$L^\infty$-ERROR ESTIMATES AND SUPERCONVERGENCE IN MAXIMUM NORM OF MIXED FINITE ELEMENT METHODS FOR NONFICKIAN FLOWS IN POROUS MEDIA

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Abstract. On the basis of the estimates for the regularized Green’s functions with memory terms, optimal order $L^\infty$-error estimates are established for the nonFickian flow of fluid in porous media by means of a mixed Ritz-Volterra projection. Moreover, local $L^\infty$-superconvergence estimates for the velocity along the Gauss lines and for the pressure at the Gauss points are derived for the mixed finite element method, and global $L^\infty$-superconvergence estimates for the velocity and the pressure are also investigated by virtue of an interpolation post-processing technique. Meanwhile, some useful a-posteriori error estimators are presented for this mixed finite element method.

Key Words. NonFickian flow, mixed finite element methods, the mixed Ritz-Volterra projection, Green’s functions, error estimates and superconvergence

1. Introduction

The nonFickian flow of fluid in porous media can be modelled by an integro-differential equation which seeks $u = u(x,t)$ such that

$$
\begin{align*}
    u_t &= \nabla \cdot \sigma + cu + f \quad \text{in } \Omega \times J, \\
    \sigma &= A(t) \cdot \nabla u - \int_0^t B(t,s) \cdot \nabla u(s) ds \quad \text{in } \Omega \times J, \\
    u &= g \quad \text{on } \partial \Omega \times J, \\
    u &= u_0(x) \quad x \in \Omega, \ t = 0.
\end{align*}
$$

(1.1)

where $\Omega \subset \mathbb{R}^d (d = 2,3)$ is an open bounded domain with smooth boundary $\partial \Omega$, $J = (0, T)$ with $T > 0$, $A(t) = A(x,t)$ and $B(t,s) = B(x,t,s)$ are two $2 \times 2$ or $3 \times 3$ matrices, and $A$ is positive definite, $c \leq 0$, $f$, $g$ and $u_0$ are known smooth functions. This kind of flow is complicated by the history effect characterizing various mixing length growth of the flow, which has been investigated, for example, in [9, 10] and references cited therein.
The numerical approximations of the problem (1.1) are available in extensive literature. See, for instance, [2, 3, 12, 13, 16, 14, 15, 20, 21, 22], where some optimal order error estimates and superconvergence have been established.

In the present paper, the solutions of (1.1) are approximated by mixed finite element methods [14, 15, 16]. Optimal order $L^\infty$-error estimates are obtained by employing a mixed Ritz-Volterra projection introduced in [16]. In addition, local $L^\infty$-superconvergence estimates for the velocity along the Gauss lines and for the pressure at the Gauss points are derived, and with the aid of an interpolation post-processing method global $L^\infty$-superconvergence estimates are also derived for the velocity and the pressure approximations. As a result of the global superconvergence, a-posteriori error indicators of the mixed finite element method are presented in the paper.

Compared with [16], where the optimal and superconvergence estimates of the mixed finite element method in $L^2$-norm have been discussed for the problem (1.1), the key point of the present paper is the introduction of the regularized Green’s functions with memory terms and the establishment of the various estimates for them and their mixed finite element approximations, which will play an important role in the forthcoming analysis in deriving the above optimal and superconvergence $L^\infty$-error estimates. As a result, the methodology and the techniques used in this paper are quite different from those in [16].

The paper is organized in the following manner. In Section 2, we give the approximate sub-space and the approximate problem. Two regularized Green’s functions and a Ritz-Volterra projection with memory terms for the mixed form for the problem (1.1) are introduced in Section 3. Also, in Section 3 the $L^1$-error estimates and related estimates for the mixed finite element approximations of the regularized Green’s functions are stated, and the $L^\infty$-error estimates for the mixed Ritz-Volterra projection are established. In Section 4, optimal order error estimates in maximum norm are given for the mixed finite element approximations. Section 5 is devoted to the local and global $L^\infty$-superconvergence analysis of the mixed finite element method, by which some a-posteriori error indicators are obtained for the mixed finite element method. Finally, the $L^1$-error estimates and related estimates for the mixed finite element approximations of the regularized Green’s functions are proved in Section 6.

2. The mixed finite element method

In this section, we give the mixed finite element approximate scheme for the parabolic integro-differential equation (1.1). For simplicity, the method will be presented on plane domains.

Let $W := L^2(\Omega)$ be the standard $L^2$ space on $\Omega$ with norm $\| \cdot \|_0$. Denote by

$$
V := H(\text{div}, \Omega) = \{ \sigma \in (L^2(\Omega))^2 \mid \nabla \cdot \sigma \in L^2(\Omega) \}
$$

the Hilbert space equipped with the following norm:

$$
\| \sigma \|_V := \left( \| \sigma \|_0^2 + \| \nabla \cdot \sigma \|_0^2 \right)^{1/2}.
$$

There are several ways to discretize the problem (1.1) based on the variables $\sigma$ and $u$; each method corresponds to a particular variational form of (1.1) [14, 22].

Let $T_h$ be a finite element partition of $\Omega$ into triangles or quadrilaterals which is quasi-uniform. Let $V_h \times W_h$ denote a pair of finite element spaces satisfying
the Brezzi-Babuska condition. Although there are now several choices for \( V_h \) and \( W_h \), here we only consider the Raviart-Thomas elements of order \( k \geq 0 \) [24]. The extension to other stable elements can be made without any difficulty.

Recall from [14] that the weak mixed formulation of (1.1) is given by finding \((u, \sigma) \in W \times V\) such that

\[
\begin{align*}
(u_t, w) - (\nabla \cdot \sigma, w) - (cu, w) &= (f, w), \\
(\alpha \sigma, v) + \int_0^t (M(t, s) \sigma(s), v) ds + (\nabla \cdot v, u) &= \langle g, v \cdot n \rangle, \quad v \in V, \\
u(0, x) &= u_0(x) \quad \text{in } L^2(\Omega),
\end{align*}
\]

where \( \alpha = A^{-1}(t) \), \( M(t, s) = R(t, s)A^{-1}(s) \) and \( R(t, s) \) is the resolvent of the matrix \( A^{-1}(t)B(t, s) \) and is given by

\[
R(t, s) = A^{-1}(t)B(t, s) + \int_s^t A^{-1}(t)B(t, \tau) R(\tau, s) ds, \quad t > s \geq 0.
\]

Here \( \langle \cdot, \cdot \rangle \) indicates the \( L^2 \)-inner product on \( \partial \Omega \).

The corresponding semi-discrete version is to seek a pair \((u_h, \sigma_h) \in W_h \times V_h\) such that

\[
\begin{align*}
(u_{h,t}, w_h) - (\nabla \cdot \sigma_h, w_h) - (cu_h, w_h) &= (f, w_h), \\
(\alpha \sigma_h, v_h) + \int_0^t (M_h(t, s) \sigma_h(s), v_h) ds + (u_h, \nabla \cdot v_h) &= \langle g, n \cdot v_h \rangle, \quad v_h \in V_h.
\end{align*}
\]

The discrete initial condition \( u_h(0, x) = u_{0,h} \), where \( u_{0,h} \in W_h \) is some appropriately chosen approximation of the initial data \( u_0(x) \), should be added to (2.2) for starting. The pair \((u_h, \sigma_h)\) is a semi-discrete approximation of the true solution of (1.1) in the finite element space \( W_h \times V_h \) [1, 6, 16, 14, 15], where \( \sigma_h(0, x) \) is chosen to satisfy the equation (2.2) with \( t = 0 \); namely, it is related to \( u_{0,h} \) as follows:

\[
(\alpha \sigma_h(0), v_h) + (u_{0,h}, \nabla \cdot v_h) = \langle g_0, n \cdot v_h \rangle,
\]

where \( g_0 = g(0, x) \) is the initial value of the boundary data.

From (2.1) and (2.2) we derive the following mixed finite element error equation:

\[
\begin{align*}
(u_t - u_{h,t}, w_h) - (\nabla \cdot (\sigma - \sigma_h), w_h) - (c(u - u_h), w_h) &= 0, \quad w_h \in W_h, \\
(\alpha(\sigma - \sigma_h), v_h) + \int_0^t (M_h(t, s)(\sigma - \sigma_h)(s), v_h) ds + (u - u_h, \nabla \cdot v_h) &= 0, \quad v_h \in V_h.
\end{align*}
\]

Throughout the paper, we often need the following Raviart-Thomas projection [7, 24],

\[
\Pi_h^k \times P_h^k : V \times W \rightarrow V_h \times W_h,
\]

which has the properties:

(i) \( P_h^k \) is the \( L^2(\Omega) \) projection.

(ii) \( \Pi_h^k \) and \( P_h^k \) satisfy

\[
(\nabla \cdot (\sigma - \Pi_h^k \sigma), w_h) = 0, \quad w_h \in W_h \quad \text{and} \quad (\nabla \cdot v_h, u - P_h^k u) = 0, \quad v_h \in V_h.
\]

(iii) the following approximation properties hold

\[
\begin{align*}
||\sigma - \Pi_h^k \sigma||_{0,p} &\leq Ch^r||\sigma||_{r,p}, \quad 1 \leq r \leq k + 1, \quad 1 \leq p \leq \infty, \\
||\nabla \cdot (\sigma - \Pi_h^k \sigma)||_{-s,p} &\leq Ch^{r+s}||\nabla \cdot \sigma||_{r,p}, \quad 0 \leq r, s \leq k + 1, \quad 1 \leq p \leq \infty, \\
||u - P_h^k u||_{-s,p} &\leq Ch^{r+s}||u||_{r,p}, \quad 0 \leq r, s \leq k + 1, \quad 1 \leq p \leq \infty.
\end{align*}
\]
Remark 2.1. \( \Pi_h^k \) is defined on a dense subspace of \( \mathbf{V} \).

3. The mixed Ritz-Volterra projection and its \( L^\infty \)-error estimates

In this section, we consider optimal order error estimates and superconvergence in \( L^\infty \)-norm for the mixed Ritz-Volterra projection. It is well-known that the regularized Green’s function plays an essential role in the analysis of maximum norm error estimates and superconvergence for finite element methods and mixed finite element methods of elliptic equations \([8, 11, 19, 26, 27, 28]\) and parabolic equations \([19]\). For the finite element method of parabolic integro-differential equations, maximum norm error estimates and superconvergence have been obtained in \([20, 21]\) using the modified regularized Green’s function with memory term. Here we consider the mixed finite element approximations for parabolic equations with memory, and it is expected that certain modification form of the standard regularized Green’s function with memory should be introduced, analyzed and used in our analysis.

