{\partial}{\partial\theta} + \tfrac{1}{r^2}\tfrac{\partial^2}{\partial\theta^2} - \tfrac{1}{r^2\sin^2\theta} \\ &= \tfrac{1}{r}\tfrac{\partial^2}{\partial r^2}(r) + \tfrac{1}{r^2}\tfrac{\partial}{\partial\theta}(\tfrac{1}{\sin\theta}\tfrac{\partial}{\partial\theta}(\sin\theta\cdot)). \end{aligned} \tag{7.10}$$

Boundary conditions are given by

$$\begin{cases} u_\phi|_{r=1} = \sin\theta, & u_\phi|_{r=\eta} = \eta\omega\sin\theta \\ \varphi = \frac{\partial\varphi}{\partial r} = 0, & r = 1, \qquad\qquad r = \eta, \\ \varphi = 0, & \text{for} \quad \theta = 0, \qquad \theta = \pi. \end{cases} \tag{7.11}$$

Let

$$u_\phi = U + u_\phi^*, \tag{7.12}$$

where u_ϕ^* is defined by (7.4). It is easy to verify that u_ϕ^* satisfy (7.11) and

$$L^2 u_\phi^* = 0. \tag{7.13}$$

Thus, we obtain the equations for U and ψ:

$$\begin{aligned} &\frac{\partial U}{\partial t} + \frac{1}{r^3\sin^2\theta}\frac{\partial(r\sin\theta U, r\sin\theta\psi)}{\partial(r,\theta)} + \frac{1}{r^3\sin^2\theta}\frac{\partial(r\sin\theta u_\phi^*, r\sin\theta\psi)}{\partial(r,\theta)} - \lambda L^2 U \\ &= 0, \end{aligned} \tag{7.14}$$

$$\begin{aligned} &\frac{\partial\xi_\phi}{\partial t} + \frac{1}{r^2\sin\theta}\frac{\partial(\xi_\phi, r\sin\theta\psi)}{\partial(r,\theta)} + (2(U+u_\phi^*)NU + 2UNu_\phi^* + \xi_\phi N\psi) - \lambda^2\xi_\phi \\ &= f, \end{aligned} \tag{7.15}$$

$$L^2\psi = \xi_\phi, \tag{7.16}$$

with boundary conditions

$$U|_{r=1} = U|_{r=\eta} = 0, \psi|_{r=1} = \frac{\partial \psi}{\partial r}|_{r=1} = \psi|_{r=\eta} = \frac{\partial \psi}{\partial r}|_{r=\eta} = 0, \tag{7.17}$$

where

$$f(r,\theta) = -3\beta(\alpha r^{-3} + \beta r^{-6}) \sin\theta. \tag{7.18}$$

Next we consider the following eigenvalue and eigenfunction

$$\lambda L^2 U + \mu U = 0, U|_{r=1} = U|_{r=0} = 0,$$

$$\lambda L^4 \psi + \mu L^2 \psi = 0, \psi|_{r=0,1} = \frac{\partial \phi}{\partial r}|_{r=0,1} = 0.$$

The calculation shows the eigenfunctions are given

$$U_{l,n} = Q_{l,n}(\alpha_{l,n}) P_l^1(\cos\theta),$$

$$\psi_{l,n}(r,\theta) = b_l(\beta_{l,n} r) P_l^1(\cos\theta),$$

with eigenvalues

$$\mu_{l,n} = \alpha_{l,n}^2 \lambda,$$

$$\mu_{l,n} = \beta_{l,n}^2 \lambda,$$

where $\alpha_{l,n}$ and $\beta_{l,n}$ are the roots of the following equations

$$Q_l(\alpha) = 0, b_l(\beta) = 0,$$

and

$$Q_l(\alpha r) = \begin{vmatrix} j_l(\alpha r) & y_l(\alpha r) \\ j_l(\alpha\eta) & y_l(\alpha\eta) \end{vmatrix},$$

$$b_l(\beta r) = \begin{vmatrix} r^l & r^{-l-1} & j_l(\beta r) & y_l(\beta r) \\ \eta^l & \eta^{-l-1} & j_l(\beta\eta) & y_l(\beta\eta) \\ l & -l-1 & \beta j_l^{'}(\beta) & \beta y_l^{'}(\beta) \\ l\eta^{l-1} & -(l+1)\eta^{-1-l} & \beta j_l^{'}(\beta\eta) & \beta y_l^{'}(\beta\eta) \end{vmatrix},$$

$$j_l(z) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{z}} J_{l+1/2}(z), y_l(z) = \sqrt{\frac{2}{\pi z}} I_{l-1/2}(z),$$

where $J_{l+1/2}(z)$ is a Bessell's function of half integer order and $j_l(z)$,$y_l(z)$ are spherical Bessel's functions, and $P_l^m(x)$ is a general Legendre's polynomial.

