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L^∞-ERROR ESTIMATES AND SUPERCONVERGENCE IN MAXIMUM NORM OF MIXED FINITE ELEMENT METHODS FOR NONFICKIAN FLOWS IN POROUS MEDIA

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Abstract. On the basis of the estimates for the regularized Green's functions with memory terms, optimal order L^∞-error estimates are established for the nonFickian flow of fluid in porous media by means of a mixed Ritz-Volterra projection. Moreover, local L^∞-superconvergence estimates for the velocity along the Gauss lines and for the pressure at the Gauss points are derived for the mixed finite element method, and global L^∞-superconvergence estimates for the velocity and the pressure are also investigated by virtue of an interpolation post-processing technique. Meanwhile, some useful a-posteriori error estimators are presented for this mixed finite element method.

Key Words. NonFickian flow, mixed finite element methods, the mixed Ritz-Volterra projection, Green's functions, error estimates and superconvergence

1. Introduction

The nonFickian flow of fluid in porous media can be modelled by an integro-differential equation which seeks $u = u(x,t)$ such that

$$
\begin{aligned}
u_t &= \nabla \cdot \sigma + cu + f && \text{in } \Omega \times J,\\
\sigma &= A(t) \cdot \nabla u - \int_0^t B(t,s) \cdot \nabla u(s) ds && \text{in } \Omega \times J,\\
u &= g && \text{on } \partial\Omega \times J,\\
u &= u_0(x) && x \in \Omega,\ t = 0,
\end{aligned}
\tag{1.1}
$$

where $\Omega \subset R^d$ $(d = 2,3)$ is an open bounded domain with smooth boundary $\partial\Omega$, $J = (0,T)$ with $T > 0$, $A(t) = A(x,t)$ and $B(t,s) = B(x,t,s)$ are two 2×2 or 3×3 matrices, and A is positive definite, $c \leq 0$, f, g and u_0 are known smooth functions. This kind of flow is complicated by the history effect characterizing various mixing length growth of the flow, which has been investigated, for example, in [9, 10] and references cited therein.

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The numerical approximations of the problem (1.1) are available in extensive literature. See, for instance, [2, 3, 12, 13, 16, 14, 15, 20, 21, 22], where some optimal order error estimates and superconvergence have been established.

In the present paper, the solutions of (1.1) are approximated by mixed finite element methods [14, 15, 16]. Optimal order L^∞-error estimates are obtained by employing a mixed Ritz-Volterra projection introduced in [16]. In addition, local L^∞-superconvergence estimates for the velocity along the Gauss lines and for the pressure at the Gauss points are derived, and with the aid of an interpolation post-processing method global L^∞-superconvergence estimates are also derived for the velocity and the pressure approximations. As a result of the global superconvergence, a-posteriori error indicators of the mixed finite element method are presented in the paper.

Compared with [16], where the optimal and superconvergence estimates of the mixed finite element method in L^2-norm have been discussed for the problem (1.1), the key point of the present paper is the introduction of the regularized Green's functions with memory terms and the establishment of the various estimates for them and their mixed finite element approximations, which will play an important role in the forthcoming analysis in deriving the above optimal and superconvergence L^∞-error estimates. As a result, the methodology and the techniques used in this paper are quite different from those in [16].

The paper is organized in the following manner. In Section 2, we give the approximate sub-space and the approximate problem. Two regularized Green's functions and a Ritz-Volterra projection with memory terms for the mixed form for the problem (1.1) are introduced in Section 3. Also, in Section 3 the L^1-error estimates and related estimates for the mixed finite element approximations of the regularized Green's functions are stated, and the L^∞-error estimates for the mixed Ritz-Volterra projection are established. In Section 4, optimal order error estimates in maximum norm are given for the mixed finite element approximations. Section 5 is devoted to the local and global L^∞-superconvergence analysis of the mixed finite element method, by which some a-posteriori error estimators are obtained for the mixed finite element method. Finally, the L^1-error estimates and related estimates for the mixed finite element approximations of the regularized Green's functions are proved in Section 6.

2. The mixed finite element method

In this section, we give the mixed finite element approximate scheme for the parabolic integro-differential equation (1.1). For simplicity, the method will be presented on plane domains.

Let $W := L^2(\Omega)$ be the standard L^2 space on Ω with norm $\|\cdot\|_0$. Denote by

$$\mathbf{V} := H(\mathrm{div}, \Omega) = \left\{\sigma \in (L^2(\Omega))^2 \mid \nabla \cdot \sigma \in L^2(\Omega)\right\}$$

the Hilbert space equipped with the following norm:

$$\|\sigma\|_{\mathbf{V}} := \left(\|\sigma\|_0^2 + \|\nabla \cdot \sigma\|_0^2\right)^{\frac{1}{2}}.$$

There are several ways to discretize the problem (1.1) based on the variables σ and u; each method corresponds to a particular variational form of (1.1) [14, 22].

Let T_h be a finite element partition of Ω into triangles or quadrilaterals which is quasi-uniform. Let $\mathbf{V}_h \times W_h$ denote a pair of finite element spaces satisfying the Brezzi-Babŭska condition. Although there are now several choices for $\mathbf{V}_h$ and W_h, here we only consider the Raviart-Thomas elements of order $k \geq 0$ [24]. The extension to other stable elements can be made without any difficulty.

Recall from [14] that the weak mixed formulation of (1.1) is given by finding $(u, \sigma) \in W \times \mathbf{V}$ such that

$$\begin{aligned} &(u_t, w) - (\nabla \cdot \sigma, w) - (cu, w) = (f, w), && w \in W, \\ &(\alpha\sigma, \mathbf{v}) + \int_0^t (M(t,s)\sigma(s), \mathbf{v})ds + (\nabla \cdot \mathbf{v}, u) = \langle g, \mathbf{v} \cdot \mathbf{n} \rangle, && \mathbf{v} \in \mathbf{V}, \\ &u(0,x) = u_0(x) \quad \text{in } L^2(\Omega), \end{aligned} \tag{2.1}$$

where $\alpha = A^{-1}(t)$, $M(t,s) = R(t,s)A^{-1}(s)$ and $R(t,s)$ is the resolvent of the matrix $A^{-1}(t)B(t,s)$ and is given by

$$R(t,s) = A^{-1}(t)B(t,s) + \int_s^t A^{-1}(t)B(t,\tau)\, R(\tau,s)ds, \quad t > s \geq 0.$$

Here $\langle \cdot, \cdot \rangle$ indicates the L^2-inner product on $\partial\Omega$.

The corresponding semi-discrete version is to seek a pair $(u_h, \sigma_h) \in W_h \times \mathbf{V}_h$ such that

$$\begin{aligned} &(u_{h,t}, w_h) - (\nabla \cdot \sigma_h, w_h) - (cu_h, w_h) = (f, w_h), && w_h \in W_h, \\ &(\alpha\sigma_h, \mathbf{v}_h) + \int_0^t (M(t,s)\sigma_h(s), \mathbf{v}_h)ds + (u_h, \nabla \cdot \mathbf{v}_h) = \langle g, \mathbf{n} \cdot \mathbf{v}_h \rangle, && \mathbf{v}_h \in \mathbf{V}_h. \end{aligned} \tag{2.2}$$

The discrete initial condition $u_h(0,x) = u_{0,h}$, where $u_{0,h} \in W_h$ is some appropriately chosen approximation of the initial data $u_0(x)$, should be added to (2.2) for starting. The pair (u_h, σ_h) is a semi-discrete approximation of the true solution of (1.1) in the finite element space $W_h \times \mathbf{V}_h$ [1, 6, 16, 14, 15], where $\sigma_h(0,x)$ is chosen to satisfy the equation (2.2) with $t = 0$; namely, it is related to $u_{0,h}$ as follows:

$$(\alpha\sigma_h(0), \mathbf{v}_h) + (u_{0,h}, \nabla \cdot \mathbf{v}_h) = \langle g_0, \mathbf{n} \cdot \mathbf{v}_h \rangle, \tag{2.3}$$

where $g_0 = g(0,x)$ is the initial value of the boundary data.

From (2.1) and (2.2) we derive the following mixed finite element error equation:

$$\begin{aligned} &(u_t - u_{h,t}, w_h) - (\nabla \cdot (\sigma - \sigma_h), w_h) - (c(u - u_h), w_h) = 0, && w_h \in W_h, \\ &(\alpha(\sigma - \sigma_h), \mathbf{v}_h) + \int_0^t (M(t,s)(\sigma - \sigma_h)(s), \mathbf{v}_h)ds + (u - u_h, \nabla \cdot \mathbf{v}_h) = 0, && \mathbf{v}_h \in \mathbf{V}_h. \end{aligned} \tag{2.4}$$

Throughout the paper, we often need the following Raviart-Thomas projection [7, 24].

$$\Pi_h^k \times P_h^k : \mathbf{V} \times W \to \mathbf{V}_h \times W_h,$$

which has the properties:

(i) P_h^k is the $L^2(\Omega)$ projection.

(ii) Π_h^k and P_h^k satisfy

$$(\nabla \cdot (\sigma - \Pi_h^k\sigma), w_h) = 0,\ w_h \in W_h \quad \text{and} \quad (\nabla \cdot \mathbf{v}_h, u - P_h^k u) = 0,\ \mathbf{v}_h \in \mathbf{V}_h. \tag{2.5}$$

(iii) the following approximation properties hold

$$\begin{aligned} &\|\sigma - \Pi_h^k\sigma\|_{0,p} \leq Ch^r\|\sigma\|_{r,p}, && 1 \leq r \leq k+1, && 1 \leq p \leq \infty, \\ &\|\nabla \cdot (\sigma - \Pi_h^k\sigma)\|_{-s,p} \leq Ch^{r+s}\|\nabla \cdot \sigma\|_{r,p}, && 0 \leq r,\ s \leq k+1, && 1 \leq p \leq \infty, \\ &\|u - P_h^k u\|_{-s,p} \leq Ch^{r+s}\|u\|_{r,p}, && 0 \leq r,\ s \leq k+1, && 1 \leq p \leq \infty. \end{aligned} \tag{2.6}$$

Remark 2.1. Π_h^k *is defined on a dense subspace of* $\mathbf{V}$.

3. The mixed Ritz-Volterra projection and its L^∞-error estimates

In this section, we consider optimal order error estimates and superconvergence in L^∞-norm for the mixed Ritz-Volterra projection. It is well-known that the regularized Green's function plays an essential role in the analysis of maximum norm error estimates and superconvergence for finite element methods and mixed finite element methods of elliptic equations [8, 11, 19, 26, 27, 28] and parabolic equations [19]. For the finite element method of parabolic integro-differential equations, maximum norm error estimates and superconvergence have been obtained in [20, 21] using the modified regularized Green's function with memory term. Here we consider the mixed finite element approximations for parabolic equations with memory, and it is expected that certain modification form of the standard regularized Green's function with memory should be introduced, analyzed and used in our analysis.

First, let us define the following two linear operators $M*$ and $M**$ for any smooth function $f(t)$ defined on $(0,T)$ by

$$(M*f)(t) := \int_0^t M(t,s)f(s)ds \quad \text{and} \quad (M**f)(t) := \int_t^T M(s,t)f(s)ds.$$

Then, from exchanging the order of integration we have

Lemma 3.1. *There holds*

$$\langle M*f, g\rangle_T := \int_0^T M*f(t)\; g(t)dt = \int_0^T f(t)\; M**g(t)dt := \langle f, M**g\rangle_T\,.$$

Lemma 3.2. *Assume that* $f(t),\ g(t) \in L^1(0,T^*)$ *and there exists* $C>0$ *such that for any non-negative* $\phi(t) \in C^\infty(0,T)$,

$$\left|\int_0^T f(t)\phi(t)dt\right| \le C\left|\int_0^T g(t)(1+\phi(t))dt\right|, \quad 0 \le T \le T^*.$$

Then, we have

$$|f(t)| \le C\left|g(t) + \int_0^t g(s)ds\right|, \quad \forall t \in (0,T),\ a.e.$$

Especially,

$$|f(t)| \le C|g(t)|, \quad \forall t \in (0,T),\ a.e. \quad if$$
$$\left|\int_0^T f(t)\phi(t)dt\right| \le C\left|\int_0^T g(t)\phi(t)dt\right|.$$

Proof. Take $\mu > 0$ and let

$$\phi_\mu(t,t_0) = \begin{cases} (C_\mu)^{-1}\exp\left(-\frac{\mu^2}{\mu^2-|t-t_0|^2}\right), & |t-t_0|<\mu, \\ 0, & |t-t_0| \ge \mu, \end{cases}$$

where t_0 is any fixed point in $(0,T)$ and $C_\mu := \mu\int_{|t|<1}\exp(-\frac{1}{1-t^2})dt$. We see easily that for almost all $t_0 \in (0,T)$,

$$f(t_0) = \lim_{\mu\to 0}\int_0^T f(t)\phi_\mu(t,t_0)dt, \quad f \in C^\infty(0,T).$$

Thus, if we take $f_n(t) \in C^\infty(0,T)$ such that $f_n(t) \to f(t)$ as $n \to \infty$ in $L^1(0,T)$, then the result is true for all $f_n(t)$. Therefore, it is true for $f(t)$ via a limiting procedure. $\square$

Now let us introduce some notations for the use later. For an arbitrary point $z_0 \in \bar{\Omega}$, let

$$\beta(z, z_0) := (|z - z_0|^2 + \theta^2)^{1/2}$$

be the weight function used in [25, 26, 28], where $z = (x, y) \in R^2$, $\theta = \gamma h$, and γ is a positive number chosen appropriately. Moreover, as usual, for any $\alpha \in R$ we define a weighted norm by

$$||u||^2_{\beta^\alpha, Q} := \int_Q \beta^\alpha u^2 dQ,$$

and $||\cdot||_{\beta^\alpha}$ is the weighted norm for $Q = \Omega$. Then, we have [26, 28]

$$\int_\Omega \beta^{-2} d\Omega \leq C|\log h|. \tag{3.1}$$

Next we shall define two regularized Green's functions with memory terms for the problem (1.1) in mixed form in the fashion analogous to that employed earlier for Galerkin methods [28]. Our results concerning the regularized Green's functions and their mixed finite element approximations are very useful for establishing L^∞-error estimates and superconvergence in maximum norm for the mixed finite element solution of (1.1).

