INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 2, Number 4, Pages 409–421

BASIS FOR THE QUADRATIC NONCONFORMING TRIANGULAR ELEMENT OF FORTIN AND SOULIE

HEEJEONG LEE AND DONGWOO SHEEN

Abstract. A basis for the quadratic (P_2) nonconforming element of Fortin and Soulie on triangles is introduced. The local and global interpolation operators are defined. Error estimates of optimal order are derived in both broken energy and $L^2(\Omega)$-norms for second-order elliptic problems. Brief numerical results are also shown.

Key Words. Quadratic nonconforming element, finite element method, error analysis

1. Introduction

Recently the nonconforming finite element method draws increasing attention from scientists and engineers as it has successfully provided stable numerical solutions to many practical fluid and solid mechanics problems: see, for instance, [1, 8, 9, 12, 15, 16, 17, 18, 28, 29, 32] for linear or nonlinear Navier-Stokes problems and [2, 5, 10, 13, 14, 21, 22, 19, 24, 23, 26, 27, 33] for elasticity related problems, and the references therein.

In order to approximate the velocity and pressure by the finite element method based on triangulations, the use of the usual P_1-P_0 conforming finite element pair lacks in stability that is required to satisfy the discrete inf-sup condition [7]. Also the P_1 conforming element suffers from numerical locking when applied to approximate elasticity problems [3, 6]. In case the triangulation is based on quadrilaterals rather than on triangles and the Q_1 element is used instead of the P_1 element accordingly, similar instability patterns are inevitable.

A common and simple solution to resolve this kind of instability problems has been made by using nonconforming finite element instead. In 1973 Crouzeix and Raviart [12] introduced the linear nonconforming finite elements for triangles or tetrahedrons and a cubic nonconforming element for triangles. The idea, at least in the P_1 nonconforming element case, is to employ the degrees of freedom associated with the values at the midpoints of edges of triangles or those at the centroids of faces of tetrahedrons, by replacing the values imposed at the vertices in the conforming element cases. These nonconforming elements were shown to supply stable finite element pairs for Stokes problems and to give optimal orders of convergence [12].

A generalization of this idea to higher degree nonconforming elements requires the patch test [20], which implies that a P_k nonconforming element needs to satisfy that on each interface the jump of adjacent polynomials be orthogonal to P_{k-1}

Received by the editors July 30, 2005.
2000 *Mathematics Subject Classification.* 35R35, 49J40, 60G40.
This research was supported in part by KRF 2003-070-C00007 and KOSEF R14-2003-019-01000-0.

polynomials on the interface. This implies that a P_2 nonconforming element must be continuous at the two Gauss points on each edge. However, to define the degrees of freedom at the two Gauss points leads to a problem due to the existence of a quadratic polynomial that vanishes at the six Gauss points of edges of any triangle. Therefore, the definition of the degrees of freedom for P_2 nonconforming elements addresses a special attention.

A successful quadratic nonconforming element has been introduced by Fortin and Soulie [17] by adding nonconforming bubble functions, "semi-loop functions", which can be eliminated by static condensation. (The three dimensional analogue has been introduced by Fortin [16].) It is shown in [17] that the global P_2 nonconforming space is identical to the union of the standard quadratic conforming element space, say, V_h, and the space of semi-loop functions, say Φ_h; moreover, their intersection $V_h \cap \Phi_h$, that is, "the set of the globally conforming and locally P_2 bubble functions", turns out to be one dimensional space. Therefore, in implementing the P_2 nonconforming element, one can modify the P_2 conforming code with a suitable addition of nonconforming bubble functions.

The purpose of the present paper is to propose another set of global basis functions for the P_2 nonconforming element, which have more nonconforming structure. For this, we define three kinds of basis functions with local supports: (1) edge-based basis (which are nonconforming), (2) vertex-based basis (which are nonconforming), and (3) triangle-based basis functions (which are bubble functions that are nonconforming). We then show that these functions form a basis for the P_2 nonconforming element space. The associated degrees of freedom and an interpolation operator are also defined. The basis functions corresponding to rectangular elements are constructed in [25].

The plan of the paper is as follows. In §2 the P_2 nonconforming basis functions are defined on triangular meshes. Interpolation and projection operators are defined in §3 and optimal order error estimates are shown. Finally in §4, brief numerical results are shown.

2. The P_2-nonconforming element on triangular meshes

In this section we introduce three kinds of basis functions for the P_2-nonconforming finite element on triangular meshes. The dimensions and basis functions are then computed for both Dirichlet and Neumann problems.

2.1. The P_2-nonconforming triangular elements. For a triangle T with the vertices $v_j, 1 \le j \le 3$, denote by $e_j, 1 \le j \le 3$, the edges from v_{j+1} to v_{j+2}, respectively, with the identification $v_1 = v_4$ and $v_2 = v_5$. Also, let m_j be the midpoint of $e_j, j = 1, 2, 3$. Throughout the paper we shall assume that the vertex indices are oriented counter clockwise. Designate by τ the unit tangent vector on the boundary ∂T with the direction from v_{j+1} to v_{j+2}, respectively, and by $\frac{\partial \varphi}{\partial \tau}$ its tangential derivative. Let g be the barycenter of T. As usual, for a nonnegative integer k, denote by $P_k(T)$ and $P_k(e_j)$ the spaces of polynomials on T and e_j, respectively, of degree $\le k$. We begin with the following fact.