First, let us define the following two linear operators \( M^* \) and \( M^{**} \) for any smooth function \( f(t) \) defined on \((0, T)\) by

\[
(M^* f)(t) := \int_0^t M(t, s) f(s) \, ds \quad \text{and} \quad (M^{**} f)(t) := \int_t^T M(s, t) f(s) \, ds.
\]

Then, from exchanging the order of integration we have

Lemma 3.1. There holds

\[
\langle M^* f, g \rangle_T := \int_0^T M^* f(t) g(t) \, dt = \int_t^T f(t) M^{**} g(t) \, dt := \langle f, M^{**} g \rangle_T.
\]

Lemma 3.2. Assume that \( f(t), g(t) \in L^1(0, T^*) \) and there exists \( C > 0 \) such that for any non-negative \( \phi(t) \in C^\infty(0, T) \),

\[
\left| \int_0^T f(t) \phi(t) \, dt \right| \leq C \left| \int_0^T g(t)(1 + \phi(t)) \, dt \right|, \quad 0 \leq T \leq T^*.
\]

Then, we have

\[
|f(t)| \leq C \left| g(t) + \int_0^t g(s) \, ds \right|, \quad \forall t \in (0, T), \ a.e.
\]

Especially,

\[
|f(t)| \leq C |g(t)|, \quad \forall t \in (0, T), \ a.e. \quad \text{if}
\]

\[
\left| \int_0^T f(t) \phi(t) \, dt \right| \leq C \left| \int_0^T g(t) \phi(t) \, dt \right|.
\]

Proof. Take \( \mu > 0 \) and let

\[
\phi_\mu(t, t_0) = \begin{cases} (C_\mu)^{-1} \exp \left( -\frac{\mu^2}{t - t_0} \right), & |t - t_0| < \mu, \\ 0, & |t - t_0| \geq \mu, \end{cases}
\]

where \( t_0 \) is any fixed point in \((0, T)\) and \( C_\mu := \mu \int_{|t| < 1} \exp \left( -\frac{1}{1 + t^2} \right) \, dt \). We see easily that for almost all \( t_0 \in (0, T) \),

\[
f(t_0) = \lim_{\mu \to 0} \int_0^T f(t) \phi_\mu(t, t_0) \, dt, \quad f \in C^\infty(0, T).
\]
Thus, if we take \( f_n(t) \in C^\infty(0,T) \) such that \( f_n(t) \to f(t) \) as \( n \to \infty \) in \( L^1(0,T) \),
then the result is true for all \( f_n(t) \). Therefore, it is true for \( f(t) \) via a limiting procedure. □

Now let us introduce some notations for the use later. For an arbitrary point \( z_0 \in \bar{\Omega} \), let
\[
\beta(z, z_0) := |z - z_0|^2 + \theta^2)^{1/2}
\]
be the weight function used in [25, 26, 28], where \( z = (x, y) \in \mathbb{R}^2 \), \( \theta = \gamma h \), and \( \gamma \)
is a positive number chosen appropriately. Moreover, as usual, for any \( \alpha \in \mathbb{R} \) we define a weighted norm by
\[
\|u\|_{\beta, \alpha}^2 := \int_Q \beta^{\alpha} u^2 dQ,
\]
and \( \| \cdot \|_{\beta^\alpha} \) is the weighted norm for \( Q = \Omega \). Then, we have [26, 28]
\[
\int_{\Omega} \beta^{-2} d\Omega \leq C|\log h|.
\]

Next we shall define two regularized Green’s functions with memory terms for the problem (1.1) in mixed form in the fashion analogous to that employed earlier for Galerkin methods [28]. Our results concerning the regularized Green’s functions and their mixed finite element approximations are very useful for establishing \( L^\infty \)-error estimates and superconvergence in maximum norm for the mixed finite element solution of (1.1).

For simplicity, we assume that \( c = 0 \). Thus, for an arbitrary point \( z_0 \in \bar{\Omega} \) the first pair of modified regularized Green’s function \((G_1, \lambda_1, \lambda_1(z, z_0))\) with memory is defined as the solution of the following system:
\[
\begin{align*}
\alpha G_1 + M * G_1 - \nabla \lambda_1 &= 0, & \text{in } \Omega \times (0,T), \\
\text{div} G_1 &= \delta^h_1 \phi(t), & \text{in } \Omega \times (0,T), \\
\lambda_1 &= 0, & \text{on } \partial \Omega \times (0,T),
\end{align*}
\]
where \( \phi_1(t) \in C^\infty(0,T) \), and \( \delta^h_1 = \delta^h_1(z, z_0) \in W_h \) is the regularized Dirac \( \delta \)-function at any fixed point \( z_0 \in \bar{\Omega} \) such that ([8, 11, 26, 27])
\[
||w_h||_{\infty} \leq C||w_h, \delta^h_1||, \quad w_h \in W_h.
\]

We also introduce the second pair of regularized Green’s function \((G_2, \lambda_2, \lambda_2(z, z_0))\) such that
\[
\begin{align*}
\alpha G_2 + M * G_2 - \nabla \lambda_2 &= \delta^h_2 \phi_2(t), & \text{in } \Omega \times (0,T), \\
\text{div} G_2 &= 0, & \text{in } \Omega \times (0,T), \\
\lambda_2 &= 0, & \text{on } \partial \Omega \times (0,T),
\end{align*}
\]
where \( \phi(t) \in C^\infty(0,T) \) and \( \delta^h_2 \) is either \( (\delta_2^h, 0) \) or \( (0, \delta_2^h) \) with \( \delta^h_2 \) being a regularized Dirac \( \delta \)-function at \( z_0 \), which depends upon the needs of our analysis, such that an analogue of (3.3) is also valid for \( \delta^h_2 \). In addition, \( \delta^h_2, \phi_1(t) \) and \( \phi_2(t) \) are required to satisfy
\[
\delta^h_2 \geq 0, \quad \int_{\Omega} \delta^h_2 d\Omega = 1; \quad \phi_i(t) \geq 0, \quad \int_0^T \phi_i(t) dt \leq 1, \quad i = 1, 2.
\]

Now and in what follows of this paper, the domain \( \Omega \) is assumed to be \( H^2 \)-regular [7]. Therefore, it is not difficult to show (see, for example, (3.6a) – (3.6d) in [26]) that the following result is true.
Theorem 3.1. There exists a positive constant $C > 0$, independent of $h, t,$ and $\phi_1(t)$, such that
\[
\|\nabla \lambda_1\|_0 \leq C |\log h|^{1/2}(1 + \phi_1(t)),
\|\nabla^2 \lambda_1\|_0 \leq C h^{-1}(1 + \phi_1(t)),
\|\nabla^2 \lambda_1\|_q \leq C |\log h|^{1/2}(1 + \phi_1(t)),
\|\nabla^2 \lambda_1\|_{L^1(\Omega)} \leq C |\log h|(1 + \phi_1(t)).
\]

Our main results regarding error estimates between $(G_1, \lambda_1)$ and $(G^h_1, \lambda^h_1)$, and $(G_2, \lambda_2)$ and $(G^h_2, \lambda^h_2)$ are contained in the following two theorems.

Theorem 3.2. Assume that $(G_1, \lambda_1)$ and $(G^h_1, \lambda^h_1)$ are the exact solution and the mixed finite element approximation of (3.2), respectively. Then, there exists a positive constant $C > 0$, independent of $h, t,$ and $\phi_1$, such that
\[
\|G^h_1 - G_1\|_0 \leq C(1 + \phi_1(t)),
\|G^h_1 - G_1\|_{L^1(\Omega)} \leq C h |\log h|(1 + \phi_1(t)),
\|\lambda^h_1 - \lambda_1\|_0 \leq C h |\log h|^{1/2}(1 + \phi_1(t)).
\]

Theorem 3.3. Assume that $(G_2, \lambda_2)$ and $(G^h_2, \lambda^h_2)$ are the exact solution and the mixed finite element approximation of (3.4), respectively. Then, there exists a positive constant $C > 0$, independent of $h, t,$ and $\phi_2$, such that
\[
\|G^h_2 - G_2\|_0 \leq C h^{-1}(1 + \phi_2(t)),
\|G^h_2 - G_2\|_{L^1(\Omega)} \leq C |\log h|^{1/2}(1 + \phi_2(t)),
\|\lambda^h_2 - \lambda_2\|_0 \leq C(1 + \phi_2(t)),
\|\nabla \lambda_2\|_0 \leq C h^{-1}(1 + \phi_2(t)),
\|\nabla \lambda_2\|_{L^1(\Omega)} \leq C |\log h|(1 + \phi_2(t)),
\|\nabla^2 \lambda_2\|_{L^1(\Omega)} \leq C h^{-1}|\log h|^{1/2}(1 + \phi_2(t)).
\]

Remark 3.1. We would like to point out that the estimate
\[
\|\nabla^2 \lambda_2\|_{L^1(\Omega)} \leq C h^{-1} \left( \log \frac{1}{h} \right)^{1/2} (1 + \phi_2(t))
\]
is not sharp, since it can be improved to
\[
(3.6) \quad \|\nabla^2 \lambda_2\|_{L^1(\Omega)} \leq C h^{-1}(1 + \phi_2(t))
\]
if the domain is smooth enough. A proof of (3.6) can be found in [25].

Remark 3.2. The proofs of Theorems 3.2 and 3.3 will be postponed to Section 6 where the weighted norm estimates are used.

Following the procedure for Theorems 3.3 and 3.4 in [26] together with the application of Gronwall’s lemma, we can also obtain the following results to be used in the superconvergence analysis.

Theorem 3.4. Assume that $\Omega$ is a plane rectangular domain and $q \in [1, \infty]$. Then, we have
\[
\|G^h_1\|_q \leq C h^{\min(0, \frac{q}{q-1})} |\log h|^{1/2}(1 + \phi_1(t)) \quad \text{and} \quad \|G_1 - G^h_1\|_q \leq (C(q) + C |\log h|) h^{1-\frac{q}{p}}(1 + \phi_1(t)), \quad 1 < q < \infty,
\]
where $p = \frac{q}{q-1}$ is the conjugate of $q$. 

Theorem 3.5. For \( q \in [1, \infty] \), there hold
\[
\|G_2^h\|_q \leq \begin{cases} 
Ch^{-\frac{2}{3}}|\log h|(1 + \phi_2(t)), & 1 \leq q < 2, \\
Ch^{-\frac{2}{3}}(1 + \phi_2(t)), & q \geq 2,
\end{cases}
\]
\[
\|G_2 - G_2^h\|_q \leq (C(q) + C|\log h|^{1/2})h^{-\frac{2}{3}}(1 + \phi_2(t)),
\]
where \( p = \frac{q}{q-1} \).

In the following we shall present the error estimates in the maximum norm for the mixed Ritz-Volterra projection. To this end, we first give its definition [16].

Definition 3.1. For \((u, \sigma) \in W \times V\) we define a pair \((\bar{u}_h, \bar{\sigma}_h) : [0, T] \to W_h \times V_h\) such that
\[
(3.7) \quad (\alpha(\sigma - \bar{\sigma}_h) + M*(\sigma - \bar{\sigma}_h), v_h) + (u - \bar{u}_h, \text{div} v_h) = 0, \quad v_h \in V_h
\]
\[
(\text{div}(\sigma - \bar{\sigma}_h), w_h) = 0, \quad w_h \in W_h,
\]
where \(\alpha = A^{-1}\). The pair \((\bar{u}_h, \bar{\sigma}_h)\) is called the mixed Ritz-Volterra projection of \((u, \sigma)\). It has been proved in [16] that the solution of (3.7) exists uniquely for a given pair \((u, \sigma)\).