For simplicity, we assume that $c = 0$. Thus, for an arbitrary point $z_0 \in \bar{\Omega}$ the first pair of modified regularized Green's function $(\mathbf{G}_1, \lambda_1) = (\mathbf{G}_1(z, z_0), \lambda_1(z, z_0))$ with memory is defined as the solution of the following system:

$$\begin{array}{rcll} \alpha \mathbf{G}_1 + M ** \mathbf{G}_1 - \nabla \lambda_1 &=& 0, & \text{in } \Omega \times (0,T), \\ \text{div}\mathbf{G}_1 &=& \delta_1^h \phi_1(t), & \text{in } \Omega \times (0,T), \\ \lambda_1 &=& 0, & \text{on } \partial\Omega \times (0,T), \end{array} \tag{3.2}$$

where $\phi_1(t) \in C^\infty(0,T)$, and $\delta_1^h = \delta_1^h(z, z_0) \in W_h$ is the regularized Dirac δ-function at any fixed point $z_0 \in \bar{\Omega}$ such that ([8, 11, 26, 27])

$$||w_h||_\infty \leq C|(w_h, \delta_1^h)|, \quad w_h \in W_h. \tag{3.3}$$

We also introduce the second pair of regularized Green's function $(\mathbf{G}_2, \lambda_2) = (\mathbf{G}_2(z, z_0),$
$\lambda_2(z, z_0))$ such that

$$\begin{array}{rcll} \alpha \mathbf{G}_2 + M ** \mathbf{G}_2 - \nabla \lambda_2 &=& \delta_2^h \phi_2(t), & \text{in } \Omega \times (0,T), \\ \text{div}\mathbf{G}_2 &=& 0, & \text{in } \Omega \times (0,T), \\ \lambda_2 &=& 0, & \text{on } \partial\Omega \times (0,T), \end{array} \tag{3.4}$$

where $\phi(t) \in C^\infty(0,T)$ and δ_2^h is either $(\delta_2^h, 0)$ or $(0, \delta_2^h)$ with δ_2^h being a regularized Dirac δ-function at z_0, which depends upon the needs of our analysis, such that an analogue of (3.3) is also valid for δ_2^h. In addition, δ_2^h, $\phi_1(t)$ and $\phi_2(t)$ are required to satisfy

$$\delta_2^h \geq 0, \ \int_\Omega \delta_2^h d\Omega = 1; \ \phi_i(t) \geq 0, \ \int_0^T \phi_i(t) dt \leq 1, \ i = 1, 2. \tag{3.5}$$

Now and in what follows of this paper, the domain Ω is assumed to be H^2-regular [7]. Therefore, it is not difficult to show (see, for example, $(3.6a) - (3.6d)$ in [26]) that the following result is true.

Theorem 3.1. *There exists a positive constant $C > 0$, independent of h, t, and $\phi_1(t)$, such that*

$$\begin{aligned}
&||\nabla \lambda_1||_0 \le C|\log h|^{1/2}(1+\phi_1(t)),\\
&||\nabla^2 \lambda_1||_0 \le Ch^{-1}(1+\phi_1(t)),\\
&||\nabla^2 \lambda_1||_{\beta^2} \le C|\log h|^{1/2}(1+\phi_1(t)),\\
&||\nabla^2 \lambda_1||_{L^1(\Omega)} \le C|\log h|(1+\phi_1(t)).
\end{aligned}$$

Our main results regarding error estimates between $(\mathbf{G}_1, \lambda_1)$ and $(\mathbf{G}_1^h, \lambda_1^h)$, and $(\mathbf{G}_2, \lambda_2)$ and $(\mathbf{G}_2^h, \lambda_2^h)$ are contained in the following two theorems.

Theorem 3.2. *Assume that $(\mathbf{G}_1, \lambda_1)$ and $(\mathbf{G}_1^h, \lambda_1^h)$ are the exact solution and the mixed finite element approximation of (3.2), respectively. Then, there exists a positive constant $C > 0$, independent of h, t, and ϕ_1, such that*

$$\begin{aligned}
&||\mathbf{G}_1^h - \mathbf{G}_1||_0 \le C(1+\phi_1(t)),\\
&||\mathbf{G}_1^h - \mathbf{G}_1||_{L^1(\Omega)} \le Ch|\log h|(1+\phi_1(t)),\\
&||\lambda_1^h - \lambda_1||_0 \le Ch|\log h|^{1/2}(1+\phi_1(t)).
\end{aligned}$$

Theorem 3.3. *Assume that $(\mathbf{G}_2, \lambda_2)$ and $(\mathbf{G}_2^h, \lambda_2^h)$ are the exact solution and the mixed finite element approximation of (3.4), respectively. Then, there exists a positive constant $C > 0$, independent of h, t, and ϕ_2, such that*

$$\begin{aligned}
&||\mathbf{G}_2^h - \mathbf{G}_2||_0 \le Ch^{-1}(1+\phi_2(t)),\\
&||\mathbf{G}_2^h - \mathbf{G}_2||_{L^1(\Omega)} \le C|\log h|^{1/2}(1+\phi_2(t)),\\
&||\lambda_2^h - \lambda_2||_0 \le C(1+\phi_2(t)),\\
&||\lambda_2||_0 \le C(1+|\log h|^{1/2})(1+\phi_2(t)),\\
&||\nabla \lambda_2||_0 \le Ch^{-1}(1+\phi_2(t)),\\
&||\nabla \lambda_2||_{L^1(\Omega)} \le C|\log h|(1+\phi_2(t)),\\
&||\nabla^2 \lambda_2||_{L^1(\Omega)} \le Ch^{-1}|\log h|^{1/2}(1+\phi_2(t)).
\end{aligned}$$

Remark 3.1. *We would like to point out that the estimate*

$$||\nabla^2 \lambda_2||_{L^1(\Omega)} \le Ch^{-1}\left(\log \frac{1}{h}\right)^{1/2}(1+\phi_2(t))$$

is not sharp, since it can be improved to

$$||\nabla^2 \lambda_2||_{L^1(\Omega)} \le Ch^{-1}(1+\phi_2(t)) \tag{3.6}$$

if the domain is smooth enough. A proof of (3.6) can be found in [25].

Remark 3.2. *The proofs of Theorems 3.2 and 3.3 will be postponed to Section 6 where the weighted norm estimates are used.*

Following the procedure for Theorems 3.3 and 3.4 in [26] together with the application of Gronwall's lemma, we can also obtain the following results to be used in the superconvergence analysis.

Theorem 3.4. *Assume that Ω is a plane rectangular domain and $q \in [1, \infty]$. Then, we have*

$$\begin{aligned}
||\mathbf{G}_1^h||_q &\le Ch^{min\{0, \frac{2}{q}-1\}}|\log h|^{1/2}(1+\phi_1(t))\\
||\mathbf{G}_1 - \mathbf{G}_1^h||_q &\le (C(q) + C|\log h|)\, h^{1-\frac{2}{p}}(1+\phi_1(t)),\ 1 < q < \infty,
\end{aligned}$$

where $p = \frac{q}{q-1}$ is the conjugate of q.

Theorem 3.5. *For $q \in [1, \infty]$, there hold*

$$\begin{aligned} ||\mathbf{G}_2^h||_q &\leq \begin{cases} Ch^{-\frac{2}{p}}|\log h|(1+\phi_2(t)), & 1 \leq q < 2, \\ Ch^{-\frac{2}{p}}(1+\phi_2(t)), & q \geq 2, \end{cases} \\ ||\mathbf{G}_2 - \mathbf{G}_2^h||_q &\leq \left(C(q) + C|\log h|^{1/2}\right) h^{-\frac{2}{p}}(1+\phi_2(t)),\ 1 < q < \infty, \end{aligned}$$

where $p = \frac{q}{q-1}$.

In the following we shall present the error estimates in the maximum norm for the mixed Ritz-Volterra projection. To this end, we first give its definition [16].

Definition 3.1 For $(u, \sigma) \in W \times \mathbf{V}$ we define a pair $(\bar{u}_h, \bar{\sigma}_h) : [0, T] \to \mathbf{W}_h \times \mathbf{V}_h$ such that

$$\begin{aligned} (\alpha(\sigma - \bar{\sigma}_h) + M * (\sigma - \bar{\sigma}_h), \mathbf{v}_h) + (u - \bar{u}_h, \mathrm{div}\mathbf{v}_h) &= 0, \quad \mathbf{v}_h \in \mathbf{V}_h \\ (\mathrm{div}(\sigma - \bar{\sigma}_h), w_h) &= 0, \quad w_h \in W_h, \end{aligned} \tag{3.7}$$

where $\alpha = A^{-1}$. The pair $(\bar{u}_h, \bar{\sigma}_h)$ is called the mixed Ritz-Volterra projection of (u, σ). It has been proved in [16] that the solution of (3.7) exists uniquely for a given pair (u, σ).

The following lemma is basic to the main results of this section.

Lemma 3.3. *Assume that $(\bar{u}_h, \bar{\sigma}_h)$ is the mixed Ritz-Volterra projection of $(u, \sigma) \in W \times \mathbf{V}$. Then we have*

$$\begin{aligned} \int_0^T (\bar{u}_h - P_h^k u, \delta_1^h)\phi_1(t)dt &= \int_0^T (\alpha(\sigma - \Pi_h^k \sigma) + M * (\sigma - \Pi_h^k \sigma),\ \mathbf{G}_1^h)dt, \\ \int_0^T (\bar{\sigma}_h - \Pi_h^k \sigma, \delta_2^h)\phi_2(t)dt &= \int_0^T (\alpha(\sigma - \Pi_h^k \sigma) + M * (\sigma - \Pi_h^k \sigma),\ \mathbf{G}_2^h)dt. \end{aligned}$$

Proof. It follows from (3.2) and its corresponding mixed finite element error equation to (2.4) that

$$(\bar{u}_h - P_h^k u, \delta_1^h \phi_1(t)) = (\bar{u}_h - P_h^k u, \mathrm{div}\mathbf{G}_1) = (\bar{u}_h - P_h^k u, \mathrm{div}\mathbf{G}_1^h).$$

Note that P_h^k is a local L^2-projection operator. Thus, we know from (2.5) that

$$(\bar{u}_h - P_h^k u, \delta_1^h \phi_1(t)) = (\bar{u}_h - u, \mathrm{div}\mathbf{G}_1^h)$$

which, together with (3.7), leads to

$$\begin{aligned} (\bar{u}_h - P_h^k u, \delta_1^h \phi_1(t)) &= (\alpha(\sigma - \bar{\sigma}_h) + M * (\sigma - \bar{\sigma}_h), \mathbf{G}_1^h) \\ &= (\alpha(\sigma - \Pi_h^k \sigma) + M * (\sigma - \Pi_h^k \sigma),\ \mathbf{G}_1^h) \\ &\quad + (\alpha(\Pi_h^k \sigma - \bar{\sigma}_h) + M * (\Pi_h^k \sigma - \bar{\sigma}_h),\ \mathbf{G}_1^h). \end{aligned}$$

Hence,

$$\begin{aligned} \int_0^T (\bar{u}_h - P_h^k u, \delta_1^h)\phi_1(t)dt &= \int_0^T (\alpha(\sigma - \Pi_h^k \sigma) + M * (\sigma - \Pi_h^k \sigma),\ \mathbf{G}_1^h)dt \\ &\quad + \int_0^T (\alpha(\Pi_h^k \sigma - \bar{\sigma}_h) + M * (\Pi_h^k \sigma - \bar{\sigma}_h),\ \mathbf{G}_1^h)dt \\ &:= K_1 + K_2. \end{aligned} \tag{3.8}$$

However, it follows from Lemma 3.1 and the mixed finite element approximation of (3.2) as well as Green's formula that

$$\begin{aligned} K_2 &= \int_0^T (\alpha \mathbf{G}_1^h + M ** \mathbf{G}_1^h,\ \Pi_h^k \sigma - \bar{\sigma}_h) dt \\ &= \int_0^T (\nabla \lambda_1^h,\ \Pi_h^k \sigma - \bar{\sigma}_h) dt \\ &= -\int_0^T (\lambda_1^h, \mathrm{div}(\Pi_h^k \sigma - \bar{\sigma}_h)) dt, \end{aligned}$$

which, together with (2.5) and (3.7), yields

$$K_2 = -\int_0^T (\lambda_1^h, \mathrm{div}(\Pi_h^k \sigma - \sigma)) dt - \int_0^T (\lambda_1^h, \mathrm{div}(\sigma - \bar{\sigma}_h)) dt = 0.$$

Thus, from (3.8) we know that the first identity in Lemma 3.3 is true.