We choose basic space

$$H = H_0^1(\Omega) \times H_0^2(\Omega)$$

and finite dimensional subspaces

$$H_N = span\{U_{l,n}, \phi_{l,n}, l, n \leq N\}.$$

Let u_L, u_H be the lower and higher components ofu_ϕ, respectively, ψ_L, ψ_H lower and higher components of ψ, respectively, Λ_L^1, Λ_H^1 index set of lower and higher components of u_ϕ, respectively, Λ_L^2, Λ_H^2 index set of lower and higher components of ψ, respectively.

Assume that

$$u_L = \sum_{l,n \in \Lambda_L^1} x_{l,n} U_{l,n}(r,\theta),\ u_H = \sum_{l,n \in \Lambda_H^2} z_{l,n}(t) U_{l,n}(r,\theta),$$

$$\psi_L = \sum_{l,n \in \Lambda_L^2} z_{l,n}(t) \psi_{l,n}(r,\theta),\ \psi_H = \sum_{l,n \in \Lambda_H^2} z_{l,n}(t) \psi_{l,n}(r,\theta),$$

$$u_H = u_L + u_H, \psi_H = \psi_L + \psi_H,$$

$$\Lambda_H^1 = \Lambda_L^1 \cup \Lambda_H^1,\ \Lambda_H^2 = \Lambda_L^2 \cup \Lambda_H^2.$$

By applying the orthogonality

$$\int_\Omega r^2 \sin\theta U_{l,n} U_{m,k} dr d\theta = N_{l,n}\delta_{l,m}\delta_{n,k},$$
$$\int_\Omega r^2 \sin\theta L^2\psi_{l,n}\psi_{k,m} dr d\theta = M_{l,n}\delta_{l,m}\delta_{n,k},$$

where

$$N_{l,n} = \int_\Omega r^2 \sin\theta U_{l,n} dr d\theta$$
$$M_{l,n} = \int_\Omega r^2 \sin\theta L^2\psi_{l,n}\psi_{l,n} dr d\theta$$

we can rewrite the SG method, the NG method and the ONG method as follows:

The SG method:

$$\frac{d}{dt}x_{k,m} + \frac{1}{Re}\alpha^2_{k,m}x_{k,m} + \frac{1}{N_{k,m}}b_1(u_H + u^*_\varphi, \psi_H, U_{k,m}) = 0, \quad \forall k,m \in \Lambda^1_N,$$

$$\begin{aligned}\frac{dz_{k,m}}{dt} + \frac{1}{Re}\beta^2_{k,m}z_{k,m} \quad &+ \quad \frac{1}{M_{k,n}}[b_1(\psi_H,\psi_H,\psi_{k,m}) + 2b_2(u_H + u^*_\varphi, u_H + u^*_\varphi, \psi_{k,m}),\\ &+ \quad 2b_2(L^2\psi_H, \psi_H, \psi_{k,m})] = 0, \quad \forall k,m \in \Lambda^2_N.\end{aligned}$$

The NG method:

$$\tfrac{d}{dt}x_{k,m} + \tfrac{1}{Re}\alpha^2_{k,m}x_{k,m} + N^{-1}_{k,m}[b_1(u_N + u^*_\varphi, \varphi_H, U_{k,m})] = 0, \quad \forall k,m \in \Lambda^1_L,$$

$$\begin{aligned}\frac{d}{dt}z_{k,m} + \frac{1}{Re}\beta^2_{k,m}z_{k,m} \quad &+ \quad M^{-1}_{k,m}[b_1(\varphi_H,\varphi_H,\varphi_{k,m}) + 2b_2(u_H + u^*_\varphi, u_H + u^*_\varphi, \varphi_{k,m})\\ &+ \quad 2b_2(L^2\varphi_H, \varphi_H, \varphi_{k,m})] = 0, \quad \forall k,m \in \Lambda^2_L,\end{aligned}$$