Lemma 2.1. *Let e be an edge of T. Then if $\varphi \in P_2(T)$ satisfies $\int_e \varphi\, ds = 0$ and $\int_e \frac{\partial \varphi}{\partial \tau}\, ds = 0$, φ vanishes at the two Gauss points on e.*

Proof. Since $\frac{\partial \varphi}{\partial \tau} \in P_1(e)$, $\int_e \frac{\partial \varphi}{\partial \tau}\, ds = |e| \frac{\partial \varphi}{\partial \tau}(m) = 0$, where $|e|$ denotes the length of e. This implies that $\varphi|_e$ is symmetric with respect to the midpoint of e. Then $\int_e \varphi\, ds = 0$ implies that $\varphi|_e \in P_2(e)$ vanishes at the two Gauss points on e. This proves the lemma. □

Recall also the following lemma:

Lemma 2.2. *Suppose that $\varphi \in P_2(T)$ vanishes at the two Gauss points on each e_j, for $j = 1, 2, 3$. Then if $\int_T \varphi\, dx = 0$ or $\varphi(g) = 0$, where g is the barycenter of T, it follows that $\varphi \equiv 0$ in T.*

Next, set

$$P_2^*(T) = \{\varphi \in P_2(T) | \int_{e_j} \varphi\, ds = 0,\ j = 1, 2, 3, \int_T \varphi\, dx = 0\}.$$

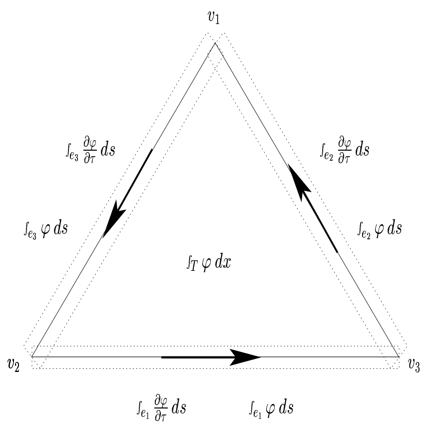
and, define $\varphi_j \in P_2^*(T)$ for $1 \le j \le 3$, by

$$\int_{e_k} \frac{\partial \varphi_j}{\partial \tau}\, ds = \begin{cases} 1, & k = j+1 \mod 3, \\ -1, & k = j+2 \mod 3, \\ 0, & k = j \mod 3. \end{cases}$$

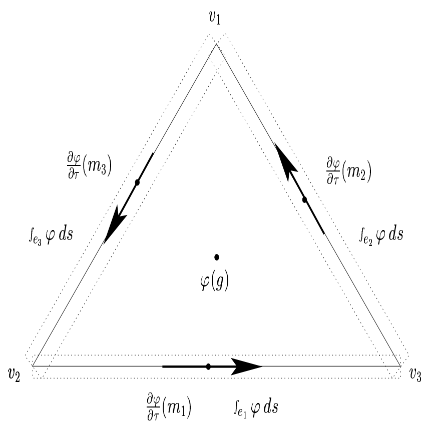
One then has the following simple lemma.

Lemma 2.3. $\mathrm{Span}\{\varphi_1, \varphi_2, \varphi_3\} = P_2^*(T)$. *Indeed, any two of $\varphi_1, \varphi_2, \varphi_3$ span $P_2^*(T)$.*

Proof. Clearly $\mathrm{Span}\{\varphi_1, \varphi_2, \varphi_3\} \subset P_2^*(T)$. Therefore it suffices to show that $P_2^*(T) \subset \mathrm{Span}\{\varphi_1, \varphi_2\}$ because of rotational symmetry. Let $\varphi \in P_2^*(T)$ be arbitrary with $\int_{e_1} \frac{\partial \varphi}{\partial \tau}\, ds = \alpha, \int_{e_2} \frac{\partial \varphi}{\partial \tau}\, ds = \beta$, and $\int_{e_3} \frac{\partial \varphi}{\partial \tau}\, ds = \gamma$ such that $\alpha + \beta + \gamma = 0$. Set $\psi = \beta\varphi_1 + (\beta + \gamma)\varphi_2$. Then it is easy to see that $\int_{e_1} \frac{\partial \psi}{\partial \tau}\, ds = \alpha, \int_{e_2} \frac{\partial \psi}{\partial \tau}\, ds = \beta$, and $\int_{e_3} \frac{\partial \psi}{\partial \tau}\, ds = \gamma$. Lemma 2.2 implies that φ is identical to ψ. This proves that $P_2^*(T) = \mathrm{Span}\{\varphi_1, \varphi_2\}$. ☐



(a) The P_2-nonconforming triangle of type 1



(b) The P_2-nonconforming triangle of type 2

FIGURE 1. The P_2-nonconforming triangles with vertices v_1, v_2, and v_3. (a) The six degrees of freedom are $\int_{e_j} \varphi\, ds$ for $j = 1, 2, 3$, $\int_{e_j} \frac{\partial \varphi}{\partial \tau}\, ds$ for $j = 1, 2$, and $\int_T \varphi\, dx$. (b) The six degrees of freedom are $\int_{e_j} \varphi\, ds$ for $j = 1, 2, 3$, $\frac{\partial \varphi}{\partial \tau}(m_j)$ for $j = 1, 2$, and $\varphi(g)$.

Due to Lemmas 2.2 and 2.3, we have the following unisolvency result, which motivates to define the local degrees of freedom for a $P_2(T)$ nonconforming element:

Proposition 2.1. *A function φ in $P_2(T)$ is uniquely determined by the six degrees of freedom: the three $\int_{e_j} \varphi\, ds$ for $j = 1, 2, 3,$ any two among the three $\int_{e_j} \frac{\partial \varphi}{\partial \tau}\, ds$ for $j = 1, 2, 3,$ and $\int_T \varphi\, dx$.*

Remark 2.1. *We remark that a function φ in $P_2(T)$ is uniquely determined by the six degrees of freedom: the three $\int_{e_j} \varphi\, ds$ for $j = 1, 2, 3,$, any two among the three $\frac{\partial \varphi}{\partial \tau}(m_j)$ for $j = 1, 2, 3,$ and $\varphi(g)$; these six values can be used as alternative degrees of freedom.*