To prove the second identity, we use (3.4) and its mixed finite element error equation to see that

$$(\bar{\sigma}_h - \Pi_h^k \sigma, \delta_2^h) \phi_2(t) = (\alpha \mathbf{G}_2^h + M ** \mathbf{G}_2^h, \bar{\sigma}_h - \Pi_h^k \sigma) + (\lambda_2^h, \mathrm{div}(\bar{\sigma}_h - \Pi_h^k \sigma)).$$

Thus, by means of Lemma 3.1, (2.5) and (3.7) we have

$$\begin{aligned} \int_0^T (\bar{\sigma}_h - \Pi_h^k \sigma, \delta_2^h) \phi_2(t) dt &= \int_0^T (\alpha(\bar{\sigma}_h - \Pi_h^k \sigma) + M * (\bar{\sigma}_h - \Pi_h^k \sigma),\ \mathbf{G}_2^h) dt \\ &\quad + \int_0^T (\lambda_2^h, \mathrm{div}(\bar{\sigma}_h - \sigma)) dt + \int_0^T (\lambda_2^h, \mathrm{div}(\sigma - \Pi_h^k \sigma)) dt \\ &= \int_0^T (\alpha(\bar{\sigma}_h - \Pi_h^k \sigma) + M * (\bar{\sigma}_h - \Pi_h^k \sigma),\ \mathbf{G}_2^h) dt \\ &= \int_0^T (\alpha(\sigma - \Pi_h^k \sigma) + M * (\sigma - \Pi_h^k \sigma), \mathbf{G}_2^h) dt \\ &\quad + \int_0^T (u - \bar{u}_h, \mathrm{div} \mathbf{G}_2^h) dt \\ &= \int_0^T (\alpha(\sigma - \Pi_h^k \sigma) + M * (\sigma - \Pi_h^k \sigma),\ \mathbf{G}_2^h) dt, \end{aligned}$$

where $\mathrm{div} \mathbf{G}_2^h = 0$ has been used. This completes the proof. □

We are now ready to show the maximum norm error estimate for the mixed Ritz-Volterra projection. First, we consider it for $\bar{u}_h - P_h^k u$.

Theorem 3.6. *Let $(\bar{u}_h, \bar{\sigma}_h)$ be the Ritz-Volterra projection of (u, σ). Then, there exists a constant $C > 0$, independent of h and t, such that*

$$||\bar{u}_h - P_h^k u||_\infty \le \begin{cases} Ch|\log h|(|||\sigma - \Pi_h^0 \sigma|||_\infty + |\log h|^{-1/2} |||(I - P_h^0)\nabla \cdot \sigma|||_0), & k = 0, \\ C(|||\sigma - \Pi_h^0 \sigma|||_0 + h|\log h|^{1/2} |||(I - P_h^0)\nabla \cdot \sigma|||_0), & k = 0, \\ Ch|\log h|(|||\sigma - \Pi_h^k \sigma|||_\infty + h|||(I - P_h^k)\nabla \cdot \sigma|||_\infty), & k \ge 1, \end{cases}$$

where $|||u|||_{r,p} := ||u(t)||_{r,p} + \int_0^t ||u(s)||_{r,p} ds,\ -\infty \le r \le \infty,\ 1 \le p \le \infty,\ t > 0$. As usual, $|||u|||_{r,p}$ is simply denoted by $|||u|||_r$ when $p = 2$.

Proof. For any point $z_0 \in \bar{\Omega}$, let δ_1^h be the regularized Dirac δ-function associated with this point z_0, and then we find from Lemma 3.3 that

$$\begin{aligned}\int_0^T (\bar{u}_h - P_h^k u, \delta_1^h)\phi_1(t)dt &= \int_0^T \big(\alpha(\sigma - \Pi_h^k\sigma) + M * (\sigma - \Pi_h^k\sigma),\ \mathbf{G}_1^h - \mathbf{G}_1\big)\, dt \\ &\quad + \int_0^T \big(\alpha(\sigma - \Pi_h^k\sigma) + M * (\sigma - \Pi_h^k\sigma),\ \mathbf{G}_1\big)\, dt \\ &:= K_{11} + K_{22}.\end{aligned}$$

It is easy to see from Lemma 3.1, (2.5) and (3.2) that

$$\begin{aligned} K_{22} &= \int_0^T (\alpha\mathbf{G}_1 + M * *\mathbf{G}_1,\ \sigma - \Pi_h^k\sigma)dt \\ &= \int_0^T (\nabla\lambda_1,\ \sigma - \Pi_h^k\sigma)dt \\ &= -\int_0^T (\lambda_1, \mathrm{div}(\sigma - \Pi_h^k\sigma))dt \\ &= -\int_0^T (\lambda_1 - P_h^k\lambda_1, \mathrm{div}(\sigma - \Pi_h^k\sigma))dt \\ &= -\int_0^T (\lambda_1 - P_h^k\lambda_1, \mathrm{div}\sigma)dt \\ &= -\int_0^T (\lambda_1 - P_h^k\lambda_1, (I - P_h^k)\mathrm{div}\sigma)dt. \end{aligned} \tag{3.9}$$

Thus, we have for $k = 0$ that

$$\left|\int_0^T (\bar{u}_h - P_h^0 u, \delta_1^h)\phi_1(t)dt\right| \le \begin{cases} C\int_0^T (|||\sigma - \Pi_h^0\sigma|||_\infty ||\mathbf{G}_1^h - \mathbf{G}_1||_{L^1(\Omega)} + ||\lambda_1 - P_h^0\lambda_1||_0 ||(I - P_h^0)\mathrm{div}\sigma||_0)dt, \\ C\int_0^T (|||\sigma - \Pi_h^0\sigma|||_0 ||\mathbf{G}_1^h - \mathbf{G}_1||_0 + ||(I - P_h^0)\lambda_1||_0 ||(I - P_h^0)\mathrm{div}\sigma||_0)dt. \end{cases}$$

Noticing that for $k = 0$ by Theorem 3.1,

$$||\lambda_1 - P_h^0\lambda_1||_0 \le Ch||\nabla\lambda_1||_0 \le Ch|\log h|^{1/2}(1 + \phi_1(t)),$$

it follows from the above inequality and Theorem 3.2 that for $k = 0$

$$\left|\int_0^T (\bar{u}_h - P_h^0 u, \delta_1^h)\phi_1(t)dt\right| \le \begin{cases} Ch|\log h|\int_0^T (|||\sigma - \Pi_h^0\sigma|||_\infty + |\log h|^{-1/2}||(I - P_h^0)\mathrm{div}\sigma||_0)(1 + \phi_1(t))dt, \\ C\int_0^T (|||\sigma - \Pi_h^0\sigma|||_0 + h|\log h|^{1/2}||(I - P_h^0)\mathrm{div}\sigma||_0)(1 + \phi_1(t))dt. \end{cases}$$

We now see from Lemma 3.2 and the arbitrariness of $\phi_1(t)$ that

$$|(\bar{u}_h - P_h^0 u, \delta_1^h)| \le \begin{cases} Ch|\log h|(|||\sigma - \Pi_h^0\sigma|||_\infty + |\log h|^{-1/2}|||(I - P_h^0)\mathrm{div}\sigma|||_0), \\ C(|||\sigma - \Pi_h^0\sigma|||_0 + h|\log h|^{1/2}|||(I - P_h^0)\mathrm{div}\sigma|||_0), \end{cases}$$

from which and (3.3) we derive that for $k = 0$

$$||\bar{u}_h - P_h^0 u||_\infty \le \begin{cases} Ch|\log h|(|||\sigma - \Pi_h^0\sigma|||_\infty + |\log h|^{-1/2}|||(I - P_h^0)\mathrm{div}\sigma|||_0), \\ C(|||\sigma - \Pi_h^0\sigma|||_0 + h|\log h|^{1/2}|||(I - P_h^0)\mathrm{div}\sigma|||_0). \end{cases}$$

Therefore, Theorem 3.6 is true for $k = 0$.

For $k \ge 1$, we have by Theorem 3.1 that

$$||(I - P_h^k)\lambda_1||_{L^1(\Omega)} \le Ch^2||\nabla^2\lambda_1||_{L^1(\Omega)} \le Ch^2|\log h|(1 + \phi_1(t)),$$

which, together with Theorem 3.2, leads to

$$\begin{aligned}
&\left|\int_0^T (\bar{u}_h - P_h^k u, \delta_1^h)\phi_1(t)dt\right| \\
&\leq C\int_0^T (|||\sigma - \Pi_h^k\sigma|||_\infty ||\mathbf{G}_1^h - \mathbf{G}_1||_{L^1(\Omega)} \\
&\qquad\qquad + ||(I-P_h^k)\lambda_1||_{L^1(\Omega)}||(I-P_h^k)\mathrm{div}\sigma||_\infty)dt \\
&\leq Ch|\log h|\int_0^T (|||\sigma - \Pi_h^k\sigma|||_\infty + h||(I-P_h^k)\mathrm{div}\sigma||_\infty)(1+\phi_1(t))dt.
\end{aligned}$$

This, together with Lemma 3.2 and (3.3), yields that for $k \geq 1$

$$||\bar{u}_h - P_h^k u||_\infty \leq Ch|\log h|(|||\sigma - \Pi_h^k\sigma|||_\infty + h|||(I-P_h^k)\mathrm{div}\sigma|||_\infty).$$

This completes the proof of Theorem 3.6. □

Theorem 3.7. *Under the same conditions as for Theorem 3.6, there exists a constant $C > 0$, independent of h and t, such that*

$$||\sigma - \bar{\sigma}_h||_\infty \leq C|\log h|^{1/2}(|||\sigma - \Pi_h^k\sigma|||_\infty + h|\log h|^{\delta_{k0}/2}|||(I-P_h^k)\,div\sigma|||_\infty),$$

where δ_{kj} is the usual Kronecker symbol.

Proof. It suffices to bound $\bar{\sigma}_h - \Pi_h^k\sigma$ in L^∞-norm. By Lemma 3.3 we have that

$$\begin{aligned}
\int_0^T (\bar{\sigma}_h - \Pi_h^k\sigma, \delta_2^h)\phi_2(t)dt &= \int_0^T (\alpha(\sigma - \Pi_h^k\sigma) + M*(\sigma - \Pi_h^k\sigma), \mathbf{G}_2^h - \mathbf{G}_2)dt \\
&\quad + \int_0^T (\alpha(\sigma - \Pi_h^k\sigma) + M*(\sigma - \Pi_h^k\sigma), \mathbf{G}_2)dt \\
&:= M_1 + M_2.
\end{aligned}$$

Similar to (3.9), it follows from Lemma 3.1, (2.5) and (3.4) that

$$\begin{aligned}
M_2 &= \int_0^T (\alpha\mathbf{G}_2 + M**\mathbf{G}_2, \sigma - \Pi_h^k\sigma)dt \\
&= \int_0^T (\nabla\lambda_2 + \delta_2^h\phi_2(t), \sigma - \Pi_h^k\sigma)dt \\
&= -\int_0^T (\lambda_2, \mathrm{div}(\sigma - \Pi_h^k\sigma))dt + \int_0^T (\delta_2^h, \sigma - \Pi_h^k\sigma)\phi_2(t)dt \\
&= \int_0^T (P_h^k\lambda_2 - \lambda_2, (I-P_h^k)\mathrm{div}\sigma)dt + \int_0^T (\delta_2^h, \sigma - \Pi_h^k\sigma)\phi_2(t)dt.
\end{aligned}$$

Thus, we have by (3.5) and Theorem 3.3 that

$$\begin{aligned}
&\left|\int_0^T (\bar{\sigma}_h - \Pi_h^k\sigma, \delta_2^h)\phi_2(t)dt\right| \\
&\leq C\int_0^T |||\sigma - \Pi_h^k\sigma|||_\infty(||\mathbf{G}_2^h - \mathbf{G}_2||_{L^1(\Omega)} + ||\delta_2^h||_{L^1(\Omega)}\phi_2(t))dt \\
&\quad + \int_0^T ||\lambda_2 - P_h^k\lambda_2||_{L^1(\Omega)}||(I-P_h^k)\mathrm{div}\sigma||_\infty dt \\
&\leq C|\log h|^{1/2}\int_0^T |||\sigma - \Pi_h^k\sigma|||_\infty(1+\phi_2(t))dt \\
&+ Ch|\log h|^{\frac{1+\delta_{k0}}{2}}\int_0^T ||(I-P_h^k)\mathrm{div}\sigma||_\infty(1+\phi_2(t))dt,
\end{aligned}$$

which implies by virtue of Lemma 3.2 and the analogue of (3.3) for δ_2^h that

$$||\bar{\sigma}_h - \Pi_h^k \sigma||_\infty \le C|\log h|^{1/2}(|||\sigma - \Pi_h^k\sigma|||_\infty + h|\log h|^{\delta_{k0}/2}|||(I-P_h^k)\mathrm{div}\sigma|||_\infty).$$

This, together with the standard triangle inequality, yields Theorem 3.7. □

Remark 3.3. *By (3.6) we have*

$$(3.10) \qquad ||\lambda_2 - P_h^k\lambda_2||_{L^1(\Omega)} \le Ch(1+\phi_2(t)), \quad k \ge 1,$$

for sufficiently regular $\partial\Omega$. Thus, Theorem 3.7 can be improved to become

$$(3.11) \qquad ||\sigma - \bar{\sigma}_h||_\infty \le C(|\log h|^{1/2}|||\sigma - \Pi_h^k\sigma|||_\infty + h|||(I-P_h^k)\,div\sigma|||_\infty)$$

for $k \ge 1$ if $\partial\Omega$ is sufficiently smooth.