The following lemma is basic to the main results of this section.

Lemma 3.3. Assume that \((\bar{u}_h, \bar{\sigma}_h)\) is the mixed Ritz-Volterra projection of \((u, \sigma) \in W \times V\). Then we have
\[
\int_0^T (\bar{u}_h - P_h^k u, \delta_1^h)\phi_1(t)dt = \int_0^T (\alpha(\sigma - \Pi_h^k \sigma) + M*(\sigma - \Pi_h^k \sigma), G_1^h)dt,
\]
\[
\int_0^T (\bar{\sigma}_h - \Pi_h^k \sigma, \delta_2^h)\phi_2(t)dt = \int_0^T (\alpha(\sigma - \Pi_h^k \sigma) + M*(\sigma - \Pi_h^k \sigma), G_2^h)dt.
\]

Proof. It follows from (3.2) and its corresponding mixed finite element error equation to (2.4) that
\[
(\bar{u}_h - P_h^k u, \delta_1^h \phi_1(t)) = (\bar{u}_h - P_h^k u, \text{div} G_1^h) = (\bar{u}_h - P_h^k u, \text{div} G_1^h).
\]

Note that \(P_h^k\) is a local \(L^2\)-projection operator. Thus, we know from (2.5) that
\[
(\bar{u}_h - P_h^k u, \delta_1^h \phi_1(t)) = (\bar{u}_h - u, \text{div} G_1^h)
\]
which, together with (3.7), leads to
\[
(\bar{u}_h - P_h^k u, \delta_1^h \phi_1(t)) = (\alpha(\sigma - \bar{\sigma}_h) + M*(\sigma - \bar{\sigma}_h), G_1^h)
\]
\[
= (\alpha(\sigma - \Pi_h^k \sigma) + M*(\sigma - \Pi_h^k \sigma), G_1^h)
\]
\[
+ (\alpha(\Pi_h^k \sigma - \bar{\sigma}_h) + M*(\Pi_h^k \sigma - \bar{\sigma}_h), G_1^h).
\]

Hence,
\[
(3.8) \quad \int_0^T (\bar{u}_h - P_h^k u, \delta_1^h)\phi_1(t)dt = \int_0^T (\alpha(\sigma - \Pi_h^k \sigma) + M*(\sigma - \Pi_h^k \sigma), G_1^h)dt
\]
\[
+ \int_0^T (\alpha(\Pi_h^k \sigma - \bar{\sigma}_h) + M*(\Pi_h^k \sigma - \bar{\sigma}_h), G_1^h)dt
\]
\[
:= K_1 + K_2.
\]
However, it follows from Lemma 3.1 and the mixed finite element approximation of (3.2) as well as Green's formula that

\[
K_2 = \int_0^T (\alpha G_h^b + M \ast \Phi_1, \Pi_h^b \sigma - \bar{\sigma}_h) dt
\]

\[
= \int_0^T (\nabla \lambda_h^b, \Pi_h^b \sigma - \bar{\sigma}_h) dt
\]

\[
= -\int_0^T (\lambda_h^b, \text{div}((\Pi_h^b \sigma - \bar{\sigma}_h))) dt,
\]

which, together with (2.5) and (3.7), yields

\[
K_2 = -\int_0^T (\lambda_h^b, \text{div}((\Pi_h^b \sigma - \bar{\sigma}_h))) dt - \int_0^T (\lambda_h^b, \text{div}(\sigma - \bar{\sigma}_h)) dt = 0.
\]

Thus, from (3.8) we know that the first identity in Lemma 3.3 is true.

To prove the second identity, we use (3.4) and its mixed finite element error equation to see that

\[
(\bar{\sigma}_h - \Pi^b_h \sigma, \delta^2_2) \phi_2(t) = (\alpha G_2^b + M \ast \Phi_2^b, \bar{\sigma}_h - \Pi^b_h \sigma) + (\lambda^b_2, \text{div}(\bar{\sigma}_h - \Pi^b_h \sigma)).
\]

Thus, by means of Lemma 3.1, (2.5) and (3.7) we have

\[
\int_0^T (\bar{\sigma}_h - \Pi^b_h \sigma, \delta^2_2) \phi_2(t) dt = \int_0^T (\alpha(\bar{\sigma}_h - \Pi^b_h \sigma) + M \ast (\bar{\sigma}_h - \Pi^b_h \sigma), G^b_2) dt
\]

\[
+ \int_0^T (\lambda^b_2, \text{div}(\bar{\sigma}_h - \sigma)) dt + \int_0^T (\lambda^b_2, \text{div}(\sigma - \Pi^b_h \sigma)) dt
\]

\[
= \int_0^T (\alpha(\bar{\sigma}_h - \Pi^b_h \sigma) + M \ast (\bar{\sigma}_h - \Pi^b_h \sigma), G^b_2) dt
\]

\[
= \int_0^T (\alpha(\sigma - \Pi^b_h \sigma) + M \ast (\sigma - \Pi^b_h \sigma), G^b_2) dt
\]

\[
+ \int_0^T (u - \bar{u}_h, \text{div}G^b_2) dt
\]

\[
= \int_0^T (\alpha(\sigma - \Pi^b_h \sigma) + M \ast (\sigma - \Pi^b_h \sigma), G^b_2) dt,
\]

where \(\text{div}G^b_2 = 0\) has been used. This completes the proof. \(\square\)

We are now ready to show the maximum norm error estimate for the mixed Ritz-Volterra projection. First, we consider it for \(\bar{u}_h - P_h^k u\).

**Theorem 3.6.** Let \((\bar{u}_h, \bar{\sigma}_h)\) be the Ritz-Volterra projection of \((u, \sigma)\). Then, there exists a constant \(C > 0\), independent of \(h\) and \(t\), such that

\[
\|\bar{u}_h - P_h^k u\|_{\infty} \leq \begin{cases}
C h \log h \|(|\sigma - \Pi_h^k \sigma|_\infty + |\log h|^{-1/2}|||I - P_h^k\| \nabla \cdot \sigma||_0), & k = 0, \\
C (|||\sigma - \Pi_h^k \sigma||_0 + h \log h|^{1/2} |||I - P_h^k\| \nabla \cdot \sigma||_0), & k = 0, \\
C h \log h \|(|\sigma - \Pi_h^k \sigma|_\infty + h\|||I - P_h^k\| \nabla \cdot \sigma||_\infty), & k \geq 1,
\end{cases}
\]

where \(||u||_{r,p} := ||u(t)||_{r,p} + \int_0^t ||u(s)||_{r,p} ds\), \(-\infty \leq r \leq \infty, 1 \leq p \leq \infty, t > 0\). As usual, \(||u||_{r,p}\) is simply denoted by \(||u||_r\), when \(p = 2\).
Proof. For any point $z_0 \in \tilde{\Omega}$, let $\delta^h_{z_0}$ be the regularized Dirac $\delta$-function associated with this point $z_0$, and then we find from Lemma 3.3 that

$$\int_0^T (\bar{u}_h - P_h^0 u, \delta^h_{z_0}) \phi_1(t) dt = \int_0^T (\alpha(\sigma - \Pi_h^k \sigma) + M \ast (\sigma - \Pi_h^k \sigma), \mathbf{G}_h^k - \mathbf{G}_1) dt + \int_0^T (\alpha(\sigma - \Pi_h^k \sigma) + M \ast (\sigma - \Pi_h^k \sigma), \mathbf{G}_1) dt := K_{11} + K_{22}.$$  

It is easy to see from Lemma 3.1, (2.5) and (3.2) that

$$K_{22} = \int_0^T (\alpha \mathbf{G}_1 + M \ast \mathbf{G}_1, \sigma - \Pi_h^k \sigma) dt$$

$$= \int_0^T (\nabla \lambda_1, \sigma - \Pi_h^k \sigma) dt$$

$$= - \int_0^T (\lambda_1, \text{div}(\sigma - \Pi_h^k \sigma)) dt$$

$$= - \int_0^T (\lambda_1 - P_h^k \lambda_1, \text{div}(\sigma - \Pi_h^k \sigma)) dt$$

$$= - \int_0^T (\lambda_1 - P_h^k \lambda_1, (I - P_h^k) \text{div} \sigma) dt.$$  

Thus, we have for $k = 0$ that

$$\left| \int_0^T (\bar{u}_h - P_h^0 u, \delta^h_{z_0}) \phi_1(t) dt \right| \leq \begin{cases} C \int_0^T (||\sigma - \Pi_h^0 \sigma||_{L^\infty} ||\mathbf{G}_h^k - \mathbf{G}_1||_{L^1(\Omega)} + ||\lambda_1 - P_h^0 \lambda_1||_0 ||(I - P_h^0) \text{div} \sigma||_0) dt, \\
C \int_0^T (||\sigma - \Pi_h^0 \sigma||_0 ||\mathbf{G}_h^k - \mathbf{G}_1||_0 + ||(I - P_h^0) \lambda_1||_0 ||(I - P_h^0) \text{div} \sigma||_0) dt. \end{cases}$$

Noticing that for $k = 0$ by Theorem 3.1,

$$||\lambda_1 - P_h^0 \lambda_1||_0 \leq Ch ||\nabla \lambda_1||_0 \leq Ch \log h^{1/2}(1 + \phi_1(t)),$$

it follows from the above inequality and Theorem 3.2 that for $k = 0$

$$\left| \int_0^T (\bar{u}_h - P_h^0 u, \delta^h_{z_0}) \phi_1(t) dt \right| \leq \begin{cases} Ch \log h \int_0^T (||\sigma - \Pi_h^0 \sigma||_{L^\infty} + ||\log h||^{-1/2} ||(I - P_h^0) \text{div} \sigma||_0)(1 + \phi_1(t)) dt, \\
C \int_0^T (||\sigma - \Pi_h^0 \sigma||_0 + h \log h^{1/2} ||(I - P_h^0) \text{div} \sigma||_0)(1 + \phi_1(t)) dt. \end{cases}$$

We now see from Lemma 3.2 and the arbitrariness of $\phi_1(t)$ that

$$|\langle \bar{u}_h - P_h^0 u, \delta^h_{z_0} \rangle| \leq \begin{cases} Ch \log h (||||\sigma - \Pi_h^0 \sigma||_{L^\infty} + ||\log h||^{-1/2} ||(I - P_h^0) \text{div} \sigma||_0), \\
C(||||\sigma - \Pi_h^0 \sigma||_0 + h \log h^{1/2} ||(I - P_h^0) \text{div} \sigma||_0), \end{cases}$$

from which and (3.3) we derive that for $k = 0$

$$||\bar{u}_h - P_h^0 u||_{L^\infty} \leq \begin{cases} Ch \log h (||||\sigma - \Pi_h^0 \sigma||_{L^\infty} + ||\log h||^{-1/2} ||(I - P_h^0) \text{div} \sigma||_0), \\
C(||||\sigma - \Pi_h^0 \sigma||_0 + h \log h^{1/2} ||(I - P_h^0) \text{div} \sigma||_0). \end{cases}$$

Therefore, Theorem 3.6 is true for $k = 0$.