$$\tfrac{1}{Re}\alpha^2_{k,m}x_{k,m} + N^{-1}_{k,m}b_1(u_L + u^*_\varphi, \varphi_L, U_{k,m}) = 0, \quad \forall k,m \in \Lambda^1_H,$$

$$\begin{aligned}\frac{1}{Re}\beta^2_{k,m}z_{k,m} \quad &+ \quad M^{-1}_{k,m}[b_1(L^2\varphi_L,\varphi_L,\varphi_{k,m}) + 2b_2(u^*_\varphi + u_L, u^*_\varphi + u_L, \varphi_{k,m})\\ &+ \quad 2b_2(L^2\varphi_L, \varphi_L, \varphi_{k,m})] = 0, \quad \forall k,m \in \Lambda^2_H.\end{aligned}$$

The ONG method:

$$\tfrac{d}{dt}x_{k,m} + \tfrac{1}{Re}\alpha^2_{k,m}x^+_{k,m}N^{-1}_{k,m}b_1(u^*_\varphi + u_H, \varphi_H, U_{k,m}) = 0 \quad \forall k,m \in \Lambda^1_L,$$

$$\begin{aligned}\frac{d}{dt}z_{k,m} \quad &+ \tfrac{1}{Re}\beta^2_{k,m}z_{k,m} + M^{-1}_{k,m}[b_1(L^2\varphi_H,\varphi_H,\varphi_{k,m}) + 2b_2(u^*_\varphi + u_H, u^*_\varphi + u_H, \varphi_{k,m})\\ &+2b_2(L^2\varphi_H, \varphi_H, \varphi_{k,m})] = 0, \quad \forall k,m \in \Lambda^2_L,\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}x_{k,m} \quad &+ \quad \frac{1}{Re}\alpha^2_{k,m}x_{k,m} + N^{-1}_{k,m}[b_1(u^*_\varphi + u_L, \varphi_L, U_{k,m})\\ &+ \quad b_1(u^*_\varphi + u_L, \varphi_H, U_{k,m})b_1(u_H, \varphi_L, U_{k,m})] = 0, \quad \forall k,m \in \Lambda^1_H,\end{aligned}$$

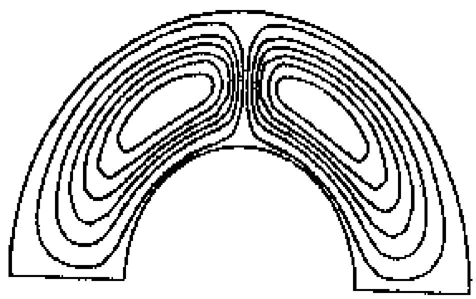
$$\begin{aligned}\frac{d}{dt}z_{k,m} \quad + \quad &\frac{1}{Re}\beta^2_{k,m}z_{k,m} + M^{-1}_{k,m}[b_1(L^2\varphi_L,\varphi_L,\varphi_{k,m})\\ &+2b_2(u_L + u^2_\varphi, u_L + u^*_\varphi, u_L + u^*_\varphi, \varphi_{k,m}) + 2b_2(L^2\varphi_L,\varphi_L,\varphi_{k,m})\\ &+b_1(L^2\varphi_L,\varphi_H,\varphi_{k,m}) + b_1(L^2\varphi_H,\varphi_L,\varphi_{k,m})\\ &+2b_2(u_L + u^*_\varphi, u_H, \varphi_{k,m}) + 2b_2(u_H, u_L, \varphi_{k,m})\\ &+2b_2(L^2\varphi_L,\varphi_H,\varphi_{k,m}) + 2b_2(L^2\varphi_H,\varphi_L,\varphi_{k,m})] = 0, \quad \forall k,m \in \Lambda^2_H.\end{aligned}$$

Our numerical computation will appear in the Fig.2-Fig.7.

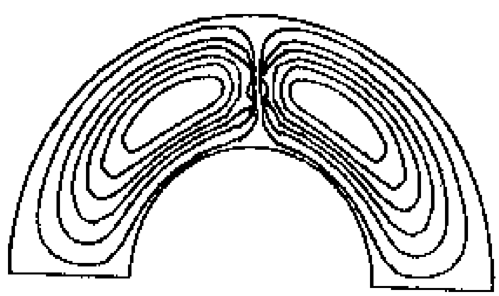
Fig.2: (r,θ)-Projection of the meridional streamlines of the 0-vortex flow at $Re = 600, \eta = 0.18$.