Corollary 3.1. *Under the assumptions of Theorem 3.6, we have*

$$||\bar{u}_h - P_h^k u||_\infty \le \begin{cases} Ch^2|\log h|(|||\sigma|||_{1,\infty} + |\log h|^{-1/2}|||\sigma|||_2), & k=0,\\ Ch^{k+2}|\log h||||\sigma|||_{k+1,\infty}, & k \ge 1.\end{cases}$$

Proof. By (2.6) we have for the interpolation operators Π_h^k and P_h^k that

$$\begin{aligned} ||\mathbf{f} - \Pi_h^k\mathbf{f}||_{0,p} &\le Ch^{k+1}||\mathbf{f}||_{k+1,p}, \qquad 1 \le p \le \infty,\\ ||g - P_h^k g||_{0,p} &\le Ch^{k+1}||g||_{k+1,p}, \qquad 1 \le p \le \infty.\end{aligned}$$

Then, we find from Theorem 3.6 that for $k=0$

$$\begin{aligned} ||\bar{u}_h - P_h^0 u||_\infty &\le Ch|\log h|(|||\sigma - \Pi_h^0\sigma|||_\infty + |\log h|^{-1/2}|||(I-P_h^0)\mathrm{div}\sigma|||_0)\\ &\le Ch^2|\log h|(|||\sigma|||_{1,\infty} + |\log h|^{-1/2}|||\sigma|||_2).\end{aligned}$$

The estimates for $k \ge 1$ can be derived along the same line. □

Similarly, from Theorem 3.7 we can establish the following result.

Corollary 3.2. *We have under the assumptions of Theorem 3.6 that*

$$||\sigma - \bar{\sigma}_h||_\infty \le Ch^{k+1}|\log h|^{(\delta_{k0}+1)/2}|||\sigma|||_{k+1,\infty}, \quad k \ge 0.$$

4. Optimal order L^∞-error estimates for mixed finite element solutions

In this section we consider error estimates in maximum norms for the mixed finite element approximation of (1.1) by means of the L^∞-error estimates for the mixed Ritz-Volterra projection and the estimates for the regularized Green's functions given in the last section. First, the following error estimate of $||u_t - u_{h,t}||$ is demonstrated for the future needs. To this purpose, we recall from [16] the following two lemmas.

Lemma 4.1. *Assume that the matrix $A(t)$ is positive define. Then, the norms $||\sigma||_0^2 := (\sigma,\sigma)$ and $||\sigma||_{A^{-1}}^2 := (A^{-1}\sigma,\sigma)$ are equivalent.*

Lemma 4.2. *Let $(\bar{u}_h, \bar{\sigma}_h)$ be the mixed Ritz-Volterra projection of $(u,\sigma) \in W \times \mathbf{V}$ defined by (3.7). Then, there is a positive constant $C > 0$, independent of $h > 0$, such that, for any positive integer m,*

$$\begin{aligned} ||D_t^m(u - \bar{u}_h)||_0 &\le C\begin{cases} h|||u(t)|||_{2,2,m}, & k=0,\\ h^r|||u(t)|||_{r,2,m}, & k\ge 1 \text{ and } 2 \le r \le k+1,\end{cases}\\ ||D_t^m(\sigma - \bar{\sigma}_h)||_0 &\le Ch^r|||u(t)|||_{r+1,2,m}, \qquad 1 \le r \le k+1,\end{aligned}$$

where $|||u(t)|||_{r,p,m} := \sum_{j=0}^m ||D_t^j u(t)||_{r,p} + \int_0^t \sum_{j=0}^m ||D_t^j u(s)||_{r,p}ds$, $-\infty \le r \le \infty$, $1 \le p \le \infty$, $t \ge 0$.

Theorem 4.1. *Assume that (u,σ) and (u_h,σ_h) are the solutions of (2.1) and (2.2), respectively, and $(u_h(0),\sigma_h(0))$ are chosen as follows:*

$$(4.1)\qquad \begin{array}{ll} (\alpha(0)(\sigma_h(0)-\sigma(0)),\mathbf{v}_h)+(div\mathbf{v}_h,u_h(0)-u_0)=0, & \mathbf{v}_h\in\mathbf{V}_h,\\ (div(\sigma_h(0)-\sigma(0)),w_h)=0, & w_h\in W_h. \end{array}$$

Then we have for $k=0$ that

$$||u_t-u_{h,t}||_0\le Ch\left\{||u||_2+||u_t||_2+\left[\int_0^t(||u||_2^2+||u_t||_2^2+||u_{tt}||_2^2)ds\right]^{1/2}\right\}$$

and for $k\ge 1$ that

$$\begin{aligned}||u_t-u_{h,t}||_0 \ \le\ & Ch^{k+1}\{||u||_{k+1}+||u_t||_{k+1}\\ &+\left[\int_0^t(||u||_{k+1}^2+||u_t||_{k+1}^2+||u_{tt}||_{k+1}^2)ds\right]^{1/2}\right\}.\end{aligned}$$

Proof. Let

$$\begin{aligned}u-u_h&=(u-\bar u_h)+(\bar u_h-u_h):=\rho+\rho_h,\\ \sigma-\sigma_h&=(\sigma-\bar\sigma_h)+(\bar\sigma_h-\sigma_h):=\theta+\theta_h,\end{aligned}$$

where $(\bar u_h,\bar\sigma_h)$ is the Ritz-Volterra projection of (u,σ). Then, by Lemma 4.2 we have

$$(4.2)\qquad \begin{aligned} ||\rho_t||_0&\le\begin{cases} Ch|||u|||_{2,2,1}, & k=0,\\ Ch^{k+1}|||u|||_{k+1,2,1}, & k\ge1,\end{cases}\\ ||\rho_{tt}||_0&\le\begin{cases} Ch|||u|||_{2,2,2}, & k=0,\\ Ch^{k+1}|||u|||_{k+1,2,2}, & k\ge1.\end{cases}\end{aligned}$$

Thus, only $||\rho_{h,t}||_0$ needs to be estimated in order to get the estimate for $||u_t-u_{h,t}||_0$. For this purpose, we first get the estimate for $\theta_h(t)$.

We derive from (3.7) and (4.1) that

$$\begin{array}{ll}(\alpha(0)\theta_h(0),\mathbf{v}_h)+(\mathrm{div}\mathbf{v}_h,\rho_h(0))=0, & \mathbf{v}_h\in\mathbf{V}_h,\\ (\mathrm{div}\theta_h(0),w_h)=0, & w_h\in W_h,\end{array}$$

which, together with the uniqueness of the solution to (3.7), implies

$$(4.3)\qquad \theta_h(0)=\rho_h(0)=0.$$

It follows from (3.7) and (2.4) that (ρ_h,θ_h) satisfies

$$(4.4)\qquad \begin{array}{ll}(\alpha\theta_h+M*\theta_h,\mathbf{v}_h)+(\mathrm{div}\mathbf{v}_h,\rho_h)=0, & \mathbf{v}_h\in\mathbf{V}_h,\\ (\rho_{h,t},w_h)-(\mathrm{div}\theta_h,w_h)=-(\rho_t,w_h), & w_h\in W_h.\end{array}$$

Differentiate (4.4) to obtain

$$(\alpha_t\theta_h+\alpha\theta_{h,t}+M(t,t)\theta_h+M_t*\theta_h,\mathbf{v}_h)+(\mathrm{div}\mathbf{v}_h,\rho_{h,t})=0,\quad \mathbf{v}_h\in\mathbf{V}_h,$$

and then we have by setting $\mathbf{v}_h=\theta_h$ in the above equation and $w_h=\rho_{h,t}$ in (4.4) that

$$(4.5)\qquad ||\rho_{h,t}||_0^2+(\alpha\theta_{h,t},\theta_h)+(\alpha_t\theta_h,\theta_h)=-(M\theta_h+M_t*\theta_h,\theta_h)-(\rho_t,\rho_{h,t}).$$

Since

$$\alpha(\theta_h^2)_t=(\alpha\theta_h^2)_t-\alpha_t\theta_h^2,$$

then

$$(\alpha\theta_{h,t},\theta_h)=\frac12\frac{d}{dt}||\theta_h||_{A^{-1}}^2-\frac12(\alpha_t\theta_h,\theta_h).$$

Hence, (4.5) can be rewritten as

$$||\rho_{h,t}||_0^2 + \frac{1}{2}\frac{d}{dt}||\theta_h||_{A^{-1}}^2 + \frac{1}{2}(\alpha_t\theta_h, \theta_h) = -(M\theta_h + M_t * \theta_h, \theta_h) - (\rho_t, \rho_{h,t}).$$

Thus, we find via integrating the above equation, and using Lemma 4.1, (4.3), Gronwall's lemma and the ϵ-inequality that

$$||\theta_h||_0^2 \le C\int_0^t ||\rho_t||_0^2 ds. \tag{4.6}$$

Next we shall obtain the estimate for $||\rho_{h,t}||_0$. To this end, we differentiate (4.4) to obtain

(4.7)

$$\begin{aligned}
&(\alpha_t\theta_h + \alpha\theta_{h,t} + M(t,t)\theta_h + M_t * \theta_h, \mathbf{v}_h) + (\mathrm{div}\mathbf{v}_h, \rho_{h,t}) = 0, && \mathbf{v}_h \in \mathbf{V}_h,\\
&(\rho_{h,tt}, w_h) - (\mathrm{div}\theta_{h,t}, w_h) = -(\rho_{tt}, w_h), && w_h \in W_h.
\end{aligned}$$

And hence, we have by setting $\mathbf{v}_h = \theta_{h,t}$ and $w_h = \rho_{h,t}$ in (4.7) and following the procedure for (4.6) that

$$||\rho_{h,t}||_0^2 \le C\left\{||\rho_{h,t}(0)||_0^2 + \int_0^t ||\rho_t||_0^2 ds + \int_0^t ||\rho_{tt}||_0^2 ds\right\}. \tag{4.8}$$

By letting $t = 0$ and $w_h = \rho_{h,t}(0)$ in (4.4) we obtain from (4.3) that

$$||\rho_{h,t}(0)||_0 \le ||\rho_t(0)||_0,$$

which, together with (4.8) and (4.2), leads to

$$\begin{aligned}
||\rho_{h,t}||_0^2 &\le C\left\{||\rho_t(0)||_0^2 + \int_0^t (||\rho_t||_0^2 + ||\rho_{tt}||_0^2) ds\right\}\\
&\le \begin{cases}
Ch^2\left[||u(0)||_2^2 + ||u_t(0)||_2^2 + \displaystyle\int_0^t (||u||_2^2 + ||u_t||_2^2 + ||u_{tt}||_2^2) ds\right], & k = 0,\\
Ch^{2k+2}\Big[||u(0)||_{k+1}^2 & \\
\quad + ||u_t(0)||_{k+1}^2 + \displaystyle\int_0^t (||u||_{k+1}^2 + ||u_t||_{k+1}^2 + ||u_{tt}||_{k+1}^2) ds\Big], & k \ge 1.
\end{cases}
\end{aligned}$$

This completes the proof of the theorem by (4.2). □

Now we are in a position to get our main theorem in this section.