For $k \geq 1$, we have by Theorem 3.1 that

$$||I(P_h^k)\lambda_1||_{L^1(\Omega)} \leq Ch^2 |||\nabla^2 \lambda_1||_{L^1(\Omega)} \leq Ch^2 ||h||(1 + \phi_1(t)),$$
which, together with Theorem 3.2, leads to
\[ \left| \int_0^T (\bar{u}_h - P_h^k u, \delta^h_i) \phi_1(t) dt \right| \leq C \int_0^T \left( ||\sigma - \Pi_h^k \sigma||_\infty \| G^h_1 - G_1 \|_{L^1(\Omega)} + ||(I - P_h^k)\lambda_1 \|_{L^1(\Omega)} ||(I - P_h^k)\text{div}\sigma||_\infty \right) dt \]
\leq C h |\log h| \int_0^T \left( ||\sigma - \Pi_h^k \sigma||_\infty + h ||(I - P_h^k)\text{div}\sigma||_\infty \right)(1 + \phi_1(t)) dt.

This, together with Lemma 3.2 and (3.3), yields that for \( k \geq 1 \)
\[ ||\bar{u}_h - P_h^k u||_\infty \leq C h |\log h| ||\sigma - \Pi_h^k \sigma||_\infty + h ||(I - P_h^k)\text{div}\sigma||_\infty. \]

This completes the proof of Theorem 3.6.

**Theorem 3.7.** Under the same conditions as for Theorem 3.6, there exists a constant \( C > 0 \), independent of \( h \) and \( t \), such that
\[ ||\sigma - \bar{\sigma}_h||_\infty \leq C |\log h|^{1/2} ||\sigma - \Pi_h^k \sigma||_\infty + h |\log h|^{5/2} ||(I - P_h^k)\text{div}\sigma||_\infty, \]
where \( \delta_{kj} \) is the usual Kronecker symbol.

**Proof.** It suffices to bound \( \bar{\sigma}_h - \Pi_h^k \sigma \) in \( L^\infty \)-norm. By Lemma 3.3 we have that
\[ \int_0^T (\bar{\sigma}_h - \Pi_h^k \sigma, \delta^h_i) \phi_2(t) dt = \int_0^T \left( \alpha (\sigma - \Pi_h^k \sigma) + M * (\sigma - \Pi_h^k \sigma), \sigma \right) dt + \int_0^T \left( \alpha (\sigma - \Pi_h^k \sigma) + M * (\sigma - \Pi_h^k \sigma), \sigma \right) dt := M_1 + M_2. \]

Similar to (3.9), it follows from Lemma 3.1, (2.5) and (3.4) that
\[ M_2 = \int_0^T (\alpha G_2 + M * G_2, \sigma - \Pi_h^k \sigma) dt = \int_0^T \left( \nabla \lambda_2 + \delta^h_2 \phi_2(t), \sigma - \Pi_h^k \sigma \right) dt = - \int_0^T (\lambda_2, \text{div}(\sigma - \Pi_h^k \sigma)) dt + \int_0^T (\delta^h_2, \sigma - \Pi_h^k \sigma) \phi_2(t) dt = \int_0^T (P_h^k \lambda_2 - \lambda_2, (I - P_h^k)\text{div}\sigma) dt + \int_0^T (\delta^h_2, \sigma - \Pi_h^k \sigma) \phi_2(t) dt.

Thus, we have by (3.5) and Theorem 3.3 that
\[ \left| \int_0^T (\bar{\sigma}_h - \Pi_h^k \sigma, \delta^h_i) \phi_2(t) dt \right| \leq C \int_0^T \left( ||\sigma - \Pi_h^k \sigma||_\infty (\| G^h_2 - G_2 \|_{L^1(\Omega)} + ||\delta^h_2\|_{L^1(\Omega)} \phi_2(t)) dt \right) + \int_0^T ||\lambda_2 - P_h^k \lambda_1 ||_{L^1(\Omega)} ||(I - P_h^k)\text{div}\sigma||_\infty dt \leq C |\log h|^{1/2} \int_0^T ||\sigma - \Pi_h^k \sigma||_\infty (1 + \phi_2(t)) dt \]
\[ + C h |\log h|^{5/2} \int_0^T ||(I - P_h^k)\text{div}\sigma||_\infty (1 + \phi_2(t)) dt, \]
which implies by virtue of Lemma 3.2 and the analogue of (3.3) for \( \delta^2 \) that
\[
\| \tilde{\sigma}_h - \Pi_k^h \sigma \|_{\infty} \leq C \log h^{1/2} (\| \sigma - \Pi_k^h \sigma \|_{\infty} + h \log h^{\delta_{\infty} / 2} (\| I - P_k^h \| \text{div}\sigma \|_{\infty})).
\]
This, together with the standard triangle inequality, yields Theorem 3.7. \( \square \)

**Remark 3.3.** By (3.6) we have
\[
\| \lambda_2 - P_k^h \lambda_2 \|_{L^2(\Omega)} \leq C h(1 + \phi_\tau(t)), \quad k \geq 1,
\]
for sufficiently regular \( \partial \Omega \). Thus, Theorem 3.7 can be improved to become
\[
\| \sigma - \tilde{\sigma}_h \|_{\infty} \leq C (\| \log h^{1/2} (\| \sigma - \Pi_k^h \sigma \|_{\infty} + h \| (I - P_k^h) \| \text{div}\sigma \|_{\infty})
\]
for \( k \geq 1 \) if \( \partial \Omega \) is sufficiently smooth.

**Corollary 3.1.** Under the assumptions of Theorem 3.6, we have
\[
\| \tilde{u}_h - P_k^h u \|_{\infty} \leq \left\{ \begin{array}{ll}
Ch^2 \| \sigma \|_{k+1,\infty} + \| \log h \|^{1/2} \| \sigma \|_{2}, & k = 0, \\
Ch^{k+2} \| \sigma \|_{k+1,\infty} & k \geq 1.
\end{array} \right.
\]

**Proof.** By (2.6) we have for the interpolation operators \( \Pi_k^h \) and \( P_k^h \) that
\[
\| f - \Pi_k^h f \|_{0,p} \leq Ch^{k+1} \| f \|_{k+1,p}, \quad 1 \leq p \leq \infty,
\]
\[
\| g - P_k^h g \|_{0,p} \leq Ch^{k+1} \| g \|_{k+1,p}, \quad 1 \leq p \leq \infty.
\]
Then, we find from Theorem 3.6 that for \( k = 0 \)
\[
\| \tilde{u}_h - P^h_0 u \|_{\infty} \leq Ch \| \log h \| (\| \sigma - \Pi^h_0 \sigma \|_{\infty} + \| \log h \|^{1/2} (\| I - P^h_0 \| \text{div}\sigma \|_{0}))
\]
\[
\leq Ch^2 \| \sigma \|_{k+1,\infty} + \| \log h \|^{1/2} \| \sigma \|_{2}.
\]
The estimates for \( k \geq 1 \) can be derived along the same line. \( \square \)

**Corollary 3.2.** We have under the assumptions of Theorem 3.6 that
\[
\| \sigma - \tilde{\sigma}_h \|_{\infty} \leq Ch^{k+1} \| \log h \|^{(\delta_{\infty} + 1)/2} \| \sigma \|_{k+1,\infty}, \quad k \geq 0.
\]

**4. Optimal order \( L^\infty \)-error estimates for mixed finite element solutions**

In this section we consider error estimates in maximum norms for the mixed finite element approximation of (1.1) by means of the \( L^\infty \)-error estimates for the mixed Ritz-Volterra projection and the estimates for the regularized Green’s functions given in the last section. First, the following error estimate of \( \| u_t - u_{h,t} \| \) is demonstrated for the future needs. To this purpose, we recall from [16] the following two lemmas.

**Lemma 4.1.** Assume that the matrix \( A(t) \) is positive definite. Then, the norms
\[
\| \sigma \|_{A_h} := (\sigma, \sigma) \quad \text{and} \quad \| \sigma \|_{A_h^{-1}} := (A^{-1} \sigma, \sigma)
\]
are equivalent.

**Lemma 4.2.** Let \((\tilde{u}_h, \tilde{\sigma}_h)\) be the mixed Ritz-Volterra projection of \((u, \sigma) \in W \times V\) defined by (3.7). Then, there is a positive constant \( C > 0 \), independent of \( h > 0 \), such that, for any positive integer \( m \),
\[
\| D^m_t (u - \tilde{u}_h) \|_0 \leq C \left\{ \begin{array}{ll}
h \| u(t) \|_{r,2,m}, & k = 0, \\
h^r \| u(t) \|_{r+1,2,m}, & k \geq 1 \text{ and } 2 \leq r \leq k + 1,
\end{array} \right.
\]
\[
\| D^m_t (\sigma - \tilde{\sigma}_h) \|_0 \leq C h^r \| u(t) \|_{r+1,2,m}, \quad 1 \leq r \leq k + 1,
\]
where
\[
\| u(t) \|_{r,p,m} := \sum_{j=0}^m \| D^j_t u(t) \|_{r,p} + \int_0^t \sum_{j=0}^m \| D^j_t u(s) \|_{r,p} ds, \quad -\infty \leq r \leq \infty, \quad 1 \leq p \leq \infty, \quad t \geq 0.
\]
Theorem 4.1. Assume that \((u, \sigma)\) and \((u_h, \sigma_h)\) are the solutions of (2.1) and (2.2), respectively, and \((u_0, \sigma_0)\) are chosen as follows:

\[
\begin{align*}
\alpha(0)(\sigma_h(0) - \sigma(0)), v_h) + (\text{div} v_h, u_h(0) - u_0) = 0, & \quad v_h \in V_h, \\
\text{div} (\sigma_h(0) - \sigma(0)), w_h) = 0, & \quad w_h \in W_h.
\end{align*}
\]

Then we have for \(k \geq 0\) that

\[
||u_t - u_{h,t}||_0 \leq Ch \left\{ ||u||_2 + ||u_t||_2 + \left[ \int_0^T (||u||_2^2 + ||u_t||_2^2) ds \right]^{1/2} \right\}
\]

and for \(k \geq 1\) that

\[
||u_t - u_{h,t}||_0 \leq Ck^{k+1} \left\{ ||u||_{k+1} + ||u_t||_{k+1} + \left[ \int_0^T (||u||_{k+1}^2 + ||u_t||_{k+1}^2) ds \right]^{1/2} \right\}.
\]

Proof. Let

\[
u - u_h = (u - \bar{u}_h) + (\bar{u}_h - u_h) := \rho + \rho_h,
\]

\[
\sigma - \sigma_h = (\sigma - \sigma_h) + (\sigma_h - \sigma_h) := \theta + \theta_h,
\]

where \((\bar{u}_h, \sigma_h)\) is the Ritz-Volterra projection of \((u, \sigma)\). Then, by Lemma 4.2 we have

\[
||\rho_t||_0 \leq \begin{cases} Ck^{2,2,1} ||u||_{2,2,1}, & k = 0, \\
Ck^{k+1} ||u||_{k+1,2,1}, & k \geq 1,
\end{cases}
\]

\[
||\rho_{t,t}||_0 \leq \begin{cases} Ch ||u||_{2,2,2}, & k = 0, \\
Ch^{k+1} ||u||_{k+1,2,2}, & k \geq 1.
\end{cases}
\]

Thus, only \(||\rho_{h,t}||_0\) needs to be estimated in order to get the estimate for \(||u_t - u_{h,t}||_0\).

