INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 2, Number 4, Pages 367–408

© 2005 Institute for Scientific
Computing and Information

NUMERICAL APPROXIMATION OF TWO-DIMENSIONAL CONVECTION-DIFFUSION EQUATIONS WITH MULTIPLE BOUNDARY LAYERS

CHANG-YEOL JUNG[1] AND ROGER TEMAM[1,2]

Abstract. In this article, we demonstrate how one can improve the numerical solution of singularly perturbed problems involving multiple boundary layers by using a combination of analytic and numerical tools. Incorporating the structures of boundary layers into finite element spaces can improve the accuracy of approximate solutions and result in significant simplifications. We discuss here convection-diffusion equations in the case where both ordinary and parabolic boundary layers are present.

Key Words. boundary layers, finite elements, singularly perturbed problem, convection-diffusion

Contents

2000 *Mathematics Subject Classification.* 65N30, 34D15, 76N20.
This work was supported in part by NSF Grant DMS 0305110, and by the Research Fund of Indiana University.

1. Introduction

In this article we consider linear singularly perturbed convection dominated boundary value problems of the following types:

$$L_\epsilon u^\epsilon := -\epsilon \triangle u^\epsilon - u^\epsilon_x = f(x,y) \quad \text{for } (x,y) \in \Omega, \tag{1.1a}$$

with boundary conditions

$$u^\epsilon = 0 \quad \text{on } \partial\Omega, \tag{1.1b}$$

or,

$$\begin{aligned} u^\epsilon &= 0 \quad \text{at } x = 0, 1, \\ \frac{\partial u^\epsilon}{\partial y} &= 0 \quad \text{at } y = 0, 1. \end{aligned} \tag{1.1c}$$

Here $0 < \epsilon << 1$, and $\Omega = (0,1) \times (0,1) \subset \mathbf{R}^2$.

It can be shown (see below) that $u^\epsilon \to u^0$ in L^2 where u^0 is the solution of the limit problem:

$$-u^0_x = f \quad \text{in } \Omega, \tag{1.2a}$$

$$u^0 = 0 \quad \text{at } x = 1, \tag{1.2b}$$

so that we have

$$u^0 = \int_x^1 f(s,y)ds. \tag{1.2c}$$

Comparison between u^ϵ and u^0 is not easy because many discrepancies between u^ϵ and u^0 appear at the boundary. Just proving the L^2- convergence of u^ϵ to u^0 (which is a byproduct of the analysis below) is not straightforward. For a comparison between u^ϵ and u^0 in smaller spaces (spaces of more regular functions), we need to introduce a number of boundary layers of different types to account for the discrepancies. The most common boundary layer appears at $x = 0$ since $u^0(0,y)$ does not vanish in general; this boundary layer is obtained using the technique of ordinary boundary layers (OBL). From (1.2c), we see also that some discrepancies appear in general at the boundaries $y = 0, 1$. These will be accounted for by a less common concept of boundary layer, namely the parabolic boundary layer (PBL).

In [11] we discussed the problem (1.1a), (1.1b) when $f(x,y) = f_{yy}(x,y) = 0$ at $y = 0, 1$. In this case we only observe the discrepancy at $x = 0$ (note that $u^0(x,0) = u^0(x,1) = 0$), and the problem was thus handled by an OBL. In [16] we discussed equation (1.1a) in a channel with (1.1b) at $y = 0, 1$ and periodicity in the x- direction; in this case we only observe parabolic boundary layers (PBL).

Here, by considering equation (1.1a) in a square, we theoretically and numerically investigate the case where both OBLs and PBLs are present. In fact some restriction (compatibility conditions) will be assumed on f; indeed, as shown in [26], in the most general case (square with no restriction on f), several other inconsistencies occur which have to be accounted for by still other boundary layers. In this article, as we said, we avoid these additional boundary layers, and consider cases where only OBLs and PBLs are present. In fact we will see that for the mixed boundary value problem (1.1a), (1.1c), the compatibility conditions on f and the effects of the PBLs are mild (see Section 4), whereas for (1.1a), (1.1b) we fully show how to overcome this compatibility condition issue.

Through the boundary layer analysis in Section 2, we will find rigorously that OBLs occur at the outflow $x = 0$ and PBLs occur at the characteristic lines $y = 0, 1$.

It turns out that OBLs and PBLs severely affect the numerical solutions because they are, respectively, of order $O(\epsilon^{-3/2})$ and $O(\epsilon^{-3/4})$ in the H^2- norm. These H^2- singularities make our discretized (approximating) system highly unstable or ill-conditioned. Furthermore, if the boundary layers are not properly handled, the discretization errors due to the OBLs at $x = 0$ pollute the whole domain Ω, whereas the effect of the PBLs remain "localized" near the characteristic lines $y = 0, 1$. More precisely, the OBL errors propagate in the x- direction due to the convective term $-u^\epsilon_x$ in (1.1) and hence if the discretizetion errors (the stiffness of the problem) are not properly accounted for, the approximate solutions display wild oscillations in the x- direction throughout the domain Ω as in the classical approximation method, see Figure 2 (a) below and see also e.g. [3], [11], [12]. [20], [23] and [25]. On the other hand, the discretization errors due to the PBLs at $y = 0, 1$ are localized only at $y = 0, 1$ because they are aligned parallel to the propagation direction x- axis, see Figure 3 (a) and 4 (a) below and see also e.g. [16], [23] and [25]. This phenomenon happens similarly in a reaction-diffusion problem in the absence of a convective term, see [13].

Our first aim in this article is thus to construct the ordinary and parabolic boundary layer elements (BLE) which, respectively, capture the singularities due to both OBLs and PBLs for the problems (1.1) under consideration. We numerically implement the BLEs in our approximating system, and thus we avoid the mesh refinements near the occurrences of each boundary layer and we are led to a significant simplification for the numerical implementations; we do not consider a (time-consuming) special mesh strategy and mesh refinement in the region of the boundary layers which are very costly in practice; we simply utilize a *uniform mesh*, Q_1- elements, that is the hat functions. See e.g. [6], [7], [10], [17], [18], [20], [21], [26] and [29] for many other developments on boundary layers and their asymptotic approximations, and see the book of [23] for the numerical aspects of singularly perturbed problems.

Because of the ordinary boundary layer (OBL) at $x = 0$, the approximate solutions are not stable in the H^1 norm, but we do estimate the H^1- error for the approximate solution once the singular H^1- part has been captured using what we call below the boundary layer elements (BLE). However, it is noteworthy that our new discretized system (3.25) below is stable in the L^2 space. More precisely, for any $f \in L^2$

$$|u_N|_{L^2} \leq \kappa |f|_{L^2}, \tag{1.3}$$

where u_N is an approximate solution obtained from (3.25), and a positive constant κ is independent of the mesh size $\bar{h}$ and the small parameter ϵ; see the numerical results in the Tables 1 and 2 in [11] for a related situation. The L^2- stability analysis and L^2- error estimates for the current problem will appear in [14]; in this case the analysis is technical due to the absence of a reaction term, e.g. u^ϵ, in (1.1).

We would like to mention that during this research we improved the results of the article [11] in two respects. Firstly we could weaken the conditions needed to avoid the occurrence of PBLs for the Dirichlet boundary value problem. More precisely, if $f = 0$ at $y = 0, 1$, PBLs do not exist in $H^2(\Omega)$, see Lemma 3.1 below. The second one is that we could weaken the compatibility conditions which appeared in [11] for the mixed boundary value problem, see (4.20) below.

We denote the mesh size by $\bar{h} = \max\{h_1, h_2\}$ where $h_1 = 1/M$, $h_2 = 1/N$, M, N are the number of elements respectively in the x-, and y- directions. Hence, the number of rectangular elements is MN.

We shall consider the Sobolev spaces $H^m(\Omega)$, m integer, equipped with the semi-norms,

$$|u|_{H^m} = \left(\sum_{|\alpha|=m} \int_\Omega |D^\alpha u|^2 dxdy \right)^{1/2}, \tag{1.4}$$

the norms,

$$\|u\|_{H^m} = \left(\sum_{j=0}^{m} |u|^2_{H^j} \right)^{1/2}, \tag{1.5}$$

and the corresponding inner products,

$$((u, v))_{H^m(\Omega)} = \sum_{|\alpha|\le m} (D^\alpha u, D^\alpha v), \tag{1.6a}$$

where

$$(u, v) = \int_\Omega uvdxdy. \tag{1.6b}$$

We will also make use of a weighted energy norm which is useful when analyzing the convection-diffusion problem in the finite element context, namely:

$$\|u\|_\epsilon = \sqrt{\epsilon|\nabla u|^2_{L^2} + |u|^2_{L^2}}. \tag{1.7}$$

As usual, when $m = 0$, H^m is the space L^2. For the Dirichlet boundary value problem (1.1a), (1.1b), we use the Sobolev space $H^1_0(\Omega)$, which is the closure in the space $H^1(\Omega)$ of C^∞ functions compactly supported in Ω; the appropriate space for (1.1a), (1.1c) will be introduced below. In the text κ, c denote generic positive constants independent of $\epsilon, h_1, h_2, \bar{h}$, which may be different at different occurrences; the c are absolute constants, the κ are constants depending on the data.

We realize of course that the problem considered here is a model problem. A number of generalizations can be considered: more general elliptic operators, more general convection operators, nonlinear or time dependant problems; such generalizations will be considered elsewhere.

This article is organized as follows: we start in Section 2 by analyzing the boundary layers for the Dirichlet boundary value problem (1.1a), (1.1b) using asymptotic expansion techniques. We continue in Section 3 by constructing the boundary layer elements (BLE) via finite element methods which incorporate the BLEs and deriving error estimates in H^1. In Section 4 we consider the mixed boundary value problem (1.1a), (1.1c) using a similar approach. It is important to identify the type of boundary layers that occur depending on the data f and the boundary conditions. In Section 5 we thus summarize two major boundary layers which are the OBLs and PBLs. Finally in Section 6 we show the numerical results that support our analysis.

2. Boundary Layer Analysis

Throughout this paper $\Omega = (0, 1) \times (0, 1)$, and $f = f(x, y)$ is assumed to be smooth on $\bar{\Omega}$.

We first consider the Dirichlet boundary value problem (1.1a), (1.1b). Its weak formulation is as follows:

To find $u \in H^1_0(\Omega)$ such that

$$a_\epsilon(u, v) = F(v),\ \forall v \in H^1_0(\Omega), \tag{2.1a}$$

where

$$a_\epsilon(u,v) = \epsilon \int_\Omega \nabla u \cdot \nabla v dxdy - \int_\Omega u_x v dxdy, \tag{2.1b}$$

$$F(v) = \int_\Omega f v dxdy. \tag{2.1c}$$

It is easy to verify the coercivity of a_ϵ on $H_0^1(\Omega)$, i.e.,

$$a_\epsilon(u,u) \geq \epsilon \|u\|^2, \ \forall u \in H_0^1(\Omega), \tag{2.2}$$

the continuity of the bilinear form a_ϵ on $H_0^1 \times H_0^1$, and the continuity of the linear form F on H_0^1. Hence, by the Lax-Milgram theorem, there exists a unique function $u \in V$ satisfying equation (2.1).

Along the asymptotic analysis, we define the outer expansion $u^\epsilon \sim \sum_{j=0}^\infty \epsilon^j u^j$. By formal identification at each power of ϵ, we find

$$O(1): \ -u_x^0 = f, \qquad u^0 = 0 \text{ at } x = 1, \tag{2.3a}$$

$$O(\epsilon^j): \ -\triangle u^{j-1} - u_x^j = 0, \ u^j = 0 \text{ at } x = 1, \tag{2.3b}$$

for $j \geq 1$. The boundary conditions in (2.3) are natural boundary conditions for the operator $-d/dx$ on $(0,1)$; this choice of the boundary condition will be justified afterwards by the convergence theorem (see Theorem 2.2).

By explicit calculations, we find for $j = 0, 1, 2$:

$$u^0(x,y) = \int_x^1 f(s,y) ds, \tag{2.4a}$$

$$u^1(x,y) = \int_x^1 \triangle u^0(s,y) ds = -f(1,y) + f(x,y) + \int_x^1 (s-x) f_{yy}(s,y) ds \tag{2.4b}$$

$$\begin{aligned} u^2(x,y) = \int_x^1 \triangle u^1(s,y) ds &= f_x(1,y) - f_x(x,y) - (1-x) f_{yy}(1,y) \\ &+ 2\int_x^1 f_{yy}(s,y) ds + \int_x^1 \frac{(s-x)^2}{2} \frac{\partial^4 f}{\partial y^4}(s,y) ds. \end{aligned} \tag{2.4c}$$

2.1. The Parabolic Boundary Layers. It is clear that the functions u^j of the outer expansion do not generally satisfy the boundary conditions (1.1b) at $x = 0$, and $y = 0, 1$. To resolve these discrepancies, we will introduce the ordinary boundary layers (OBLs) (for $x = 0$), and the so-called parabolic boundary layers (PBLs), for $y = 0, 1$. We start with the parabolic boundary layers which are defined by the inner expansion $u^\epsilon \sim \sum_{j=0}^\infty \epsilon^j \varphi_l^j$ at $y = 0$, where $\varphi_l^j = \varphi_l^j(x, \bar{y})$, $\bar{y} = y/\sqrt{\epsilon}$. Then we find:

$$-\sum_{j=0}^\infty \left\{ \epsilon^{j+1} \varphi_{lxx}^j + \epsilon^j \varphi_{l\bar{y}\bar{y}}^j \right\} - \sum_{j=0}^\infty \epsilon^j \varphi_{lx}^j = 0.$$

By formal identification at each power of ϵ, we obtain the following heat equations in which $-x$ is the timelike variable:

$$\begin{aligned} &O(1): \ -\varphi_{l\bar{y}\bar{y}}^0 - \varphi_{lx}^0 = 0, \\ &O(\epsilon^j): \ -\varphi_{l\bar{y}\bar{y}}^j - \varphi_{lx}^j = \varphi_{lxx}^{j-1}, \ \text{for } j \geq 1. \end{aligned} \tag{2.5a}$$

The "initial" and boundary conditions that we choose (and that are justified below afterwards) are:

$$\varphi_l^j(x,\bar{y}) = 0, \text{ at } x = 1, \tag{2.5b}$$

$$\varphi_l^j(x,0) = r_j(x), \tag{2.5c}$$

$$\varphi_l^j(x,\bar{y}) \to 0 \text{ as } \bar{y} \to \infty, \tag{2.5d}$$

where $r_j(x) = -u^j(x,0),\ j \geq 0$.

2.2. Construction and Properties of the φ_l^j. Firstly, we consider the following heat equation in a semi-strip, see Theorem 20.3.1 in [2]. Let

$$D = \{(x,y) \in \mathbf{R}^2;\ 0 < x < 1, y > 0\}. \tag{2.6}$$

We are given f^* which is uniformly Hölder continuous in x and y for each compact subset of D and satisfies

$$|f^*(x,y)| \leq \kappa \exp(-\gamma y), \tag{2.7}$$

for some $\gamma > 0$, and all $0 < x < 1$ and $y > 0$; we are also given g^* which is continuous on $[0,1]$. Then we look for u satisfying:

$$\begin{cases} -\dfrac{\partial u}{\partial x} - \dfrac{\partial^2 u}{\partial y^2} = f^*, \text{ for } (x,y) \in D, \\ u(x,0) = g^*(x),\ 0 < x < 1, \\ u(x,y) \to 0 \text{ as } y \to \infty, 0 < x < 1, \\ u(1,y) = 0. \end{cases} \tag{2.8}$$

Compatibility Conditions. We will assume the following smoothness and compatibility conditions on the data f^*, g^* which guarantee that $u \in C^l(\bar{D})$, $l \geq 0$, see e.g. [26], [27]:

(2.9a) $f^*(x,y)$ and $g^*(x)$ are sufficiently smooth[1] on $\bar{D}$ and $[0,1]$, respectively,

and

$$\frac{\partial^i}{\partial x^i} f^*(1,y) = \frac{\partial^i}{\partial x^i} g^*(1) = 0, \text{ for } 0 \leq i \leq l. \tag{2.9b}$$

Let us recall the motivation for (2.9b): assume that $u \in C^l(\bar{D})$ is a solution of (2.8), we then firstly notice that from the first equation (2.8), since $\partial^2 u/\partial y^2 = -\partial u/\partial x - f^*$, we recursively find

$$\begin{aligned} \frac{\partial^{2k}}{\partial y^{2k}} u &= \frac{\partial^{2(k-1)}}{\partial y^{2(k-1)}} \left(\frac{\partial^2}{\partial y^2} u \right) = -\frac{\partial^{2k-1}}{\partial x \partial y^{2(k-1)}} u - \frac{\partial^{2k-2}}{\partial y^{2(k-1)}} f^* \\ &= (-1)^k \frac{\partial^k}{\partial x^k} u + \sum_{s=0}^{k-1} (-1)^{s+1} \frac{\partial^{2k-s-2}}{\partial x^s \partial y^{2(k-s-1)}} f^*. \end{aligned} \tag{2.10}$$

Since $u(1,y) = 0$, setting $x = 1$ in (2.10), we then easily find

$$(-1)^k \frac{\partial^k}{\partial x^k} g^*(1) + \sum_{s=0}^{k-1} (-1)^{s+1} \frac{\partial^{2k-s-2}}{\partial x^s \partial y^{2(k-s-1)}} f^*(1,y) = 0,\ k = 0, 1, \cdots, l, \tag{2.11}$$

[1] The level of smoothness is unspecified for the sake of simplicity since smoothness is not the main issue in this article: we generally assume $C^\infty(\bar{\Omega})$ regularity of the data, and from time to time we will mention weaker regularity assumptions which are sufficient; e.g. for (2.10),(2.15), we require $f^* \in C^k(\bar{D})$ with $k = \max\{2l-2,\ l+1\}$.

which is necessary for $u \in C^l(\bar{D})$; conditions (2.9) are much stronger than (2.11) and it is proven in e.g. [27] that the conditions (2.9) guarantee that $u \in C^l(\bar{D})$.

The case where these conditions (2.9) are not satisfied is more involved and will be considered elsewhere; we expect corner singularities at $(1, 0)$, see (7.6) below or see [26].

From now on we thus assume that the following compatibility conditions between the "initial" and boundary conditions for φ_l^j hold: for $0 \le i \le 2n + d - 2j$, $d = 0, 1$, and $0 \le j \le n$,

$$r_j^{(i)}(1) = -\frac{\partial^i}{\partial x^i} u^j(1, 0) = 0. \tag{2.12}$$

To derive below the error estimate in the context of the standard finite elements method, we will need further estimates on the spatial derivatives of φ_l^j. To derive these estimates and for later purpose, it is useful to obtain the expression of the φ_l^j to be provided by the following lemma.

Lemma 2.1. *Let $u = u(x, y)$ be the solution of the heat equation (2.8) in D. Then the solution u is unique and it admits the integral representation:*

(2.13)
$$\begin{aligned} u(x,y) &= \sqrt{\frac{2}{\pi}} \int_{y/\sqrt{2(1-x)}}^{\infty} \exp\left(-\frac{t^2}{2}\right) g^*\left(x + \frac{y^2}{2t^2}\right) dt \\ &+ \frac{1}{2\sqrt{\pi}} \int_0^{1-x} \int_0^{\infty} \frac{1}{\sqrt{s}} \left\{ \exp\left[-\frac{(y-t)^2}{4s}\right] - \exp\left[-\frac{(y+t)^2}{4s}\right] \right\} f^*(x+s, t)\, dt ds, \end{aligned}$$

and

$$|u(x, y)| \le \kappa \exp(-\gamma y), \text{ for the same } \gamma \text{ as in (2.7).} \tag{2.14}$$

If the conditions (2.9) hold, then $u \in C^l(\bar{D})$, $l \ge 0$. Furthermore, if the following decay conditions hold:

$$\left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} f^*(x, y) \right| \le \kappa \exp(-\gamma y), \text{ for } 0 \le i + m \le l + 1,\ \gamma > 0 \text{ as before,} \tag{2.15}$$

then the following pointwise estimates for u and its derivatives hold: for each i and m, there exists a constant κ_{im} which depends only on f^* and g^* such that

$$\left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} u(x, y) \right| \le \kappa_{im} \exp(-\gamma y), \forall (x, y) \in D, \tag{2.16}$$

for $0 \le i + m \le l + 1$, the same γ as in (2.15).

For the proof, see the Appendix.