Theorem 4.2. *We have under the assumptions of Theorem 4.1 that for $k = 0$*

$$\begin{aligned}
||u - u_h||_\infty &\le Ch\left[||u||_{1,\infty} + |\log h|^{1/2}(||u||_2 + ||u_t||_2)\right]\\
&\quad + Ch|\log h|^{1/2}\left[\int_0^t (||u||_2^2 + ||u_t||_2^2 + ||u_{tt}||_2^2) ds\right]^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
||\sigma - \sigma_h||_\infty &\le Ch|\log h|^{1/2}\left(|\log h|^{1/2}|||u|||_{2,\infty} + ||u||_2 + ||u_t||_2\right)\\
&\quad + Ch|\log h|^{1/2}\left[\int_0^t (||u||_2^2 + ||u_t||_2^2 + ||u_{tt}||_2^2) ds\right]^{1/2};
\end{aligned}$$

for $k \ge 1$

$$\begin{aligned}
||u - u_h||_\infty &\le Ch^{k+1}|\log h|^{1/2}\left(|\log h|^{1/2}|||u|||_{k+1,\infty} + ||u||_{k+1} + ||u_t||_{k+1}\right)\\
&\quad + Ch^{k+1}|\log h|^{1/2}\left[\int_0^t (||u||_{k+1}^2 + ||u_t||_{k+1}^2 + ||u_{tt}||_{k+1}^2) ds\right]^{1/2}
\end{aligned}$$

and

$$\begin{aligned}||\sigma-\sigma_h||_\infty &\le Ch^{k+1}|\log h|^{1/2}\left(|||u|||_{k+2,\infty}+||u||_{k+1}+||u_t||_{k+1}\right)\\ &\quad +Ch^{k+1}|\log h|^{1/2}\left[\int_0^t(||u||_{k+1}^2+||u_t||_{k+1}^2+||u_{tt}||_{k+1}^2)ds\right]^{1/2}.\end{aligned}$$

Proof. With the same decomposition of the errors as that in Theorem 4.1, we know from Corollaries 3.1 and 3.1 that

$$\begin{aligned}||\rho||_\infty &\le ||u-P_h^k u||_\infty+||P_h^k u-\bar{u}_h||_\infty\\ &\le \begin{cases} Ch\left(||u||_{1,\infty}+|\log h|^{1/2}|||u|||_2\right), & k=0,\\ Ch^{k+1}|\log h||||u|||_{k+1,\infty}, & k\ge 1,\end{cases}\\ ||\theta||_\infty &\le Ch^{k+1}|\log h|^{(\delta_{k0}+1)/2}|||\sigma|||_{k+1,\infty}.\end{aligned}\tag{4.9}$$

Therefore, only $||\rho_h||_\infty$ and $||\theta_h||_\infty$ are left to be estimated.

Set $\mathbf{v}_h=\mathbf{G}_1^h$ in (4.4) to obtain from the mixed finite element approximation of (3.2) that

$$(\delta_1^h\phi_1(t),\rho_h)=(\text{div}\mathbf{G}_1^h,\rho_h)=-(\alpha\theta_h+M*\theta_h,\mathbf{G}_1^h),$$

so that it follows from the integration, Lemma 3.1, and the mixed finite element solution of (3.2) that

$$\begin{aligned}\int_0^T(\delta_1^h,\rho_h)\phi_1(t)dt &= -\int_0^T(\alpha\mathbf{G}_1^h+M**\mathbf{G}_1^h,\theta_h)dt\\ &=\int_0^T(\lambda_1^h,\text{div}\theta_h)dt\\ &=\int_0^T(\lambda_1^h-\lambda_1,\text{div}\theta_h)dt+\int_0^T(\lambda_1,\text{div}\theta_h)dt.\end{aligned}\tag{4.10}$$

Since $\lambda_1|_{\partial\Omega}=0$, it follows from Theorems 3.1 and 3.2 that

$$\begin{aligned}&||\lambda_1||_0\le C||\nabla\lambda_1||_0\le C|\log h|^{1/2}(1+\phi_1(t)),\\ &||\lambda_1-\lambda_1^h||_0\le Ch|\log h|^{1/2}(1+\phi_1(t)).\end{aligned}$$

Hence, we find from (4.10), Lemma 3.2 and (3.3) that

$$||\rho_h||_\infty\le C|\log h|^{1/2}(h+1)|||\text{div}\theta_h|||_0.\tag{4.11}$$

We know from (3.7) and the mixed finite element error equation (2.4) that

$$\begin{aligned}&(\text{div}(\sigma-\bar{\sigma}_h),w_h)=0, && w_h\in W_h,\\ &(\text{div}(\sigma-\sigma_h),w_h)=(u_t-u_{h,t},w_h), && w_h\in W_h.\end{aligned}$$

This implies

$$(\text{div}\theta_h,w_h)=(\text{div}(\bar{\sigma}_h-\sigma_h),w_h)=(\text{div}(\sigma-\sigma_h),w_h)=(u_t-u_{h,t},w_h),\ \ w_h\in W_h,$$

from which we have by means of the arbitrariness of $w_h\in W_h$ that

$$||\text{div}\theta_h||_0\le||u_t-u_{h,t}||_0.\tag{4.12}$$

Combining (4.11) with (4.12) and Theorem 4.1 leads to
(4.13)

$$||\rho_h||_\infty \leq \begin{cases} Ch|\log h|^{1/2}\,[||u_0||_2+||u_t(0)||_2+||u||_2+||u_t||_2] \\ +Ch|\log h|^{1/2}\left[\int_0^t(||u||_2^2+||u_t||_2^2+||u_{tt}||_2^2)ds\right]^{1/2}, & k=0, \\ Ch^{k+1}|\log h|^{1/2}[||u_0||_{k+1}+||u_t(0)||_{k+1}+||u||_{k+1}+||u_t||_{k+1}] \\ +Ch^{k+1}|\log h|^{1/2}\left[\int_0^t(||u||_{k+1}^2+||u_t||_{k+1}^2+||u_{tt}||_{k+1}^2)ds\right]^{1/2}, & k\geq 1. \end{cases}$$

Next we shall give the estimate for $||\sigma-\sigma_h||_\infty$. For this purpose, we let $\mathbf{v}_h=\mathbf{G}_2^h$ in (4.4) to get according to (3.4) that

$$\begin{aligned}(\alpha\theta_h+M*\theta_h,\mathbf{G}_2^h) &= -(\mathrm{div}\mathbf{G}_2^h,\rho_h)\\ &= -(\mathrm{div}(\mathbf{G}_2^h-\mathbf{G}_2),\rho_h)-(\mathrm{div}\mathbf{G}_2,\rho_h)\\ &= 0.\end{aligned}$$

This yields by Lemma 3.1 and Green's formula that

$$\begin{aligned}0=\int_0^T(\alpha\theta_h+M*\theta_h,\mathbf{G}_2^h)dt &= \int_0^T(\alpha\mathbf{G}_2^h+M**\mathbf{G}_2^h,\theta_h)dt\\ &= \int_0^T(\delta_2^h,\theta_h)\phi_2(t)dt-\int_0^T(\lambda_2^h,\mathrm{div}\theta_h)dt,\end{aligned}$$

which, together with Theorem 3.3, implies that

$$\begin{aligned}\left|\int_0^T(\delta_2^h,\theta_h)\phi_2(t)dt\right| &= \left|\int_0^T(\lambda_2^h-\lambda_2,\mathrm{div}\theta_h)dt+\int_0^T(\lambda_2,\mathrm{div}\theta_h)dt\right|\\ &\leq \int_0^T C||\mathrm{div}\theta_h||_0(1+\phi_2(t))dt\\ &\quad+\int_0^T C(1+|\log h|^{1/2})||\mathrm{div}\theta_h||_0(1+\phi_2(t))dt.\end{aligned}$$

Thus, we derive from Lemma 3.2 and (4.12) that

$$\left|(\delta_2^h,\theta_h)\right|\leq C|\log h|^{1/2}|||\mathrm{div}\theta_h|||_0\leq C|\log h|^{1/2}|||u_t-u_{h,t}|||_0$$

which, together with Theorem 4.1 and an analogue of (3.3) for δ_2^h, shows that

$$||\theta_h||_\infty \leq \begin{cases} Ch|\log h|^{1/2}(||u||_2+||u_t||_2) \\ +Ch|\log h|^{1/2}\left[\int_0^t(||u||_2^2+||u_t||_2^2+||u_{tt}||_2^2)ds\right]^{1/2}, & k=0, \\ Ch^{k+1}|\log h|^{1/2}(||u||_{k+1}+||u_t||_{k+1}) \\ +Ch^{k+1}|\log h|^{1/2}\left[\int_0^t(||u||_{k+1}^2+||u_t||_{k+1}^2+||u_{tt}||_{k+1}^2)ds\right]^{1/2}, & k\geq 1. \end{cases}$$

This, together with (4.9) and (4.13), completes the proof of the theorem. □

5. Superconvergence in L^∞-norm and a-posteriori error estimates

The aim of this section is to give local and global maximum norm superconvergence and a-posteriori error estimators for the mixed finite element approximation of (2.1). First of all, we consider the local superconvergence. To this end, let us define some seminorms as follows.

Following [11] we assume that $\Omega \subset R^2$ is a rectangle and $e = [a, b] \times [c, d] \in T_h$ is an arbitrary element of the partition T_h. We denote by $(g_1, g_2, \cdots, g_{k+1})$ the Gauss points in $[a, b]$ and $(\hat{g}_1, \hat{g}_2, \cdots, \hat{g}_{k+1})$ the Gauss points in $[c, d]$, and define

$$\begin{aligned} |||w|||_{*,\infty} &:= \max_{e \in T_h} \max_{1 \le i,j \le k+1} |w(g_i, \hat{g}_j)|, \\ |||\mathbf{v}|||_{*,\infty} &:= |||v_1|||^*_{\infty,1} + |||v_2|||^*_{\infty,2}, \end{aligned}$$

where

$$\begin{aligned} |||v_1|||^*_{\infty,1} &:= \max_{e \in T_h} \max_{1 \le j \le k+1} \max_{(x,\hat{g}_j) \in e} |v_1(x, \hat{g}_j)|, \\ |||v_2|||^*_{\infty,2} &:= \max_{e \in T_h} \max_{1 \le i \le k+1} \max_{(g_i,y) \in e} |v_2(g_i, y)|. \end{aligned}$$

By assuming that the matrices A and B are diagonal or the partition T_h is uniform in this section, we recall from [8, 19] and [16] the following Lemma 5.1 and Lemma 5.2, respectively.

Lemma 5.1. *Let σ be a sufficiently smooth vector-valued function, $B = (b_{ij})$ be a 2×2 matrix with $b_{ij} \in W^{1,\infty}(\Omega)$ and Ω be a rectangular domain. Then, we have*

$$|(B \cdot (\sigma - \Pi_h^k \sigma), \mathbf{v}_h)| \le C h^{k+2} |\sigma|_{k+2,p} ||\mathbf{v}_h||_{0,p'}, \quad \forall \mathbf{v}_h \in \mathbf{V}_h,$$

where $|f|_{m,q} := \left(\sum_{|i|=m} ||D^i f||^q_{0,q,\Omega} \right)^{1/q}$, $1 \le q < \infty$, $|f|_{m,\infty} := \max_{|i|=m} \{ ess \sup_{\Omega} |D^i f| \}$ *and* $p' = \frac{p}{p-1}$ *is the conjugate of* $p \ge 1$.

Lemma 5.2. *Let $(\bar{u}_h, \bar{\sigma}_h)$ be the mixed Ritz-Volterra projection of (u, σ). Then, we have*

$$||\bar{u}_h - P_h^k u||_W + ||\bar{\sigma}_h - \Pi_h^k \sigma||_{\mathbf{V}} \le C h^{k+2} (||u||_{k+1} + |||\sigma|||_{k+2}),$$

where $||u||_W := ||u||_0$ *and* $||\sigma||_{\mathbf{V}} := \left(||\sigma||_0^2 + ||\nabla \cdot \sigma||_0^2 \right)^{1/2}$.

From now on, for convenience in writing, we shall refrain from tracing the exact dependence on the smoothness of u and σ, and only give the order of errors.

Theorem 5.1. *Assume that (u, σ) and (u_h, σ_h) are the solutions of (2.1) and (2.2), respectively, and $(u_h(0), \sigma_h(0))$ are chosen to satisfy (4.1). If the exact solution u and σ satisfies σ, $\sigma_t \in (H^{k+2}(\Omega))^2$, then we have*

$$||u_h - P_h^k u||_0 + ||(u_h - P_h^k u)_t||_0 + ||\sigma_h - \Pi_h^k \sigma||_0 \le C(u, \sigma) h^{k+2}.$$

Proof. Let $\rho_h^* := u_h - P_h^k u$ and $\theta_h^* := \sigma_h - \Pi_h^k \sigma$. Then, it follows from (2.4) and (2.5) that

(5.1)
$$\begin{aligned} &(\alpha \theta_h^* + M * \theta_h^*, \mathbf{v}_h) + (\rho_h^*, \nabla \cdot \mathbf{v}_h) = (\alpha(\sigma - \Pi_h^k \sigma) + M * (\sigma - \Pi_h^k \sigma), \mathbf{v}_h), && \mathbf{v}_h \in \mathbf{V}_h, \\ &(\rho_{h,t}^*, w_h) - (\nabla \cdot \theta_h^*, w_h) = 0, && w_h \in W_h. \end{aligned}$$

Thus, letting $w_h = \rho_h^*$ and $\mathbf{v}_h = \theta_h^*$ in (5.1) we obtain from Lemmas 4.1, 5.1, the ϵ-type inequality, the integration and Gronwall's lemma that

$$||\rho_h^*||_0^2 + \int_0^t ||\theta_h^*||_0^2 ds \le C(u, \sigma) \{ ||\rho_h^*(0)||_0^2 + h^{2(k+2)} \}. \tag{5.2}$$

From (4.3) we know

$$\bar{u}_h(0) - u_h(0) = \bar{\sigma}_h(0) - \sigma_h(0) = 0. \tag{5.3}$$

Therefore, from Lemma 5.2 we have

$$||\rho_h^*(0)||_0 = ||\bar{u}_h(0) - P_h^k u_0||_0 \le C(u,\sigma)h^{k+2},$$

and then from (5.2) we further obtain

$$||\rho_h^*||_0 \le C(u,\sigma)h^{k+2}. \tag{5.4}$$

Again, we have according to (5.3) and Lemma 5.2 that

$$||\theta_h^*(0)||_0 = ||\bar{\sigma}_h(0) - \Pi_h^k\sigma(0)||_0 \le C(u,\sigma)h^{k+2}. \tag{5.5}$$

The second equation in (5.1) implies

$$\rho_{h,t}^* = \nabla\cdot\theta_h^*, \tag{5.6}$$

which, together with (5.3) and Lemma 5.2, implies

$$||\rho_{h,t}^*(0)||_0 = ||\nabla\cdot\theta_h^*(0)||_0 = ||\nabla\cdot(\bar{\sigma}_h - \Pi_h^k\sigma)(0)||_0 \le C(u,\sigma)h^{k+2}. \tag{5.7}$$

Following the steps for θ_h and $\rho_{h,t}$ in Theorem 4.1 and using the initial approximations (5.5) and (5.7) we obtain

$$||\theta_h^*||_0 + ||\rho_{h,t}^*||_0 \le C(u,\sigma)h^{k+2}. \tag{5.8}$$

The proof of Theorem 5.1 is completed by (5.4) and (5.8). □

Now we are ready to obtain our superapproximation theorem.