For this purpose, we first get the estimate for \(\theta_h(t)\).

We derive from (3.7) and (4.1) that

\[
(\alpha(0)\theta_h(0), v_h) + (\text{div} v_h, \rho_h(0)) = 0, \quad v_h \in V_h, \\
(\text{div} \theta_h(0), w_h) = 0, \quad w_h \in W_h,
\]

which, together with the uniqueness of the solution to (3.7), implies

\[
\theta_h(0) = \rho_h(0) = 0.
\]

It follows from (3.7) and (2.4) that \((\rho_h, \theta_h)\) satisfies

\[
(\alpha \theta_h + M * \theta_h, v_h) + (\text{div} v_h, \rho_h) = 0, \quad v_h \in V_h, \\
(\rho_{h,t}, w_h) - (\text{div} \theta_h, w_h) = -(\rho_t, w_h), \quad w_h \in W_h.
\]

Differentiate (4.4) to obtain

\[
(\alpha \theta_h + \alpha \rho_{h,t} + M(t)\theta_h + M(t) * \theta_h, v_h) + (\text{div} v_h, \rho_{h,t}) = 0, \quad v_h \in V_h,
\]

and then we have by setting \(v_h = \theta_h \) in the above equation and \(w_h = \rho_{h,t}\) in (4.4) that

\[
||\rho_{h,t}||_0^2 + (\alpha \theta_{h,t}, \theta_h) + (\alpha \theta_{h,t}, \theta_h) = -(M \theta_h + M \theta_h, \theta_h) - (\rho_t, \rho_{h,t}).
\]

Since

\[
\alpha(\theta_{h,t}^2) = (\alpha \theta_{h,t}^2) - \alpha t \theta_h^2,
\]

then

\[
(\alpha \theta_{h,t}, \theta_h) = \frac{1}{2} \frac{d}{dt} ||\theta_h||_A^{-1} - \frac{1}{2} (\alpha_t \theta_h, \theta_h).
\]
Hence, (4.5) can be rewritten as

\[ \|\rho_h(t)\|^2 + \frac{1}{2} \frac{d}{dt} \|\theta_h\|_{L^2}^2 + \frac{1}{2} (\alpha t \theta_h, \theta_h) = -(M \theta_h + M_t \ast \theta_h, \theta_h) - (\rho_t, \rho_h). \]

Thus, we find via integrating the above equation, and using Lemma 4.1, (4.3), Gronwall's lemma and the \( \epsilon \)-inequality that

\[ \|\theta_h\|^2 \leq C \int_0^t \|\rho_t\|^2 ds. \]

Next we shall obtain the estimate for \( \|\rho_h, t\|_0 \). To this end, we differentiate (4.4) to obtain

\[ (\alpha_t \theta_h + \alpha \theta_{h, t} + M(t) \theta_h + M_t \ast \theta_h, v_h) + (\text{div} \, \rho_h, \rho_{h, t}) = 0, \quad v_h \in V_h, \]
\[ (\rho_{h, t}, w_h) - (\text{div} \, \theta_{h, t}, w_h) = - (\rho_t, w_h), \quad w_h \in W_h. \]

And hence, we have by setting \( v_h = \theta_{h, t} \) and \( w_h = \rho_{h, t} \) in (4.7) and following the procedure for (4.6) that

\[ \|\rho_{h, t}\|^2 \leq C \left\{ \|\rho_{h, t}(0)\|^2 + \int_0^t \|\rho_t\|^2 ds + \int_0^t \|\rho_{h, t}\|^2 ds \right\}. \]

By letting \( t = 0 \) and \( w_h = \rho_{h, t}(0) \) in (4.4) we obtain from (4.3) that

\[ \|\rho_{h, t}(0)\|_0 \leq \|\rho_t(0)\|_0, \]

which, together with (4.8) and (4.2), leads to

\[ \|\rho_{h, t}\|^2 \leq C \left\{ \|\rho_t(0)\|^2 + \int_0^t \left( \|\rho_t\|^2 + \|\rho_{h, t}\|^2 \right) ds \right\} \]

\[ \leq \begin{cases} Ch^2 \left[ \|u(0)\|^2 + \|u_t(0)\|^2 + \int_0^t \left( \|u\|^2 + \|u_t\|^2 \right) ds \right], & k = 0, \\
Ch^{2k+2} \left[ \|u(0)\|^2 + \int_0^t \left( \|u\|^2 + \|u_t\|^2 \right) ds \right] \] + \|u_t(0)\|^2_{k+1} + \int_0^t \left( \|u\|^2 + \|u_t\|^2 \right) ds, & k \geq 1. \]

This completes the proof of the theorem by (4.2).

Now we are in a position to get our main theorem in this section.

**Theorem 4.2.** We have under the assumptions of Theorem 4.1 that for \( k = 0 \)

\[ \|u - u_h\|_\infty \leq Ch \left[ \|u\|_{1, \infty} + \|\log h\|^{1/2} (\|u\|_2 + \|u_t\|_2) \right] \]

\[ + Ch \|\log h\|^{1/2} \left[ \int_0^t \left( \|u\|^2 + \|u_t\|^2 \right) ds \right]^{1/2} \]

and

\[ \|\sigma - \sigma_h\|_\infty \leq Ch \|\log h\|^{1/2} \left( \|\log h\|^{1/2} (\|u\|_{2, \infty} + \|u\|_2 + \|u_t\|_2) \right) \]

\[ + Ch \|\log h\|^{1/2} \left[ \int_0^t \left( \|u\|^2 + \|u_t\|^2 \right) ds \right]^{1/2} \];

for \( k \geq 1 \)

\[ \|u - u_h\|_\infty \leq Ch^{k+1} \|\log h\|^{1/2} \left( \|\log h\|^{1/2} (\|u\|_{k+1, \infty} + \|u\|_{k+1} + \|u_t\|_{k+1}) \right) \]

\[ + Ch^{k+1} \|\log h\|^{1/2} \left[ \int_0^t \left( \|u\|^2 + \|u_t\|^2 \right) ds \right]^{1/2} \]
and

\[ \|\sigma - \sigma_h\|_\infty \leq Ch^{k+1} |\log h|^{1/2} (\|u\|_{k+2, \infty} + \|u\|_{k+1} + \|u_t\|_{k+1}) \]
\[ + Ch^{k+1} |\log h|^{1/2} \int_0^t \left( \|u\|_{k+1}^2 + \|u_t\|_{k+1}^2 + \|u_{tt}\|_{k+1}^2 \right) ds \]

Proof. With the same decomposition of the errors as that in Theorem 4.1, we know from Corollaries 3.1 and 3.1 that

\[ \|\rho\|_\infty \leq \|u - P_h u\|_\infty + \|P_h u - \bar{u}_h\|_\infty \]
\[ \leq \begin{cases} Ch (\|u\|_{1, \infty} + |\log h|^{1/2} \|u\|_2), & k = 0, \\ Ch^{k+1} |\log h|^{1/2} \|u\|_{k+1, \infty}, & k \geq 1, \end{cases} \]
\[ \|\theta\|_\infty \leq Ch^{k+1} |\log h|^{(\alpha + 1)/2} \|\sigma\|_{k+1, \infty}. \]

Therefore, only \( \|\rho_h\|_\infty \) and \( \|\theta_h\|_\infty \) are left to be estimated.

Set \( v_h = G_1^h \) in (4.4) to obtain from the mixed finite element approximation of (3.2) that

\[ (\delta_1^h \phi_1(t), \rho_h) = (\text{div} G_1^h, \rho_h) = - (\alpha \theta_h + M * \theta_h, G_1^h), \]

so that it follows from the integration, Lemma 3.1, and the mixed finite element solution of (3.2) that

\[ \int_0^T (\delta_1^h, \rho_h) \phi_1(t) \, dt = - \int_0^T (\alpha G_1^h + M * \theta_h) \, dt \]
\[ = \int_0^T (\lambda_1^h, \text{div} \theta_h) \, dt \]
\[ = \int_0^T (\lambda_1^h - \lambda_1, \text{div} \theta_h) \, dt + \int_0^T (\lambda_1, \text{div} \theta_h) \, dt. \]

Since \( \lambda_1|_{\partial \Omega} = 0 \), it follows from Theorems 3.1 and 3.2 that

\[ \|\lambda_1\|_0 \leq C|\nabla \lambda_1|_0 \leq C|\log h|^{1/2} (1 + \phi_1(t)), \]
\[ \|\lambda_1 - \lambda_1^h\|_0 \leq Ch |\log h|^{1/2} (1 + \phi_1(t)). \]

Hence, we find from (4.10), Lemma 3.2 and (3.3) that

\[ \|\rho_h\|_\infty \leq C |\log h|^{1/2} (h+1) \|\text{div} \theta_h\|_0. \]

We know from (3.7) and the mixed finite element error equation (2.4) that

\[ (\text{div}(\sigma - \sigma_h), w_h) = 0, \quad w_h \in W_h, \]
\[ (\text{div}(\sigma - \sigma_h), w_h) = (u_t - u_{h,t}, w_h), \quad w_h \in W_h. \]

This implies

\[ (\text{div} \theta_h, w_h) = (\text{div}(\sigma_h - \sigma_h), w_h) = (\text{div}(\sigma - \sigma_h), w_h) = (u_t - u_{h,t}, w_h), \quad w_h \in W_h, \]

from which we have by means of the arbitrariness of \( w_h \in W_h \) that

\[ \|\text{div} \theta_h\|_0 \leq \|u_t - u_{h,t}\|_0. \]
Combining (4.11) with (4.12) and Theorem 4.1 leads to (4.13)

\[ \| \rho_h \|_\infty \leq \begin{cases} 
C h | \log h |^{1/2} \left[ \| u_0 \|_2 + \| u_t (0) \|_2 + \| u \|_2 + \| u_t \|_2 \right] \\
+ C h | \log h |^{1/2} \left[ \int_0^t \left( \| u \|_2^2 + \| u_t \|_2^2 + \| u_{tt} \|_2^2 \right) ds \right]^{1/2}, & k = 0, \\
C h^{k+1} | \log h |^{1/2} \left[ \| u_0 \|_{k+1} + \| u_t (0) \|_{k+1} + \| u \|_{k+1} + \| u_t \|_{k+1} \right] \\
+ C h^{k+1} | \log h |^{1/2} \left[ \int_0^t \left( \| u \|_{k+1}^2 + \| u_t \|_{k+1}^2 + \| u_{tt} \|_{k+1}^2 \right) ds \right]^{1/2}, & k \geq 1.
\end{cases} \]