Remark 2.1. From Lemma 2.1, if (2.9) and (2.15) hold, it is obvious that

$$u \in C^l(\bar{D}) \cap H^{l+1}(D), \tag{2.17a}$$

$$u(x, \cdot) \in C^l([0, 1]) \cap H^{l+1}(0, 1),\ \forall x \in [0, 1], \tag{2.17b}$$

$$u(\cdot, y) \in C^l([0, \infty)) \cap H^{l+1}(0, \infty),\ \forall y \ge 0. \tag{2.17c}$$

From Lemma 2.1 setting $u = \varphi_l^k$ and $y = \bar{y}$, we find the solutions φ_l^j for equation (2.5) recursively:

(2.18a)

$$\varphi_l^0(x,\bar{y}) = \sqrt{\frac{2}{\pi}} \int_{\bar{y}/\sqrt{2(1-x)}}^{\infty} \exp\left(-\frac{t^2}{2}\right) r_0 \left(x + \frac{\bar{y}^2}{2t^2}\right) dt,$$

(2.18b)

$$\varphi_l^j(x,\bar{y}) = \sqrt{\frac{2}{\pi}} \int_{\bar{y}/\sqrt{2(1-x)}}^{\infty} \exp\left(-\frac{t^2}{2}\right) r_j \left(x + \frac{\bar{y}^2}{2t^2}\right) dt$$
$$+ \frac{1}{2\sqrt{\pi}} \int_0^{1-x} \int_0^{\infty} \frac{1}{\sqrt{s}} \left\{ \exp\left[-\frac{(\bar{y}-t)^2}{4s}\right] - \exp\left[-\frac{(\bar{y}+t)^2}{4s}\right] \right\} \frac{\partial^2}{\partial x^2} \varphi_l^{j-1}(x+s,t) dt ds,$$

for $1 \le j \le n$. Furthermore, thanks to the compatibility conditions (2.12), we find that for $0 \le j \le n$, $\varphi_l^j(x,\bar{y}) = \varphi_l^j(x, y/\sqrt{\epsilon})$ satisfies the regularities (2.17) with $l = 2n + d - 2j$, and $y = \bar{y}$.

The following lemmas easily follow from (2.16); the lemmas provide pointwise and norm estimates on the derivatives of φ_l^j which will be used below.

Lemma 2.2. *Assume that the conditions (2.12) hold. Then there exist a positive constant κ_{ijm} independent of ϵ such that the following inequalities hold*

$$\left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \varphi_l^j \left(x, \frac{y}{\sqrt{\epsilon}}\right) \right| \le \kappa_{ijm} \epsilon^{-m/2} \exp\left(-\frac{y}{\sqrt{\epsilon}}\right), \forall (x,y) \in \bar{\Omega}^2, \tag{2.19}$$

for $0 \le i + m \le 2n + d + 1 - 2j$, and $0 \le j \le n$.

The following L^2- estimates are immediate consequences of Lemma 2.2.

Lemma 2.3. *Assume that the conditions (2.12) hold. Let, for $0 \le \sigma < 1$,*

$$\Omega^\sigma = (0,1) \times (\sigma, 1).$$

Then there exists a positive constant κ_{ijm} independent of ϵ such that the following inequalities hold: for $0 \le i + m \le 2n + d + 1 - 2j$,

$$\left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \varphi_l^j \right|_{L^2(\Omega^\sigma)} \le \kappa_{ijm} \epsilon^{-m/2+1/4} \exp\left(-\frac{\sigma}{\sqrt{\epsilon}}\right); \tag{2.20a}$$

in particular,

$$\left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \varphi_l^j \right|_{L^2(\Omega)} \le \kappa_{ijm} \epsilon^{-m/2+1/4}. \tag{2.20b}$$

Remark 2.2. Similarly, at $y = 1$, we introduce another PBL, φ_u^j, having the same structure as φ_l^j with the role of $\bar{y}$ and $\tilde{y} = (1-y)/\sqrt{\epsilon}$ being exchanged. We then need (assume) the following compatibility conditions, similar to (2.12):

$$-\frac{\partial^i}{\partial x^i} u^j(1,1) = 0, \text{ for } 0 \le i \le 2n + d - 2j,\ d = 0, 1, \text{ and } 0 \le j \le n. \tag{2.21}$$

Under the hypothesis (2.21), the results of Lemma 2.2 and Lemma 2.3 are valid with φ_l^j replaced by φ_u^j, $\bar{y}$ by $\tilde{y}$, and $(\sigma, 1)$ by $(0, 1-\sigma)$. We also notice that

[2] $\exp\left(-y/\sqrt{\epsilon}\right)$ can be replaced by $\exp\left(-cy/\sqrt{\epsilon}\right)$ for any $c > 0$ with then κ depending on c. Note that, by the estimate (2.16) applied to φ_l^0 with $f^* = 0$, we find that $\gamma > 0$ in (2.15) (and thus $c > 0$) can be chosen arbitrarily. For $j \ge 1$, we apply (2.16) to φ_l^j with $f^* = \partial^2 \varphi_l^{j-1}/\partial x^2$, and then we find, recursively, that the same c is valid.

$\varphi_u^j(x,\tilde{y}) = \varphi_u^j(x,(1-y)/\sqrt{\epsilon})$ satisfies the regularities (2.17) with $l = 2n+d-2j$, and $y = \tilde{y}$. If $\sigma = \kappa\epsilon^\alpha$ with $\alpha < 1/2$, the parabolic boundary layers, φ_l^j and φ_u^j, as indicated in the estimate (2.20), are exponentially small on $(0,1)\times(\sigma,1-\sigma)$. This implies that we only need to take care of the parabolic boundary layers near the boundaries $y=0$ and $y=1$ in the finite element solutions. □

Before we go further, it is convenient here to recall the definition of exponentially small functions.

Definition 2.1. *A function $\tilde{g}^\epsilon$ is called an exponentially small term, denoted e.s.t., if there exists $\alpha \in (0,1)$ and $\alpha' > 0$ such that for any $k \geq 0$, there exists a constant $c_{\alpha,\alpha',k} > 0$ independent of ϵ with*

$$\|\tilde{g}^\epsilon\|_{H^k} \leq c_{\alpha,\alpha',k} e^{-\alpha/\epsilon^{\alpha'}}. \tag{2.22}$$

An e.s.t.(n) is a function $\tilde{g}^\epsilon$ for which (2.22) holds for $0 \leq k \leq n$; $e^{-(1+x)/\epsilon}$ is an example of e.s.t. $g(x)e^{-1/\epsilon}$ with $g(x) = x\log x$ is an example of e.s.t.(1); note that $g(x) \in H_0^1(0,1)$ but $g(x) \notin H^2(0,1)$.

2.3. The Ordinary Boundary Layers. At this stage the function u^ϵ is tentatively approximated by $\sum_{j=0}^\infty \epsilon^j(u^j + \varphi_l^j + \varphi_u^j)$. However, with the definitions above of u^j, φ_l^j, and φ_u^j, for each j, $0 \leq j \leq n$, the function $-u^j(x,y) - \varphi_l^j(x,\bar{y}) - \varphi_u^j(x,\tilde{y})$ is 0 at $x=1$ because of the boundary conditions (2.3), and (2.5b), and this function is exponentially small at $y=0$ and $y=1$ by the boundary conditions (2.5c) and Lemma 2.2. We now want to take care of the discrepancies at the boundary $x=0$ where

$$\begin{aligned} g^j(y) &= -u^j(0,y) - \varphi_l^j\left(0,\frac{y}{\sqrt{\epsilon}}\right) - \varphi_u^j\left(0,\frac{1-y}{\sqrt{\epsilon}}\right) \\ &\in C^{2n+d-2j}([0,1]) \cap H^{2n+d+1-2j}(0,1) \end{aligned} \tag{2.23}$$

does not vanish unlike u^ϵ. We will handle these discrepancies with an ordinary boundary layer. Note that, in general, one cannot resolve these discrepancies with one single boundary layer, since while "repairing" the boundary condition at $x=1$, we do not want to "damage" again the boundary conditions at $y=0,1$, which were "repaired" by the PBLs. In general, as we said above and in the Introduction, we cannot do this with one single boundary layer (see [26]); we can do this here because of the simplifying assumptions (2.12) and (2.21) which follow from (2.38) below.

For that purpose, we now introduce the so-called ordinary boundary layer functions θ^j which are defined by the inner expansion $u^\epsilon \sim \sum_{j=0}^\infty \epsilon^j\theta^j$ at $x=0$, where $\theta^j = \theta^j(\bar{x},y)$, $\bar{x} = x/\epsilon$. By formal identification at each power of ϵ, we find

$$O(\epsilon^{-1}):\ -\theta^0_{\bar{x}\bar{x}} - \theta^0_{\bar{x}} = 0, \tag{2.24a}$$

$$O(1):\ -\theta^1_{\bar{x}\bar{x}} - \theta^1_{\bar{x}} = 0, \tag{2.24b}$$

$$O(\epsilon^{j-1}):\ -\theta^j_{\bar{x}\bar{x}} - \theta^j_{\bar{x}} = \theta^{j-2}_{yy}, \tag{2.24c}$$

for $2 \leq j \leq n$. The boundary conditions are, for $0 \leq j \leq n$,

$$\theta^j = g^j(y),\ \text{at } x=0, \text{ and }\ \theta^j = 0,\ \text{at } x=1,\ g^j \text{ as in (2.23)}. \tag{2.24d}$$

By explicit calculations, we find: for $j = 0, 1$,

$$\begin{aligned}\theta^j\left(\frac{x}{\epsilon}, y\right) &= g^j(y)\left(\frac{e^{-x/\epsilon} - e^{-1/\epsilon}}{1 - e^{-1/\epsilon}}\right)\\ &= g^j(y)e^{-x/\epsilon} + e.s.t.(2n + d + 1 - 2j),\end{aligned} \tag{2.25a}$$

and

$$\begin{aligned}\theta^2\left(\frac{x}{\epsilon}, y\right) &= g^2(y)\left(\frac{e^{-x/\epsilon} - e^{-1/\epsilon}}{1 - e^{-1/\epsilon}}\right)\\ &\quad + \epsilon^{-1} g^0_{yy}(y)\left(\frac{x(e^{-x/\epsilon} + e^{-1/\epsilon})}{1 - e^{-1/\epsilon}} - \frac{2e^{-1/\epsilon}}{1 - e^{-1/\epsilon}} \cdot \frac{1 - e^{-x/\epsilon}}{1 - e^{-1/\epsilon}}\right)\\ &= g^2(y)e^{-x/\epsilon} + \epsilon^{-1} g^0_{yy}(y) x e^{-x/\epsilon} + e.s.t.(2n + d - 3).\end{aligned} \tag{2.25b}$$

2.4. Properties of the θ^j. We now derive the pointwise and norm estimates for the θ^j. For that purpose, it is useful to obtain the expression of the θ^j, $\forall j \geq 0$, which is provided by the following lemma.

Lemma 2.4. *We are given real numbers a, b, $a > 0$, and*

$$f^* = f^{*,l}(x) = \sum_{n=0}^{l} f^*_n x^n \exp(-ax), \tag{2.26a}$$

with $f^*_n \in \mathbf{R}$, $l \geq 0$ integer. Let $u = u(x)$ be the solution of the ordinary differential equation in the region $x > 0$:

$$-\frac{d^2u}{dx^2} - a\frac{du}{dx} = f^*, \ x > 0, \tag{2.26b}$$

$$u(0) = b, \tag{2.26c}$$

$$u(x) \to 0 \ \text{as} \ x \to \infty. \tag{2.26d}$$

Then

$$u = u(x) = b\exp(-ax) + \sum_{n=0}^{l} u_n x^{n+1} \exp(-ax), \tag{2.27}$$

where the $u_n \in \mathbf{R}$ are specified in the proof. Furthermore, we have the following pointwise estimates for u and its derivatives: for every $i \geq 0$ and for any $0 < c < a$, there exist a positive constant κ_{il}, depending only on f^*, c such that

$$\left|\frac{d^i}{dx^i} u(x)\right| \leq \kappa_{il} \exp(-cx). \tag{2.28}$$

Proof. The solution to the homogeneous Eq (2.26b) (i.e., when $f^* = 0$) is of the form:

$$u^h = u^h(x) = c_1 \exp(-ax) + c_2, \ c_1, c_2 \in \mathbf{R}. \tag{2.29}$$

We then look for a particular solution of the nonhomogeneous Eq (2.26b):

$$u^p = u^p(x) = \sum_{n=0}^{l} u_n x^{n+1} \exp(-ax). \tag{2.30}$$

By substituting u^p for u in Eq (2.26b), we find the coefficients u_n recursively as follows:

$$\begin{cases} u_l = \dfrac{f_l^*}{a(l+1)} & \text{for } n = l, \\ u_n = a^{-1}\left\{\dfrac{f_n^*}{n+1} + (n+2)u_{n+1}\right\} & \text{for } n = l-1, \cdots, 0. \end{cases} \tag{2.31}$$

To comply with the boundary conditions, we set $u = u^h + u^p$ and find $c_1 = b$, $c_2 = 0$ which leads to (2.27). The pointwise estimates (2.28) follow promptly from (2.27) writing

$$\left|\frac{d^i}{dx^i}P(x)\right| \exp(-ax) \le \kappa_{il} \exp\left(-cx\right), \tag{2.32}$$

where $P(x)$ is a polynomial in x of degree $\le l$, and c is any positive constant with $c < a$, $\kappa_{il} > 0$ is an appropriate constant depending on $P(x)$ and c. $\square$

Remark 2.3. If instead of being constant $f^* \in L^2(\mathbf{R}_+)$, we set $\tilde{u} = u - be^{-x}$, where u is the solution of (2.26). Eq (2.26) is changed into:

$$\begin{cases} -\dfrac{d^2\tilde{u}}{dx^2} - a\dfrac{d\tilde{u}}{dx} = f^* + (1-a)be^{-x} =: \tilde{f}^*, \ x > 0, \\ \\ \tilde{u}(0) = 0, \\ \tilde{u}(x) \to 0 \text{ as } x \to \infty. \end{cases} \tag{2.33}$$

Then $\tilde{f}^* \in L^2(\mathbf{R}_+)$, and from the Lax-Milgram theorem, there exists a unique solution $\tilde{u} = u - be^{-x} \in H_0^1(\mathbf{R}_+)$ of Eq (2.33), and hence $u \in H_0^1(\mathbf{R}_+)$. In fact, $\tilde{u}$ and thus u belong to $H^2(\mathbf{R}_+)$, and since $H^1(\mathbf{R}_+) \subset C^{0,1/2}(\mathbf{R}_+)$, u is also in $C^{1,1/2}(\mathbf{R}_+)$.

Using Lemma 2.4, we now derive the pointwise and norm estimates for the OBLs in the following lemmas.

Lemma 2.5. *Assume that the conditions (2.12) and (2.21) hold. For any* $0 < c < 1$*, there exist a positive constant* κ_{ijm}*, depending on* c *and on the data but independent of* ϵ*, such that the following inequalities hold: for* $0 \le i + m \le 2n + d + 1 - 2j$*, and* $j = 2k$ *or* $j = 2k+1$ *with* $k \ge 0$ *integer, for all* $(x, y) \in \bar{\Omega}$*,*

$$\begin{aligned} \left|\frac{\partial^{i+m}}{\partial x^i \partial y^m}\theta^j\left(\frac{x}{\epsilon}, y\right)\right| &\le \kappa_{ijm}\epsilon^{-i}\exp\left(-c\frac{x}{\epsilon}\right)\left\{1 + \epsilon^{-k-m/2}\exp\left(-\frac{y}{\sqrt{\epsilon}}\right)\right. \\ &\quad \left. + \epsilon^{-k-m/2}\exp\left(-\frac{1-y}{\sqrt{\epsilon}}\right)\right\} + e.s.t.(2n+d+1-2j). \end{aligned} \tag{2.34}$$

Proof. We consider the following equation for $\tilde{\theta}^j = \tilde{\theta}^j(\bar{x}, y)$: for $j = 2k$ with $k \ge 0$ integer,

$$-\tilde{\theta}^j_{\bar{x}\bar{x}} - \tilde{\theta}^j_{\bar{x}} = \tilde{\theta}^{j-2}_{yy}, (\tilde{\theta}^{-2} = 0 \text{ for convenience}) \tag{2.35a}$$

with the boundary conditions:

$$\tilde{\theta}^j(\bar{x} = 0, y) = g^j(y), \quad \tilde{\theta}^j(\bar{x}, y) \to 0 \text{ as } \bar{x} \to \infty, \tag{2.35b}$$

where $g^j(y)$ is defined in (2.23). Then using Lemma 2.4 with x replaced by $\bar{x}$ and u by $\tilde{\theta}^j$, we find the solutions $\tilde{\theta}^j$ for $j = 2k$ recursively, i.e., $\tilde{\theta}^j = P(\bar{x}, y)\exp(-\bar{x})$, where $P(\bar{x}, y)$ is a polynomial in $\bar{x}$ of degree k whose coefficients are linear combinations of the $\partial^{2s}g^{2k-2s}(y)/\partial y^{2s}$, $s = 0, \cdots, k$. Hence, using the estimates in

Lemma 2.2 and Remark 2.2, we find that for any $0 < c < 1$, there exist positive constants κ_{ijm} depending on c and the data but independent of ϵ such that

$$\begin{aligned}\left|\frac{\partial^{i+m}}{\partial \bar{x}^i \partial y^m}\tilde{\theta}^j(\bar{x}, y)\right| &\leq \kappa_{ijm} \exp(-c\bar{x}) \max_{s=0,\cdots,k}\left\{\left|\frac{\partial^{2s+m}}{\partial y^{2s+m}} g^{2k-2s}(y)\right|\right\}\\ &\leq \kappa_{ijm}\exp(-c\bar{x})\left\{1 + \epsilon^{-(2k+m)/2}\exp\left(-\frac{y}{\sqrt{\epsilon}}\right) + \epsilon^{-(2k+m)/2}\exp\left(-\frac{1-y}{\sqrt{\epsilon}}\right)\right\}.\end{aligned}$$

Comparing to Eq (2.24), we easily find that $\theta^j(x, y) = \tilde{\theta}^j(\bar{x}, y) + e.s.t.(2n+d+1-2j)$, hence the estimate (2.34) follows. For $j = 2k + 1$, the proof is similar. □

The following norm estimate is deduced immediately from Lemma 2.5.

Lemma 2.6. *For* $0 \leq \sigma_1, \sigma_2 < 1$, *let*

$$\Omega^{\sigma_1,\sigma_2} = (\sigma_1, 1) \times (\sigma_2, 1 - \sigma_2).$$

Assume that the conditions (2.12) and (2.21) hold. Then, for any $0 < c < 1$, *there exist a positive constant* κ_{ijm}, *depending on* c *and on the data but independent of* ϵ, *such that the following inequalities hold: for* $0 \leq i + m \leq 2n + d + 1 - 2j$, *and* $j = 2k$ *or* $j = 2k + 1$ *with* $k \geq 0$ *integer,*

$$\begin{aligned}\left|\frac{\partial^{i+m}}{\partial x^i \partial y^m}\theta^j\right|_{L^2(\Omega^{\sigma_1,\sigma_2})} &\leq \kappa_{ijm}\epsilon^{-i+1/2}\\ &\cdot\left(1 + \epsilon^{-k-m/2+1/4}\exp\left(-\frac{\sigma_2}{\sqrt{\epsilon}}\right)\right)\exp\left(-c\frac{\sigma_1}{\epsilon}\right);\end{aligned} \tag{2.36a}$$

in particular,

$$\left|\frac{\partial^{i+m}}{\partial x^i \partial y^m}\theta^j\right|_{L^2(\Omega)} \leq \kappa_{ijm}\epsilon^{-i+1/2}\big(1 + \epsilon^{-k-m/2+1/4}\big). \tag{2.36b}$$

Remark 2.4. Let

$$\sigma_1 = \kappa\epsilon^{\alpha_1} \text{ with } \alpha_1 < 1,\ \sigma_2 = \kappa\epsilon^{\alpha_2} \text{ with } \alpha_2 < 1/2, \tag{2.37a}$$

and

$$\Omega_1 = (0, \sigma_1) \times \{(0, \sigma_2) \cup (1 - \sigma_2, 1)\},\ \Omega_2 = (0, \sigma_1) \times (\sigma_2, 1 - \sigma_2), \tag{2.37b}$$

$$\Omega_3 = (\sigma_1, 1) \times \{(0, \sigma_2) \cup (1 - \sigma_2, 1)\},\ \Omega_4 = (\sigma_1, 1) \times (\sigma_2, 1 - \sigma_2). \tag{2.37c}$$

Then under the situation that OBLs and PBLs are present, as indicated in Lemma 2.6, we need to take care of the large variations of the derivatives due to the OBLs and PBLs in the subdomain Ω_1, due to the OBLs in the subdomain Ω_2, and, as indicated in Lemma 2.3 and Remark 2.2, due to the PBLs in the subdomain Ω_3. Notice that the OBLs and PBLs are both exponentially small on Ω_4.