Summarizing our P_2 triangular nonconforming elements as shown in Figure 1, let us proceed to define the nonconforming element space for the triangulation of a simply connected polygonal domain Ω in $\mathbb{R}^2$ with boundary Γ. Let $(\mathcal{T}_h)_{h>0}$ be a *regular* family of triangulations of Ω into triangles $T_j\, j = 1, \cdots, N_T$, where $h = \max_{T \in \mathcal{T}_h} h_T$ with $h_T = \mathrm{diam}(T)$. For a given triangulation $\mathcal{T}_h$ of Ω, let N_V, N_E, and N_T denote the numbers of vertices, edges, and triangles, respectively. Then set

$$
\begin{aligned}
\mathcal{T}_h &= \{T_1, T_2, \cdots, T_{N_T}\}; \quad \bigcup_{j=1}^{N_T} \overline{T}_j = \overline{\Omega}, \\
\mathcal{V}_h &= \{v_1, v_2, \cdots, v_{N_V}\} : \text{ the set of all vertices in } \mathcal{T}_h, \\
\mathcal{E}_h &= \{e_1, e_2, \cdots, e_{N_E}\} : \text{ the set of all edges in } \mathcal{T}_h, \\
\mathcal{G}_h &= \{g_1, g_2, \cdots, g_{N_T}\} : \text{ the set of all barycenters of triangles in } \mathcal{T}_h.
\end{aligned}
$$

In particular, let N_V^i and N_E^i denote the numbers of interior vertices and edges in $\mathcal{T}_h$, respectively. For a function f defined in Ω, denote by f_j its restriction to T_j. Similarly, $\tau_{j_k}, k = 1, 2, 3,$ will mean the positively oriented unit tangent vector on ∂T_j. Denoting e_{jk} the interface between elements T_j and T_k, we are now in a position to define the following nonconforming finite element spaces.

$$
\begin{aligned}
\mathcal{NC}^h &= \Big\{\varphi : \Omega \to \mathbb{R} \mid \varphi|_T \in P_2(T) \text{ for all } T \in \mathcal{T}_h, \int_{e_{jk}} \varphi_j\, ds = \int_{e_{jk}} \varphi_k\, ds, \\
&\qquad \int_{e_{jk}} \frac{\partial \varphi_j}{\partial \tau_j}\, ds + \int_{e_{jk}} \frac{\partial \varphi_k}{\partial \tau_k}\, ds = 0 \text{ on each interface } e_{jk} \in \mathcal{E}_h \Big\} \\
&= \{\varphi : \Omega \to \mathbb{R} \mid \varphi|_T \in P_2(T) \text{ for all } T \in \mathcal{T}_h, \varphi \text{ is continuous at} \\
&\qquad \text{the Gauss points of each interior edge } \in \mathcal{E}_h\}, \qquad (2.1) \\
\mathcal{NC}_0^h &= \Big\{\varphi \in \mathcal{NC}^h \mid \int_e \varphi\, ds = 0, \int_e \frac{\partial \varphi}{\partial \tau}\, ds = 0 \text{ for all } e \subset \partial\Omega, \text{ and } e \in \mathcal{E}_h \Big\} \\
&= \Big\{\varphi \in \mathcal{NC}^h \mid \varphi \text{ is vanish at the Gauss points of boundary edges} \\
&\qquad \text{of } \mathcal{E}_h \Big\}. \qquad (2.2)
\end{aligned}
$$

Notice that the equalities (2.1) and (2.2) follow from Lemma 2.1. Notice that the nonconforming spaces are identical to those defined by Fortin and Soulie [17].

2.2. Three types of basis functions. We first define the three types of basis functions in $\mathcal{NC}^h$, which will serve as global bases for the nonconforming finite element spaces.

Definition 2.1. *The first type of basis functions are associated with edges: define* $\varphi_j^E \in \mathcal{NC}^h, j = 1, 2, \cdots, N_E,$ *by*

$$\int_{e_k} \varphi_j^E \, ds = \delta_{jk} \text{ for all } k = 1, 2, \cdots, N_E,$$

$$\int_{e_k} \frac{\partial \varphi_j^E}{\partial \tau} \, ds = 0 \text{ for all } k = 1, 2, \cdots, N_E,$$

$$\int_{T_k} \varphi_j^E \, dx = 0 \text{ for all } k = 1, 2, \cdots, N_T.$$

The second type of basis functions are associated with vertices: define $\varphi_j^V \in \mathcal{NC}^h, j = 1, 2, \cdots, N_V,$ *by*

$$\int_{e_k} \varphi_j^V \, ds = 0 \text{ for all } k = 1, 2, \cdots, N_E,$$

$$\int_{e_k} \frac{\partial \varphi_j^V}{\partial \tau} \, ds = \begin{cases} 1 & \text{if } e \text{ is an edge from the vertex } v_j \text{ to an adjacent vertex} \\ & \quad v_k \in \mathcal{V}_h \text{ with } \tau \text{ being the unit tangent vector on } e \\ & \quad \text{with the direction from } v_k \text{ to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{for all } k = 1, 2, \cdots, N_E,$$

$$\int_{T_k} \varphi_j^V \, dx = 0 \text{ for all } k = 1, \cdots, N_T.$$

The last type of local function is associated with triangles: define $\varphi_j^T \in \mathcal{NC}^h, j = 1, \cdots, N_T,$ *by*

$$\int_{e_k} \varphi_j^T \, ds = 0 \text{ for all } k = 1, \cdots, N_E,$$

$$\int_{e_k} \frac{\partial \varphi_j^T}{\partial \tau} \, ds = 0 \text{ for all } k = 1, \cdots, N_E,$$

$$\int_{T_k} \varphi_j^T \, dx = \delta_{jk} \text{ for all } k = 1, \cdots, N_T.$$