Theorem 5.2. *In addition to the conditions of Theorem 5.1, if the exact solution σ is such that $\sigma \in \left(W^{k+2,\infty}(\Omega)\right)^2$, then we have*

$$|\log h|^{1/2}||u_h - P_h^k u||_\infty + ||\sigma_h - \Pi_h^k\sigma||_\infty \le C(u,\sigma)h^{k+2}|\log h|.$$

Proof. Taking $\mathbf{v}_h = \mathbf{G}_1^h$ in (5.1) we see from the mixed finite element approximation of (3.2), Theorems 3.4 and 5.1, and Lemmas 5.1 and 3.1 that

$$\begin{aligned}\left|\int_0^T(\rho_h^*,\delta_1^h)\phi_1(t)dt\right| &\le \left|\int_0^T(\alpha\mathbf{G}_1^h + M**\mathbf{G}_1^h,\theta_h^*)dt\right| \\ &\quad + C(u,\sigma)h^{k+2}|\log h|^{1/2}\int_0^T(1+\phi_1(t))dt \\ &\le C(u,\sigma)h^{k+2}|\log h|^{1/2}\int_0^T(1+\phi_1(t))dt.\end{aligned}$$

Thus, Lemma 3.2 and (3.3) imply

$$||\rho_h^*||_\infty \le C(u,\sigma)h^{k+2}|\log h|^{1/2}.$$

Next we shall obtain the superapproximation estimate for θ_h^* in the L^∞-norm. Taking $\mathbf{v}_h = \mathbf{G}_2^h$ in (5.1) and noticing $\nabla\cdot\mathbf{G}_2^h = 0$ by the mixed finite element approximation of (3.4), we have by Lemmas 3.1, 5.1 and Theorem 3.5 that

$$\left|\int_0^T(\alpha\mathbf{G}_2^h + M**\mathbf{G}_2^h,\theta_h^*)dt\right| \le C(u,\sigma)h^{k+2}|\log h|\int_0^T(1+\phi_2(t))dt,$$

which, together with Theorem 3.3, (5.6) and (5.8), leads to

$$\begin{aligned}\left|\int_0^T(\delta_2^h,\theta_h^*)\phi_2(t)dt\right| &\le \int_0^T||\lambda_2^h||_0||\rho_{h,t}^*||_0 dt + C(u,\sigma)h^{k+2}|\log h|\int_0^T(1+\phi_2(t))dt \\ &\le C(u,\sigma)h^{k+2}|\log h|\int_0^T(1+\phi_2(t))dt.\end{aligned}$$

This implies by Lemma 3.2 that

$$||\theta_h^*||_\infty \le C(u,\sigma)h^{k+2}|\log h|.$$

□

Remark 5.1. *From Lemmas 3.3 and 5.1 we can derive the following L^∞-norm superapproximation for the mixed Ritz-Volterra projection of (u,σ):*

$$|\log h|^{1/2}||\bar{u}_h - P_h^k u||_\infty + ||\bar{\sigma}_h - \Pi_h^k\sigma||_\infty \le C(u,\sigma)h^{k+2}|\log h|.$$

Hence, there holds the L^∞-superapproximation estimate under the conditions of Theorem 5.2,

$$|\log h|^{1/2}||\bar{u}_h - u_h||_\infty + ||\bar{\sigma}_h - \sigma_h||_\infty \le C(u,\sigma)h^{k+2}|\log h|.$$

In order to obtain the local superconvergence for the mixed finite element solution (u_h,σ_h), we need the following lemmas which come from [11] and [8], respectively.

Lemma 5.3. *Assume that $u \in W^{k+2,\infty}(\Omega)$. Then,*

$$|||u - P_h^k u|||_{*,\infty} \le C(u)h^{k+2}.$$

Lemma 5.4. *If $\sigma \in \left(W^{k+2,\infty}(\Omega)\right)^2$, then we have*

$$|||\sigma - \Pi_h^k\sigma|||_{*,\infty} \le C(\sigma)h^{k+2}.$$

We are now in the position to get our local superconvergence on the Gauss points for the approximation of the pressure field and along the Gauss lines for the approximation of the velocity field, respectively.

Theorem 5.3. *In addition to the conditions of Theorem 5.2, if the exact solution u is such that $u \in W^{k+2,\infty}(\Omega)$, then we have*

$$|\log h|^{1/2}|||u - u_h|||_{*,\infty} + |||\sigma - \sigma_h|||_{*,\infty} \le C(u,\sigma)h^{k+2}|\log h|.$$

Proof. From Lemma 5.3 and Theorem 5.2 we have

$$\begin{aligned} |||u - u_h|||_{*,\infty} &\le |||u - P_h^k u|||_{*,\infty} + |||P_h^k u - u_h|||_{*,\infty} \\ &\le C(u)h^{k+2} + C(u,\sigma)h^{k+2}|\log h|^{1/2} \\ &\le C(u,\sigma)h^{k+2}|\log h|^{1/2}. \end{aligned}$$

Similarly, we obtain by means of Theorem 5.2 and Lemma 5.4 that

$$|||\sigma - \Pi_h^k\sigma|||_{*,\infty} \le C(u,\sigma)h^{k+2}|\log h|.$$

□

Next we shall consider the global superconvergence for the pressure and the velocity fields by virtue of post-processing methods. Analogous to [16] we need to construct two post-processing interpolation operators Π_{2h}^{k+1} and P_{2h}^{k+1} to satisfy

$$\begin{array}{ll} \Pi_{2h}^{k+1}\Pi_h^k = \Pi_{2h}^{k+1}, & \\ ||\Pi_{2h}^{k+1}\mathbf{v}_h||_{0,p} \le C||\mathbf{v}_h||_{0,p}, & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ ||\Pi_{2h}^{k+1}\sigma - \sigma||_{0,p} \le Ch^{k+2}||\sigma||_{k+2,p}, & \forall \sigma \in (W^{k+2,p}(\Omega))^2, \\ P_{2h}^{k+1}P_h^k = P_{2h}^{k+1}, & \\ ||P_{2h}^{k+1}w_h||_{0,p} \le C||w_h||_{0,p}, & \forall w_h \in W_h, \\ ||P_{2h}^{k+1}u - u||_{0,p} \le Ch^{k+2}||u||_{k+2,p}, & \forall u \in W^{k+2,p}(\Omega), \end{array} \tag{5.9}$$

where $1 \le p \le \infty$ and $||\cdot||_{0,\infty} = ||\cdot||_\infty$. Here we take for example $k = 3$ to demonstrate the construction of the projection interpolation operators Π_{2h}^{k+1} and

P_{2h}^{k+1} satisfying (5.9). To this purpose, we assume that the rectangular partition T_h has been obtained from $T_{2h} = \{\tau\}$ with mesh size $2h$ by subdividing each element of T_{2h} into four small congruent rectangles. Let $\tau := \bigcup_{i=1}^{4} e_i$ with $e_i \in T_h$. Thus, we can define two projection operators Π_{2h}^4 and P_{2h}^4 associated with T_{2h} of degree at most 4 in x and y on τ, respectively, according to the following conditions:

$$\begin{aligned}
&\Pi_{2h}^4\sigma|_\tau \in (Q_{4,4}(\tau))^2, && P_{2h}^4 u|_\tau \in Q_{4,4}(\tau),\\
&\int_{l_i} (\sigma - \Pi_{2h}^4\sigma)\cdot \mathbf{n} q ds = 0, && \forall q \in P_2(l_i),\ i = 1, 2, \cdots, 12,\\
&\int_{e_i} (\sigma - \Pi_{2h}^4\sigma) = 0, && i = 1, 2, 3, 4,\\
&\int_{\tau} (\sigma - \Pi_{2h}^4\sigma)\cdot\phi = 0, && \forall \phi \in (Q_{1,1}(\tau)\backslash Q_{0,0}(\tau))^2, \text{ and}\\
&\int_{e_i} (u - P_{2h}^4 u)\psi = 0, && \forall \psi \in Q_{2,1}(e_i),\ i = 1, 2, 3, 4,\\
&\int_{\tau} (u - P_{2h}^4 u)\psi = 0, && \forall \psi \in Q_{3,0}(\tau)\backslash Q_{2,0}(\tau), \text{ respectively,}
\end{aligned}$$

where l_i $(i = 1, 2, \cdots, 12)$ is one of the twelve sides of the four small elements e_i $(i = 1, 2, 3, 4)$.

Similarly, we can also define Π_{2h}^{k+1} and P_{2h}^{k+1} for the case of $k \neq 3$ such that (5.9) is satisfied.

By the two projection interpolation operators Π_{2h}^{k+1} and P_{2h}^{k+1} we can immediately gain the following global superconvergence result.

Theorem 5.4. *Assume that (u, σ) and (u_h, σ_h) are the solutions of (2.1) and (2.2), respectively. Then, we have under the conditions of Theorem 5.3 that*

$$|\log h|^{1/2} ||P_{2h}^{k+1}u_h - u||_\infty + ||\Pi_{2h}^{k+1}\sigma_h - \sigma||_\infty \le C(u, \sigma)h^{k+2}|\log h|.$$

Proof. We see from one of the properties of the operator P_{2h}^{k+1} described in (5.9) that

$$P_{2h}^{k+1}u_h - u = P_{2h}^{k+1}(u_h - P_h^k u) + (P_{2h}^{k+1}u - u).$$

Therefore, it follows from Theorem 5.2 and (5.9) that

$$||P_{2h}^{k+1}u - u||_\infty \le C||u_h - P_h^k u||_\infty + C(u)h^{k+2} \le C(u, \sigma)h^{k+2}|\log h|^{1/2}.$$

Analogously, we can obtain

$$||\Pi_{2h}^{k+1}\sigma - \sigma||_\infty \le C(u, \sigma)h^{k+2}|\log h|.$$

□

Remark 5.2. *From the superapproximation estimates of $||\bar{u}_h - P_h^k u||_\infty$ and $||\bar{\sigma}_h - \Pi_h^k\sigma||_\infty$ indicated in Remark 5.1 we can obtain the following global superconvergence under the conditions of Theorem 5.3 by the post-processing method:*

$$|\log h|^{1/2} ||P_{2h}^{k+1}\bar{u}_h - u||_\infty + ||\Pi_{2h}^{k+1}\bar{\sigma}_h - \sigma||_\infty \le C(u, \sigma)h^{k+2}|\log h|.$$

As a by-product, Theorem 5.4 can be employed to construct a-posteriori error estimators to assess the accuracy of the mixed finite element solution in applications.

Theorem 5.5. *We have under the conditions of Theorem 5.3 that*

$$||u - u_h||_\infty = ||P_{2h}^{k+1}u_h - u_h||_\infty + O(h^{k+2}|\log h|^{1/2}), \tag{5.10}$$

$$||\sigma - \sigma_h||_\infty = ||\Pi_{2h}^{k+1}\sigma_h - \sigma_h||_\infty + O(h^{k+2}|\log h|). \tag{5.11}$$

In addition, if there exist positive constants C_1, C_2 and small ϵ_1, $\epsilon_2 \in (0,1)$ such that

$$||u - u_h||_\infty \geq C_1 h^{k+2-\epsilon_1}, \tag{5.12}$$

$$||\sigma - \sigma_h||_\infty \geq C_2 h^{k+2-\epsilon_2}, \tag{5.13}$$

then there hold

$$\lim_{h\to 0} \frac{||u - u_h||_\infty}{||P_{2h}^{k+1} u_h - u_h||_\infty} = 1, \tag{5.14}$$

$$\lim_{h\to 0} \frac{||\sigma - \sigma_h||_\infty}{||\Pi_{2h}^{k+1} \sigma_h - \sigma_h||_\infty} = 1. \tag{5.15}$$

Proof. Following the procedure for Theorem 5.3 in [16] we can immediately obtain the desired results. □

We see from (5.10) that the computable error quantity $||P_{2h}^{k+1} u_h - u_h||_\infty$ is the principal part of the mixed finite element error $||u - u_h||_\infty$. Moreover, by (5.14) it can be used as a reliable a-posteriori error indicator to assess the accuracy of the mixed finite element solution under the condition (5.12). Meanwhile, (5.12) seems to be a reasonable assumption since $O(h^{k+1})$ is the optimal convergence rate of the mixed finite element solution in L^∞-norm subject to the conditions of Theorem 5.3. The same comments are also valid for (5.11), (5.13) and (5.15).