2.5. Asymptotic Error Analysis. We conclude this study with the following theorems, which provide the asymptotic approximations and *which justify, on the theoretical side, the formal expansions we introduced before.* Below we focus on the H^2- asymptotic error which needs to be an $O(1)$ quantity so as to absorb all H^2- singularities due to the boundary layers. This provides the justification for the construction of the boundary layer elements in the finite elements context, see Section 3. To avoid the singularities of the derivatives of PBLs at the vertices, we need the following compatibility conditions on f (see (2.39) below):

$$f(1, 0) = f_x(1, 0) = f_{xx}(1, 0) = 0, \tag{2.38a}$$

$$f(1, 1) = f_x(1, 1) = f_{xx}(1, 1) = 0. \tag{2.38b}$$

From the explicit expressions of the u^j as in (2.4), we then have thanks to (2.38):

$$-\frac{\partial^i}{\partial x^i}u^j(1,0) = -\frac{\partial^i}{\partial x^i}u^j(1,1) = 0, \quad \text{for } 0 \le i \le 3-2j,\ j=0,1, \tag{2.39}$$

which are exactly the compatibility conditions (2.12) and (2.21); we note here that $n = d = 1$. Hence, φ_l^0, φ_u^0, θ^0 satisfy the regularity (2.17) with $l = 3$ and φ_l^1, φ_u^1, θ^1 satisfy the regularity (2.17) with $l = 1$.

To obtain the asymptotic error estimate, we set

$$w_{\epsilon n} = u^\epsilon - u_{\epsilon n} - \varphi_{l\epsilon n} - \varphi_{u\epsilon n} - \theta_{\epsilon n}, \tag{2.40a}$$

where

$$u_{\epsilon n} = \sum_{j=0}^n \epsilon^j u^j,\ \varphi_{l\epsilon n} = \sum_{j=0}^n \epsilon^j \varphi_l^j,\ \varphi_{u\epsilon n} = \sum_{j=0}^n \epsilon^j \varphi_u^j,\ \theta_{\epsilon n} = \sum_{j=0}^n \epsilon^j \theta^j. \tag{2.40b}$$

We firstly notice that $w_{\epsilon n}$ vanishes at $x = 0, 1$ and hence, setting

$$\vartheta_{\epsilon n} = w_{\epsilon n}(x,0)(1-y) + w_{\epsilon n}(x,1)y, \tag{2.41}$$

we find that $W_{\epsilon n} := w_{\epsilon n} - \vartheta_{\epsilon n}$ satisfies the boundary condition (1.1b), namely $W_{\epsilon n} = 0$ on $\partial\Omega$. We can here verify that $\vartheta_{\epsilon n}$ is exponentially small. Indeed, from (2.41) and the explicit solution θ^j in (2.25), and from Lemma 2.2, Remark 2.2 and Lemma 2.5, we find that for $n = 0, 1$,

$$\begin{aligned} \left|\frac{\partial^{i+m}}{\partial x^i \partial y^m}\vartheta_{\epsilon n}\right| &\le \kappa \sum_{j=0}^n \epsilon^j \left\{ \left|\frac{\partial^i}{\partial x^i}\varphi_l^j(x, y=1)\right| + \left|\frac{\partial^i}{\partial x^i}\varphi_u^j(x,y=0)\right| \right. \\ &\quad \left. + \left(\left|\varphi_l^j(0,y=1)\right| + \left|\varphi_u^j(0,y=0)\right|\right)\left|\frac{\partial^i}{\partial x^i}e^{-x/\epsilon}\right|\right\} + \frac{\partial^{i+m}}{\partial x^i\partial y^m}(e.s.t.(2)) \\ &\le \kappa(c)\exp\left(-c\frac{1}{\sqrt{\epsilon}}\right), \quad \text{for } 0 \le i+m \le 2, \text{ and for any } 0 < c < 1. \end{aligned} \tag{2.42}$$

From the outer expansion in (2.3), we have

$$-\epsilon\triangle u_{\epsilon n} - u_{\epsilon n x} = f - \epsilon^{n+1}\triangle u^n, \tag{2.43}$$

For the parabolic boundary layers defined in (2.5) and Remark 2.2, we have

$$-\epsilon\triangle\varphi_{l\epsilon n} - \varphi_{l\epsilon n x} = -\epsilon^{n+1}\varphi_{lxx}^n, \tag{2.44a}$$

$$-\epsilon\triangle\varphi_{u\epsilon n} - \varphi_{u\epsilon n x} = -\epsilon^{n+1}\varphi_{uxx}^n, \tag{2.44b}$$

and for the ordinary boundary layers defined in (2.24), we see that

$$-\epsilon\triangle\theta_{\epsilon n} - \theta_{\epsilon n x} = -\epsilon^n\theta_{yy}^{n-1} - \epsilon^{n+1}\theta_{yy}^n. \tag{2.45}$$

Subtracting (2.43), (2.44a), (2.44b), and (2.45) from (1.1a) and setting $\theta^{-1} = 0$ for convenience, we write for $W_{\epsilon n} = w_{\epsilon n} - \vartheta_{\epsilon n}$,

$$L_\epsilon W_{\epsilon n} = -\epsilon\triangle W_{\epsilon n} - W_{\epsilon n x} = R^n + \tilde{R}^n \quad \text{in } \Omega, \tag{2.46a}$$

with

$$R^n = \epsilon^{n+1}\left\{\triangle u^n + \varphi_{lxx}^n + \varphi_{uxx}^n + \theta_{yy}^n\right\} + \epsilon^n\theta_{yy}^{n-1}, \quad \tilde{R}^n = -L_\epsilon\vartheta_{\epsilon n} \tag{2.46b}$$

and

$$W_{\epsilon n} = 0 \text{ on } \partial\Omega; \tag{2.46c}$$

note that thanks to (2.42), we easily see that the term $\tilde{R}^n$ is exponentially small. Then the asymptotic error estimates are provided in the next theorem and corollary.

Before we proceed, we mention the following simple regularity results, which will be used repeatedly later on, for Dirichlet or mixed boundary conditions:

Lemma 2.7. *Let*

$$V = H_0^1(\Omega) \quad \textit{for (2.48b), or} \tag{2.47a}$$

$$V = \Big\{ v \in H^1(\Omega);\ v = 0 \ \textit{at}\ x = 0, 1 \Big\} \quad \textit{for (2.48c).} \tag{2.47b}$$

Let $(V', \|\cdot\|_{V'})$ *be the dual space of* $(V, \|\cdot\|_V)$, $f^* = f^*(x,y) \in V'$, *and* u *the solution of equation:*

$$L_\epsilon u = -\epsilon \triangle u - u_x = f^* \ \textit{in}\ \Omega = (0,1)\times(0,1), \tag{2.48a}$$

supplemented with either the boundary condition

$$u = 0 \qquad \textit{on}\ \partial\Omega, \tag{2.48b}$$

or with

$$\begin{aligned} u &= 0 \qquad \textit{at}\ x = 0, 1, \\ \frac{\partial u}{\partial y} &= 0 \qquad \textit{at}\ y = 0, 1. \end{aligned} \tag{2.48c}$$

Then the following regularity results hold.
If $f^* = \epsilon f_1^* + f_2^*$ *with* $f_1^* \in V'$, $f_2^* \in L^2(\Omega)$, *then there exists a constant* κ *independent of* ϵ *such that*

$$\|u\|_\epsilon \le \kappa \epsilon^{1/2} \|f_1^*\|_{V'} + \kappa |f_2^*|_{L^2(\Omega)}, \tag{2.49}$$

and if $f^* \in L^2(\Omega)$,

$$\|u\|_\epsilon \le \kappa |f^*|_{L^2(\Omega)}, \tag{2.50a}$$

$$|u|_{H^2} \le \kappa \epsilon^{-3/2} |f^*|_{L^2(\Omega)}. \tag{2.50b}$$

Proof. The weak formulation of the problem is as follows: $u \in V$ and

$$\tilde a_\epsilon(u,v) = \tilde F(v), \tag{2.51a}$$

where

$$\tilde a_\epsilon(u,v) = \epsilon \int_\Omega \nabla u \cdot \nabla v d\Omega - \int_\Omega u_x v d\Omega, \quad \tilde F(v) = < f^*, v >; \tag{2.51b}$$

if $f^* \in L^2$,

$$< f^*, v > = (f^*, v) = \int_\Omega f^* v d\Omega. \tag{2.51c}$$

Using elementary manipulations, and setting $v = e^x u$ in (2.51a), we derive (2.49) from (2.51); (2.50a) is a particular case of (2.49). Finally, (2.50b) follows from (2.48a), observing that

$$|\triangle v|_{L^2}^2 = |v_{xx}|_{L^2}^2 + |v_{yy}|_{L^2}^2 + 2\int_\Omega v_{xx} v_{yy} d\Omega, \tag{2.52}$$

and that, for both spaces V:

$$\int_\Omega v_{xx} v_{yy} d\Omega = \int_\Omega (v_{xy})^2 d\Omega,\ \forall v \in V. \tag{2.53}$$

□

Remark 2.5. We infer from (2.50b) that to make the solution u of equation (2.48) absorb the H^2- singularities, we will need an f^* of order $\epsilon^{3/2}$ in $L^2(\Omega)$.

Theorem 2.2. *Assume that the compatibility conditions (2.38) hold. Then for* $n = 0, 1$,

$$|w_{\epsilon n} - \delta_{\epsilon n}|_{L^2(\Omega)} \le \kappa\epsilon^{n+3/4}, \tag{2.54a}$$

$$\|w_{\epsilon n} - \delta_{\epsilon n}\|_{H^1(\Omega)} \le \kappa\epsilon^{n+1/4}, \tag{2.54b}$$

$$\|w_{\epsilon n} - \delta_{\epsilon n}\|_{H^2(\Omega)} \le \kappa\epsilon^{n-3/4}, \tag{2.54c}$$

where the function $\delta_{\epsilon n} \in H_0^1(\Omega)$ *is specified in the proof and satisfies: for* $n = 0$, $\delta_{\epsilon n} = 0$, *and for* $n = 1$,

$$|\delta_{\epsilon n}|_{L^2(\Omega)} \le \kappa\epsilon^{3/4}, \quad \|\delta_{\epsilon n}\|_{H^1(\Omega)} \le \kappa\epsilon^{1/4}. \tag{2.55}$$

Proof. Firstly, we derive some estimates for $R^n = R_1^n + R_2^n$ in the "error" equation (2.46). We write

$$R_1^n = \epsilon^{n+1}\big\{\triangle u^n + \varphi_{lxx}^n + \varphi_{uxx}^n + \theta_{yy}^n\big\}, \tag{2.56a}$$

$$R_2^n = \epsilon^n \theta_{yy}^{n-1}. \tag{2.56b}$$

Let $\delta_{\epsilon n}$ be the solution of:

$$L_\epsilon \delta_{\epsilon n} = R_2^n \qquad \text{in } \Omega, \tag{2.57a}$$

$$\delta_{\epsilon n} = 0 \qquad \text{on } \partial\Omega. \tag{2.57b}$$

Then

$$L_\epsilon(W_{\epsilon n} - \delta_{\epsilon n}) = R_1^n + \tilde{R}^n \qquad \text{in } \Omega, \tag{2.58a}$$

$$W_{\epsilon n} - \delta_{\epsilon n} = 0 \qquad \text{on } \partial\Omega. \tag{2.58b}$$

From the norm estimates of Lemma 2.3 and Lemma 2.6, we find

$$\begin{aligned} |R_1^n|_{L^2} &\le \epsilon^{n+1}\big\{|\triangle u^n|_{L^2} + |\varphi_{lxx}^n|_{L^2} + |\varphi_{uxx}^n|_{L^2} + |\theta_{yy}^n|_{L^2}\big\} \\ &\le \kappa\epsilon^{n+3/4}, \\ |R_2^n|_{L^2} &\le \epsilon^n|\theta_{yy}^{n-1}|_{L^2} \le \kappa \begin{cases} 0 & \text{for } n = 0, \\ \epsilon^{3/4} & \text{for } n = 1. \end{cases} \end{aligned} \tag{2.59}$$

Furthermore, from (2.42), since $\vartheta_{\epsilon n}$ and $\tilde{R}^n$ are exponentially small, these terms can be absorbed in other norms and we may drop them. Then from Lemma 2.7 applied to equations (2.57) and (2.58) with $f^* = R_2^n$, R_1^n, respectively, the estimates (2.54) and (2.55) follow. $\square$

If instead of (2.38) we impose the following stronger conditions (2.60) on $f = f(x, y)$:

$$f(x, 0) = f(x, 1) = 0, \tag{2.60}$$

we can remove the first parabolic boundary layers φ_l^0, φ_u^0, see Corollary 2.1 below. The parabolic boundary layers, $\epsilon\varphi_l^1$, $\epsilon\varphi_u^1$, are still present but they are very mild and their contributions are absorbed in the other H^2- terms. This is an improvement over the condition used in [11] to remove the parabolic boundary layers, that is

$$f(x, 0) = f_{yy}(x, 0) = f(x, 1) = f_{yy}(x, 1) = 0. \tag{2.61}$$

With the conditions (2.61), we proved in [11] that $\|w_{\epsilon n}\|_{H^2(\Omega)} \le \kappa$ for $n = 1$. We here prove the same result with the conditions (2.60), see Corollary 2.1 below.

Corollary 2.1. *Assume that the condition (2.60) holds. Then* $\varphi_l^0(x, y/\sqrt{\epsilon}) = \varphi_u^0(x, (1-y)/\sqrt{\epsilon}) = 0$ *and for* $n = 0, 1$,

$$|w_{\epsilon n}|_{L^2(\Omega)} \leq \kappa \epsilon^{(3n+3)/4}, \tag{2.62a}$$

$$\|w_{\epsilon n}\|_{H^1(\Omega)} \leq \kappa \epsilon^{(3n+1)/4}, \tag{2.62b}$$

$$\|w_{\epsilon n}\|_{H^2(\Omega)} \leq \kappa \epsilon^{(3n-3)/4}. \tag{2.62c}$$

Proof. Because of (2.60), (2.4a) and (2.18a), $\varphi_l^0(x, \bar{y}) = \varphi_u^0(x, \tilde{y}) = 0$. The explicit expression of θ^0 in (2.25) then yields

$$\theta^0(x, y) = g^0(y) e^{-x/\epsilon} + e.s.t.(4) = -u^0(0, y) e^{-x/\epsilon} + e.s.t.(4). \tag{2.63}$$

Hence,

$$|R_2^n|_{L^2} \leq \epsilon^n |\theta_{\epsilon n}^{n-1}|_{L^2} \leq \kappa \begin{cases} 0 & \text{for } n = 0, \\ \epsilon^{3/2} & \text{for } n = 1, \end{cases} \tag{2.64}$$

and by (2.59),

$$|R^n| = |R_1^n + R_2^n|_{L^2} \leq |R_1^n|_{L^2} + |R_2^n|_{L^2} \leq \kappa \epsilon^{(3n+3)/4}. \tag{2.65}$$

Then from Lemma 2.7 applied to equation (2.46) with $f^* = R^n = R_1^n + R_2^n$, the lemma follows. □

Remark 2.6. If we assume the conditions (2.38), from (2.54) with $n = 0$, using the norm estimates of Lemmas 2.3 and 2.6, we obtain

$$u^\epsilon = u^0 + O(\epsilon^{1/4}) \quad \text{in } L^2.$$

But if we assume the conditions (2.60), from Corollary 2.1 with $n = 0$ and the estimate (2.63), we obtain

$$u^\epsilon = u^0 + O(\epsilon^{1/2}) \quad \text{in } L^2.$$

If we do not assume the compatibility conditions (2.38), we only obtain the following result at the order 0:

Theorem 2.3. *For the Dirichlet boundary value problem (1.1a)-(1.1b), let the function* $f = f(x, y)$ *be any smooth function on* $\bar{\Omega}$ *(not necessarily satisfying (2.38)). Then*

$$|u^\epsilon - u^0 - \varphi_l^0 - \varphi_u^0 - \theta^0|_{L^2(\Omega)} \leq \kappa \epsilon^{3/4}, \tag{2.66a}$$

$$\|u^\epsilon - u^0 - \varphi_l^0 - \varphi_u^0 - \theta^0\|_{H^1(\Omega)} \leq \kappa \epsilon^{1/4}. \tag{2.66b}$$

Proof. We use equation (2.46) with $n = 0$, and since we do not require the compatibility conditions (2.38), we have

$$L_\epsilon(W_{\epsilon 0}) = \epsilon R_1 + R_2 \qquad \text{in } \Omega, \tag{2.67a}$$

$$W_{\epsilon 0} = 0 \qquad \text{on } \partial\Omega, \tag{2.67b}$$

where

$$R_1 = \varphi_{lxx}^0 + \varphi_{uxx}^0 + \theta_{yy}^0 - \epsilon^{-1} L_\epsilon \vartheta^0 \in H^{-1}(\Omega), \tag{2.68a}$$

$$R_2 = \epsilon \triangle u^0 \in C^\infty(\bar{\Omega}). \tag{2.68b}$$

Note here that since $-u^0(1, 0) = -u^0(1, 1) = 0$ from the explicit expression of u^0 in (2.4), the compatibility conditions (2.12) and (2.21) hold with $n = d = 0$. We then find that $\varphi_l^0,\ \varphi_u^0,\ \theta^0,\ \vartheta^0 \in H^1(\Omega)$.

We are now able to find that

$$\|R_1\|_{H^{-1}} \le \kappa\epsilon^{1/4}. \tag{2.69}$$

Indeed, for $w \in H_0^1(\Omega)$, $\|w\|_{H^1} = 1$,

$$\begin{aligned} < R_1, w > &= < \varphi^0_{lxx} + \varphi^0_{uxx} + \theta^0_{yy} + \triangle\vartheta^0 + \epsilon^{-1}\vartheta^0_x, w > \\ &= \ -\int_\Omega \Big(\varphi^0_{lx}w_x + \varphi^0_{ux}w_x + \theta^0_y w_y + \nabla\vartheta^0 \cdot \nabla w + \epsilon^{-1}\vartheta^0 w_x\Big) d\Omega. \end{aligned}$$

Hence, from Lemmas 2.3 and 2.6,

$$| < R_1, w > | \le |\varphi^0_{lx}|_{L^2} + |\varphi^0_{ux}|_{L^2} + |\theta^0_y|_{L^2} \le \kappa\epsilon^{1/4}; \tag{2.70}$$

we dropped the exponentially small term ϑ^0 which is absorbed in other H^1- terms, and

$$\|R_1\|_{H^{-1}} = \sup_{\|w\|_{H^1}=1, w\in H_0^1} | < R_1, w > | \le \kappa\epsilon^{1/4}. \tag{2.71}$$

Hence, the theorem follows from Lemma 2.7 applied to equation (2.67) for $W_{\epsilon 0}$ with $f^* = \epsilon f_1^* + f_2^*$, $f_1^* = R_1$, and $f_2^* = R_2$, since $\|f_1^*\|_{H^{-1}} \le \|R_1\|_{H^{-1}} \le \kappa\epsilon^{1/4}$ from (2.69) and $|f_2^*|_{L^2} \le |R_2|_{L^2} \le \kappa\epsilon$; we here drop ϑ^0 too. □

Remark 2.7. From Theorem 2.3 applied with $n = 0$, for any smooth function f which does not necessarily satisfy the compatibility conditions (2.38), using the norm estimates of Lemmas 2.3 and 2.6, we obtain that

$$u^\epsilon = u^0 + O(\epsilon^{1/4}) \quad \text{in } L^2.$$

3. Approximation via Finite Elements

In this section, we introduce the Boundary Layer Elements (BLEs) suitable for our problems, see ϕ_0, ψ^n_l, ψ^n_u, ψ^n_0, ψ^n_N below; these elements are closely related to the correctors in the terminology of Lions [18]. We will show that these functions absorb the H^2 singularity of u^ϵ. We will incorporate ϕ_0, ψ^n_l, ψ^n_u, ψ^n_0, ψ^n_N in the finite element spaces in which we will seek the approximate solutions; they will appear as special finite element functions - also called splines sometimes [24], [25].