Remark 2.2. *In Figure 2.2 the values of* $\int_e \frac{\partial \varphi_j^V}{\partial \tau} \, ds$ *are demonstrated. Indeed,* φ_j^V *has value* $\frac{2}{3}$ *at the vertex* v_j, $-\frac{1}{3}$ *at the neighboring vertices, and 0 at all other vertices. Its values at the Gauss points on the edges joining to* v_j *are* $\pm\frac{\sqrt{3}}{6}$ *with the signatures* $\pm$ *are* $+$ *if the points are nearer to the vertex* v_j *and* $-$ *otherwise. All the other Gauss point values of the remaining edges are 0.*

Due to Proposition 2.1, it is immediate to see that the above three types of functions have local supports as stated below:

Remark 2.3. *The three types of basis functions,* $\varphi_j^E, j = 1, \cdots, N_E$, $\varphi_j^V, j = 1, \cdots, N_V$, *and* $\varphi_j^T, j = 1, \cdots, N_T$, *have local supports. Moreover,* $\varphi_j^E, j = 1, \cdots, N_E$, $\varphi_j^V, j = 1, \cdots, N_V$, *and* $\varphi_j^T, j = 1, \cdots, N_T$, *are nonconforming;* $\varphi_j^T, j = 1, \cdots, N_T$, *are bubble functions that can be eliminated by static condensation at the implementation stage.*

Remark 2.4. *Alternatively the basis functions can be defined by replacing* $\int_{e_k} \frac{\partial \varphi_j^i}{\partial \tau} \, ds$ *and* $\int_{T_k} \varphi_j^i \, dx$ *by* $\frac{\partial \varphi_j^i}{\partial \tau}(m_k)$ *and* $\varphi_j^i(g_k)$, *respectively, for* $i = E, V, T$.

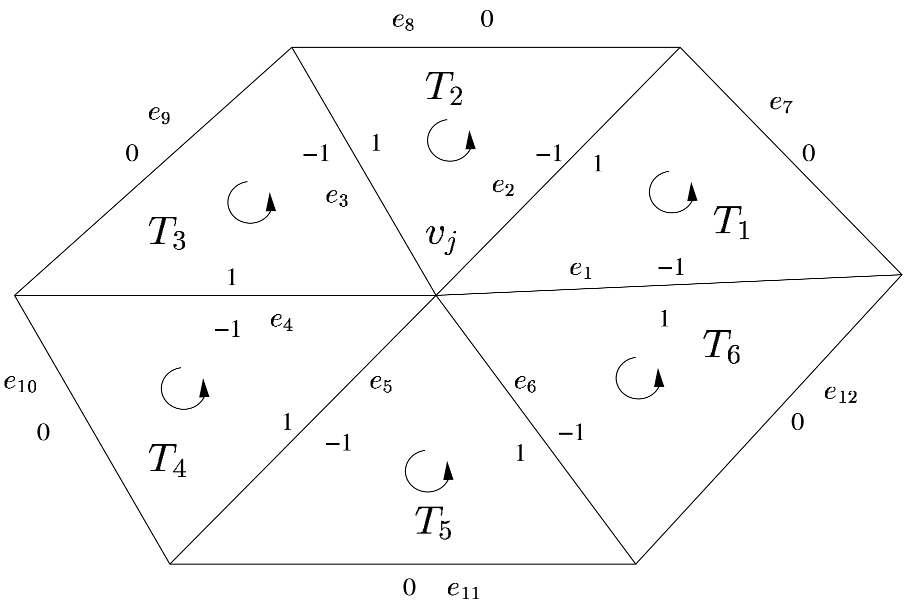


FIGURE 2. The values of $\int_e \frac{\partial \varphi_j^V}{\partial \tau}\, ds$ are equal to 1 if $e = e_k$ is regarded as an edge of ∂T_k, $k = 1, \cdots, 6$; -1 if $e = e_k$ is regarded as an edge of T_{k-1}, $k = 1, \cdots, 6$; and 0 if $e = e_k, k = 7, \cdots, 12$. Here, the identification $T_0 = T_6$ is assumed and the direction of the unit tangent vector τ is positively oriented on ∂T_k.

If e is a common edge of T_j and T_k with $j < k$, and τ_j and τ_k are positively oriented unit tangent vectors on ∂T_j and ∂T_k, respectively, we have $\int_e \frac{\partial \varphi_j}{\partial \tau_j}\, ds + \int_e \frac{\partial \varphi_k}{\partial \tau_k}\, ds = 0$ for all $\varphi \in \mathcal{N}C^h$; in this case it is convenient to represent the integral values of $\int_e \frac{\partial \varphi_j}{\partial \tau_j}\, ds$ and $-\int_e \frac{\partial \varphi_k}{\partial \tau_k}\, ds$ by $\left[\int_e \frac{\partial \varphi}{\partial \tau}\, ds\right]$, and this convention will be assumed in what follows. We are now ready to investigate on the dimension and basis functions for $\mathcal{N}C^h$ and $\mathcal{N}C_0^h$.

2.3. The dimension and basis functions for $\mathcal{N}C^h$. The dimension of $\mathcal{N}C^h$ is given by Fortin and Soulie in [17].

Lemma 2.4.
$$\dim(\mathcal{N}C^h) = 2N_E = N_V + N_E + N_T - 1.$$

Proof. According to Fortin and Soulie [17], $\mathcal{N}C^h$ consists of the union of the standard P_2 conforming element space of dimension $N_V + N_E$ and the space of nonconforming bubbles of dimension N_T. Since the standard P_2 conforming element space and the space of nonconforming bubbles have a one dimensional intersection space of the global conforming bubbles, the dimension of $\mathcal{N}C^h$ is $N_E + N_V + N_T - 1$. Due to Euler's formula, $\dim(\mathcal{N}C^h) = N_V + N_E + N_T - 1 = 2N_E$. □

The basis for $\mathcal{N}C^h$ is given in the following theorem.