Next we shall give the estimate for \(| \sigma - \sigma_h \|_\infty \). For this purpose, we let \( v_h = G^h_2 \) in (4.4) to get according to (3.4) that

\[ (\alpha \theta_h + M * \theta_h, G^h_2) = - (\text{div} G^h_2, \rho_h) \]

\[ = -(\text{div}(G^h_2 - G_2), \rho_h) - (\text{div} G_2, \rho_h) \]

\[ = 0. \]

This yields by Lemma 3.1 and Green’s formula that

\[ 0 = \int_0^T (\alpha \theta_h + M * \theta_h, G^h_2) dt = \int_0^T (\alpha G^h_2 + M * G^h_2, \theta_h) dt \]

\[ = \int_0^T (\delta^h_2, \theta_h) \phi_2 (t) dt - \int_0^T (\lambda^h_2, \text{div} \theta_h) dt, \]

which, together with Theorem 3.3, implies that

\[ \left| \int_0^T (\delta^h_2, \theta_h) \phi_2 (t) dt \right| \leq \int_0^T \left( \lambda^h_2 - \lambda_2, \text{div} \theta_h \right) dt + \int_0^T \left( \lambda_2, \text{div} \theta_h \right) dt \]

\[ \leq \int_0^T C |\text{div} \theta_h|_0 (1 + \phi_2 (t)) dt \]

\[ + \int_0^T C (1 + |\log h|^{1/2}) |\text{div} \theta_h|_0 (1 + \phi_2 (t)) dt. \]

Thus, we derive from Lemma 3.2 and (4.12) that

\[ |(\delta^h_2, \theta_h)| \leq C |\log h|^{1/2} |\text{div} \theta_h|_0 \leq C |\log h|^{1/2} || u_t - u_{ht,t} ||_0 \]

which, together with Theorem 4.1 and an analogue of (3.3) for \( \delta^h_2 \), shows that

\[ \| \theta_h \| \leq \begin{cases} 
C h | \log h |^{1/2} (\| u \|_2 + \| u_t \|_2) \\
+ C h | \log h |^{1/2} \left[ \int_0^t \left( \| u \|_2^2 + \| u_t \|_2^2 + \| u_{tt} \|_2^2 \right) ds \right]^{1/2}, & k = 0, \\
C h^{k+1} | \log h |^{1/2} (\| u \|_{k+1} + \| u_t \|_{k+1}) \\
+ C h^{k+1} | \log h |^{1/2} \left[ \int_0^t \left( \| u \|_{k+1}^2 + \| u_t \|_{k+1}^2 + \| u_{tt} \|_{k+1}^2 \right) ds \right]^{1/2}, & k \geq 1.
\end{cases} \]

This, together with (4.9) and (4.13), completes the proof of the theorem. \( \square \)

5. Superconvergence in \( L^\infty \)-norm and a-posteriori error estimates

The aim of this section is to give local and global maximum norm superconvergence and a-posteriori error estimators for the mixed finite element approximation of (2.1). First of all, we consider the local superconvergence. To this end, let us define some seminorms as follows.
Following [11] we assume that $\Omega \subset R^2$ is a rectangle and $e = [a, b] \times [c, d] \in T_h$ is an arbitrary element of the partition $T_h$. We denote by $(\bar{g}_1, \bar{g}_2, \cdots, \bar{g}_{k+1})$ the Gauss points in $[a, b]$ and $(\bar{g}_1, \bar{g}_2, \cdots, \bar{g}_{k+1})$ the Gauss points in $[c, d]$, and define
\[
\|v\|_{\infty, \infty} := \max_{e \in T_h} \max_{1 \leq j \leq k+1} |v(x, \bar{g}_j)|,
\|v\|_{\infty, \infty} := \|v_1\|_{\infty, 1} + \|v_2\|_{\infty, 2},
\]
where
\[
\|v_1\|_{\infty, 1} := \max_{e \in T_h} \max_{1 \leq j \leq k+1} |v_1(x, \bar{g}_j)|,
\|v_2\|_{\infty, 2} := \max_{e \in T_h} \max_{1 \leq j \leq k+1} |v_2(g_j, y)|.
\]

By assuming that the matrices $A$ and $B$ are diagonal or the partition $T_h$ is uniform in this section, we recall from [8, 19] and [16] the following Lemma 5.1 and Lemma 5.2, respectively.

**Lemma 5.1.** Let $\sigma$ be a sufficiently smooth vector-valued function, $B = (b_{ij})$ be a $2 \times 2$ matrix with $b_{ij} \in W^{1, \infty}(\Omega)$ and $\Omega$ be a rectangular domain. Then, we have
\[
|(B : (\sigma - \Pi_h^k \sigma), \sigma_h)| \leq C h^{k+2} |\sigma|_{k+2, p} \|\sigma_h\|_{0, p'}, \quad \forall \sigma_h \in \mathbf{V}_h,
\]
where $|f|_{m, q} := \left( \sum_{|i|=m} |D^i f|_{0, q, \Omega}^q \right)^{1/q}$, $1 \leq q < \infty$, $|f|_{m, \infty} := \max_{|i|=m} \sup_{\Omega} |D^i f|$ and $p' = \frac{p}{p-1}$ is the conjugate of $p \geq 1$.

**Lemma 5.2.** Let $(\bar{u}_h, \bar{\sigma}_h)$ be the mixed Ritz-Volterra projection of $(u, \sigma)$. Then, we have
\[
\|\bar{u}_h - P_h^k u\|_W + \|\bar{\sigma}_h - \Pi_h^k \sigma\|_W \leq C h^{k+2} (\|u\|_{k+1} + \|\sigma\|_{k+2}),
\]
where $\|u\|_W := \|u\|_0$ and $\|\sigma\|_W := \left(\|\sigma\|_0^2 + \|\nabla \cdot \sigma\|_0^2\right)^{1/2}$.

From now on, for convenience in writing, we shall refrain from tracing the exact dependence on the smoothness of $u$ and $\sigma$, and only give the order of errors.

**Theorem 5.1.** Assume that $(u, \sigma)$ and $(u_h, \sigma_h)$ are the solutions of (2.1) and (2.2), respectively, and $(u_h(0), \sigma_h(0))$ are chosen to satisfy (4.1). If the exact solution $u$ and $\sigma$ satisfies $\sigma, \sigma_t \in (H^{k+2}(\Omega))^2$, then we have
\[
\|u_h - P_h^k u\|_0 + \|u_h - P_h^k u\|_t + \|\sigma_h - \Pi_h^k \sigma\|_0 \leq C(u, \sigma) h^{k+2}.
\]

**Proof.** Let $\rho_h^* := u_h - P_h^k u$ and $\theta_h^* := \sigma_h - \Pi_h^k \sigma$. Then, it follows from (2.4) and (2.5) that
\[
(a \theta_h^* + M * \theta_h^* + \nabla \cdot \phi_h) = (\alpha (\sigma - \Pi_h^k \sigma) + M * (\sigma - \Pi_h^k \sigma), \phi_h), \quad \forall \phi_h \in \mathbf{V}_h,
\]
Thus, letting $w_h = \rho_h^*$ and $v_h = \theta_h^*$ in (5.1) we obtain from Lemmas 4.1, 5.1, the $\epsilon$-type inequality, the integration and Gronwall’s lemma that
\[
\|\rho_h^*\|_0^2 + \int_0^t \|\theta_h^*\|_0^2 ds \leq C(u, \sigma) \left\{\|\rho_h^*(0)\|_0^2 + h^{2(k+2)}\right\}.
\]
From (4.3) we know
\[
\bar{u}_h(0) = u_h(0) = \bar{\sigma}_h(0) = \sigma_h(0) = 0.
\]
Therefore, from Lemma 5.2 we have
\[ ||\rho_h^*(0)||_0 = ||\bar{u}_h(0) - P_h^k u_0||_0 \leq C(u, \sigma)h^{k+2}, \]
and then from (5.2) we further obtain
\[ (5.4) \quad ||\rho_h^*||_0 \leq C(u, \sigma)h^{k+2}. \]
Again, we have according to (5.3) and Lemma 5.2 that
\[ (5.5) \quad ||\theta_h^*||_0 = ||\bar{\sigma}_h(0) - \Pi_h^k \sigma(0)||_0 \leq C(u, \sigma)h^{k+2}. \]
The second equation in (5.1) implies
\[ (5.6) \quad \rho_{h,t}^* = \nabla \cdot \theta_h^*, \]
which, together with (5.3) and Lemma 5.2, implies
\[ (5.7) \quad ||\rho_{h,t}^*(0)||_0 = ||\nabla \cdot \theta_h^*(0)||_0 = ||\nabla \cdot (\bar{\sigma}_h - \Pi_h^k \sigma(0))||_0 \leq C(u, \sigma)h^{k+2}. \]

Following the steps for \( \theta_h \) and \( \rho_{h,t} \) in Theorem 4.1 and using the initial approximations (5.5) and (5.7) we obtain
\[ (5.8) \quad ||\theta_h^*||_0 + ||\rho_{h,t}^*||_0 \leq C(u, \sigma)h^{k+2}. \]

The proof of Theorem 5.1 is completed by (5.4) and (5.8).

\[ \square \]

Now we are ready to obtain our superapproximation theorem.

\textbf{Theorem 5.2.} In addition to the conditions of Theorem 5.1, if the exact solution \( \sigma \) is such that \( \sigma \in (W_h^{k+2,\infty}(\Omega))^2 \), then we have
\[ ||\log h||^{1/2} ||u_h - P_h^k u||_\infty + ||\sigma_h - \Pi_h^k \sigma||_\infty \leq C(u, \sigma)h^{k+2}||\log h||. \]

\textit{Proof.} Taking \( v_h = G^h_1 \) in (5.1) we see from the mixed finite element approximation of (3.2), Theorems 3.4 and 5.1, and Lemmas 5.1 and 3.1 that
\[ \left| \int_0^T (\rho_h^* - \delta_t^h)\phi_1(t)dt \right| \leq \left| \int_0^T (\alpha G^h_1 + M * G^h_1, \theta_h^*)dt \right| + \left| \int_0^T (\Phi_h^h)\phi_1(t)dt \right| \leq C(u, \sigma)h^{k+2}||\log h||^{1/2} \int_0^T (1 + \phi_1(t))dt. \]

Thus, Lemma 3.2 and (3.3) imply
\[ ||\rho_h^*||_\infty \leq C(u, \sigma)h^{k+2}||\log h||^{1/2}. \]

Next we shall obtain the superapproximation estimate for \( \theta_h^* \) in the \( L^\infty \)-norm. Taking \( v_h = G^h_2 \) in (5.1) and noticing \( \nabla \cdot G^h_2 = 0 \) by the mixed finite element approximation of (3.4), we have by Lemmas 3.1, 5.1 and Theorem 3.5 that
\[ \left| \int_0^T (\alpha G^h_2 + M * G^h_2, \theta_h^*)dt \right| \leq C(u, \sigma)h^{k+2}||\log h|| \int_0^T (1 + \phi_2(t))dt, \]
which, together with Theorem 3.3, (5.6) and (5.8), leads to
\[ \left| \int_0^T (\delta_t^h, \theta_h^*)\phi_2(t)dt \right| \leq \left| \int_0^T ||\lambda^h_2||_0||\rho_{h,t}^*||_0dt \right| + C(u, \sigma)h^{k+2}||\log h|| \int_0^T (1 + \phi_2(t))dt \leq C(u, \sigma)h^{k+2}||\log h|| \int_0^T (1 + \phi_2(t))dt. \]
This implies by Lemma 3.2 that

$$||\theta_h||_{\infty} \leq C(u, \sigma)h^{k+2}|\log h|.$$ 

\[ \square \]

**Remark 5.1.** From Lemmas 3.3 and 5.1 we can derive the following $L^\infty$-norm superapproximation for the mixed Ritz-Volterra projection of $(u, \sigma)$:

$$|\log h|^{1/2}||u_h - P_h^k u||_{\infty} + ||\sigma_h - \Pi_h^k \sigma||_{\infty} \leq C(u, \sigma)h^{k+2}|\log h|.$$ 

Hence, there holds the $L^\infty$-superapproximation estimate under the conditions of Theorem 5.2,

$$|\log h|^{1/2}||u_h - u||_{\infty} + ||\sigma_h - \sigma_h||_{\infty} \leq C(u, \sigma)h^{k+2}|\log h|.$$ 

In order to obtain the local superconvergence for the mixed finite element solution $(u_h, \sigma_h)$, we need the following lemmas which come from [11] and [8], respectively.