3.1. The Boundary Layer Elements : Constructions. We now construct the Boundary Layer Elements (BLEs) ϕ_0, ψ^n_l, ψ^n_u, ψ^n_0, ψ^n_N which will be shown by the two following lemmas to absorb the H^2- singularity of the solutions; we are interested in two cases: the case where only *ordinary boundary layers* appear, and the case where *both ordinary* and *parabolic boundary layers* appear.

If the conditions (2.60) hold, then by Corollary 2.1 and Lemma 3.1 below, φ^0_l and φ^0_u do not appear, and, essentially, the singular terms only appear in the ordinary boundary layers θ^j. Furthermore, we can extract the singular terms and slightly modify them so that they belong to the space V, that is, we derive the conforming ordinary boundary layer element:

$$\phi_0(x) = -e^{-x/\epsilon} - (1 - e^{-1/\epsilon})x + 1 \in H_0^1(0,1); \tag{3.1}$$

we easily verify that

$$\|\phi_0\|_{H^m(0,1)} \le \kappa(1 + \epsilon^{-m+1/2}), \quad \forall\ m \ge 0. \tag{3.2}$$

We will then approximate the exact solution u^ϵ by the linear system (3.25) below.

If we do not impose the conditions (2.60), as in Theorem 2.3, the *parabolic boundary layers* φ^0_l and φ^0_u as well as the *ordinary boundary layer* θ^0 play an essential role in the singular behavior of the solutions. To capture the ordinary boundary layers, we will use the same ϕ_0 as in (3.1). We now construct the parabolic boundary layer elements which capture the functions φ_l^0 and φ_u^0. Unlike for ϕ_0 we cannot extract simply the singular terms from φ_l^0 and φ_u^0. Instead we start from the explicit expression of $\varphi_l^0(x, \bar{y})$ and use it to construct the corresponding conforming parabolic boundary layer elements; the parabolic BLE at $y = 1$ is then deduced by symmetry at the axis $y = 1/2$. In order to construct all the necessary BLEs, we thus decompose the data f as follows so that we can treat the OBLs and PBLs independently; we write:

$$f(x, y) = f_1(x, y) + f_2(x, y) + f_3(x, y), \tag{3.3a}$$

where

$$f_1(x, y) = f(x, y) - f(x, 0)(1 - y) - f(x, 1)y, \tag{3.3b}$$

$$f_2(x, y) = \left(\sum_{j=0}^{n_0} \alpha_j x^j\right)(1 - y) + \left(\sum_{j=0}^{n_0} \beta_j x^j\right) y, \tag{3.3c}$$

$$f_3(x, y) = \left(f(x, 0) - \sum_{j=0}^{n_0} \alpha_j x^j\right)(1 - y) + \left(f(x, 1) - \sum_{j=0}^{n_0} \beta_j x^j\right) y. \tag{3.3d}$$

The two polynomials in x appearing in (3.3c) are the Lagrange interpolating polynomials of degree n_0 for $f(x, 0)$ and $f(x, 1)$ with nodes at the roots of the Chebyshev polynomial of degree $n_0 + 1$. Hence, we explicitly obtain the constants α_j and β_j for $j = 0, \cdots, n_0$ from $f(x, 0)$ and $f(x, 1)$, that is:

$$\sum_{j=0}^{n_0} \alpha_j x^j = \sum_{j=0}^{n_0} f(x_j, 0) L_j(x), \quad \sum_{j=0}^{n_0} \beta_j x^j = \sum_{j=0}^{n_0} f(x_j, 1) L_j(x), \tag{3.4a}$$

where

$$L_j(x) = \prod_{i=0,\ i\neq j}^{n_0} \frac{(x - x_i)}{(x_j - x_i)}, \tag{3.4b}$$

with

$$x_i = \frac{1}{2}\left(\cos\left(\frac{2i + 1}{2(n_0 + 1)}\pi\right) + 1\right). \tag{3.4c}$$

For more details, see Lemma 3.3 and Corollary 3.1 below; see also (6.5d) - (6.5f).

For each j, $j = 1, 2, 3$, we then consider the solution $u_j = u_j^\epsilon$ of the following boundary value problem, particular case of (1.1):

$$L_\epsilon u_j = f_j \quad \text{in } \Omega, \tag{3.5a}$$

$$u_j = 0 \quad \text{on } \partial\Omega; \tag{3.5b}$$

as in (2.1) the weak formulation of this problem is: $u_j \in H_0^1(\Omega)$ and

$$a_\epsilon(u_j, v) = (f_j, v), \quad \forall v \in H_0^1(\Omega). \tag{3.6}$$

For u_1, since $f_1(x, 0) = f_1(x, 1) = 0$, Corollary 2.1 and Lemma 3.1 below show that the parabolic boundary layers do not appear; only the OBLs are present. Hence, we will approximate u_1 with the linear system (3.25) with $f = f_1$ below.

For u_2, if $f_2(x, 0) \not\equiv 0$ or $f_2(x, 1) \not\equiv 0$, some of the constants α_j or β_j are nonzero, and hence due to the discrepancies between the outer solution, see (2.4a), and the boundary condition, $u_2 = 0$ at $y = 0, 1$, we expect the presence of parabolic boundary layers as described before. To derive the functions $\varphi_l^0(x,\bar{y})$ and $\varphi_u^0(x,\tilde{y})$ corresponding to f_2 and u_2, we first consider the case where $f_2(x,0)$ and $f_2(x,1)$ are the basic monomials

$$f_2(x,0) = f_2(x,1) = x^j. \tag{3.7}$$

We obtain $\varphi_l^{0,j}(x,\bar{y})$ using the explicit expression in (2.18a), see (3.8e) below; $\varphi_u^{0,j}(x,\tilde{y})$ is derived similarly. We then modify these functions to obtain the corresponding conforming BLEs, ψ_l^j, ψ_u^j belonging to $H_0^1(\Omega)$, and ψ_0^j, ψ_N^j belonging to $H_0^1(0,1)$, that is

$$\begin{aligned}\psi_l^j = \psi_l^j(x,y) = \varphi_l^{0,j}(x,\bar{y}) &+ \varphi_l^{0,j}(0,\bar{y})(x-1)\\ &+ (j+1)^{-1}(x^{j+1}-x)(y-1) + e.s.t.(1),\end{aligned} \tag{3.8a}$$

$$\psi_u^j = \psi_u^j(x,y) = \psi_l^j(x,1-y), \tag{3.8b}$$

$$\psi_0^j = \psi_0^j(y) = \varphi_l^{0,j}(0,\bar{y}) + (j+1)^{-1}(1-y) + e.s.t.(1), \tag{3.8c}$$

$$\psi_N^j = \psi_N^j(y) = \psi_0^j(1-y), \tag{3.8d}$$

where

$$\varphi_l^{0,j}(x,\bar{y}) = \frac{\sqrt{2}}{\sqrt{\pi}(j+1)} \int_{\bar{y}/\sqrt{2(1-x)}}^{\infty} \exp\left(-\frac{t^2}{2}\right) \left[\left(x + \frac{\bar{y}^2}{2t^2}\right)^{j+1} - 1\right] dt. \tag{3.8e}$$

See Lemma 3.2 below for the justification of these choices. Notice here that in (3.8) the $e.s.t.(1)$ belong to $H^1(\Omega)$ and to $H^1(0,1)$ respectively. They are introduced in the analysis so that ψ_l^j, ψ_u^j belong to $H_0^1(\Omega)$ and ψ_0^j, ψ_N^j belong to $H_0^1(0,1)$, respectively, but they will be neglected in the numerical computations.

From Lemmas 2.2 and 2.3, we easily verify that, for $m = 0,1$,

$$\|\psi_l^j(x,y)\|_{H^m(\Omega)},\ \|\psi_u^j(x,y)\|_{H^m(\Omega)} \le \kappa(1+\epsilon^{-m/2+1/4}), \tag{3.9a}$$

$$\|\psi_0^j(y)\|_{H^m(0,1)},\ \|\psi_N^j(y)\|_{H^m(0,1)} \le \kappa(1+\epsilon^{-m/2+1/4}). \tag{3.9b}$$

We then obtain all the necessary parabolic boundary layer elements $\varphi_l^{0,j}$ (and $\varphi_u^{0,j}$), $j = 0,\cdots,n_0$ as in (3.8). We thus approximate u_2 by u^* defined in (3.16b) below; the convergence errors are provided in Lemma 3.2 below.

Finally, for f_3, we truncate this function, that is the remainder of the Lagrange polynomial using all after n_0^{th} term; note that the error u_3 due to f_3 is independent of ϵ. The precise error estimate in H^1 will be given hereafter in Theorem 3.2.

The two following lemmas basically justify our constructions. They respectively prove that ϕ_0 absorbs the H^2- singularities, and they provide the asymptotic approximation error using the PBLs $\varphi_l^{0,j}$, $\varphi_u^{0,j}$, and the overlapping of the PBLs and OBLs. Lemma 3.1 will be used for $f = f_1$ (and $u^\epsilon = u_1$) and Lemma 3.2 will be used for $f = f_2$ (and $u^\epsilon = u_2$).

Lemma 3.1. *Assume that*

$$f(x,0) = f(x,1) = 0, \quad \forall x \in [0,1]. \tag{3.10}$$

Then there exist a positive constant κ independent of ϵ, and a smooth function $g = g^\epsilon(y) \in H_0^1(0,1)$ with $|g|_{H^2(0,1)} \le \kappa$ such that

$$\|u^\epsilon - g\phi_0\|_{H^2(\Omega)} \le \kappa. \tag{3.11}$$

Proof. We infer from Corollary 2.1, with $n = 1$, that

$$\left\| u^\epsilon - u^0 - \theta^0 - \epsilon\left\{u^1 + \varphi_l^1 + \varphi_u^1 + \theta^1\right\}\right\|_{H^2} \leq \kappa. \tag{3.12}$$

Since the u^j are independent of ϵ, we find with Lemma 2.3,

$$\left\| u^\epsilon - \theta^0 - \epsilon\theta^1 \right\|_{H^2} \leq \kappa, \tag{3.13}$$

and hence,

$$\| u^\epsilon + g e^{-x/\epsilon} \|_{H^2} \leq \kappa, \tag{3.14}$$

where

$$g = g^\epsilon(y) = u^0(0,y) + \epsilon\big(u^1(0,y) + \varphi_l^1(0,\bar{y}) + \varphi_u^1(0,\tilde{y})\big) + e.s.t.(2) \in H_0^1(0,1).$$

Note that $u^0(0,y) \in H_0^1(0,1)$ because of (3.10). The role of the $e.s.t.(2)$ is to make g belong to $H_0^1(0,1)$. Then by Lemma 2.2, we easily find that $|g|_{H^2(0,1)} \leq \kappa$, and by the definition of ϕ_0 in (3.1), the lemma follows. □

Lemma 3.2. *Assume that*

$$f(x,y) = \left(\sum_{j=0}^{n_0} \alpha_j x^j\right)(1-y) + \left(\sum_{j=0}^{n_0} \beta_j x^j\right) y, \tag{3.15}$$

for some fixed constants α_j, $\beta_j \in \mathbf{R}$, independent of ϵ. Then there exist a positive constant $\kappa(n_0)$ independent of ϵ such that

$$\| u^\epsilon - u^* \|_{H^m(\Omega)} \leq \kappa(n_0) \begin{cases} \epsilon^{3/4} & \text{for } m = 0, \\ \epsilon^{1/4} & \text{for } m = 1, \end{cases} \tag{3.16a}$$

where

$$u^* = \sum_{j=0}^{n_0} \alpha_j \psi_l^j + \sum_{j=0}^{n_0} \beta_j \psi_u^j + \sum_{j=0}^{n_0} \alpha_j \psi_0^j \phi_0 + \sum_{j=0}^{n_0} \beta_j \psi_N^j \phi_0. \tag{3.16b}$$

Proof. Let

$$f = f_l^j(x,y) = x^j(1-y). \tag{3.17}$$

Then we easily find that $\varphi_u^0 = 0$, and from Theorem 2.3, we find

$$\left\| u_l^{\epsilon,j} - u^0 - \varphi_l^0 - \theta^0 \right\|_{H^m} \leq \kappa(j)\epsilon^\gamma, \tag{3.18}$$

where $\gamma = 3/4$, or $1/4$, for $m = 0$, or 1, respectively; $u_l^{\epsilon,j}$ is the solution of Eq. (2.1) corresponding to $f = f_l^j$ as in (3.17); $u^0 = u_l^{0,j}$, $\varphi_l^0 = \varphi_l^{0,j}$, $\theta^0 = \theta_l^{0,j}$ are the corresponding outer solutions and boundary layer functions constructed as before. Notice that from the explicit expressions of u^0, θ^0 in (2.4), (2.25), and from the expression of f in (3.17), we find

$$u^0(x,y) = \int_x^1 f(s,y)ds = (j+1)^{-1}(1 - x^{j+1})(1-y); \tag{3.19}$$

in particular,

$$u^0(0,y) = (j+1)^{-1}(1-y). \tag{3.20}$$

Hence, we write

$$\begin{aligned} u^0 + \varphi_l^0 + \theta^0 &= u^0(x,y) + \varphi_l^0(x,\bar{y}) - (u^0(0,y) + \varphi_l^0(0,\bar{y}))e^{-x/\epsilon} + e.s.t.(1) \\ &= \psi_l^j(x,y) + \psi_0^j(y)\phi_0(x) + e.s.t.(1) \ \in H_0^1(\Omega), \end{aligned} \tag{3.21}$$

where ϕ_0, ψ_l^j, ψ_0^j are defined in (3.1), (3.8a) - (3.8d). We then infer from (3.18), for $m = 0, 1$, that

$$\|u_l^{\epsilon,j} - \psi_l^j(x,y) - \psi_0^j(y)\phi_0(x)\|_{H^m} \leq \kappa(j)\epsilon^\gamma. \tag{3.22}$$

By considering the symmetry at the axis $y = 1/2$, we can deduce similar estimates for the solution $u_u^{\epsilon,j}$ corresponding to the data $f = f_u^j(x,y) = x^j y$. Finally, by linearity and superposition of the solutions, we see that, for f as in (3.15),

$$u^\epsilon = u^{\epsilon,n_0} = \sum_{j=0}^{n_0} \alpha_j u_l^{\epsilon,j} + \sum_{j=0}^{n_0} \beta_j u_u^{\epsilon,j}, \tag{3.23}$$

and

$$\|u^\epsilon - u^*\|_{H^m(\Omega)} \leq \sum_{j=0}^{n_0} \alpha_j \|u^{\epsilon,j} - \psi_l^j(x,y) - \psi_0^j(y)\phi_0(x)\|_{H^m(\Omega)}$$
$$+ \sum_{j=0}^{n_0} \beta_j \|u^{\epsilon,j} - \psi_u^j(x,y) - \psi_N^j(y)\phi_0(x)\|_{H^m(\Omega)} \leq \kappa(n_0)\epsilon^\gamma,$$

where γ is defined in (3.18); the estimate (3.16) follows from (3.22) and the similar bound at $y = 1$. □

Remark 3.1. In Lemmas 3.1 and 3.2, the term $g\phi_0$ is due to the OBLs, whereas $\sum_{j=0}^{n_0} \alpha_j \psi_l^j + \sum_{j=0}^{n_0} \beta_j \psi_u^j$ is due to the PBLs, and $\sum_{j=0}^{n_0} \alpha_j \psi_0^j \phi_0 + \sum_{j=0}^{n_0} \beta_j \psi_N^j \phi_0$ is due to the PBLs and OBLs. Figure 1 gives the graphs of the boundary layer elements (3.1) and (3.8a) - (3.8d).

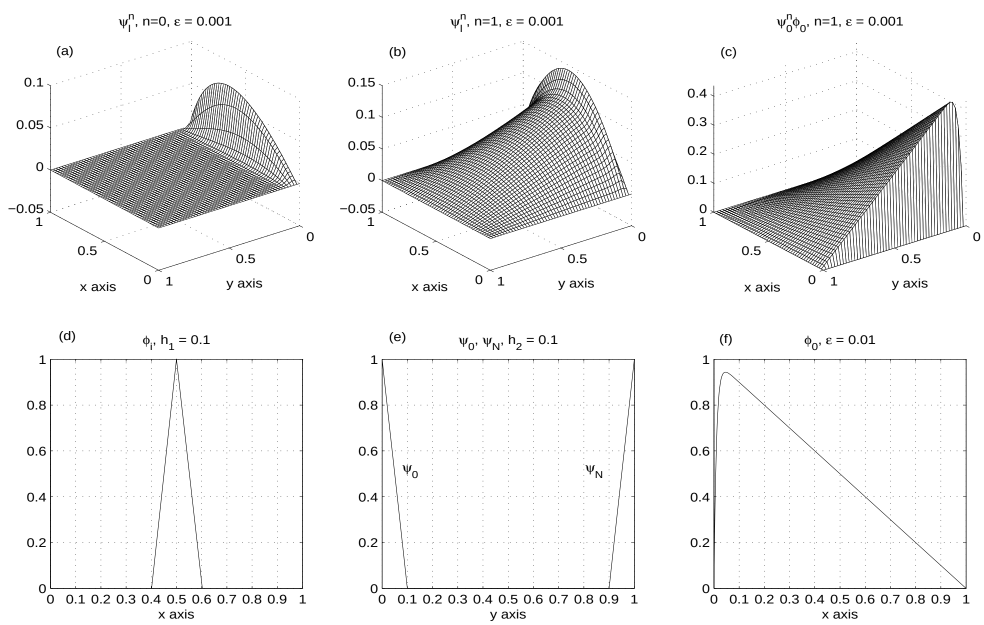

FIGURE 1. (a)(b) PBL elements $\psi_l^n(x,y)$; (c) the overlapping of PBL and OBL $\psi_0^n(y)\phi_0(x)$; (d)(e) bilinear elements, $\phi_i(x)$, $\psi_0(y)$, $\psi_N(y)$; (f) OBL element, $\phi_0(x)$.

3.2. Finite Element Spaces, Schemes, and Approximation Errors. We now define the finite element spaces and consider the new schemes making use of the classical Q_1 elements and the ordinary boundary layer element ϕ_0 as in (3.1) for V_N or adding to them the parabolic boundary layer elements as in (3.8) $\psi_l^0, \cdots, \psi_l^{n_0}$, $\psi_u^0, \cdots, \psi_u^{n_0}$, $\psi_0^0, \cdots, \psi_0^{n_0}$, $\psi_N^0, \cdots, \psi_N^{n_0}$ for $\tilde{V}_N$; that is we introduce the spaces:

$$V_N := \left\{ \sum_{j=1}^{N-1} c_{0j}\phi_0\psi_j + \sum_{i=1}^{M-1}\sum_{j=1}^{N-1} c_{ij}\phi_i\psi_j \right\} \subset H_0^1(\Omega), \tag{3.24a}$$

$$\begin{aligned} \tilde{V}_N := \Bigg\{ &\sum_{j=0}^{n_0} \alpha_j\psi_l^j + \sum_{j=0}^{n_0} \beta_j\psi_u^j + \sum_{j=0}^{n_0} \alpha_j\psi_0^j\phi_0 \\ &+ \sum_{j=0}^{n_0} \beta_j\psi_N^j\phi_0 + v;\ v \in V_N \Bigg\} \subset H_0^1(\Omega), \end{aligned} \tag{3.24b}$$

where ϕ_0, ψ_l^j, ψ_u^j, ψ_0^j, and ψ_N^j are defined in (3.1), and (3.8); the constants α_j and β_j are as in (3.4); ϕ_i, ψ_j, for $i = 1, \cdots, M-1$, $j = 1, \cdots, N-1$, are bilinear elements w.r.t x and y, respectively, i.e., hat functions, see (d)(e) in Figure 1.

We now consider the three types of approximations corresponding to the three types of functions f in (3.3b)-(3.3d), $f = f_1,\ f_2,\ f_3$ and the corresponding solutions u_1, u_2, u_3 in (3.5). The general case follows by superposition. If f satisfies $f(x,0) = f(x,1) = 0$ (i.e. $f = f_1$), we look for an approximate solution $u_N \in V_N$ such that

$$a_\epsilon(u_N, v) = \int_\Omega f_1 v d\Omega, \forall v \in V_N. \tag{3.25}$$

Thanks to (3.11), we derive the result of H^1- approximation error in Theorem 3.1 below quoted from [11]: note that we use the conditions (3.10) only, same as (2.60) as explained after (2.61).