Theorem 2.1. *For $i = E, V, T$, $j = 1, \cdots, M_i$ with $M_E = N_E, M_V = N_V - 1, M_T = N_T$, let φ_j^i be the functions defined in Definition 2.1. Then $\{\varphi_1^E, \cdots, \varphi_{N_E}^E, \varphi_1^V, \cdots, \varphi_{N_V-1}^V, \varphi_1^T, \cdots, \varphi_{N_T}^T\}$ forms a basis for $\mathcal{N}C^h$.*

Proof. It suffices to show that $\{\varphi_1^E, \cdots, \varphi_{N_E}^E, \varphi_1^V, \cdots, \varphi_{N_V-1}^V, \varphi_1^T, \cdots, \varphi_{N_T}^T\}$ is linearly independent. For this, suppose that $\sum_{i=E,V,T}\sum_{j=1}^{M_i} c_j^i \varphi_j^i = 0$. For any $k, k = 1, \cdots, N_E$, by integrating on e_k, one sees that $0 = \sum_{i=E,V,T}\sum_{j=1}^{M_i} c_j^i \int_{e_k} \varphi_j^i\, ds = c_k^E$. Similarly, for any $k, k = 1, \cdots, N_T$, an integration on T_k leads to $0 = \sum_{i=E,V,T}\sum_{j=1}^{M_i} c_j^i \int_{T_k} \varphi_j^i\, dx = c_k^T$. It remains to show that all c_j^V vanishes for $j = 1, \cdots, N_V - 1$. Let v_k be a vertex connected to v_{N_V} by an edge e_{kN_V}. Then it follows that

$$0 = \left|\sum_{j=1}^{N_V-1} c_j^V \left[\int_{e_{kj}} \frac{\partial \varphi_j^V}{\partial \tau}\, ds\right]\right| = |c_k^V|.$$

In this fashion one sees that all the coefficients c_k^V's associated with such vertices v_k's that are connected by an edge to v_{N_V} vanish. Then delete all the triangles whose vertices are connected to v_{N_V} by an edge, and call the remaining domain Ω_1. We proceed with a deleted vertex v_k and let v_l be a vertex in Ω_1 that is connected by an edge, say e_m. By repeating the above argument, we see that $c_l^V = 0$, and so on. Consequently, we have all the coefficients c_l^V's associated with the vertices v_l's that are connected by an edge to the already deleted vertices v_k's must vanish. Then strip out such vertices v_l's again, and repeat the argument until the domain is exhausted. This shows that $c_j^V = 0$ for all $j = 1, \cdots, N_V - 1$. Therefore, $\{\varphi_1^E, \cdots, \varphi_{N_E}^E, \varphi_1^V, \cdots, \varphi_{N_V-1}^V, \varphi_1^T, \cdots, \varphi_{N_T}^T\}$ is linearly independent, and forms a basis for $\mathcal{NC}^h$ by Lemma 2.4. □

Remark 2.5. *The proof of Remark 2.4 suggests an alternative choice of basis for $\mathcal{NC}^h$ using the P_2 conforming basis. Choose any triangle, say T_{N_T}. Then a basis for $\mathcal{NC}^h$ can be chosen as the union of P_2 conforming basis and the nonconforming bubbles consisting of semi-loop functions based on the triangles $T_1, \cdots, T_{N_T-1}$.*

2.4. The dimension and basis functions for $\mathcal{NC}_0^h$. We now consider the case of $\mathcal{NC}_0^h$ whose dimension and basis functions can be obtained from the case of $\mathcal{NC}^h$.

Lemma 2.5.

$$\dim(\mathcal{NC}_0^h) = N_E + N_V + N_T - 1 - 2(N_E - N_E^i).$$

Proof. This follows from the dimension of $\mathcal{NC}^h$ and the definition of $\mathcal{NC}_0^h$ that has two restrictions at each boundary edge of Ω_h. □

The global basis functions are given in the following theorem.

Theorem 2.2. *For $i = E, V, T$, $j = 1, \cdots, M_i$ with $M_E = N_E^i, M_V = N_V^i, M_T = N_T$, let φ_j^i be the function defined in Definition 2.1. Then*

$$\{\varphi_1^E, \cdots, \varphi_{N_E^i}^E, \varphi_1^V, \cdots, \varphi_{N_V^i}^V, \varphi_1^T, \cdots, \varphi_{N_T}^T\}$$

forms a basis for $\mathcal{NC}_0^h$. Consequently, $\dim(\mathcal{NC}_0^h) = N_E^i + N_V^i + N_T = 2N_E^i + 1$.

Proof. Suppose that $\sum_{i=E,V,T}\sum_{j=1}^{M_i} c_j^i \varphi_j^i = 0$. For any $k, k = 1, \cdots, N_E^i$, by integrating on e_k, one sees that $0 = \sum_{i=E,V,T}\sum_{j=1}^{M_i} c_j^i \int_{e_k} \varphi_j^i\, ds = c_k^E$. Similarly, for any $k, k = 1, \cdots, N_T$, an integration on T_k leads to $0 = \sum_{i=E,V,T}\sum_{j=1}^{M_i} c_j^i \int_{T_k} \varphi_j^i\, dx = c_k^T$. It remains to show that all c_j^V vanishes for $j = 1, \cdots, N_V^i$. For this, let v_k and v_l be two interior and boundary vertices, respectively, which are connected by an edge e_{kl} in $\mathcal{T}_h$. Without loss of generality let τ be the positive oriented unit tangent vector on e_{kl} Then,

$$0 = \left| \sum_{j=1}^{N_V^i} c_j^V \left[\int_{e_{kl}} \frac{\partial \varphi_j^V}{\partial \tau} \, ds \right] \right| = |c_k^V|.$$

In this way one sees that all the coefficients c_k^V's associated with such interior vertices v_k's that are connected to boundary vertices vanish. Then strip out all the triangles whose vertices meet the boundary Γ. Apply the same argument to the stripped domain and repeat until the domain is exhausted. This shows that $c_j^V = 0$ for all $j = 1, \cdots, N_V^i$. Therefore, $\{\varphi_1^E, \cdots, \varphi_{N_E^i}^E, \varphi_1^V, \cdots, \varphi_{N_V^i}^V, \varphi_1^T, \cdots, \varphi_{N_T}^T\}$ is linearly independent, and thus forms a basis for $\mathcal{NC}_0^h$ by Lemma 2.5. $\square$

3. The interpolation and projection operators and convergence analysis

In this section we define an interpolation operator and analyze convergence in the case of Dirichlet problem. The case of Neumann or Robin problem is quite similar and we omit the details, but state the results.