6. Estimates for the regularized Green's functions

In the previous sections, we have seen that the regularized Green's functions play an important role in the analysis of convergence and superconvergence estimates in maximum norms for the mixed finite element method of (1.1). We present the proofs of Theorems 3.2 and 3.3 in this section. The proofs are based on a series of lemmas. First, we prove the following result.

Lemma 6.1. *We have under the assumptions of Theorem 3.2 that*

$$||\mathbf{G}_1^h - \mathbf{G}_1||_0 \leq C(1 + \phi_1(t)).$$

Proof. It follows from (3.2), Gronwall's lemma and Theorem 3.1 that

$$||\mathbf{G}_1||_0 \leq C||\nabla\lambda_1||_0 \leq C|\log h|^{1/2}(1 + \phi_1(t)),$$

which yields via using the estimate for $||\nabla^2\lambda_1||_0$ in Theorem 3.1 that

$$||\text{div}\mathbf{G}_1||_0 \leq Ch^{-1}(1 + \phi_1(t)) + C|\log h|^{1/2}(1 + \phi_1(t)) \leq Ch^{-1}(1 + \phi_1(t)).$$

Decompose the error $\mathbf{G}_1 - \mathbf{G}_1^h$ as follows:

$$\mathbf{G}_1 - \mathbf{G}_1^h = (\mathbf{G}_1 - \Pi_h^k \mathbf{G}_1) + (\Pi_h^k \mathbf{G}_1 - \mathbf{G}_1^h) := \theta^{**} + \theta_h^{**}.$$

Then, θ_h^{**} satisfies the following equation by (2.5) and the mixed finite element error equation of (3.2) that

$$\begin{aligned}(\alpha\theta_h^{**} + M**\theta_h^{**}, \mathbf{v}_h) &= -(\alpha\theta^{**} + M**\theta^{**}, \mathbf{v}_h) - (\lambda_1 - \lambda_1^h, \nabla\cdot\mathbf{v}_h) \\ &= -(\alpha\theta^{**} + M**\theta^{**}, \mathbf{v}_h) - (P_h^k\lambda_1 - \lambda_1^h, \nabla\cdot\mathbf{v}_h), \mathbf{v}_h \in \mathbf{V}_h.\end{aligned}$$

Since

$$(P_h^k\lambda_1 - \lambda_1^h, \nabla\cdot\theta_h^{**}) = 0,$$

by (2.5) and the mixed finite element error equation of (3.2), taking $\mathbf{v}_h = \theta_h^{**}$ in the above equation leads to

$$(\alpha\theta_h^{**} + M**\theta_h^{**}, \theta_h^{**}) = -(\alpha\theta^{**} + M**\theta^{**}, \theta_h^{**}).$$

Thus, we know from Lemma 4.1 and Gronwall's lemma that

$$||\theta_h^{**}||_0 \le C\left(||\theta^{**}||_0 + \int_t^T ||\theta^{**}||_0 ds\right).$$

Hence, we obtain by virtue of the above estimate for $\mathrm{div}\mathbf{G}_1$ in L^2-norm and (3.5) that

$$\begin{aligned} ||\mathbf{G}_1 - \mathbf{G}_1^h||_0 &\le C\left(||\theta^{**}||_0 + \int_t^T ||\theta_h^{**}||_0 ds\right) \\ &\le Ch\left(||\mathrm{div}\mathbf{G}_1||_0 + \int_t^T ||\mathrm{div}\mathbf{G}_1||_0 ds\right) \\ &\le C(1+\phi_1(t)). \end{aligned}$$

This completes the proof of Lemma 6.1. □

Lemma 6.2. *Under the assumptions of Theorem 3.2,*

$$\begin{aligned} ||\lambda_1^h - P_h^k\lambda_1||_0 &\le Ch(1+\phi_1(t)), \\ ||\lambda_1^h - \lambda_1||_0 &\le Ch|\log h|^{\delta_{k0}/2}(1+\phi_1(t)). \end{aligned}$$

Proof. Let $(\mathbf{w}, \lambda) \in \mathbf{V} \times L^2(\Omega)$ be defined such that

$$\begin{aligned} \alpha\mathbf{w} + M*\mathbf{w} - \nabla\lambda &= 0, && \text{in } \Omega\times(0,T), \\ \mathrm{div}\mathbf{w} &= (\lambda_1^h - P_h^k\lambda_1)\phi(t), && \text{in } \Omega\times(0,T), \\ \lambda &= 0, && \text{on } \partial\Omega\times(0,T), \end{aligned} \tag{6.1}$$

where $\phi(t) \ge 0$ and $\int_0^T \phi(t)dt \le 1$. Clearly, $(\mathbf{w}, \lambda)$ is well defined and satisfies

$$||\nabla^2\lambda||_0 \le C\left(||\lambda_1^h - P_h^k\lambda_1||_0\phi(t) + \int_0^t ||\lambda_1^h - P_h^k\lambda_1||_0\phi(s)ds\right)$$

by the regularity assumption on Ω. Now, it follows from (2.5), the mixed finite element error equation of (3.2) and Lemma 3.1 that

$$\begin{aligned} \int_0^T ||\lambda_1^h - P_h^k\lambda_1||_0^2\phi(t)dt &= \int_0^T (\lambda_1^h - P_h^k\lambda_1, \mathrm{div}\Pi_h^k\mathbf{w})dt \\ &= \int_0^T (\lambda_1^h - \lambda_1, \mathrm{div}\Pi_h^k\mathbf{w})dt \\ &= \int_0^T (\alpha(\mathbf{G}_1 - \mathbf{G}_1^h) + M**(\mathbf{G}_1 - \mathbf{G}_1^h), \Pi_h^k\mathbf{w})dt \\ &= \int_0^T (\alpha(\mathbf{G}_1 - \mathbf{G}_1^h) + M**(\mathbf{G}_1 - \mathbf{G}_1^h), \Pi_h^k\mathbf{w} - \mathbf{w})dt \\ &\quad + \int_0^T (\lambda, \mathrm{div}(\mathbf{G}_1^h - \mathbf{G}_1))dt := N_1 + N_2. \end{aligned} \tag{6.2}$$

Obviously, we have by using (3.2) and (2.5) that

$$\begin{aligned} N_2 &= \int_0^T (P_h^k\lambda, \mathrm{div}\mathbf{G}_1^h)dt - \int_0^T (\lambda, \delta_1^h)\phi_1(t)dt \\ &= \int_0^T (P_h^k\lambda, P_h^k\delta_1^h)\phi_1(t)dt - \int_0^T (\lambda, \delta_1^h)\phi_1(t)dt \\ &= \int_0^T (P_h^k\lambda - \lambda, \delta_1^h)\phi_1(t)dt. \end{aligned}$$

Thus, we have for $k \geq 1$ that

$$\begin{aligned} |N_2| &\leq \int_0^T Ch^2||\nabla^2\lambda||_0||\delta_1^h||_0\phi_1(t)dt \\ &\leq Ch\int_0^T \left(||\lambda_1^h - P_h^k\lambda_1||_0\phi(t) + \int_0^t ||\lambda_1^h - P_h^k\lambda_1||_0\phi(s)ds\right)\phi_1(t)dt \\ &\leq Ch\int_0^T \left(\phi_1(t) + \int_t^T \phi_1(s)ds\right)||\lambda_1^h - P_h^k\lambda_1||_0\phi(t)dt \\ &\leq Ch\int_0^T (\phi_1(t)+1)||\lambda_1^h - P_h^k\lambda_1||_0\phi(t)dt. \end{aligned} \tag{6.3}$$

Similarly, we have for N_1 by virtue of Lemma 6.1 and (6.1) that

$$\begin{aligned} |N_1| &\leq Ch\int_0^T (1+\phi_1(t))||\mathrm{div}\mathbf{w}||_0 dt \\ &\leq Ch\int_0^T (1+\phi_1(t))||\lambda_1^h - P_h^k\lambda_1||_0\phi(t)dt. \end{aligned} \tag{6.4}$$

We have by combining (6.2) with (6.3) and (6.4), and using Lemma 3.2 that

$$||\lambda_1^h - P_h^k\lambda_1||_0 \leq Ch(1+\phi_1(t)), \quad \text{for } k \geq 1.$$

It remains to treat N_2 for $k=0$. Since

$$\int_0^T (P_h^0\lambda - P_h^1\lambda, \delta_1^h)\phi_1(t)dt = 0 \quad \text{(see [26, 27])},$$

we know from the same arguments as those for (6.3) that

$$\begin{aligned} |N_2| &= \left|\int_0^T (P_h^1\lambda - \lambda, \delta_1^h)\phi_1(t)dt\right| \\ &\leq Ch\int_0^T (1+\phi_1(t))||\lambda_1^h - P_h^0\lambda_1||_0\phi(t)dt. \end{aligned}$$

Finally, the second inequality in Lemma 6.2 is a result of the first inequality in the same lemma and Theorem 3.1 together with the standard triangle inequality. □

Remark 6.1. *Using the similar duality argument to that as above we can easily obtain* [26]

$$||\lambda_2^h - P_h^k\lambda_2||_0 + ||\lambda_2^h - \lambda_2||_0 \leq C(1+\phi_2(t)).$$

Here we omit the details.

Lemma 6.3. *We have under the assumptions of Theorem 3.2 that*

$$||\mathbf{G}_1^h - \mathbf{G}_1||_{L^1(\Omega)} \leq Ch|\log h|(1+\phi_1(t)).$$

Proof. By Schwartz inequality and (3.1) we have

$$||\mathbf{G}_1^h - \mathbf{G}_1||_{L^1(\Omega)} \le C|\log h|^{1/2}||\mathbf{G}_1^h - \mathbf{G}_1||_{\beta^2}. \tag{6.5}$$

Let

$$\mathbf{\Psi}_1 := \beta^2(\mathbf{G}_1 - \mathbf{G}_1^h).$$

Then, we derive from Lemma 4.1 and (3.2) that

$$\begin{aligned} ||\mathbf{G}_1^h - \mathbf{G}_1||_{\beta^2}^2 &\le C_0(\alpha(\mathbf{G}_1 - \mathbf{G}_1^h), \mathbf{\Psi}_1 - \Pi_h^k\mathbf{\Psi}_1) \\ &\quad + C_0(\alpha(\mathbf{G}_1 - \mathbf{G}_1^h) + M**(\mathbf{G}_1 - \mathbf{G}_1^h), \Pi_h^k\mathbf{\Psi}_1) \\ &\quad - C_0(M**(\mathbf{G}_1 - \mathbf{G}_1^h), \Pi_h^k\mathbf{\Psi}_1) \\ &= C_0(\alpha(\mathbf{G}_1 - \mathbf{G}_1^h), \mathbf{\Psi}_1 - \Pi_h^k\mathbf{\Psi}_1) - C_0(\lambda_1 - \lambda_1^h, \mathrm{div}\Pi_h^k\mathbf{\Psi}_1) \\ &\quad - C_0(M**(\mathbf{G}_1 - \mathbf{G}_1^h), \Pi_h^k\mathbf{\Psi}_1) \\ &:= M_1 + M_2 + M_3. \end{aligned} \tag{6.6}$$

Now we consider M_i's individually. First, it follows from Lemma 6.5 below that

$$\begin{aligned} |M_1| &\le C_0||\alpha(\mathbf{G}_1 - \mathbf{G}_1^h)||_{\beta^2} \cdot ||\mathbf{\Psi}_1 - \Pi_h^k\mathbf{\Psi}_1||_{\beta^{-2}} \\ &\le \epsilon||\mathbf{G}_1 - \mathbf{G}_1^h||_{\beta^2}^2 + Ch^2|\log h|(1+\phi_1(t))^2. \end{aligned} \tag{6.7}$$

We know from (2.5) that

$$(\mathrm{div}\Pi_h^k\sigma, w_h) = (\mathrm{div}\sigma, w_h), \quad \forall\ w_h \in W_h,$$

which, together with Lemma 6.2, implies

$$\begin{aligned} |M_2| &= C_0|(P_h^k\lambda_1 - \lambda_1^h, \mathrm{div}\Pi_h^k\mathbf{\Psi}_1)| \\ &= C_0|(P_h^k\lambda_1 - \lambda_1^h, \mathrm{div}\mathbf{\Psi}_1)| \\ &\le Ch(1+\phi_1(t))||\mathrm{div}\mathbf{\Psi}_1||_0. \end{aligned} \tag{6.8}$$