Theorem 3.1. *Assume that the conditions (3.10) on $f = f(x,y)$ hold, namely, $f = f_1$. Let $u = u_1^\epsilon$ be the exact solution of (2.1), and u_N the solution of (3.25). Then*

$$|u - u_N|_{H^1(\Omega)} \le \kappa(\bar{h} + \bar{h}^2\epsilon^{-1}). \tag{3.26}$$

If (3.10) is not satisfied, $f \neq f_1$, then parabolic boundary layers appear and we account for them by considering f_2 and f_3.

We firstly notice that if $f_2(x,0) \not\equiv 0$ or $f_2(x,1) \not\equiv 0$, then some coefficients α_j or β_j are not zero and the parabolic boundary layers φ_l^0 and φ_u^0 corresponding to f_2 and u_2 appear as indicated from their explicit expression (2.18a). To handle them, we consider the approximate solution $u_N^* \in \tilde{V}_N$ such that

$$u_N^* = u_N + u^*, \tag{3.27a}$$

where u_N is the solution of equation (3.25), and

$$u^* = \sum_{j=0}^{n_0} \alpha_j\psi_l^j + \sum_{j=0}^{n_0} \beta_j\psi_u^j + \sum_{j=0}^{n_0} \alpha_j\psi_0^j\phi_0 + \sum_{j=0}^{n_0} \beta_j\psi_N^j\phi_0. \tag{3.27b}$$

If $f = f_2$, then $f_1 = f_3 = 0$, and $u_N = 0$, we actually do not need to solve the linear system (3.25), and the approximate solution u_N^* can be found easily and explicitly from the data f as in (3.3c) and (3.4). We then expect that from Lemma 3.2,

$$|u - u_N^*|_{L^2} = |u - u^*|_{L^2} \leq \kappa\epsilon^{3/4}; \tag{3.28}$$

notice also that the approximation errors due to the parabolic boundary layers do not affect the approximating system (3.25); the errors are totally independent of the discretization errors which arise in (3.25). To approximate the parabolic boundary layers φ_l^0 and φ_u^0, a piecewise uniform mesh, which is only refined near $y = 0, 1$ based on the pointwise estimate in Lemma 2.2, can be considered; but this will appear elsewhere. Here, as we mentioned before, we instead approximate the PBLs using the Lagrange interpolating polynomials as in (3.3c) and (3.4); they can be computed separately and independently of the discretized system (3.25).

Finally, to handle the term f_3 corresponding to the truncating error of a Lagrange interpolation, we will need the following classical results on Lagrange interpolations, see e.g. [22], or Corollary 8.11 in [1].

Lemma 3.3. *If $P(x)$ is the Lagrange interpolating polynomial of degree at most n of a function $g \in C^{n+1}([-1, 1])$ with nodes at the roots of the Chebyshev polynomial of degree $n + 1$, i.e.,*

$$z_k = \cos\left(\frac{2k+1}{2(n+1)}\pi\right), \text{ for } k = 0, 1, \cdots, n, \tag{3.29}$$

then

$$\max_{x\in[-1,1]} \big|g(x) - P(x)\big| \leq \frac{1}{2^n(n+1)!} \max_{x\in[-1,1]} \big|g^{(n+1)}(x)\big|. \tag{3.30}$$

By the change of variable $\tilde{x} = (x + 1)/2$, we obtain the similar result on $[0, 1]$:

Corollary 3.1. *If $P(x)$ is the Lagrange interpolating polynomial of degree at most n of $g \in C^{n+1}([0, 1])$ with the nodes at*

$$z_k' = \frac{z_k + 1}{2}, \text{ for } k = 0, 1, \cdots, n, \tag{3.31}$$

then

$$\max_{x\in[0,1]} \big|g(x) - P(x)\big| \leq \frac{1}{2\cdot 4^n(n+1)!} \max_{x\in[0,1]} \big|g^{(n+1)}(x)\big|. \tag{3.32}$$

Theorem 3.2. *For any $f = f(x, y) \in C^\infty(\bar{\Omega})$, let $u = u^\epsilon$ be the exact solution of (2.1), and let u_N^* be defined as in (3.27). Then there exist positive constants κ independent of n_0 and ϵ, and $\kappa(n_0)$ independent of ϵ such that*

$$\begin{aligned} |u - u_N^*|_{H^1(\Omega)} \leq{}& \kappa(\bar{h} + \bar{h}^2\epsilon^{-1}) + \kappa(n_0)\epsilon^{1/4} \\ &+ \frac{\kappa}{2\cdot 4^{n_0}(n_0+1)!\epsilon^{1/2}} \max_{x\in[0,1], y\in\{0,1\}} \left|f^{(n_0+1)}(x, y)\right|. \end{aligned} \tag{3.33}$$

Proof. By the linearity of equation (2.1) and the uniqueness of solutions, we find

$$u = u_1 + u_2 + u_3 \in H_0^1(\Omega), \tag{3.34}$$

where u_j, $j = 1, 2, 3$ are as in (3.5) and (3.6). We have already obtained the approximation results for u_1 and u_2 in Theorem 3.1 and Lemma 3.2, respectively. We now majorize the norm $|u_3|_{H^m}$. From Lemma 2.7 applied to (3.6) with $j = 3$, we find that

$$\big\|u_3\big\|_\epsilon \leq \kappa|f_3|_{L^2}. \tag{3.35}$$

Having chosen the polynomials $\sum_{j=0}^{n_0} \alpha_j x^j$ and $\sum_{j=0}^{n_0} \beta_j x^j$ as the Lagrange interpolating polynomials for $f(x,0)$ and $f(x,1)$, respectively, as in Corollary 3.1 or (3.3c) and (3.4), we find

$$\begin{aligned}
|f_3(x,y)| &\leq (1-y)\Big|f(x,0) - \sum_{j=0}^{n_0} \alpha_j x^j\Big| + y\Big|f(x,1) - \sum_{j=0}^{n_0} \beta_j x^j\Big| \\
&\leq \frac{1}{2\cdot 4^{n_0}(n_0+1)!}\left\{(1-y)\max_{x\in[0,1]}\Big|f^{(n_0+1)}(x,0)\Big| + y\max_{x\in[0,1]}\Big|f^{(n_0+1)}(x,1)\Big|\right\} \\
&\leq \frac{1}{2\cdot 4^{n_0}(n_0+1)!}\max_{x\in[0,1],y\in\{0,1\}}\Big|f^{(n_0+1)}(x,y)\Big|.
\end{aligned}$$

Writing

$$|u-u_N^*|_{H^1} \leq |u_1+u_2+u_3-u_N-u^*|_{H^1} \leq |u_1-u_N|_{H^1} + |u_2-u^*|_{H^1} + |u_3|_{H^1},$$

the theorem now follows from Theorem 3.1, and from Lemma 3.2. □

Remark 3.2. From Theorem 3.1 and Theorem 3.2, we find that for the schemes (3.25) and (3.27) to be effective, we require the space mesh to be of order $\bar{h} = o(\epsilon^{1/2})$ in the H^1 approximation. The L^2- error estimate can be derived via an L^2- stability analysis which will appear in [14].

4. A Mixed Boundary Value Problem

We now consider another type of boundary conditions for which the effects of the parabolic boundary layers at $y = 0, 1$ are milder. We consider the mixed boundary value problem (1.1a),(1.1c). Its weak formulation is as in (2.1), $H_0^1(\Omega)$ being replaced by

$$V = \Big\{v \in H^1(\Omega);\ v = 0 \text{ at } x = 0, 1\Big\}. \tag{4.1}$$

The error estimates for the approximate solutions and numerical simulations of the mixed boundary value problem are shown in [11] under strong conditions on f, namely, $f_y, f_{yyy} = 0$ at $y = 0, 1$. These conditions make the normal derivatives of u^0 and u^1, which are obtained from the explicit solutions in (2.4), vanish at $y = 0, 1$. This suppresses the occurrences of parabolic boundary layers, see [11]. But in general, we do not expect that the normal derivatives of u^j vanish at $y = 0, 1$. When this happens, to remove the discrepancies with the second condition (1.1c) at $y = 0$, we consider the following parabolic equations for $\bar{\varphi}_l^j$:

$$\begin{aligned}
O(1):&\ -\bar{\varphi}^0_{l\bar{y}\bar{y}} - \bar{\varphi}^0_{lx} = 0, \\
O(\epsilon^j):&\ -\bar{\varphi}^j_{l\bar{y}\bar{y}} - \bar{\varphi}^j_{lx} = \bar{\varphi}^{j-1}_{lxx},\ \text{for } j \geq 1.
\end{aligned} \tag{4.2a}$$

The boundary conditions are

$$\bar{\varphi}_l^j(x,\bar{y}) = 0,\ \text{at } x = 1, \tag{4.2b}$$

$$\frac{\partial \bar{\varphi}_l^j}{\partial \bar{y}}(x,\bar{y}=0) = \epsilon^{1/2} R_j(x)\ ^3, \tag{4.2c}$$

$$\bar{\varphi}_l^j(x,\bar{y}) \to 0 \text{ as } \bar{y} \to \infty, \tag{4.2d}$$

where $R_j(x) = -\partial u^j/\partial y(x,0)$.

[3] The boundary condition (4.2c) is equivalent to: $\partial\bar{\varphi}_l^j/\partial y(x,y=0) = R_j(x)$, and then we easily see that $\partial\bar{\varphi}_l^j/\partial y$ resolves the discrepancies of $-\partial u^j/\partial y$ at $y = 0$.

The explicit form of the $\bar{\varphi}_l^j$, and its pointwise and norm estimates are provided by the following lemma. As before, we consider a heat equation in a semi-strip, but this time with a flux boundary condition, see Theorem 20.3.2 in [2]. Let

$$D = \{(x, y) \in \mathbf{R}^2;\ 0 < x < 1, y > 0\}. \tag{4.3}$$

We are given f^* which is uniformly Hölder continuous in x and y for each compact subset of D and satisfies

$$|f^*(x, y)| \leq \kappa\epsilon^{1/2} \exp(-\gamma y), \tag{4.4}$$

for some $\gamma > 0$, and all $0 < x < 1$ and $y > 0$; we are also given g^* which is continuous on $[0, 1]$. We look for u satisfying:

$$\begin{cases} -\dfrac{\partial u}{\partial x} - \dfrac{\partial^2 u}{\partial y^2} = \epsilon^{1/2} f^*,\ \text{for } (x, y) \in D, \\ \dfrac{\partial u}{\partial y}(x, 0) = \epsilon^{1/2} g^*(x),\ 0 < x < 1, \\ u(x, y) \to 0 \text{ as } y \to \infty,\ 0 < x < 1, \\ u(1, y) = 0. \end{cases} \tag{4.5}$$

Compatibility Conditions. We consider as before the following smoothness and compatibility conditions on the data f^*, g^* to attain $u \in C^l(\bar{D})$, $l \geq 0$:

(4.6a) $f^*(x, y)$ and $g^*(x)$ are sufficiently smooth on $\bar{D}$ and $[0, 1]$, respectively,

and

$$\frac{\partial^i}{\partial x^i} f^*(1, y) = \frac{\partial^i}{\partial x^i} g^*(1) = 0,\ \text{for } 0 \leq i \leq l. \tag{4.6b}$$

Differentiating (2.10) in y with f^* being replaced by $\epsilon^{1/2} f^*$, and setting $x = 1$, from the boundary conditions of (4.5), we then find

$$(-1)^k \frac{\partial^k}{\partial x^k} g^*(1) + \sum_{s=0}^{k-1} (-1)^{s+1} \frac{\partial^{2k-s-1}}{\partial x^s \partial y^{2(k-s-1)+1}} f^*(1, y) = 0,\ k = 0, 1, \cdots, l, \tag{4.7}$$

which is necessary for $u \in C^l(\bar{D})$; conditions (4.6) are much stronger than (4.7).

From now on we thus assume that: for $0 \leq i \leq 2n + d - 2j$, $d = 0, 1$, and $0 \leq j \leq n$,

$$R_j^{(i)}(1) = -\frac{\partial^{i+1}}{\partial x^i \partial y} u^j(1, 0) = 0. \tag{4.8}$$

We then find results similar to Lemma 2.1, that is:

Lemma 4.1. *Let $u = u(x, y)$ be the solution of the heat equation (4.5) in D: Then the solution u is unique and it admits the integral representation:*

(4.9)

$$u(x, y) = -\frac{\epsilon^{1/2}}{\sqrt{\pi}} \int_0^{1-x} \frac{1}{\sqrt{t}} \exp\left(-\frac{y^2}{4t}\right) g^*(x + t)\, dt$$
$$+ \frac{\epsilon^{1/2}}{2\sqrt{\pi}} \int_0^{1-x} \int_0^{\infty} \frac{1}{\sqrt{s}} \left\{ \exp\left[-\frac{(y-t)^2}{4s}\right] + \exp\left[-\frac{(y+t)^2}{4s}\right] \right\} f^*(x + s, t) dt ds,$$

and

$$|u(x, y)| \leq \kappa \exp(-\gamma y),\ \textit{for the same } \gamma \textit{ as in (4.4)}. \tag{4.10}$$

If the conditions (4.6) hold, then $u \in C^l(\bar{D})$, $l \geq 0$. *Furthermore, if the following decay conditions hold:*

$$\left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} f^*(x,y) \right| \leq \kappa \epsilon^{1/2} \exp(-\gamma y), \text{ for } 0 \leq i+m \leq l+1, \text{ some } \gamma > 0, \tag{4.11}$$

then the following pointwise estimates for u *and its derivatives hold: for each* i *and* m, *there exists a constant* κ_{im} *which depends only on* f^* *and* g^* *such that*

$$\left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} u(x,y) \right| \leq \kappa_{im} \epsilon^{1/2} \exp(-\gamma y), \forall (x,y) \in D, \tag{4.12}$$

for $0 \leq i+m \leq l+1$, *the same* γ *as in (4.11).*

For the proof, see the Appendix.

Remark 4.1. From Lemma 4.1, if (4.6) and (4.11) hold, it is obvious that, as before, the regularity properties (2.17) hold. □

We find the solutions $\bar{\varphi}_l^j$ of equation (4.2) recursively:

(4.13a)

$$\bar{\varphi}_l^0(x,\bar{y}) = -\frac{\epsilon^{1/2}}{\sqrt{\pi}} \int_0^{1-x} \frac{1}{\sqrt{t}} \exp\left(-\frac{y^2}{4t}\right) R_0(x+t) dt.$$

(4.13b)

$$\bar{\varphi}_l^j(x,\bar{y}) = -\frac{\epsilon^{1/2}}{\sqrt{\pi}} \int_0^{1-x} \frac{1}{\sqrt{t}} \exp\left(-\frac{y^2}{4t}\right) R_j(x+t) dt$$
$$+ \frac{\epsilon^{1/2}}{2\sqrt{\pi}} \int_0^{1-x} \int_0^{\infty} \frac{1}{\sqrt{s}} \left\{ \exp\left[-\frac{(\bar{y}-t)^2}{4s}\right] + \exp\left[-\frac{(\bar{y}+t)^2}{4s}\right] \right\} \frac{\partial^2}{\partial x^2} \bar{\varphi}_l^{j-1}(x+s,t) dt ds,$$

for $1 \leq j \leq n$.

Notice that the $\bar{\varphi}_l^j$ resolve the discrepancies of the normal derivatives of u^j at $y = 0$. Furthermore, thanks to the compatibility conditions (4.8), we find that for $0 \leq j \leq n$, $\bar{\varphi}_l^j(x,\bar{y}) = \bar{\varphi}_l^j(x, y/\sqrt{\epsilon})$ satisfies the regularities (2.17) with $l = 2n + d - 2j$, and $y = \bar{y}$.

The following lemma can be deduced from (4.12) directly and this lemma provides the derivative estimates for $\bar{\varphi}_l^j$ which will be used for asymptotic error estimates later on. Furthermore, as indicated in the pointwise and norm estimates in the two subsequent lemmas, it turns out that these boundary layers are not crucial, i.e. they are mild, unlike the parabolic boundary layers in the Dirichlet boundary value problem (1.1a),(1.1b).

Lemma 4.2. *Assume that the conditions (4.8) hold. Then there exist a positive constants* κ *independent of* ϵ *such that the following inequalities hold*

$$\left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \bar{\varphi}_l^j \left(x, \frac{y}{\sqrt{\epsilon}}\right) \right| \leq \kappa_{ijm} \epsilon^{-m/2+1/2} \exp\left(-\frac{y}{\sqrt{\epsilon}}\right), \forall (x,y) \in \bar{\Omega}, \tag{4.14}$$

for $0 \leq i+m \leq 2n+d+1-2j$, *and* $0 \leq j \leq n$.

The following norm estimates are deduced from Lemma 4.2.

Lemma 4.3. *Assume that the conditions (4.8) hold. Let, for* $0 \leq \sigma < 1$,

$$\Omega^\sigma = (0,1) \times (\sigma, 1).$$

Then there exist positive constants κ and c independent of ϵ such that the following inequalities hold: for $0 \leq i + m \leq 2n + d + 1 - 2j$,

$$\left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \bar{\varphi}_l^j \right|_{L^2(\Omega^\sigma)} \leq \kappa_{ijm} \epsilon^{-m/2+3/4} \exp\left(-\frac{\sigma}{\sqrt{\epsilon}}\right); \tag{4.15a}$$

in particular,

$$\left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \bar{\varphi}_l^j \right|_{L^2(\Omega)} \leq \kappa_{ijm} \epsilon^{-m/2+3/4}. \tag{4.15b}$$

Remark 4.2. Similarly, at $y = 1$, we may introduce $\bar{\varphi}_u^j$ which have the same structure as $\bar{\varphi}_l^j$. Then similarly to (4.8), we will need the following conditions:

$$-\frac{\partial^{i+1}}{\partial x^i \partial y} u^j(1,1) = 0, \text{ for } 0 \leq i \leq 2n + d - 2j,\ d = 0, 1, \text{ and } 0 \leq j \leq n. \tag{4.16}$$

We also notice that $\bar{\varphi}_u^j(x, \tilde{y}) = \bar{\varphi}_u^j(x, (1-y)/\sqrt{\epsilon})$ satisfies the regularities (2.17) with $l = 2n + d - 2j$, and $y = \tilde{y}$. Similarly Lemma 4.2 and Lemma 4.3 are valid with $\bar{\varphi}_l^j$ replaced by $\bar{\varphi}_u^j$, $\bar{y}$ by $\tilde{y}$, and $(\sigma, 1)$ by $(0, 1 - \sigma)$. □

We now have to resolve the discrepancies at $x = 0$ due to u^j, $\bar{\varphi}_l^j$, and $\bar{\varphi}_u^j$; we thus define $\bar{\theta}^j = \bar{\theta}^j(\bar{x}, y)$ as θ^j before, and we can derive the pointwise and norm estimates in the following lemmas as in Lemma 2.5 and Lemma 2.6; the proof is similar. The explicit solutions can be found as before: for $j = 0, 1$,

$$\begin{aligned} \bar{\theta}^j\left(\frac{x}{\epsilon}, y\right) = &-\left(u^j(0,y) + \bar{\varphi}_l^j\left(0, \frac{y}{\sqrt{\epsilon}}\right) + \bar{\varphi}_u^j\left(0, \frac{1-y}{\sqrt{\epsilon}}\right)\right) \exp\left(-\frac{x}{\epsilon}\right) \\ &+ e.s.t.(2n + d + 1 - 2j). \end{aligned} \tag{4.17}$$

Lemma 4.4. *Assume that the conditions (4.8) and (4.16) hold. For any $0 < c < 1$, there exist a positive constant κ_{ijm} independent of ϵ such that the following inequalities hold: for $0 \leq i + m \leq 2n + d + 1 - 2j$, and $j = 2k$ or $j = 2k + 1$ with $k \geq 0$, for all $(x, y) \in \bar{\Omega}$,*

$$\begin{aligned} \left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \bar{\theta}^j\left(\frac{x}{\epsilon}, y\right) \right| \leq \kappa_{ijm} \epsilon^{-i} \exp\left(-c\frac{x}{\epsilon}\right) &\left\{ 1 + \epsilon^{-k-m/2+1/2} \exp\left(-\frac{y}{\sqrt{\epsilon}}\right) \right. \\ &\left. + \epsilon^{-k-m/2+1/2} \exp\left(-\frac{1-y}{\sqrt{\epsilon}}\right) \right\} + e.s.t.(2n + d + 1 - 2j). \end{aligned} \tag{4.18}$$

Lemma 4.5. *For $0 \leq \sigma_1, \sigma_2 < 1$, let*

$$\Omega^{\sigma_1, \sigma_2} = (\sigma_1, 1) \times (\sigma_2, 1 - \sigma_2).$$

Assume that the conditions (4.8) and (4.16) hold. For any $0 < c < 1$, there exist a positive constant κ_{ijm} independent of ϵ such that the following inequalities hold, for $0 \leq i \leq 2n + d + 1 - 2j$, and $j = 2k$ or $j = 2k + 1$ with $k \geq 0$,

$$\begin{aligned} \left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \bar{\theta}^j \right|_{L^2(\Omega^{\sigma_1,\sigma_2})} &\leq \kappa_{ijm} \epsilon^{-i+1/2} \\ &\cdot \left(1 + \epsilon^{-k-m/2+3/4} \exp\left(-\frac{\sigma_2}{\sqrt{\epsilon}}\right)\right) \exp\left(-c\frac{\sigma_1}{\epsilon}\right); \end{aligned} \tag{4.19a}$$

in particular,

$$\left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \bar{\theta}^j \right|_{L^2(\Omega)} \leq \kappa_{ijm} \epsilon^{-i+1/2} \big(1 + \epsilon^{-k-m/2+3/4}\big). \tag{4.19b}$$

We now consider the following compatibility conditions:

$$\frac{\partial}{\partial y} f(1,0) = \frac{\partial}{\partial y} f(1,1) = 0; \tag{4.20}$$

note that the compatibility conditions are much weaker than in [11].