To begin with, consider the following Dirichlet problem:

$$\left\{ \begin{array}{rcll} -\nabla \cdot \alpha \nabla u + \beta u & = & f, & \Omega, \\ u & = & 0, & \Gamma, \end{array} \right. \tag{3.1}$$

with $\alpha = (\alpha_{jk}), \alpha_{jk}, \beta \in L^\infty(\Omega), j, k = 1, 2,\ 0 < \alpha_* |\xi|^2 \le \xi^t \alpha(x) \xi \le \alpha^* |\xi|^2 < \infty$, $\xi \in \mathbb{R}^2, \beta(x) \ge 0, x \in \Omega$, and $f \in H^1(\Omega)$. Moreover, we will assume that the data and the domain are sufficiently smooth so that the elliptic problem (3.1) be $H^3(\Omega)$- regular. Let $(\cdot, \cdot)$ be the $L^2(\Omega)$ inner product and (f, v) is understood as the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ which is an extension of the duality paring between $L^2(\Omega)$. The weak problem is then given as usual: find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v), \quad v \in H_0^1(\Omega), \tag{3.2}$$

where a is the bilinear form defined by $a(u, v) = (\alpha \nabla u, \nabla v) + (\beta u, v)$ for all $u, v \in H_0^1(\Omega)$.

Our P_2-nonconforming method for Problem (3.1) states as follows: find $u_h \in \mathcal{NC}_0^h$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad v_h \in \mathcal{NC}_0^h, \tag{3.3}$$

where

$$a_h(u, v) = \sum_{T \in \mathcal{T}_h} a_T(u, v),$$

with a_T being the restriction of a to T. From (3.2) and (3.3), we have the error equation.

$$a_h(u - u_h, v_h) = 0, \quad v_h \in \mathcal{NC}_0^h. \tag{3.4}$$

3.1. The interpolation and projection operators.

For a triangle $T \in \mathcal{T}_h$, define the *local interpolation operator* $\Pi_T : H^1(T) \to P_2(T)$ by

$$\int_{e_j} (v - \Pi_T v) \, ds = 0, \quad \int_{e_j} \frac{\partial (v - \Pi_T v)}{\partial \tau} \, ds = 0, \quad \int_T (v - \Pi_T v) \, dx = 0,$$

for all edges e_j of the T. The global *interpolation operator* $\Pi_h : H_0^1(\Omega) \to \mathcal{NC}_0^h$ is then defined through the local interpolation operator Π_T by $\Pi_h|_T = \Pi_T$ for all

$T \in \mathcal{T}_h$. Since Π_h preserves $P_2(T)$ for all $T \in \mathcal{T}_h$, it follows from the Bramble-Hilbert Lemma [4, 11] that

$$\sum_{T\in\mathcal{T}_h} ||\varphi - \Pi_h\varphi||_{L^2(T)} + h \sum_{T\in\mathcal{T}_h} ||\varphi - \Pi_h\varphi||_{H^1(T)} \le Ch^k||\varphi||_{H^k(\Omega)}, \qquad \varphi \in H^k(\Omega) \cap H_0^1(\Omega),\ 1 \le k \le 3. \tag{3.5}$$

Denote the intersection $\partial T_j \cap \partial T_k$ by e_{jk} for all $T_j, T_k \in \mathcal{T}_h$ and let Γ_j's be the boundary edges of $\mathcal{T}_h$. Then define

$$\Lambda^h = \{\lambda \in \Pi_{j,k}P_1(e_{jk}) \,|\, \lambda_{jk} + \lambda_{kj} = 0, \text{ where } \lambda_{jk} = \lambda|_{e_{jk}} \text{ for all } j,k\},$$

where $P_1(e)$ denotes the set of linear functions on the edge e, and $\Pi_{j,k}P_1(e_{jk})$ means that we have two copies of linear functions λ_{jk} and λ_{kj} on e_{jk} whose signs are opposite. Also define the projection $\mathcal{P}_1 : H^{\frac{3}{2}}(\Omega) \to \Lambda^h$ such that

$$\left\langle \alpha\frac{\partial v_j}{\partial \nu_j} - \mathcal{P}_1 v_j, z \right\rangle_e = 0 \quad \text{for all } z \in P_1(e) \text{ for all } e \in \mathcal{E}_h, \tag{3.6}$$

where $v_j = v|_{T_j}$ and ν_j is the unit outward normal to T_j. Then we have the following standard polynomial approximation result.

$$\left\{ \sum_j ||\alpha\frac{\partial v_j}{\partial \nu_j} - \mathcal{P}_1 v_j||^2_{L^2(\partial T_j)} \right\}^{\frac{1}{2}} \le Ch^{k-\frac{3}{2}}\|v\|_{H^k(\Omega)},\ k = 2,3. \tag{3.7}$$

Since $w_j - w_k$ has zero values at the Gauss points on e_{jk} for all $w \in \mathcal{NC}_0^h$ and the two-point Gaussian quadrature is exact on polynomials of degree 3, the following useful orthogonalities hold.