**Lemma 5.3.** Assume that $u \in W^{k+2,\infty}(\Omega)$. Then,

$$||u - P_h^k u||_{\infty} \leq C(u)h^{k+2}.$$ 

**Lemma 5.4.** If $\sigma \in (W^{k+2,\infty}(\Omega))^2$, then we have

$$||\sigma - \Pi_h^k \sigma||_{\infty} \leq C(\sigma)h^{k+2}.$$ 

We are now in the position to get our local superconvergence on the Gauss points for the approximation of the pressure field and along the Gauss lines for the approximation of the velocity field, respectively.

**Theorem 5.3.** In addition to the conditions of Theorem 5.2, if the exact solution $u$ is such that $u \in W^{k+2,\infty}(\Omega)$, then we have

$$|\log h|^{1/2}||u - u_h||_{\ast, \infty} + ||\sigma - \sigma_h||_{\ast, \infty} \leq C(u, \sigma)h^{k+2}|\log h|.$$ 

**Proof.** From Lemma 5.3 and Theorem 5.2 we have

$$||u - u_h||_{\ast, \infty} \leq ||u - P_h^k u||_{\ast, \infty} + ||P_h^k u - u_h||_{\ast, \infty} \leq C(u)h^{k+2} + C(u, \sigma)h^{k+2}|\log h|^{1/2} \leq C(u, \sigma)h^{k+2}|\log h|^{1/2}.$$ 

Similarly, we obtain by means of Theorem 5.2 and Lemma 5.4 that

$$||\sigma - \Pi_h^k \sigma||_{\ast, \infty} \leq C(u, \sigma)h^{k+2}|\log h|.$$ 

\[ \square \]

Next we shall consider the global superconvergence for the pressure and the velocity fields by virtue of post-processing methods. Analogous to [16] we need to construct two post-processing interpolation operators $\Pi_{2h}^{k+1}$ and $P_{2h}^{k+1}$ to satisfy

$$\Pi_{2h}^{k+1} = \Pi_{2h}^k, \quad P_{2h}^{k+1} = P_{2h}^k,$$

$$||\Pi_{2h}^{k+1} v_h||_{0,p} \leq C||v_h||_{0,p}, \quad \forall v_h \in V_h,$$

$$||\Pi_{2h}^{k+1} \sigma - \sigma||_{0,p} \leq Ch^{k+2}||\sigma||_{k+2,p}, \quad \forall \sigma \in (W^{k+2,p}(\Omega))^2,$$

$$||P_{2h}^{k+1} w_h||_{0,p} \leq C||w_h||_{0,p}, \quad \forall w_h \in W_h,$$

$$||P_{2h}^{k+1} u - u||_{0,p} \leq Ch^{k+2}||u||_{k+2,p}, \quad \forall u \in W^{k+2,p}(\Omega),$$

where $1 \leq p \leq \infty$ and $|| \cdot ||_{0, \infty} = || \cdot ||_{\infty}$. Here we take for example $k = 3$ to demonstrate the construction of the projection interpolation operators $\Pi_{2h}^{k+1}$ and
$P_{2h}^{k+1}$ satisfying (5.9). To this purpose, we assume that the rectangular partition $T_h$ has been obtained from $T_{2h} = \{T\}$ with mesh size $2h$ by subdividing each element of $T_{2h}$ into four small congruent rectangles. Let $\tau := \bigcup_{i=1}^4 c_i$ with $c_i \in T_h$. Thus, we can define two projection operators $\Pi_{2h}^4$ and $P_{2h}^4$ associated with $T_{2h}$ of degree at most 4 in $x$ and $y$ on $\tau$, respectively, according to the following conditions:

$$\begin{align*}
\Pi_{2h}^4 \sigma |_{\tau} & \in (Q_{4,4}(\tau))^2, \\
\int_{\tau} (\sigma - \Pi_{2h}^4 \sigma) \cdot n ds = 0, & \quad \forall q \in P_2(l_i), \ i = 1, 2, \cdots, 12, \\
\int_{\tau} (\sigma - \Pi_{2h}^4 \sigma) = 0, & \quad i = 1, 2, 3, 4, \\
\int_{\tau} (\sigma - \Pi_{2h}^4 \sigma) \cdot \phi = 0, & \quad \forall \phi \in (Q_{1,1}(\tau) \backslash Q_{0,0}(\tau))^2, \text{ and} \\
\int_{\tau} (u - P_{2h}^4 u) \psi = 0, & \quad \forall \psi \in Q_{2,1}(c_i), \ i = 1, 2, 3, 4, \\
\int_{\tau} (u - P_{2h}^4 u) \psi = 0, & \quad \forall \psi \in Q_{3,0}(\tau) \backslash Q_{2,0}(\tau), \text{ respectively,}
\end{align*}$$

where $l_i$ ($i = 1, 2, \cdots, 12$) is one of the twelve sides of the four small elements $c_i$ ($i = 1, 2, 3, 4$).

Similarly, we can also define $\Pi_{2h}^k$ and $P_{2h}^k$ for the case of $k \neq 3$ such that (5.9) is satisfied.

By the two projection interpolation operators $\Pi_{2h}^k$ and $P_{2h}^k$ we can immediately gain the following global superconvergence result.

**Theorem 5.4.** Assume that $(u, \sigma)$ and $(u_h, \sigma_h)$ are the solutions of (2.1) and (2.2), respectively. Then, we have under the conditions of Theorem 5.3 that

$$|\log h|^{1/2}||P_{2h}^{k+1} u_h - u||_\infty + ||\Pi_{2h}^{k+1} \sigma_h - \sigma||_\infty \leq C(u, \sigma) h^{k+2} |\log h|.$$  

**Proof.** We see from one of the properties of the operator $P_{2h}^{k+1}$ described in (5.9) that

$$P_{2h}^{k+1} u_h - u = P_{2h}^{k+1} (u_h - P_{2h}^k u) + (P_{2h}^{k+1} u - u).$$

Therefore, it follows from Theorem 5.2 and (5.9) that

$$||P_{2h}^{k+1} u - u||_\infty \leq C||u_h - P_{2h}^k u||_\infty + C(u) h^{k+2} \leq C(u, \sigma) h^{k+2} |\log h|^{1/2}.$$ 

Analogously, we can obtain

$$||\Pi_{2h}^{k+1} \sigma - \sigma||_\infty \leq C(u, \sigma) h^{k+2} |\log h|.$$ 

\[\square\]

**Remark 5.2.** From the superapproximation estimates of $||\bar{u}_h - P_{2h}^k u||_\infty$ and $||\bar{\sigma}_h - \Pi_{2h}^k \sigma||_\infty$ indicated in Remark 5.1 we can obtain the following global superconvergence under the conditions of Theorem 5.3 by the post-processing method:

$$|\log h|^{1/2}||P_{2h}^{k+1} \bar{u}_h - \bar{u}||_\infty + ||\Pi_{2h}^{k+1} \bar{\sigma}_h - \bar{\sigma}||_\infty \leq C(u, \sigma) h^{k+2} |\log h|.$$  

As a by-product, Theorem 5.4 can be employed to construct a-posteriori error estimators to assess the accuracy of the mixed finite element solution in applications.

**Theorem 5.5.** We have under the conditions of Theorem 5.3 that

$$||u - u_h||_\infty = ||P_{2h}^{k+1} u_h - u_h||_\infty + O(h^{k+2} |\log h|^{1/2}),$$

$$||\sigma - \sigma_h||_\infty = ||\Pi_{2h}^{k+1} \sigma_h - \sigma_h||_\infty + O(h^{k+2} |\log h|).$$
In addition, if there exist positive constants $C_1$, $C_2$ and small $\epsilon_1$, $\epsilon_2 \in (0, 1)$ such that

\begin{align}
&\|u - u_h\|_\infty \geq C_1 h^{k+2-\epsilon_1}, \\
&\|\sigma - \sigma_h\|_\infty \geq C_2 h^{k+2-\epsilon_2},
\end{align}

then there hold

\begin{align}
\lim_{h \to 0} \frac{\|u - u_h\|_\infty}{\|P_{2h}^k u_h - u_h\|_\infty} &= 1, \\
\lim_{h \to 0} \frac{\|\sigma - \sigma_h\|_\infty}{\|\Pi_{2h}^{k+1} \sigma_h - \sigma_h\|_\infty} &= 1.
\end{align}

Proof. Following the procedure for Theorem 5.3 in [16] we can immediately obtain the desired results.

We see from (5.10) that the computable error quantity $\|P_{2h}^k u_h - u_h\|_\infty$ is the principal part of the mixed finite element error $\|u - u_h\|_\infty$. Moreover, by (5.14) it can be used as a reliable a-posteriori error indicator to assess the accuracy of the mixed finite element solution under the condition (5.12). Meanwhile, (5.12) seems to be a reasonable assumption since $O(h^{k+1})$ is the optimal convergence rate of the mixed finite element solution in $L^\infty$-norm subject to the conditions of Theorem 5.3.

The same comments are also valid for (5.11), (5.13) and (5.15).

6. Estimates for the regularized Green’s functions

In the previous sections, we have seen that the regularized Green’s functions play an important role in the analysis of convergence and superconvergence estimates in maximum norms for the mixed finite element method of (1.1). We present the proofs of Theorems 3.2 and 3.3 in this section. The proofs are based on a series of lemmas. First, we prove the following result.