Lemma 4.6. *Assume that (4.20) hold. Then*

$$|u^\epsilon - u^0 - \bar{\varphi}_l^0 - \bar{\varphi}_u^0 - \bar{\theta}^0 - \bar{\delta}^\epsilon|_{L^2(\Omega)} \leq \kappa \epsilon^{7/4}, \tag{4.21a}$$

$$\|u^\epsilon - u^0 - \bar{\varphi}_l^0 - \bar{\varphi}_u^0 - \bar{\theta}^0 - \bar{\delta}^\epsilon\|_{H^1(\Omega)} \leq \kappa \epsilon^{5/4}, \tag{4.21b}$$

$$\|u^\epsilon - u^0 - \bar{\varphi}_l^0 - \bar{\varphi}_u^0 - \bar{\theta}^0 - \bar{\delta}^\epsilon\|_{H^2(\Omega)} \leq \kappa \epsilon^{1/4}, \tag{4.21c}$$

where the function $\bar{\delta}^\epsilon \in V$ is described in the proof and such that:

$$|\bar{\delta}^\epsilon|_{L^2(\Omega)} \leq \kappa \epsilon, \quad \|\bar{\delta}^\epsilon\|_{H^1(\Omega)} \leq \kappa \epsilon^{1/2}. \tag{4.22}$$

Proof. For $n = 0$, from the conditions (4.20), we can easily verify that

$$-\frac{\partial^{i+1}}{\partial x^i \partial y} u^0(1,0) = -\frac{\partial^{i+1}}{\partial x^i \partial y} u^0(1,1) = 0, \text{ for } 0 \leq i \leq 1;$$

we find that $n = 0$, $d = 1$ in (4.8) and (4.16) and hence, $\bar{\varphi}_l^0, \bar{\varphi}_u^0, \bar{\theta}^0 \in H^2(\Omega)$. Let

$$\bar{w}_{\epsilon 0} = u^\epsilon - u^0 - \bar{\varphi}_l^0 - \bar{\varphi}_u^0 - \bar{\theta}^0. \tag{4.23}$$

Then similarly to (2.41) and (2.46), setting

$$\bar{\vartheta}^0 = -\bar{w}_{\epsilon 0}(x,0) \frac{(y-1)^2}{2} + \bar{w}_{\epsilon 0}(x,1) \frac{y^2}{2}, \tag{4.24}$$

we find that $\bar{w}_{\epsilon 0} - \bar{\vartheta}^0$ satisfies the boundary condition (1.1c) on $\partial\Omega$. Since from Lemma 4.2, Remark 4.2, and Lemma 4.4, we easily find that, similarly to (2.42), $\bar{\vartheta}^0$ is exponentially small and it is absorbed in other norms; we may drop the $\bar{\vartheta}^0$. Hence, we write

$$L_\epsilon \bar{w}_{\epsilon 0} = \bar{R}_1 + \bar{R}_2, \tag{4.25a}$$

$$\bar{w}_{\epsilon 0} = 0 \text{ at } x = 0, 1, \tag{4.25b}$$

$$\frac{\partial \bar{w}_{\epsilon 0}}{\partial y} = 0 \text{ at } y = 0, 1, \tag{4.25c}$$

where

$$\bar{R}_1 = \epsilon \{ \bar{\varphi}_{lxx}^0 + \bar{\varphi}_{uxx}^0 \}, \tag{4.26a}$$

$$\bar{R}_2 = \epsilon \{ \triangle u^0 + \bar{\theta}_{yy}^0 \}. \tag{4.26b}$$

Let $\bar{\delta}$ be the solution of:

$$L_\epsilon \bar{\delta}^\epsilon = \bar{R}_2 \text{ in } \Omega, \tag{4.27a}$$

$$\bar{\delta}^\epsilon = 0 \text{ at } x = 0, 1, \tag{4.27b}$$

$$\frac{\partial \bar{\delta}^\epsilon}{\partial y} = 0 \text{ at } y = 0, 1. \tag{4.27c}$$

Then we easily find that $|\bar{R}_2|_{L^2} \leq \kappa \epsilon$, and hence, using Lemma 2.7, the estimate (4.22) follows. Furthermore, we find

$$L_\epsilon (\bar{w}_{\epsilon 0} - \bar{\delta}^\epsilon) = \bar{R}_1 \text{ in } \Omega, \tag{4.28a}$$

$$\bar{w}_{\epsilon 0} - \bar{\delta}^\epsilon = 0 \text{ at } x = 0, 1, \tag{4.28b}$$

$$\frac{\partial}{\partial y} (\bar{w}_{\epsilon 0} - \bar{\delta}^\epsilon) = 0 \text{ at } y = 0, 1. \tag{4.28c}$$

Since $|\bar{R}_1|_{L^2} \leq \kappa\epsilon^{7/4}$, by applying Lemma 2.7 to $\bar{w}_{\epsilon 0} - \bar{\delta}^\epsilon$ again, the lemma follows. $\square$

Remark 4.3. Assume that (4.20) holds. Then from Lemma 4.6 using the norm estimates of Lemma 4.3 and Lemma 4.5, we obtain

$$u^\epsilon = u^0 + O(\epsilon^{1/2}) \text{ in } L^2. \qquad \square$$

In the following lemma we will see that the boundary layer element ϕ_0 introduced in (3.1) absorbs the H^2- singularity of u^ϵ.

Lemma 4.7. *Assume that (4.20) hold. Then there exist a positive constant κ independent of ϵ, and a smooth function $\bar{g} = \bar{g}^\epsilon(y)$ with $|\bar{g}|_{H^2(0,1)} \leq \kappa\epsilon^{-1/4}$ such that*

$$\left\| u^\epsilon - \bar{g}\phi_0 - \tilde{\delta}^\epsilon - \bar{\delta}^\epsilon \right\|_{H^2(\Omega)} \leq \kappa, \tag{4.29}$$

where $\bar{\delta}^\epsilon \in V$ is as in Lemma 4.6, and the function $\tilde{\delta}^\epsilon \in V$ and its derivatives are estimated as follows:

$$\left| \frac{\partial^{i+m}\tilde{\delta}^\epsilon}{\partial x^i \partial y^m} \right|_{L^2(\Omega)} \leq \kappa \begin{cases} \epsilon^{3/4} & \textit{for } m = 0,\ i = 0,1,2, \\ \epsilon^{1/4} & \textit{for } m = 1,\ i = 0,1, \\ \epsilon^{-1/4} & \textit{for } m = 2,\ i = 0. \end{cases} \tag{4.30}$$

Proof. From the asymptotic error (4.21c), we find

$$\|u^\epsilon - \bar{\varphi}_l^0 - \bar{\varphi}_u^0 - \bar{\theta}^0 - \bar{\delta}^\epsilon\|_{H^2} \leq \kappa. \tag{4.31}$$

Notice that from the condition (4.20), $\bar{\varphi}_l^0, \bar{\varphi}_u^0, \bar{\theta}^0 \in H^2(\Omega)$, see the proof in Lemma 4.6. Then from the explicit solution $\bar{\theta}^0$ in (4.17), with the definition ϕ_0 in (3.1), we find

$$\bar{\varphi}_l^0 + \bar{\varphi}_u^0 + \bar{\theta}^0 = \bar{g}\phi_0(x) + \tilde{\delta}^\epsilon + u^0(0,y)(x-1) + e.s.t.(2), \tag{4.32a}$$

where

$$\bar{g} = \bar{g}^\epsilon(y) = u^0(0,y) + \bar{\varphi}_l^0(0,\bar{y}) + \bar{\varphi}_u^0(0,\tilde{y}), \tag{4.32b}$$

$$\tilde{\delta}^\epsilon = (\bar{\varphi}_l^0(0,\bar{y}) + \bar{\varphi}_u^0(0,\tilde{y}))(x-1) + \bar{\varphi}_l^0(x,y) + \bar{\varphi}_u^0(x,y) \in V; \tag{4.32c}$$

note that from the boundary condition $\bar{\varphi}_l^0 = \bar{\varphi}_u^0 = 0$ at $x = 1$. The estimate (4.30) and the estimate for $|g|_{H^2}$ follow from Lemma 4.2 - 4.3, and hence the lemma follows. $\square$

To approximate the solutions of (4.1), we introduce the following finite element space below.

$$\bar{V}_N := \left\{ \sum_{j=0}^{N} c_{0j}\phi_0\psi_j + \sum_{i=1}^{M-1}\sum_{j=0}^{N} c_{ij}\phi_i\psi_j \right\} \subset V, \tag{4.33}$$

where ϕ_0 is defined in (3.1), ϕ_i, ψ_j are piecewise bilinear elements w.r.t x and y, respectively, on a *uniform mesh* for $i = 1 \cdots M-1, j = 0, 1 \cdots N-1, N$ as in (d)(e) in Figure 1, and V is defined in (4.1). We look for an approximate solution $u_N \in \bar{V}_N$ such that

$$a_\epsilon(u_N, v) = F(v), \forall v \in \bar{V}_N. \tag{4.34}$$

Notice that the approximating system (4.34) has ordinary boundary layer element ϕ_0, and two hat functions ψ_0, ψ_N as in (e) in Figure 1.

As before, we will need the following interpolation inequalities to derive the approximation errors.

Lemma 4.8. *Assume that (4.20) hold. Then there exists an interpolant $\tilde{u}_N \in \bar{V}_N$ such that*

$$\|u^\epsilon - \tilde{u}_N - \bar{\delta}^\epsilon\|_{L^2(\Omega)} \le \kappa(h_1^2 + h_2^2\epsilon^{-1/4}), \tag{4.35}$$

$$\|u^\epsilon - \tilde{u}_N - \bar{\delta}^\epsilon\|_{H^1(\Omega)} \le \kappa(h_1 + h_2^2\epsilon^{-3/4} + h_2\epsilon^{-1/4}). \tag{4.36}$$

Proof. By the classical interpolation results, see e.g., [16], [5], [24], applied to $u^\epsilon - \bar{g}\phi_0 - \tilde{\delta}^\epsilon - \bar{\delta}^\epsilon \in V$ using Lemma 4.7, there exist the c_{ij} such that for $m = 0, 1$,

$$\begin{aligned} I_1(m) := &\left\| u^\epsilon - \bar{g}\phi_0 - \tilde{\delta}^\epsilon - \bar{\delta}^\epsilon - \sum_{i=1}^{M-1}\sum_{j=0}^{N} c_{ij}\phi_i\psi_j \right\|_{H^m(\Omega)} \\ &\le \kappa\bar{h}^{2-m}|u^\epsilon - \bar{g}\phi_0|_{H^2(\Omega)} \le \kappa\bar{h}^{2-m}. \end{aligned} \tag{4.37}$$

Then using Lemma 4.7 again, we derive the following estimates. There exist the c_{0j} such that for $m = 0, 1$,

$$\left\| \bar{g} - \sum_{j=0}^{N} c_{0j}\psi_j \right\|_{H^m(\Omega)} \le \kappa h_2^{2-m}|\bar{g}|_{H^2(0,1)} \le \kappa h_2^{2-m}\epsilon^{-1/4}. \tag{4.38}$$

Hence, we find that for $m = 0, 1$,

$$\begin{aligned} I_2(m) := &\left| \bar{g}\phi_0 - \sum_{j=0}^{N} c_{0j}\phi_0\psi_j \right|_{H^m(\Omega)} \\ &\le \kappa \begin{cases} \left|\bar{g} - \sum_{j=0}^{N} c_{0j}\psi_j\right|_{L^2(0,1)} \\ \epsilon^{-1/2}\left|\bar{g} - \sum_{j=0}^{N} c_{0j}\psi_j\right|_{L^2(0,1)} + \left|\bar{g} - \sum_{j=0}^{N} c_{0j}\psi_j\right|_{H^1(0,1)} \end{cases} \\ &\le \kappa \begin{cases} h_2^2\epsilon^{-1/4} & \text{for } m = 0, \\ h_2^2\epsilon^{-3/4} + h_2\epsilon^{-1/4} & \text{for } m = 1. \end{cases} \end{aligned} \tag{4.39}$$

By the classical interpolation results, see e.g., [24], or Lemma 3.3 in [16], applied to $\tilde{\delta} = \tilde{\delta}^\epsilon \in V$ with the estimates (4.30), we find that there exist the c_{ij} such that for $m = 0, 1$,

$$\begin{aligned} I_3(m) := &\left\| \tilde{\delta} - \sum_{i=1}^{M-1}\sum_{j=0}^{N} c_{ij}\phi_i\psi_j \right\|_{L^2(\Omega)} \\ &\le \kappa \begin{cases} h_1^2|\partial^2\tilde{\delta}/\partial x^2|_{L^2} + h_2^2|\partial^2\tilde{\delta}/\partial y^2|_{L^2} + h_1h_2|\partial^2\tilde{\delta}/\partial x\partial y|_{L^2} \\ h_1|\partial^2\tilde{\delta}/\partial x^2|_{L^2} + h_2|\partial^2\tilde{\delta}/\partial y^2|_{L^2} + \bar{h}|\partial^2\tilde{\delta}/\partial x\partial y|_{L^2} \end{cases} \\ &\le \kappa \begin{cases} h_1^2\epsilon^{3/4} + h_2^2\epsilon^{-1/4} + h_1h_2\epsilon^{1/4} & \text{for } m = 0, \\ h_1\epsilon^{3/4} + h_2\epsilon^{-1/4} + \bar{h}\epsilon^{1/4} & \text{for } m = 1. \end{cases} \end{aligned} \tag{4.40}$$

Hence, we easily verify that

$$\left\| u^\epsilon - \sum_{j=0}^{N} c_{0j}\phi_0\psi_j - \sum_{i=1}^{M-1}\sum_{j=0}^{N} c_{ij}\phi_i\psi_j - \bar{\delta}^\epsilon \right\|_{H^m} \le I_1(m) + I_2(m) + I_3(m), \tag{4.41}$$

and setting

$$\tilde{u}_N = \sum_{j=0}^{N} c_{0j}\phi_0\psi_j + \sum_{i=1}^{M-1}\sum_{j=0}^{N} c_{ij}\phi_i\psi_j, \tag{4.42}$$

the lemma follows. □

We are now able to derive the error estimates below using Lemma 4.8. But by the subtlety of the a priori estimates involving the small parameter ϵ, the term $\bar{\delta}^\epsilon$ should be dealt with caution, see (4.45) below.

Theorem 4.1. *Assume that the conditions (4.20) on $f = f(x, y)$ hold. Let u_N be the solution of (4.34) and $\tilde{u}_N$ the interpolant in $\bar{V}_N$ defined in Lemma 4.8. Then*

$$\begin{aligned} \left|u_N - \tilde{u}_N\right|_{H^1(\Omega)} &\leq \kappa(h_1 + h_1^2\epsilon^{-1} + h_2\epsilon^{-1/4} + h_2^2\epsilon^{-5/4} + \epsilon^{1/2}) \\ &\quad + \kappa\epsilon^{-1/2}\left|u_N - \tilde{u}_N\right|_{L^2(\Omega)}. \end{aligned} \tag{4.43}$$

Proof. Subtracting (4.34) from (4.1), we find

$$a_\epsilon(u - u_N, v) = 0, \ \forall v \in \bar{V}_N, \tag{4.44}$$

and hence

$$\begin{aligned} a_\epsilon(u_N - \tilde{u}_N, u_N - \tilde{u}_N) &= a_\epsilon(u - \tilde{u}_N, u_N - \tilde{u}_N) \\ &= (\text{from equation (4.27) for } \bar{\delta}^\epsilon) \\ &= a_\epsilon(u - \tilde{u}_N - \bar{\delta}, u_N - \tilde{u}_N) + (\bar{R}_2, u_N - \tilde{u}_N). \end{aligned} \tag{4.45}$$

Then we find

$$\begin{aligned} \epsilon|\nabla(u_N - \tilde{u}_N)|_{L^2}^2 &\leq \kappa\epsilon|\nabla(u - \tilde{u}_N - \bar{\delta})|_{L^2}^2 + \kappa\epsilon^{-1}|u - \tilde{u}_N - \bar{\delta}|_{L^2}^2 \\ &\quad + \kappa|\bar{R}_2|_{L^2}^2 + \kappa|u_N - \tilde{u}_N|_{L^2}^2. \end{aligned} \tag{4.46}$$

Since $|\bar{R}_2|_{L^2} \leq \kappa\epsilon$, we easily verify that

$$\begin{aligned} |u_N - \tilde{u}_N|_{H^1} &\leq \kappa|u - \tilde{u}_N - \bar{\delta}|_{H^1} + \kappa\epsilon^{-1}|u - \tilde{u}_N - \bar{\delta}|_{L^2} \\ &\quad + \kappa\epsilon^{1/2} + \kappa\epsilon^{-1/2}|u_N - \tilde{u}_N|_{L^2}; \end{aligned} \tag{4.47}$$

the theorem follows from the interpolation inequalities in Lemma 4.8. □

Remark 4.4. The L^2- error, $\left|u_N - \tilde{u}_N\right|_{L^2(\Omega)}$, will be derived using the L^2- stability analysis which will appear in [14]. Then the H^1- error, $\left|u_N - \tilde{u}_N\right|_{H^1(\Omega)}$, will be easily found from Theorem 4.1; thus $\left|u^\epsilon - \tilde{u}_N\right|_{H^m(\Omega)}$, $m = 0, 1$, will follow.

5. Occurrence of Boundary Layers

In this section, we summarize the type of the occurrences of boundary layers using the model equation:

$$-\epsilon\triangle u - u_x = f \qquad \text{in } \Omega = (0,1)\times(0,1); \tag{5.1}$$

the boundary conditions are specified in the table below. More general singularly perturbed equations will appear elsewhere. But this simple model equation covers two major boundary layers which essentially affect numerical computations in general cases.

From the lower-order asymptotic analysis, i.e. with $n = 0, 1$, as we did in previous sections, we are able to detect two major boundary layers which are ordinary and parabolic boundary layers at the outflow and at the characteristic boundaries, respectively. We notice that these are determined from the data f and the boundary conditions. More precisely, for the Dirichlet boundary value problem (1.1a), (1.1b), if

$$f = 0 \quad \text{at } y = 0, 1, \tag{5.2}$$

parabolic boundary layers are suppressed. In this case, we only observe ordinary boundary layers at the outflow (i.e. at $x = 0$ in problem (5.1)). Then the discretizetion errors due to them are propagated in the whole domain due to the convective term (i.e. in the x- direction due to the term u_x in problem (5.1)). If the discretizetion errors are not properly handled, the approximate solutions display wild oscillations in the propagation direction, see Figure 2 below.

Another way to suppress parabolic boundary layers is imposing the boundary conditions as in the mixed boundary value problem (1.1a), (1.1c), see Section 4; obviously, we can expect that from condition (1.1c) (the normal derivatives of the exact solution u^ϵ vanish at the characteristic boundaries), the u^ϵ varies slowly at those characteristic boundaries and thus the parabolic boundary layers are mild.

For the case where parabolic boundary layers stand out, since the discretizetion errors due to them are localized near the characteristic boundaries, i.e. $y = 0, 1$, we can approximate them using a Lagrange interpolating polynomial or using a refined mesh only near at their occurrences, see Section 3.2, and see Figure 3, Figure 4.