Lemma 3.1. *If $u \in H^{\frac{3}{2}}(\Omega)$, then the following equality hold:*

$$\langle \mathcal{P}_1 u_j, w_j\rangle_{e_{jk}} + \langle \mathcal{P}_1 u_k, w_k\rangle_{e_{kj}} = \langle \mathcal{P}_1 u_j, w_j - w_k\rangle_{e_{jk}} = 0 \text{ for all } w \in \mathcal{NC}_0^h. \tag{3.8}$$

3.2. The energy-norm error estimate. Denote the broken energy norm

$$\|\varphi\|_h = a_h(\varphi,\varphi)^{\frac{1}{2}}.$$

We next recall the following Strang lemma [30, 31].

Lemma 3.2. *Let $u \in H^1(\Omega)$ and $u_h \in \mathcal{NC}_0^h$ be the solutions of* (3.2) *and* (3.3), *respectively. Then,*

$$\|u - u_h\|_h \le C\left\{ \inf_{v\in\mathcal{NC}_0^h} \|u - v\|_h + \sup_{w\in\mathcal{NC}_0^h} \frac{|a_h(u,w) - \langle f, w\rangle\,|}{\|w\|_h} \right\}. \tag{3.9}$$

Assume that $u \in H^3(\Omega) \cap H_0^1(\Omega)$. Due to (3.5), the first term in the right side of (3.9) is bounded by

$$\inf_{v\in\mathcal{NC}_0^h} \|u - v\|_h \le Ch^2\|u\|_{H^3(\Omega)}. \tag{3.10}$$

Proceed to estimate the second term in the right side of (3.9). A simple calculation shows that

$$a_h(u,w) - \langle f, w\rangle = \sum_j \langle \alpha\frac{\partial u_j}{\partial \nu_j}, w\rangle_{\partial T_j\setminus\Gamma_j},$$

and the two orthogonalities (3.6) and (3.8) imply that

$$a_h(u,w) - \langle f, w\rangle = \sum_j \left\langle \alpha\frac{\partial u_j}{\partial \nu_j} - \mathcal{P}_1 u_j, w - m_j \right\rangle_{\partial T_j\setminus\Gamma_j}, \tag{3.11}$$

where m_j is chosen to be the P_1 projection of w on ∂T_j. Applying the trace theorem, (3.5), and (3.7), we get

$$\left|\sum_j \left\langle \alpha\frac{\partial u_j}{\partial \nu_j} - \mathcal{P}_1 u_j, w - m_j\right\rangle_{\partial T_j}\right|$$

$$\leq Ch^{\frac{3}{2}}\|u\|_{H^3(\Omega)}\left(\sum_j ||w-m_j||_{L^2(T_j)}||\nabla(w-m_j)||_{L^2(T_j)}\right)^{\frac{1}{2}}$$

$$\leq Ch^{\frac{3}{2}}\|u\|_{H^3(\Omega)}h^{\frac{1}{2}}\|\nabla w\|_h. \tag{3.12}$$

Equations (3.11) and (3.12) result in

$$\sup_{w\in\mathcal{N}C_0^h}\frac{|a_h(u,w)-\langle f,w\rangle\,|}{\|w\|_h}\leq Ch^2\|u\|_{H^3(\Omega)},$$

which, combined with (3.10) in Lemma 3.2, gives the following energy-norm error estimate.

Theorem 3.1. *Let $u\in H^3(\Omega)\cap H^1_0(\Omega)$ and $u_h\in\mathcal{N}C_0^h$ be the solutions of* (3.2) *and* (3.3)*, respectively. Then we have*

$$||u-u_h||_h \quad\leq\quad Ch^2||u||_{H^3(\Omega)}.$$

3.3. The L^2 Error Estimate. In order to apply the duality argument, let $\eta = u-u_h$ and $\psi\in H^2(\Omega)$ be the solution of the dual problem:

$$\begin{aligned} L^*(\psi) = -\nabla\cdot\alpha\nabla\psi+\beta\psi &= \eta, \quad \Omega,\\ \psi &= 0, \quad \Gamma. \end{aligned}$$

Owing to the elliptic regularity,

$$\|\psi\|_{H^2(\Omega)}\leq C\|\eta\|_{L^2(\Omega)}. \tag{3.13}$$

Recalling the error equation (3.4) and the orthogonality (3.6), we have

$$\begin{aligned} \|\eta\|^2 &= (L^*\psi,\eta) = (-\nabla\cdot(\alpha\nabla\psi)+\beta\psi,\eta)\\ &= \sum_j(\alpha\nabla\psi_j,\nabla\eta_j)_{T_j}+(\beta\psi,\eta)-\sum_j\left\langle\alpha\frac{\partial\psi_j}{\partial\nu_j},\eta_j\right\rangle_{\partial T_j\setminus e_j}\\ &= a_h(\psi,\eta)-\sum_j\left\langle\alpha\frac{\partial\psi_j}{\partial\nu_j}-\mathcal{P}_1\psi_j,\eta_j\right\rangle_{\partial T_j\setminus e_j}\\ &= a_h(\eta,\psi-\Pi_h\psi)-\sum_j\left\langle\alpha\frac{\partial\psi_j}{\partial\nu_j}-\mathcal{P}_1\psi_j,\eta_j\right\rangle_{\partial T_j\setminus e_j}. \end{aligned}\tag{3.14}$$

Owing to (3.5) the first term on the right-hand side of (3.14) can be bounded as follows:

$$|a_h(\eta,\psi-v)|\leq Ch^3\|u\|_{H^3(\Omega)}\|\eta\|_{L^2(\Omega)} \tag{3.15}$$