Fig.3: The meridional velocity of the 0-vortex flow with pinches at $Re = 650, \eta = 0.l8$.
Fig.4: The meridional streamlines of the 1-vortex flow at $Re = 750, \eta = 0.18$ Taylor-vortex separated from large basic vortices.
Fig.5: The meridional streamlines of the 2-vortex flow at $Re = 1100, \eta = 0.18$.
Fig.6: When one applies time evolution method to approximate stead state. The energy curves with time evolution are showed by using the SG method it is blow up at finite time.
Fig.7: The energy curves are showed by using the ONG method it converges to steady state as $t \to \infty$ without blow up.

The calculations show that though extra cost of the ONG method and the NG method consists in solving the similar linear subproblem in $H_N \backslash H_n$ the ONG method is superior to the NG method and the NG method is superior to the SG method.



(2)



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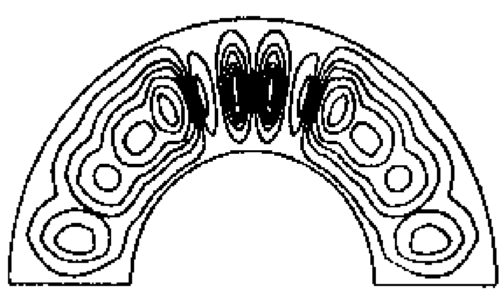

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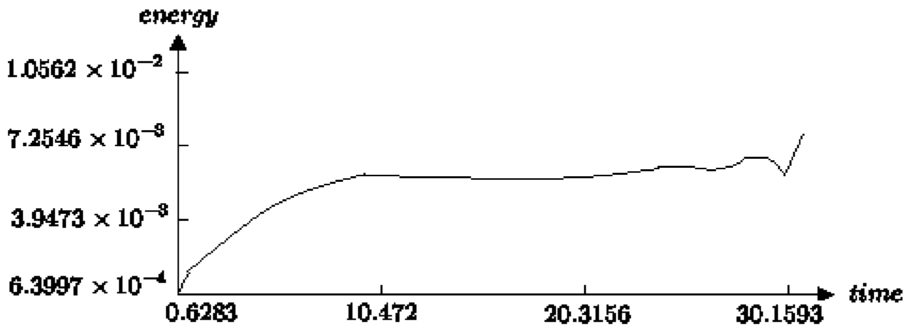

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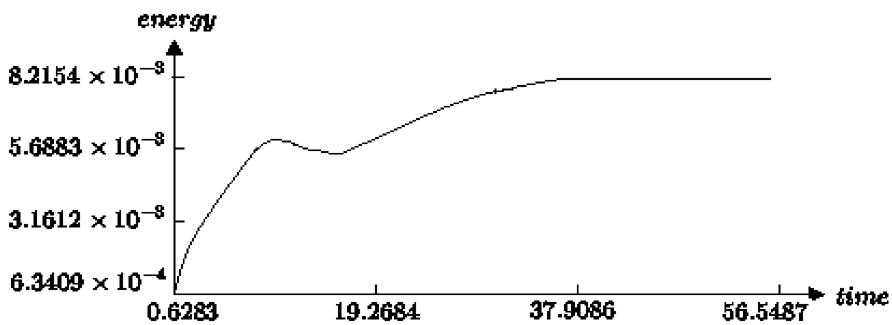

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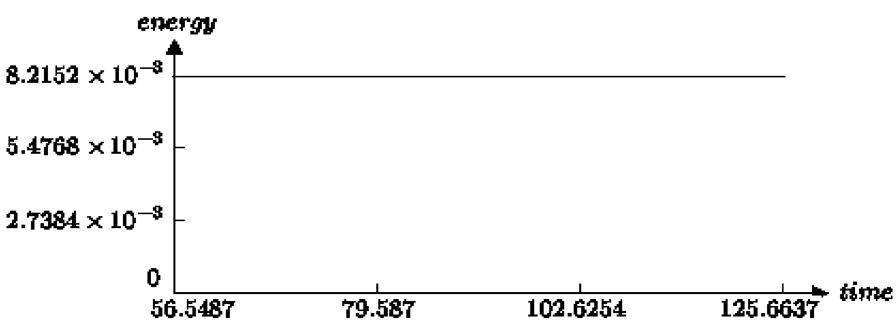

(7)

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