For a channel problem, see [16], if

$$\int_0^1 f(s, y)ds = 0, \tag{5.3a}$$

and

$$f(x, y) \text{ is 1- periodic in } x, \tag{5.3b}$$

ordinary boundary layers are suppressed. In the Dirichlet boundary value problem, if (5.3a) holds, then the ordinary boundary layers will be mild; note that $u^0(0, y) = 0$ from (2.4a) and hence from (2.25) we find

$$|\theta^0|_{L^2} \leq \kappa\epsilon^{3/4}, \; |\partial\theta^0/\partial x|_{L^2} \leq \kappa\epsilon^{-1/4}, \; |\partial\theta^0/\partial y|_{L^2} \leq \kappa\epsilon^{1/4}. \tag{5.4}$$

We thus summarize the occurrences of the boundary layers and their norm estimates.

- Occurrences of boundary layers

	Dirichlet boundary problem $u = 0$ on $\partial\Omega$		mixed boundary problem $\left\{ \begin{array}{l} u = 0 \text{ at } x = 0, 1 \\ \partial u/\partial y = 0 \text{ at } y = 0, 1 \end{array} \right.$	channel problem $\left\{ \begin{array}{l} u = 0 \text{ at } y = 0, 1 \\ u \text{ is 1- periodic in } x \end{array} \right.$
conditions on f	$f = 0$ at $y = 0, 1$.	no conditions	compatibility (4.20)	$\int_0^1 f(s, y)ds = 0$ f is 1- periodic in x
OBLs	YES at $x = 0$	YES at $x = 0$	YES at $x = 0$	NO
PBLs	NO	YES at $y = 0, 1$	Mild PBLs at $y = 0, 1$ $\sim \left\{ \begin{array}{l} O(\epsilon^{3/4}) \text{ in } L^2 \\ O(\epsilon^{1/4}) \text{ in } H^1 \end{array} \right.$	YES at $y = 0, 1$

• Norm estimates

	O B L θ^j	P B L $\varphi^j \sim \varphi^j_l + \varphi^j_u$	OBL element ϕ_0	PBL elements ψ^j_l, ψ^j_u, ψ^j_0, ψ^j_N
local (or, stretched) variables	$\bar{x} = x/\epsilon$	$\bar{y} = y/\sqrt{\epsilon}$, $\tilde{y} = (1-y)/\sqrt{\epsilon}$	·	·
L^2- norm	$O(\epsilon^{1/2})$	$O(\epsilon^{1/4})$	$O(1)$	$O(1)$
H^1- norm	$O(\epsilon^{-1/2})$	$O(\epsilon^{-1/4})$	$O(\epsilon^{-1/2})$	$O(\epsilon^{-1/4})$

6. Numerical Simulations

In this section we present the numerical results of the new scheme (NS) and, the new scheme with PBLs (NSP), which are corresponding to the problems (3.25) and (3.27). Each uses V_N, and $\tilde{V}_N$, respectively, as the finite element spaces.

The numerical calculations were carried out on 4CPU POWER3+ 375 Mhz with 2GB memory running AIX. We solve the linear system directly by using Gaussian elimination. The numerical simulations for the mixed boundary value problem (1.1a)(1.1c) with its approximating system (4.34) have been shown in [11].

6.1. Numerical Implementations. To compute the approximating systems (3.25), (3.27) and (4.34), we must evaluate integrations of singular functions, which are ordinary and parabolic boundary layer elements. Since our approximating systems have a small coercivity (= ϵ), we need to take care of computations involving ϵ with a caution; the explicit integration formulas are available using MAPLE, and in particular, the integrations involving ϕ_0. We would like to mention that even after we get those explicit formulas, we have to modify them as follows to avoid the overflow of numeric limits in computers, e.g., if $\epsilon = 10^{-4}$, $e^{1/\epsilon} = e^{10000}$ is out of the range of a double precision. For example, in the formula we have the following terms which need to be modified as below:

$$\frac{1}{e^{h_1/\epsilon}} \to e^{-h_1/\epsilon}, \ \frac{e^{-h_1/\epsilon}}{\epsilon^8} \to \left(\frac{e^{-h_1/(8\epsilon)}}{\epsilon}\right)^8. \tag{6.1}$$

6.2. Numerical Results : Examples. For the function $f = \sin(\pi y)$, we have tested several cases of M, N and ϵ, and obtained L^2-, and L^∞- errors in [11]. This function f clearly satisfies the condition $f(x,0) = f(x,1) = 0$, and hence the parabolic boundary layer will not appear. Then we see that the ordinary boundary layer element ϕ_0 will play an important role to approximate solutions as indicated in previous sections. For the classical schemes, i.e., without boundary layer elements, we have observed wild oscillations in the x- direction, i.e., in the propagation direction due to the convective term u_x. But the ordinary boundary layer element ϕ_0 stabilizes the approximate solutions and thus the approximation errors, see [11] and Figure 2 below for the classical and new schemes. In particular, we have simulated for $f = y(1-y)$ and $f = x^2(1-e^{-y})(1-e^{-(1-y)})$ in Figure 2 which satisfy $f(x,0) = f(x,1) = 0$.

In Figure 3, we notice, for $f = \cos(\pi y)$ and $\epsilon = 10^{-6}$, that the parabolic boundary layers appear in (a) at $y = 0$ and $y = 1$; note that $f(x,0) \neq 0$ and $f(x,1) \neq 0$. But since the propagation direction, i.e., the x- direction, is parallel to the place of the occurrences of parabolic boundary layers, the approximation errors due to them are localized near $y = 0$ and $y = 1$ as shown in (a) of Figure 3 and in (a) of Figure 4,

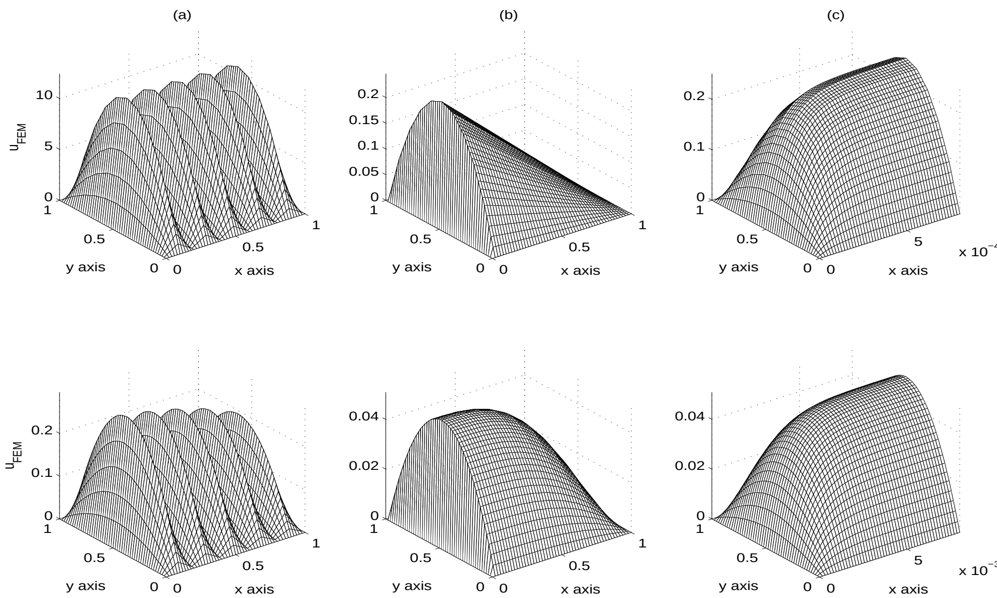

FIGURE 2. $-\epsilon\triangle u - u_x = f$, $f = y(1-y)$ in the first row with Dirichlet boundary condition, $\epsilon = 10^{-4}, M = 10, N = 10$, $f = x^2(1-e^{-y})(1-e^{-(1-y)})$ in the second row with Dirichlet boundary condition, $\epsilon = 10^{-3}, M = 10, N = 20$: (a) Classical Scheme without BL elements; (b) New Scheme (3.25) with OBL element ϕ_0; (c) Zooming of (b) near $x = 0$.

see also [11] for the errors due to PBLs in numerical simulations. To resolve these errors, as in (b) we added the parabolic boundary layer elements. More precisely, we decompose $f = \cos(\pi y)$ as below: since $f(x,0) = 1$ and $f(x,1) = -1$,

$$f = f_1 + f_2, \tag{6.2a}$$

where

$$f_1 = \cos(\pi y) - f_2, \tag{6.2b}$$

$$\begin{aligned} f_2 &= f(x,0)(1-y) + f(x,1)y \\ &= 1 - 2y. \end{aligned} \tag{6.2c}$$

Hence, we find from f_2,

$$u^* = \psi_l^0 - \psi_u^0 + \psi_0^0\phi_0 - \psi_N^0\phi_0, \tag{6.3}$$

and the approximate solution u_N^* as in (3.27):

$$u_N^* = u_N + u^*, \tag{6.4}$$

where u_N is the solution of the approximating system (3.25) with f_1 in (6.2b). Then as indicated in Theorem 3.2, we can expect that the approximation errors due to PBLs is $O(\epsilon^{3/4}) = O(10^{-9/2})$ in L^2. In Figure 3, one might notice that the approximate solution u_N^* is composed of (d), (e), and (f), and we see that the PBL elements and the overlappings of PBL and OBL elements in (e) contribute to the approximate solution the most, the OBL elements in (d) the next, and the classical elements in (f) the least.

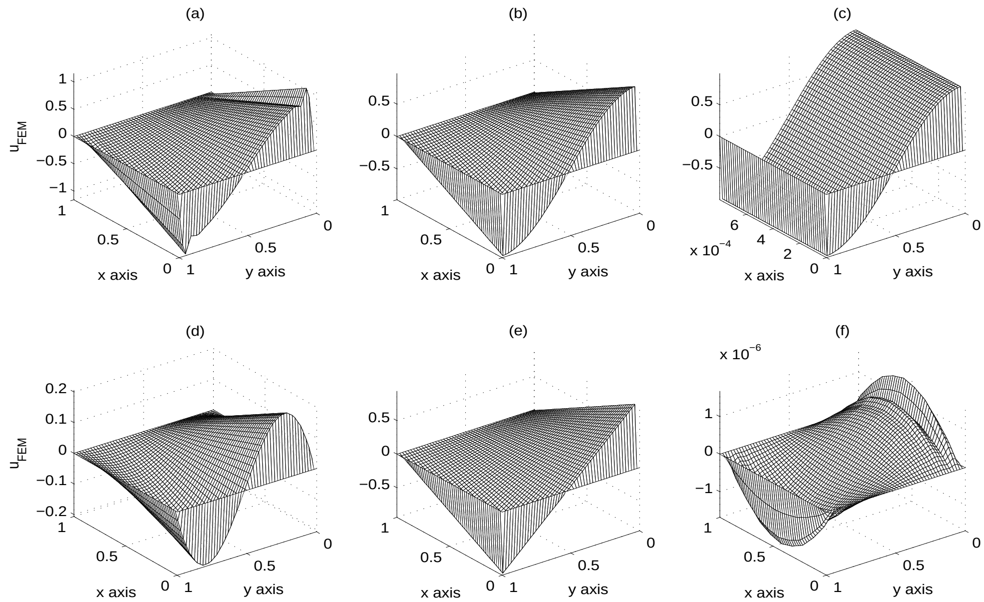

FIGURE 3. $-\epsilon\triangle u - u_x = \cos(\pi y)$, with Dirichlet boundary condition, $\epsilon = 10^{-6}, M = 10, N = 20$: (a) New Scheme (3.25) with only OBL element, f_1 being replaced by $\cos(\pi y)$; (b) New Scheme (3.27) with OBL and PBL elements; (c) Zooming of (b) near $x = 0$; (b) = (d) + (e) + (f) where (d) OBL elements $\phi_0\psi_j$, (e) PBL elements ψ_l^0, ψ_u^0, and the overlappings of PBL and OBL elements $\psi_0^0\phi_0$, $\psi_N^0\phi_0$, and (f) classical elements (or bilinear elements) $\phi_i\psi_j$.

If $f(x,0)$ and $f(x,1)$ are not polynomials, we have to allow a truncation error f_3 which is defined in (3.3). In Figure 4, we have tested the case for $f = \exp(x+y)$. For f_2, we use a Lagrange interpolating polynomial of degree 3 defined below. We write

$$f = f_1 + f_2 + f_3, \tag{6.5a}$$

where

$$f_1 = \exp(x+y) - f_2 - f_3, \tag{6.5b}$$

$$f_2 = f(x,0)(1-y) + f(x,1)y, \tag{6.5c}$$

with

$$f(x,0) = \sum_{k=0}^{3} \exp(x_k)L_k(x), \;\; f(x,1) = \sum_{k=0}^{3} \exp(x_k+1)L_k(x), \tag{6.5d}$$

where, for $k = 0, \cdots, 3$,

$$x_k = \frac{1}{2}\left[\cos\left(\frac{2k+1}{8}\pi\right) + 1\right], \tag{6.5e}$$

$$L_k(x) = \prod_{i=0,\; i\neq k}^{3} \frac{(x-x_i)}{(x_k-x_i)}, \tag{6.5f}$$

and f_3 is a truncation error: see Lemma 3.3 and Corollary 3.1 for Chebyshev node points x_k. Then we easily find using e.g. MAPLE,

$$f_2 = (0.27823967x^3 + 0.42430102x^2 + 1.01563252x + 0.9995086147)(1 + 1.718281828y),$$

and hence we can easily derive u^* similarly as before. It is not hard to find the truncation error, $\max_{x\in[0,1],\ y\in\{0,1\}} |f_3| = O(10^{-3})$, and the approximation errors due to f_3 is also $O(10^{-3})$ in L^2. From Theorem 3.2, we can expect that the approximation errors due to PBLs is $O(\epsilon^{3/4}) = O(10^{-15/4})$ in L^2. We also notice in Figure 4 that u_N^* is composed of (d), (e), (f), and (g), and we see that the overlappings of PBL and OBL elements in (f) contribute to the approximate solution the most, the OBL and PBL elements in (d)(e) the next, and the classical elements in (g) the least.

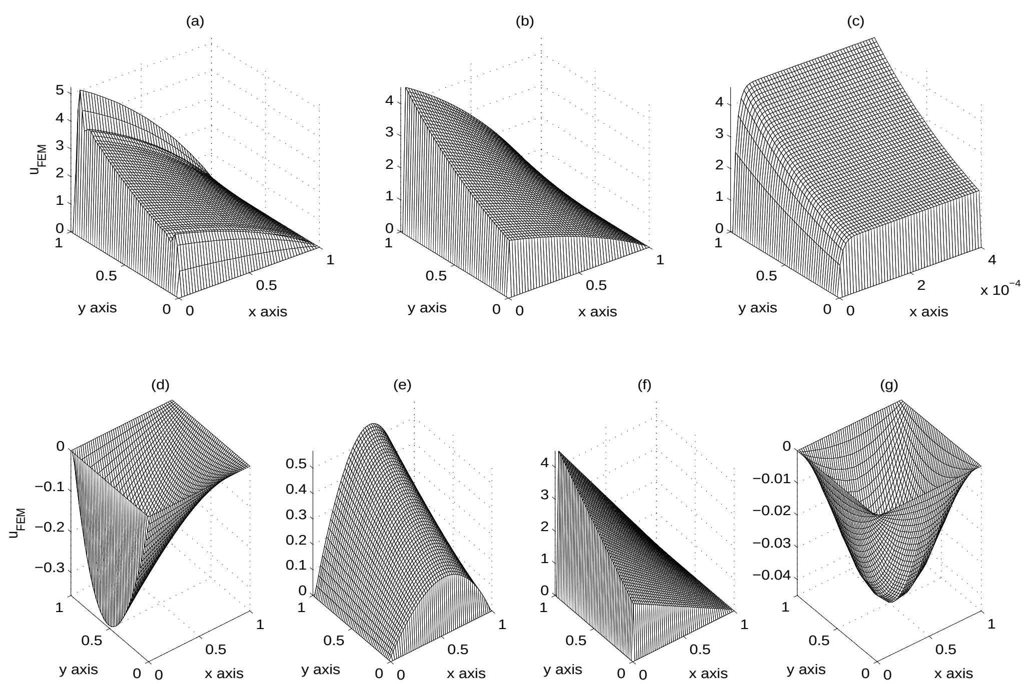

FIGURE 4. $-\epsilon\triangle u - u_x = \exp(x+y)$, with Dirichlet boundary condition, $\epsilon = 10^{-5}, M = 10, N = 20$: (a) New Scheme (3.25) with only OBL element, f_1 being replaced by $\exp(x+y)$; (b) New Scheme (3.27) with OBL and PBL elements; (c) Zooming of (b) near $x = 0$; (b) = (d) + (e) + (f) + (g) where (d) OBL elements $\phi_0\psi_j$, (e) PBL elements ψ_l^n, ψ_u^n, $n = 0, \cdots, 3$, (f) the overlappings of PBL and OBL elements $\psi_0^n\phi_0$, $\psi_N^n\phi_0$, $n = 0, \cdots, 3$, and (g) classical elements (or bilinear elements) $\phi_i\psi_j$.

7. Appendix

Proof of Lemma 2.1. By a simple change of variable, we can derive the solution u of equation (2.8) as follows. Let

$$\tilde{x} = 1 - x. \tag{7.1}$$

We then have the following classical heat equation in the semi-strip $D = (0,1) \times (0,\infty)$:

$$\frac{\partial u}{\partial \tilde{x}} - \frac{\partial^2 u}{\partial y^2} = f^*(1-\tilde{x}, y), \text{ for } (\tilde{x}, y) \in D, \tag{7.2a}$$

$$u(\tilde{x}, 0) = g^*(1-\tilde{x}), \ 0 < \tilde{x} < 1, \tag{7.2b}$$

$$u(\tilde{x}, y) \to 0 \text{ as } y \to \infty, \ 0 < \tilde{x} < 1, \tag{7.2c}$$

$$u(\tilde{x} = 0, y) = 0. \tag{7.2d}$$

From Theorem 20.3.1 in [2], see also [26], we find that the solution u is unique and is explicitly expressed as:

$$u(\tilde{x}, y) = I_1 + I_2, \tag{7.3a}$$

with

$$I_1 = -2 \int_0^{\tilde{x}} \frac{\partial K}{\partial y}(\tilde{x} - \tau, y) g^*(1-\tau) d\tau, \tag{7.3b}$$

$$I_2 = \int_0^{\tilde{x}} \int_0^{\infty} G(\tilde{x} - \tau, y; t) f^*(1-\tau, t) dt d\tau, \tag{7.3c}$$

where the fundamental solution $K = K(\tilde{x}, y)$ and the Green function $G = G(\tilde{x}, y; t)$ are

$$K(\tilde{x}, y) = \frac{1}{\sqrt{4\pi\tilde{x}}} \exp\left(-\frac{y^2}{4\tilde{x}}\right), \tag{7.3d}$$

$$G(\tilde{x}, y; t) = K(\tilde{x}, y-t) - K(\tilde{x}, y+t). \tag{7.3e}$$

Then, since

$$\frac{\partial}{\partial y} K(\tilde{x} - \tau, y) = -\frac{y}{4\sqrt{\pi}(\tilde{x}-\tau)^{3/2}} \exp\left(-\frac{y^2}{4(\tilde{x}-\tau)}\right), \tag{7.4}$$

we find

$$\begin{aligned} I_1 &= \frac{y}{2\sqrt{\pi}} \int_0^{\tilde{x}} \frac{1}{(\tilde{x}-\tau)^{3/2}} \exp\left(-\frac{y^2}{4(\tilde{x}-\tau)}\right) g^*(1-\tau) d\tau \\ &= (\text{setting } t = y/\sqrt{2(\tilde{x}-\tau)}, \ \tilde{x} = 1-x) \\ &= \sqrt{\frac{2}{\pi}} \int_{y/\sqrt{2(1-x)}}^{\infty} \exp\left(-\frac{t^2}{2}\right) g^*\left(x + \frac{y^2}{2t^2}\right) dt, \end{aligned} \tag{7.5a}$$

and we also find
(7.5b)

$$\begin{aligned} I_2 = &\int_0^{\tilde{x}} \int_0^{\infty} \frac{1}{\sqrt{4\pi(\tilde{x}-\tau)}} \left\{ \exp\left[-\frac{(y-t)^2}{4(\tilde{x}-\tau)}\right] - \exp\left[-\frac{(y+t)^2}{4(\tilde{x}-\tau)}\right] \right\} f^*(1-\tau, t) dt d\tau \\ &= (\text{setting } s = \tilde{x} - \tau, \ \tilde{x} = 1-x) \\ &= \frac{1}{2\sqrt{\pi}} \int_0^{1-x} \int_0^{\infty} \frac{1}{\sqrt{s}} \left\{ \exp\left[-\frac{(y-t)^2}{4s}\right] - \exp\left[-\frac{(y+t)^2}{4s}\right] \right\} f^*(x+s, t) dt ds; \end{aligned}$$

hence (2.13) follows.