Choosing q_j to be the P_1 projection of η_j on each edge of $T_j\subset\Gamma$, again due to Theorem 3.1, (3.7), (3.13), the latter term on the right-hand side of (3.14) can be bounded as follows:

$$\begin{aligned} \left|\sum_j\left\langle\alpha\frac{\partial\psi_j}{\partial\nu_j}-\mathcal{P}_1\psi_j,\eta_j\right\rangle_{\partial T_j\setminus\Gamma}\right| &= \left|\sum_j\left\langle\alpha\frac{\partial\psi_j}{\partial\nu_j}-\mathcal{P}_1\psi_j,\eta_j-q_j\right\rangle_{\partial T_j\setminus\Gamma}\right|\\ &\leq Ch\|\psi\|_{H^2(\Omega)}\|\eta\|_h\leq Ch^3\|\eta\|_{L^2(\Omega)}\|u\|_{H^3(\Omega)}. \end{aligned}\tag{3.16}$$

Combining (3.14), (3.15), and (3.16), we obtain the following $L^2(\Omega)$ error estimate:

Theorem 3.2. *Let the elliptic problem* (3.1) *be sufficiently regular and* $u \in H^3(\Omega)\cap H^1_0(\Omega)$ *and* $u_h \in \mathcal{NC}^h_0$ *be the solutions of* (3.2) *and* (3.3), *respectively.*

$$||u - u_h||_{L^2(\Omega)} \le Ch^3||u||_{H^3(\Omega)}.$$

3.4. The Robin boundary value problem. Instead of the Dirichlet problem (3.1), if the following Robin boundary value problem:

$$\left\{ \begin{array}{rcll} -\nabla\cdot\alpha\nabla u + \beta u &=& f, & \Omega, \\ \alpha\frac{\partial u}{\partial n} + \gamma u &=& g, & \Gamma, \end{array} \right. \tag{3.17}$$

is considered, The weak problem is then replaced by finding $u \in H^1(\Omega)$ such that

$$a^R(u,v) = (f,v) + \langle g, v\rangle\,, \quad v \in H^1(\Omega), \tag{3.18}$$

where a^R is the bilinear form defined by $a^R(u,v) = (\alpha\nabla u,\nabla v)+(\beta u, v)+\langle\gamma u, v\rangle$ for all $u, v \in H^1(\Omega)$, and $\langle\cdot,\cdot\rangle$ is the duality paring between $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$. The P_2-nonconforming method for Problem (3.17) then states as follows: find $u_h \in \mathcal{NC}^h$ such that

$$a^R_h(u_h, v_h) = (f, v_h) + \langle g, v_h\rangle\,, \quad v_h \in \mathcal{NC}^h. \tag{3.19}$$

Then all the arguments given above for the Dirichlet case hold analogously, whose details are omitted here,

Theorem 3.3. *Let* $u \in H^3(\Omega)$ *and* $u_h \in \mathcal{NC}^h$ *be the solutions of* (3.18) *and* (3.19), *respectively. Then we have*

$$||u - u_h||_h \quad\le\quad Ch^2||u||_{H^3(\Omega)}.$$

Theorem 3.4. *Let the elliptic problem* (3.17) *be sufficiently regular,* $u \in H^3(\Omega)$ *and* $u_h \in \mathcal{NC}^h$ *be the solutions of* (3.18) *and* (3.19), *respectively. Then we have*

$$||u - u_h||_{L^2(\Omega)} \le Ch^3||u||_{H^3(\Omega)}.$$

4. Numerical Examples

In this section we illustrate two numerical examples. First, consider the following Dirichlet problem:

$$\begin{aligned} -\triangle u = f, &\qquad \Omega, \\ u = 0, &\qquad \Gamma, \end{aligned}$$

where $\Omega =]0,1[^2$ and the source term f is calculated from the the exact solution $u(x,y) = \sin(2\pi x)\sin(2\pi y)(x^3 - y^4 + x^2y^3)$. Table 1 shows the numerical results, where the error reduction ratios in $L^2(\Omega)$ and broken energy norm are optimal.

Next, turn to the following Neumann problem:

$$\begin{array}{rcll} -\triangle u + u &=& f, & \Omega, \\ \dfrac{\partial u}{\partial n} &=& g, & \Gamma, \end{array}$$

with the same domain $\Omega =]0,1[^2$ as above and the source terms f and g are generated from the same exact solution as in the above Dirichlet problem case. Again Table 2 shows the numerical results, where the error reduction ratios in $L^2(\Omega)$ and broken energy norm are optimal.

h	D.O.F	$\|u-u_h\|_{L^2(\Omega)}$	ratio	$\|u-u_h\|_h$	ratio
1/4	81	0.193226E-01	-	0.570856E+00	-
1/8	353	0.212760E-02	3.18	0.154522E+00	1.97
1/16	1473	0.244224E-03	3.12	0.393285E-01	1.99
1/32	6017	0.296812E-04	3.18	0.987574E-02	1.99
1/64	24321	0.368206E-05	3.04	0.247166E-02	2.00
1/128	97793	0.459366E-06	3.00	0.618087E-03	2.00
1/256	392193	0.573927E-07	3.00	0.154532E-03	2.00

TABLE 1. The Dirichlet problem: The apparent L^2 and broken energy norm errors and their reduction ratios.

h	D.O.F	$\|u-u_h\|_{L^2(\Omega)}$	ratio	$\|u-u_h\|_h$	ratio
1/4	112	0.140301E-01	-	0.518300E+00	-
1/8	416	0.170392E-02	3.04	0.140810E+00	1.88
1/16	1600	0.212157E-03	3.01	0.359839E-01	1.97
1/32	6272	0.266717E-04	2.99	0.905582E-02	1.99
1/64	24832	0.334975E-05	2.99	0.226905E-02	2.00
1/128	98816	0.419889E-06	3.00	0.567752E-03	2.00
1/256	394240	0.525646E-07	3.00	0.141990E-03	2.00

TABLE 2. The Neumann problem: The apparent L^2 and broken energy norm errors and their reduction ratios.

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Department of Mathematics, Seoul National University, Seoul 151-747, Korea
E-mail: hlee@nasc.snu.ac.kr and sheen@snu.ac.kr
URL: http://www.nasc.snu.ac.kr/hlee and http://www.nasc.snu.ac.kr/sheen