Since there holds by (3.2)

$$\mathrm{div}\mathbf{\Psi}_1 = \nabla(\beta^2)\cdot(\mathbf{G}_1 - \mathbf{G}_1^h) + \beta^2(\delta_1^h - P_h^k\delta_1^h)\phi_1(t),$$

we have

$$||\mathrm{div}\mathbf{\Psi}_1||_0 \le C||\mathbf{G}_1 - \mathbf{G}_1^h||_{\beta^2} + Ch\phi_1(t).$$

Thus, we obtain from (6.8) that

$$|M_2| \le Ch^2(1+\phi_1(t))^2 + \epsilon||\mathbf{G}_1 - \mathbf{G}_1^h||_{\beta^2}^2. \tag{6.9}$$

It follows from Schwartz inequality and Lemma 6.5 that

$$\begin{aligned} |M_3| &\le C\left(\int_t^T ||\mathbf{G}_1 - \mathbf{G}_1^h||_{\beta^2} ds\right)||\Pi_h^k\mathbf{\Psi}_1 - \mathbf{\Psi}_1||_{\beta^{-2}} \\ &\quad + C\left(\int_t^T ||\mathbf{G}_1 - \mathbf{G}_1^h||_{\beta^2} ds\right)||\mathbf{\Psi}_1||_{\beta^{-2}} \\ &\le C\left(\int_t^T ||\mathbf{G}_1 - \mathbf{G}_1^h||_{\beta^2} ds\right)h|\log h|^{1/2}(1+\phi_1(t)) \\ &\quad + C\left(\int_t^T ||\mathbf{G}_1 - \mathbf{G}_1^h||_{\beta^2} ds\right)||\mathbf{G}_1 - \mathbf{G}_1^h||_{\beta^2} \\ &\le \epsilon||\mathbf{G}_1 - \mathbf{G}_1^h||_{\beta^2}^2 + C\left(\int_t^T ||\mathbf{G}_1 - \mathbf{G}_1^h||_{\beta^2} ds\right)^2 \\ &\quad + Ch^2|\log h|(1+\phi_1(t))^2. \end{aligned} \tag{6.10}$$

Combining (6.6) with (6.7), (6.9) and (6.10) gives for small and fixed $\epsilon > 0$ that

$$||\mathbf{G}_1 - \mathbf{G}_1^h||_{\beta^2}^2 \le Ch^2|\log h|(1+\phi_1(t))^2 + C\left(\int_t^T ||\mathbf{G}_1 - \mathbf{G}_1^h||_{\beta^2} ds\right)^2,$$

so that Gronwall's lemma yields

$$||\mathbf{G}_1 - \mathbf{G}_1^h||_{\beta^2} \le Ch|\log h|^{1/2}(1+\phi_1(t)). \tag{6.11}$$

Hence, Lemma 6.3 follows from (6.5) and (6.11). □.

Lemma 6.4. *Under the assumptions of Theorem 3.3, we have*

$$\begin{aligned}
&||\mathbf{G}_2 - \mathbf{G}_2^h||_{L^1(\Omega)} \le C|\log h|^{1/2}(1+\phi_2(t)),\\
&||\mathbf{G}_2 - \mathbf{G}_2^h||_0 \le Ch^{-1}(1+\phi_2(t)),\\
&||\nabla\lambda_2||_0 \le Ch^{-1}(1+\phi_2(t)).
\end{aligned}$$

Proof. We have by virtue of Schwartz inequality and (3.1) that

$$||\mathbf{G}_2 - \mathbf{G}_2^h||_{L^1(\Omega)} \le C|\log h|^{1/2}||\mathbf{G}_2 - \mathbf{G}_2^h||_{\beta^2}. \tag{6.12}$$

Let

$$\Psi_2 := \beta^2(\mathbf{G}_2 - \mathbf{G}_2^h).$$

Then, it follows from a similar argument to that for Lemma 6.3 that

$$\begin{aligned}
||\mathbf{G}_2 - \mathbf{G}_2^h||_{\beta^2}^2 &\le C_0(\alpha(\mathbf{G}_2 - \mathbf{G}_2^h), \Psi_2 - \Pi_h^k\Psi_2) - C_0(\lambda_2 - \lambda_2^h, \mathrm{div}\Pi_h^k\Psi_2)\\
&\quad -C_0(M**(\mathbf{G}_2 - \mathbf{G}_2^h), \Pi_h^k\Psi_2)\\
&:= M_1' + M_2' + M_3'.
\end{aligned} \tag{6.13}$$

Thus, we know from Lemma 6.5 below that

$$|M_1'| \le \epsilon||\mathbf{G}_2 - \mathbf{G}_2^h||_{\beta^2}^2 + C(1+\phi_2(t))^2. \tag{6.14}$$

Moreover, we see from Remark 6.1 and the same arguments as those for (6.8) that

$$|M_2'| \le C(1+\phi_2(t))||\mathrm{div}\Psi_2||_0. \tag{6.15}$$

It follows from (3.4) that

$$\mathrm{div}\Psi_2 = \nabla(\beta^2)\cdot(G_2 - G_2^h),$$

which yields by (6.15) that

$$|M_2'| \le \epsilon||\mathbf{G}_2 - \mathbf{G}_2^h||_{\beta^2}^2 + C(1+\phi_2(t))^2. \tag{6.16}$$

Also, we obtain according to the similar steps for (6.10) that

$$|M_3'| \le \epsilon||\mathbf{G}_2 - \mathbf{G}_2^h||_{\beta^2}^2 + C\left(\int_t^T ||\mathbf{G}_2 - \mathbf{G}_2^h||_{\beta^2} ds\right)^2 + C(1+\phi_2(t))^2. \tag{6.17}$$

Combining (6.14), (6.16) and (6.17) with (6.13), we have via using Gronwall's lemma that

$$||\mathbf{G}_2 - \mathbf{G}_2^h||_{\beta^2} \le C(1+\phi_2(t)).$$

Hence, from (6.12) we obtain

$$||\mathbf{G}_2 - \mathbf{G}_2^h||_{L^1(\Omega)} \le C|\log h|^{1/2}(1+\phi_2(t)).$$

By the H^2-regularity assumption, we have

$$||\nabla\lambda_2||_0 \le Ch^{-1}(1+\phi_2(t)).$$

Thus, from [26] we see that

$$||\mathbf{G}_2 - \mathbf{G}_2^h||_0 \leq Ch^{-1}(1+\phi_2(t)).$$

□

Lemma 6.5. *Let Ψ_i $(i=1,2)$ be the functions defined as before. Then, we have*

$$\begin{aligned} ||\Psi_1 - \Pi_h^k \Psi_1||_{\beta^{-2}} &\leq Ch|\log h|^{1/2}(1+\phi_1(t)), \\ ||\Psi_2 - \Pi_h^k \Psi_2||_{\beta^{-2}} &\leq C(1+\phi_2(t)). \end{aligned}$$

Proof. Recall

$$\Psi_i = \beta^2(\mathbf{G}_i - \mathbf{G}_i^h), \quad i=1,2,$$

and rewrite them as

$$\Psi_i = \beta^2(\mathbf{G}_i - \Pi_h^k \mathbf{G}_i) + \beta^2(\Pi_h^k \mathbf{G}_i - \mathbf{G}_i^h) := \Psi_{i1} + \Psi_{i2}.$$

Thus,

$$||\Psi_i - \Pi_h^k \Psi_i||_{\beta^{-2}} \leq ||\Psi_{i1} - \Pi_h^k \Psi_{i1}||_{\beta^{-2}} + ||\Psi_{i2} - \Pi_h^k \Psi_{i2}||_{\beta^{-2}}. \tag{6.18}$$

Since Π_h^k is a local projection operator, it follows from [26] that

$$||\Psi_{i1} - \Pi_h^k \Psi_{i1}||_{\beta^{-2}} \leq C||\Psi_{i1}||_{\beta^{-2}} \leq C||\mathbf{G}_i - \Pi_h^k \mathbf{G}_i||_{\beta^2} \leq Ch||\nabla^2 \lambda_i||_{\beta^2}.$$

Then, Theorem 3.1 and (6.28) below lead to

$$||\Psi_{i1} - \Pi_h^k \Psi_{i1}||_{\beta^{-2}} \leq \begin{cases} Ch|\log h|^{1/2}(1+\phi_1(t)), & \text{for } i=1, \\ C(1+\phi_2(t)), & \text{for } i=2. \end{cases} \tag{6.19}$$

Following [26] we obtain from Lemmas 6.1 and 6.4 that

$$||\Psi_{i2} - \Pi_h^k \Psi_{i2}||_{\beta^{-2}} \leq \begin{cases} Ch(1+\phi_1(t)), & \text{for } i=1, \\ C(1+\phi_2(t)), & \text{for } i=2. \end{cases} \tag{6.20}$$

Now, (6.19) and (6.20) lead (6.18) to

$$||\Psi_i - \Pi_h^k \Psi_i||_{\beta^{-2}} \leq \begin{cases} Ch|\log h|^{1/2}(1+\phi_1(t)), & \text{for } i=1, \\ C(1+\phi_2(t)), & \text{for } i=2, \end{cases}$$

which verifies the conclusions of Lemma 6.5. □

Lemma 6.6. *Under the assumptions of Theorem 3.3 there hold*

$$\begin{aligned} ||\lambda_2||_0 &\leq C|\log h|^{1/2}(1+\phi_2(t)), \\ ||\nabla\lambda_2||_{L^1(\Omega)} &\leq C|\log h|(1+\phi_2(t)), \\ ||\nabla^2\lambda_2||_{L^1(\Omega)} &\leq Ch^{-1}|\log h|^{1/2}(1+\phi_2(t)). \end{aligned}$$

Proof. From Schwarz's inequality and (3.1) we have

$$||\nabla\lambda_2||_{L^1(\Omega)} \leq C|\log h|^{1/2}||\nabla\lambda_2||_{\beta^2}. \tag{6.21}$$

Furthermore, it follows from (3.4) and Green's formula that

$$\begin{aligned} ||\nabla\lambda_2||_{\beta^2}^2 &= (\nabla\lambda_2, \beta^2\nabla\lambda_2) = -(\Delta\lambda_2, \beta^2\lambda_2) + \frac{1}{2}(\lambda_2, \Delta(\beta^2)\lambda_2) \\ &\leq |(\mathrm{div}\delta_2^h\phi_2(t), \beta^2\lambda_2)| + C||\lambda_2||_0^2 \\ &\leq C(\phi_2^2(t) + ||\lambda_2||_0^2). \end{aligned} \tag{6.22}$$

Now, let us consider the following auxiliary Dirichlet problem to bound $||\lambda_2||_0$:

$$\begin{aligned} -\Delta r = \lambda_2 \quad & \text{in } \Omega, \\ r = 0 \quad & \text{on } \partial\Omega. \end{aligned}$$

From the regularity assumption on the domain Ω we have

$$||\nabla^2 r||_0 \leq C||\lambda_2||_0. \tag{6.23}$$

In addition, it follows from (3.4) and Green's formula that

$$\begin{aligned} ||\lambda_2||_0^2 &= (\nabla\lambda_2, \nabla r) = -(\nabla^2\lambda_2, r) \\ &= (\mathrm{div}\delta_2^h, r)\phi_2(t) = -(\delta_2^h, \nabla r)\phi_2(t) \\ &:= N_*. \end{aligned} \tag{6.24}$$

Following the procedure in [26], we have, according to (3.5), (6.23) and the standard inverse estimate, that

$$||(\nabla r)^I||_\infty \leq C|\log h|^{1/2}||(\nabla r)^I||_1 \leq C|\log h|^{1/2}||\nabla^2 r||_0,$$

and

$$\begin{aligned} |N_*| &\leq \{|(\delta_2^h, \nabla r - (\nabla r)^I)| + |(\delta_2^h, (\nabla r)^I)|\}\phi_2(t) \\ &\leq C\left(||\nabla^2 r||_0 + ||\delta_2^h||_{L^1(\Omega)}||(\nabla r)^I||_\infty\right)\phi_2(t) \\ &\leq C\left(1 + |\log h|^{1/2}\right)||\lambda_2||_0\phi_2(t), \end{aligned} \tag{6.25}$$

where f^I stands for the standard locally regularized piecewise linear interpolation of f (see, for example, [26]).

Combining (6.25) with (6.24) yields

$$||\lambda_2||_0 \leq C\left(1 + |\log h|^{1/2}\right)\phi_2(t). \tag{6.26}$$

Now, (6.26) and (6.22) lead (6.21) to

$$||\nabla\lambda_2||_{L^1(\Omega)} \leq C|\log h|(1 + \phi_2(t)).$$

Again, we use Schwarz's inequality and (3.1) to obtain

$$||\nabla^2\lambda_2||_{L^1(\Omega)} \leq C|\log h|^{1/2}||\nabla^2\lambda_2||_{\beta^2}. \tag{6.27}$$

Following [26] we further have

$$||\nabla^2\lambda_2||_{\beta^2} \leq Ch^{-1}(1 + \phi_2(t)). \tag{6.28}$$

Thus,

$$||\nabla^2\lambda_2||_{L^1(\Omega)} \leq Ch^{-1}|\log h|^{1/2}(1 + \phi_2(t)).$$

□

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