We will see that if conditions (2.9) hold, from (7.7), (7.12), (7.15), (7.17), and (7.23), since u is $l+1$ times differentiable on $\bar{D}$, we easily find that $u \in C^l(\bar{D})$ [4].

We now prove the estimates (2.14) and (2.16); we perform calculations similar to those in the proof of Theorem 3.8 in [26]. For $m = 0$, thanks to (2.9) and the explicit form of u in (2.13), we find

$$\frac{\partial^i}{\partial x^i} I_1 = \sqrt{\frac{2}{\pi}} \int_{y/\sqrt{2(1-x)}}^{\infty} \exp\left(-\frac{t^2}{2}\right) \frac{\partial^i}{\partial x^i} g^*\left(x + \frac{y^2}{2t^2}\right) dt, \text{ for } 0 \le i \le l+1. \tag{7.7}$$

Since g^* is sufficiently smooth and $0 \le x + y^2/(2t^2) \le 1$, we have

$$\left| \frac{\partial^i}{\partial x^i} g^*\left(x + \frac{y^2}{2t^2}\right) \right| \le \kappa, \tag{7.8}$$

and since

$$\exp\left(-\frac{t^2}{2}\right) \le \exp\left(\frac{c^2}{2} - ct\right) = \exp\left(\frac{c^2}{2}\right)\exp(-ct), \text{ for any } c > 0, \tag{7.9}$$

we can write

$$\exp\left(-\frac{t^2}{2}\right) \le \kappa(c)\exp(-ct), \text{ for any } c > 0. \tag{7.10}$$

Hence

$$\begin{aligned} \left|\frac{\partial^i}{\partial x^i} I_1\right| &\le \kappa(c) \int_{y/\sqrt{2(1-x)}}^{\infty} \exp(-ct)dt \le \kappa(c)\exp\left(-\frac{cy}{\sqrt{2}}\right) \\ &\le (\text{setting } c = \sqrt{2}\gamma) \le \kappa(\gamma)\exp(-\gamma y). \end{aligned} \tag{7.11}$$

We can also differentiate I_2 using (2.15) as follows: for $0 \le i \le l+1$,

(7.12)
$$\frac{\partial^i}{\partial x^i} I_2 = \frac{1}{2\sqrt{\pi}} \int_0^{1-x} \int_0^{\infty} \frac{1}{\sqrt{s}} \left\{ \exp\left[-\frac{(y-t)^2}{4s}\right] - \exp\left[-\frac{(y+t)^2}{4s}\right] \right\} \frac{\partial^i}{\partial x^i} f^*(x+s,t)dtds.$$

Then since $0 \le s \le 1$, from the condition (2.15) we find

$$\begin{aligned} \left|\frac{\partial^i}{\partial x^i} I_2\right| &\le \kappa \left[\int_0^{1-x} \frac{1}{\sqrt{s}} ds\right] \int_0^{\infty} \exp\left(-\frac{(y-t)^2}{4}\right) \exp(-\gamma t)\, dt \\ &\le \kappa \int_0^{\infty} \exp\left(-\frac{(t+(2\gamma-y))^2}{4} - \gamma y + \gamma^2\right) dt \\ &\le \kappa \exp(-\gamma y) \int_0^{\infty} \exp\left(-\frac{(t+(2\gamma-y))^2}{4}\right) dt \le \kappa\exp(-\gamma y). \end{aligned} \tag{7.13}$$

Hence, from (7.11) and (7.13), we conclude that

$$\left|\frac{\partial^i}{\partial x^i} u(x,y)\right| \le \kappa\exp(-\gamma y), \forall (x,y) \in D, \tag{7.14}$$

for $0 \le i \le l+1$.

---

[4] If g^* does not satisfy (2.9b), e.g. $g^*(1) \neq 0$ with $f^* = 0$, we then find

(7.6)
$$\frac{\partial u}{\partial x} = \sqrt{\frac{2}{\pi}} \left[\int_{y/\sqrt{2(1-x)}}^{\infty} \exp\left(-\frac{t^2}{2}\right) g^{*\prime}\left(x + \frac{y^2}{2t^2}\right) dt - y(2(1-x))^{-3/2} g^*(1) \exp\left(-\frac{y^2}{4(1-x)}\right) \right],$$

see [26]. Note that the second term in (7.6) does not belong to $C^0(\bar{\Omega})$.

For $m \geq 1$, we treat repeatedly the case where $m = 2k$ and where $m = 2k+1$. Firstly, for $m = 2k$, $k \geq 1$, differentiating equation (2.10) i times in x, we then find

$$\frac{\partial^{i+2k}}{\partial x^i \partial y^{2k}} u = (-1)^k \frac{\partial^{i+k}}{\partial x^{i+k}} u + \sum_{s=0}^{k-1} (-1)^{s+1} \frac{\partial^{i+2k-s-2}}{\partial x^{i+s} \partial y^{2(k-s-1)}} f^*; \tag{7.15}$$

note that $i + k \leq l + 1 - k \leq l + 1$, and hence

$$\left| \frac{\partial^{i+2k}}{\partial x^i \partial y^{2k}} u \right| \leq \left| \frac{\partial^{i+k}}{\partial x^{i+k}} u \right| + \sum_{s=0}^{k-1} \left| \frac{\partial^{i+2k-s-2}}{\partial x^{i+s} \partial y^{2(k-s-1)}} f^* \right|. \tag{7.16}$$

We then easily find from (2.15) and (7.14) that the estimates (2.16) with $m = 2k$ hold.

If $m = 2k+1$, $k \geq 0$, we proceed as follows. Firstly we find similarly

$$\begin{aligned} \frac{\partial^{1+i}}{\partial x^i \partial y} I_1 &= \frac{\partial^{1+i}}{\partial y \partial x^i} I_1 \\ &= \sqrt{\frac{2}{\pi}} \int_{y/\sqrt{2(1-x)}}^{\infty} \exp\left(-\frac{t^2}{2}\right) \frac{\partial^{1+i}}{\partial x^{1+i}} g^* \left(x + \frac{y^2}{2t^2}\right) \frac{y}{t^2} dt, \end{aligned} \tag{7.17}$$

for $0 \leq i \leq l$. Since

$$\left| \frac{\partial^{1+i}}{\partial x^{1+i}} g^* \left(x + \frac{y^2}{2t^2}\right) \right| \leq \kappa, \tag{7.18}$$

we find

$$\begin{aligned} \left| \frac{\partial^{1+i}}{\partial y \partial x^i} I_1 \right| &\leq \kappa y \int_{y/\sqrt{2(1-x)}}^{\infty} \exp\left(-\frac{t^2}{2}\right) d\left(-t^{-1}\right) \\ &\leq \text{(integrating by parts)} \\ &\leq \kappa \exp\left(-\frac{y^2}{4}\right) + \kappa y \int_{y/\sqrt{2(1-x)}}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt \\ &\leq \kappa \exp\left(-\frac{(y-1)^2 - 1}{4}\right) \exp\left(-\frac{y}{2}\right) \\ &\quad + \kappa y \int_{y/\sqrt{2(1-x)}}^{\infty} \exp\left(-\frac{(t-1)^2 - 1}{2}\right) \exp(-t)\, dt \\ &\leq \kappa \exp\left(-\frac{y}{2}\right) + \kappa y \int_{y/\sqrt{2(1-x)}}^{\infty} \exp(-t)\, dt \\ &\leq \kappa \exp\left(-\frac{y}{2}\right) + \kappa y \exp\left(-\frac{y}{\sqrt{2}}\right) \leq \kappa \exp\left(-\frac{y}{2}\right), \end{aligned} \tag{7.19}$$

for $0 \leq i \leq l$. We also find

$$\begin{aligned} \frac{\partial^{i+1}}{\partial x^i \partial y} I_2 = \frac{\partial^{i+1}}{\partial y \partial x^i} I_2 = \frac{1}{2\sqrt{\pi}} \int_0^{1-x} \int_0^{\infty} \frac{1}{\sqrt{s}} \frac{\partial}{\partial y} \bigg\{ &\exp\left[-\frac{(y-t)^2}{4s}\right] \\ &- \exp\left[-\frac{(y+t)^2}{4s}\right] \bigg\} \frac{\partial^i}{\partial x^i} f^*(x+s, t) dt ds, \end{aligned} \tag{7.20}$$

and

$$\left| \frac{\partial^{1+i}}{\partial y \partial x^i} I_2 \right| \leq \kappa \left[\int_0^{1-x} \frac{1}{\sqrt{s}} \cdot I_3 ds \right], \tag{7.21}$$

where

$$
\begin{aligned}
I_3 &= \left|\int_0^\infty \frac{\partial}{\partial y}\left\{\exp\left[-\frac{(y-t)^2}{4s}\right]-\exp\left[-\frac{(y+t)^2}{4s}\right]\right\}\frac{\partial^i}{\partial x^i}f^*(x+s,t)dt\right| \\
&= \left|\int_0^\infty \frac{\partial}{\partial t}\left\{-\exp\left[-\frac{(y-t)^2}{4s}\right]-\exp\left[-\frac{(y+t)^2}{4s}\right]\right\}\frac{\partial^i}{\partial x^i}f^*(x+s,t)dt\right| \\
&= (\text{integrating by parts}) \leq 2\exp\left[-\frac{y^2}{4s}\right]\left|\frac{\partial^i}{\partial x^i}f^*(x+s,0)\right| \\
&\quad + \int_0^\infty \left\{\exp\left[-\frac{(y-t)^2}{4s}\right]+\exp\left[-\frac{(y+t)^2}{4s}\right]\right\}\left|\frac{\partial^{1+i}}{\partial t\partial x^i}f^*(x+s,t)\right|dt \\
&\leq (\text{by } (2.15)) \leq \kappa\exp(-\gamma y)+\kappa\int_0^\infty \exp\left(-\frac{(y-t)^2}{4}\right)\exp\left(-\gamma t\right)dt \\
&\leq (\text{similarly to } (7.13)) \leq \kappa\exp\left(-\gamma y\right).
\end{aligned}
$$

Hence, we conclude

$$
\left|\frac{\partial^{1+i}}{\partial y\partial x^i}u\right| \leq \kappa\exp\left(-\gamma y\right), \text{ for } 0\leq i\leq l. \tag{7.22}
$$

Differentiating (7.15) in y, we easily find

$$
\frac{\partial^{i+2k+1}}{\partial x^i\partial y^{2k+1}}u = (-1)^k\frac{\partial^{i+k+1}}{\partial y\partial x^{i+k}}u+\sum_{s=0}^{k-1}(-1)^{s+1}\frac{\partial^{i+2k-s-1}}{\partial x^{i+s}\partial y^{2(k-s)-1}}f^*; \tag{7.23}
$$

note that $i+k\leq l-k\leq l$, and hence

$$
\left|\frac{\partial^{i+2k+1}}{\partial x^i\partial y^{2k+1}}u\right| \leq \left|\frac{\partial^{i+k+1}}{\partial y\partial x^{i+k}}u\right|+\sum_{s=0}^{k-1}\left|\frac{\partial^{i+2k-s-1}}{\partial x^{i+s}\partial y^{2(k-s)-1}}f^*\right|. \tag{7.24}
$$

We then easily find from (2.15) and (7.22) that the estimates (2.16) with $m=2k+1$ hold. Hence, the estimates (2.16) follow. □

Proof of Lemma 4.1. As in the proof of Lemma 2.1, we set

$$
\tilde{x} = 1-x. \tag{7.25}
$$

Then we have the following classical heat equation in a semi-strip D:

$$
\frac{\partial u}{\partial \tilde{x}}-\frac{\partial^2 u}{\partial y^2} = \epsilon^{1/2}f^*(1-\tilde{x},y), \text{ for } (\tilde{x},y)\in D, \tag{7.26a}
$$

$$
\frac{\partial u}{\partial y}(\tilde{x},0) = \epsilon^{1/2}g^*(1-\tilde{x}),\ 0<\tilde{x}<1, \tag{7.26b}
$$

$$
u(\tilde{x},y)\to 0 \text{ as } y\to\infty,\ 0<\tilde{x}<1, \tag{7.26c}
$$

$$
u(\tilde{x}=0,y)=0. \tag{7.26d}
$$

From Theorem 20.3.2 in [2], we find that the solution u is unique and it is explicitly expressed as:

$$
u(\tilde{x},y) = I_4+I_5, \tag{7.27a}
$$

with

$$
I_4 = -2\epsilon^{1/2}\int_0^{\tilde{x}} K(\tilde{x}-\tau,y)g^*(1-\tau)d\tau, \tag{7.27b}
$$

$$
I_5 = \epsilon^{1/2}\int_0^{\tilde{x}}\int_0^\infty N(\tilde{x}-\tau,y;t)f^*(1-\tau,t)dtd\tau, \tag{7.27c}
$$

where the fundamental solution $K = K(\tilde{x}, y)$ of the heat equation is as in (7.3d) and the Green function $N = N(\tilde{x}, y; t)$ is

$$N(\tilde{x}, y; t) = K(\tilde{x}, y - t) + K(\tilde{x}, y + t). \tag{7.27d}$$

Then

$$\begin{aligned} I_4 &= -2\epsilon^{1/2} \int_0^{\tilde{x}} \frac{1}{\sqrt{4\pi(\tilde{x} - \tau)}} \exp\left(-\frac{y^2}{4(\tilde{x} - \tau)}\right) g^*(1 - \tau) d\tau \\ &= (\text{setting } t = \tilde{x} - \tau,\ \tilde{x} = 1 - x) \\ &= -\frac{\epsilon^{1/2}}{\sqrt{\pi}} \int_0^{1-x} \frac{1}{\sqrt{t}} \exp\left(-\frac{y^2}{4t}\right) g^*(x + t) dt; \end{aligned} \tag{7.28a}$$

we also find
(7.28b)

$$\begin{aligned} I_5 &= \epsilon^{1/2} \int_0^{\tilde{x}} \int_0^{\infty} \frac{1}{\sqrt{4\pi(\tilde{x} - \tau)}} \left\{ \exp\left[-\frac{(y - t)^2}{4(\tilde{x} - \tau)}\right] + \exp\left[-\frac{(y + t)^2}{4(\tilde{x} - \tau)}\right] \right\} f^*(1 - \tau, t) dt d\tau \\ &= (\text{setting } s = \tilde{x} - \tau,\ \tilde{x} = 1 - x) \\ &= \frac{\epsilon^{1/2}}{2\sqrt{\pi}} \int_0^{1-x} \int_0^{\infty} \frac{1}{\sqrt{s}} \left\{ \exp\left[-\frac{(y - t)^2}{4s}\right] + \exp\left[-\frac{(y + t)^2}{4s}\right] \right\} f^*(x + s, t) dt ds; \end{aligned}$$

hence (4.9) follows.

Similarly to the proof in Lemma 2.1, we find that conditions (4.6) are sufficient for $u \in C^l(\bar{D})$ and for the estimates (4.10) and (4.12), we can also proceed similarly; we use here the fact that since g^* is sufficiently smooth,

$$\begin{aligned} \left| \int_0^{1-x} \frac{1}{\sqrt{t}} \exp\left(-\frac{y^2}{4t}\right) \frac{\partial^i}{\partial x^i} g^*(x + t) dt \right| &\leq \kappa \left[\int_0^1 \frac{1}{\sqrt{t}} dt \right] \exp\left(-\frac{y^2}{4}\right) \\ &\leq (\text{by } (7.10)) \leq \kappa \exp(-\gamma y), \end{aligned} \tag{7.29}$$

and

$$\frac{\partial}{\partial y} I_4 = \epsilon^{1/2} I_1, \tag{7.30}$$

where I_1 is as in (7.3b). □

References

[1] Richard L. Burden and J.Douglas Faires, *Numerical analysis.* Brooks/Cole, CA, 2001.

[2] J. R. Cannon, *The one-dimensional heat equation.* Encyclopedia of Mathematics and Its Application, Vol. 23, Addison-Wesley, Reading, MA, 1984.

[3] W. Cheng and R. Temam, *Numerical approximation of one-dimensional stationary diffusion equations with boundary layers.* Computers and Fluids. 31(2002), pp 453-466.

[4] W. Cheng, R. Temam and X. Wang, *New approximation algorithms for a class of partial differential equations displaying boundary layer behavior.* Methods and Applications of Analysis. 7(2000), no. 2, pp 363-390.

[5] P. G. Ciarlet, *The finite element method for elliptic problems.* North-Holland, New York, 1978.

[6] W. Eckhaus, *Boundary layers in linear elliptic singular perturbations.* SIAM Rev. 14(1972), pp 225-270.

[7] W. Eckhaus and E. M. De Jager, *Asymptotic solutions of singular perturbation problems for linear differential equations of elliptic type.* Arch. Rational Mech. Anal. 23(1966), pp 26-86.

[8] L. C. Evans, *Partial differential equations.* AMS, Providence, Rhode Island, 1998.

[9] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order.* Springer- Verlag, New York, 1998.

[10] M. H. Holmes, *Introduction to perturbation methods.* Springer- Verlag, New York, 1995.

[11] C. Jung, *Numerical approximation of two-dimensional convection-diffusion equations with boundary layers.* Numer. Methods Partial Differential Eq. 21(2005), pp 623-648.

[12] C. Jung, *Numerical approximation of convection-diffusion equations in a channel using boundary layer elements*. Applied Numerical Mathematics. To appear.
[13] C. Jung, *Numerical approximation of two-dimensional reaction-diffusion equations using boundary layer elements*. In preparation.
[14] C. Jung and R. Temam, *Construction of boundary layer elements for singularly perturbed convection-diffusion equations and L^2- stability analysis*. Submitted.
[15] C. Jung and R. Temam, *Double-gyre problems and singular perturbation with numerical applications*. In preparation.
[16] C. Jung and R. Temam, *On parabolic boundary layers for convection-diffusion equations in a channel: analysis and numerical applications*. Journ. of Scientific Computing. To appear.
[17] K. R. Knaub and R. E. O'Malley, *The motion of internal layers in singularly perturbed advection-diffusion-reaction equations*. Stud. Appl. Math. 112(2004), pp 1-15.
[18] J. L. Lions, *Perturbations singulières dans les problèmes aux limites et en contrôle optimal*. (French) Lecture Notes in Mathematics, Vol. 323. Springer-Verlag, Berlin-New York, 1973.
[19] S. Larsson and V. Thomée, *Partial differential equations with numerical methods*. Springer-Verlag, Berlin, 2003.
[20] R. E. O'Malley, *Singular perturbation analysis for ordinary differential equations*. Communications of the Mathematical Institute, Rijksuniversiteit Utrecht, 5. Rijksuniversiteit Utrecht, Mathematical Institute, Utrecht, 1977.
[21] R. E. O'Malley, *Naive singular perturbation theory*. Special issue in memory of Richard Weiss. Math. Models Methods Appl. Sci. 11(2001), pp 119-131.
[22] R. Plato, *Concise numerical mathematics*. AMS, Providence, Rhode Island, 2003.
[23] H.-G.Roos, M. Stynes and L. Tobiska, *Numerical methods for singularly perturbed differential equations*. Springer- Verlag, Berlin, 1996.
[24] M. H. Schultz, *Spline analysis*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1973.
[25] M. Stynes, *Steady-state convection-diffusion problems*. Acta Numerica. (2005), pp 445-508.
[26] S. Shih and R. B. Kellogg, *Asymptotic analysis of a singular perturbation problem*. Siam J. Math. Anal. 18(1987), pp 1467-1511.
[27] R. Temam, *Behaviour at time $t = 0$ of the solutions of semi-linear evolution equations*, Journal of Differential Equations, 43(1982), pp 73-92.
[28] R. Temam, *Infinite dimensional dynamical systems in mechanics and physics*, Springer-Verlag, New-York, Applied Mathematical Sciences Series, Vol. 68, 1997.
[29] M.I. Vishik and L.A. Lyusternik, *Regular degeneration and boundary layer for linear differential equations with small parameter*, Uspekki Mat. Nauk. 12(1957), pp 3-122.

[1] The Institute for Scientific Computing and Applied Mathematics, Indiana University, Bloomington, IN 47405, USA; [2] Laboratoire d'Analyse Numérique, Université Paris-Sud, Bâtiment 425, 91405 Orsay, France

E-mail: `chajung@indiana.edu and temam@indiana.edu`

URL: `http://mypage.iu.edu/~chajung/ and http://mypage.iu.edu/~temam/`