**Lemma 6.1.** We have under the assumptions of Theorem 3.2 that

$$\|G^h_1 - G_1\|_0 \leq C(1 + \phi_1(t)).$$

*Proof.* It follows from (3.2), Grönwall’s lemma and Theorem 3.1 that

$$\|G_1\|_0 \leq C\|\nabla \lambda_1\|_0 \leq C(1 + \phi_1(t)),$$

which yields via using the estimate for $\|\nabla^2 \lambda_1\|_0$ in Theorem 3.1 that

$$\||\text{div}G_1||_0 \leq Ch^{-1}(1 + \phi_1(t)) + C(1 + \phi_1(t)) \leq C h^{-1}(1 + \phi_1(t)).$$

Decompose the error $G_1 - G^h_1$ as follows:

$$G_1 - G^h_1 = (G_1 - \Pi_h^k G_1) + (\Pi_h^k G_1 - G^h_1) := \theta^* + \theta^*_h.$$

Then, $\theta^*_h$ satisfies the following equation by (2.5) and the mixed finite element error equation of (3.2) that

\begin{align}
(\alpha \theta^*_h + M * \theta^*_h, v_h) &= -(\alpha \theta^* + M * \theta^*, v_h) - (\lambda_1 - \lambda^h_1, \nabla \cdot v_h) \\
&= -(\alpha \theta^* + M * \theta^*, v_h) - (P^k_h \lambda_1 - \lambda^h_1, \nabla \cdot v_h), v_h \in V_h.
\end{align}

Since

$$\langle P^k_h \lambda_1 - \lambda^h_1, \nabla \cdot \theta^*_h \rangle = 0,$$
by (2.5) and the mixed finite element error equation of (3.2), taking \( v_h = \theta_h^{**} \) in the above equation leads to

\[
(\alpha \theta_h^{**} + M \ast \theta_h^{**}, \theta_h^{**}) = -(\alpha \theta^{**} + M \ast \theta^{**}, \theta_h^{**}).
\]

Thus, we know from Lemma 4.1 and Gronwall’s lemma that

\[
\|\theta_h^{**}\|_0 \leq C \left( \|\theta^{**}\|_0 + \int_t^T \|\theta^{**}\|_0 ds \right).
\]

Hence, we obtain by virtue of the above estimate for \( \text{div} G_1 \) in \( L^2 \)-norm and (3.5) that

\[
\|G_1 - G^h_1\|_0 \leq C \left( \|\theta^{**}\|_0 + \int_t^T \|\theta^{**}\|_0 ds \right)
\]

\[
\leq C h \left( \|\text{div} G_1\|_0 + \int_t^T \|\text{div} G_1\|_0 ds \right)
\]

\[
\leq C (1 + \phi_1(t)).
\]

This completes the proof of Lemma 6.1. \( \square \)

**Lemma 6.2.** Under the assumptions of Theorem 3.2,

\[
\|\lambda^h_1 - P_h^k \lambda_1\|_0 \leq C h(1 + \phi_1(t)),
\]

\[
\|\lambda^h_1 - \lambda_1\|_0 \leq C h(\log h)^{3/n/2}(1 + \phi_1(t)).
\]

*Proof.* Let \( (w, \lambda) \in V \times L^2(\Omega) \) be defined such that

\[
\begin{align*}
\alpha w + M \ast w - \nabla \lambda &= 0, & \text{in } \Omega \times (0, T), \\
\text{div} w &= (\lambda^h_1 - P_h^k \lambda_1) \phi(t), & \text{in } \Omega \times (0, T), \\
\lambda &= 0, & \text{on } \partial \Omega \times (0, T),
\end{align*}
\]

where \( \phi(t) \geq 0 \) and \( \int_0^T \phi(t) dt \leq 1 \). Clearly, \( (w, \lambda) \) is well defined and satisfies

\[
\|\nabla^2 \lambda\|_0 \leq C \left( \|\lambda^h_1 - P_h^k \lambda_1\|_0 \phi(t) + \int_0^T \|\lambda^h_1 - P_h^k \lambda_1\|_0 \phi(s) ds \right)
\]

by the regularity assumption on \( \Omega \). Now, it follows from (2.5), the mixed finite element error equation of (3.2) and Lemma 3.1 that

\[
\begin{align*}
\int_0^T \|\lambda^h_1 - P_h^k \lambda_1\|_0^2 \phi(t) dt &= \int_0^T (\lambda^h_1 - P_h^k \lambda_1, \text{div} \Pi_h^k w) dt \\
&= \int_0^T (\lambda^h_1 - \lambda_1, \text{div} \Pi_h^k w) dt \\
&= \int_0^T (\alpha(G_1 - G^h_1) + M \ast (G_1 - G^h_1), \Pi_h^k w) dt \\
&= \int_0^T (\alpha(G_1 - G^h_1) + M \ast (G_1 - G^h_1), \Pi_h^k w - w) dt \\
&\quad + \int_0^T (\lambda, \text{div}(G^h_1 - G_1)) dt := N_1 + N_2.
\end{align*}
\]
Obviously, we have by using (3.2) and (2.5) that
\[
N_2 = \int_0^T (P^k_h \lambda, \text{div} G^h_1) dt - \int_0^T (\lambda, \delta^h_1) \phi_1(t) dt
\]
\[
= \int_0^T (P^k_h \lambda, P^k_h \delta^h_1) \phi_1(t) dt - \int_0^T (\lambda, \delta^h_1) \phi_1(t) dt
\]
\[
= \int_0^T (P^k_h \lambda - \lambda, \delta^h_1) \phi_1(t) dt.
\]
Thus, we have for \( k \geq 1 \) that
\[
|N_2| \leq \int_0^T Ch^2 ||\nabla^2 \lambda||_0 ||\delta^h_1||_0 \phi_1(t) dt
\]
\[
\leq Ch \int_0^T \left( ||\lambda^h_1 - P^k_h \lambda_1||_0 \phi(t) + \int_0^t ||\lambda^h_1 - P^k_h \lambda_1||_0 \phi(s) ds \right) \phi_1(t) dt
\]
\[
\leq Ch \int_0^T (\phi_1(t) + 1)||\lambda^h_1 - P^k_h \lambda_1||_0 \phi(t) dt.
\]
Similarly, we have for \( N_1 \) by virtue of Lemma 6.1 and (6.1) that
\[
|N_1| \leq Ch \int_0^T (1 + \phi_1(t)) ||\text{div} w||_0 dt
\]
\[
\leq Ch \int_0^T (1 + \phi_1(t)) ||\lambda^h_1 - P^k_h \lambda_1||_0 \phi(t) dt.
\]
We have by combining (6.2) with (6.3) and (6.4), and using Lemma 3.2 that
\[
||\lambda^h_1 - P^k_h \lambda_1||_0 \leq Ch(1 + \phi_1(t)), \quad \text{for } k \geq 1.
\]
It remains to treat \( N_2 \) for \( k = 0 \). Since
\[
\int_0^T (P^0_h \lambda - P^1_h \lambda, \delta^h_1) \phi_1(t) dt = 0 \quad \text{(see [26, 27])},
\]
we know from the same arguments as those for (6.3) that
\[
|N_2| = \left| \int_0^T (P^0_h \lambda - \lambda, \delta^h_1) \phi_1(t) dt \right|
\]
\[
\leq Ch \int_0^T (1 + \phi_1(t)) ||\lambda^h_1 - P^0_h \lambda_1||_0 \phi(t) dt.
\]
Finally, the second inequality in Lemma 6.2 is a result of the first inequality in the same lemma and Theorem 3.1 together with the standard triangle inequality.

\[\square\]

**Remark 6.1.** Using the similar duality argument to that as above we can easily obtain [26]
\[
||\lambda^h - P^k_h \lambda||_0 + ||\lambda^h - \lambda||_0 \leq C(1 + \phi_2(t)).
\]
Here we omit the details.

**Lemma 6.3.** We have under the assumptions of Theorem 3.2 that
\[
||G^h_1 - G_1||_{L^1(\Omega)} \leq Ch \log h(1 + \phi_1(t)).
\]
Proof. By Schwartz inequality and (3.1) we have
\[
|\|G_1^h - G_1\|_{L^1(\Omega)}| \leq C|\log h|^{1/2}|G_1^h - G_1|_{\beta_2}.
\]
Let
\[
\Psi_1 := \beta_2^2(G_1 - G_1^h).
\]
Then, we derive from Lemma 4.1 and (3.2) that
\[
|\|G_1^h - G_1\|_{\beta_2}^2 \leq C_0(\alpha(G_1 - G_1^h), \Psi_1 - \Pi^h_k \Psi_1)
+ C_0(\alpha(G_1 - G_1^h) + M * (G_1 - G_1^h), \Pi^h_k \Psi_1)
- C_0(M * (G_1 - G_1^h), \Pi^h_k \Psi_1)
= C_0(\alpha(G_1 - G_1^h), \Psi_1 - \Pi^h_k \Psi_1) - C_0(\lambda_1 - \lambda_1^h, \text{div}\Pi^h_k \Psi_1)
- C_0(M * (G_1 - G_1^h), \Pi^h_k \Psi_1)
:= M_1 + M_2 + M_3.
\]
Now we consider $M_i$'s individually. First, it follows from Lemma 6.5 below that
\[
|M_1| \leq C_0|\alpha(G_1 - G_1^h)|_{\beta_2} \cdot |\|\Psi_1 - \Pi^h_k \Psi_1\|_{\beta_2} - 2
\leq \epsilon|\|G_1^h - G_1\|_{\beta_2}^2 + C\epsilon^2 \log h|(1 + \phi_1(t))^2).
\]
We know from (2.5) that
\[
(\text{div}\Pi^h_k \sigma, w_h) = (\text{div}\sigma, w_h), \quad \forall w_h \in W_h,
\]
which, together with Lemma 6.2, implies
\[
|M_2| = C_0(P_h^k \lambda_1 - \lambda_1^h, \text{div}\Pi^h_k \Psi_1)
= C_0(P_h^k \lambda_1 - \lambda_1^h, \text{div}\Psi_1)
\leq Ch(1 + \phi_2(t))|\|\text{div}\Psi_1\|_0.
\]
Since there holds by (3.2)
\[
\text{div}\Psi_1 = \nabla(\beta^2) \cdot (G_1 - G_1^h) + \beta^2(\delta_1^h - P_h^k \delta_1^h) \phi_1(t),
\]
we have
\[
|\|\text{div}\Psi_1\|_0 \leq C|\|G_1 - G_1^h\|_{\beta_2} + Ch \phi_1(t).
\]
Thus, we obtain from (6.8) that
\[
|M_2| \leq Ch^2(1 + \phi_1(t))^2 + \epsilon|\|G_1^h - G_1^h\|_{\beta_2}.
\]
It follows from Schwartz inequality and Lemma 6.5 that
\[
|M_3| \leq C \left( \int_t^T |\|G_1 - G_1^h\|_{\beta_2} ds \right) |\|\Pi^h_k \Psi_1 - \Psi_1\|_{\beta_2 - 2}
+ C \left( \int_t^T |\|G_1 - G_1^h\|_{\beta_2} ds \right) |\|\Psi_1\|_{\beta_2 - 2}
\leq C \left( \int_t^T |\|G_1 - G_1^h\|_{\beta_2} ds \right) h \log h|^{1/2}(1 + \phi_1(t))
+ C \left( \int_t^T |\|G_1 - G_1^h\|_{\beta_2} ds \right) |\|G_1 - G_1^h\|_{\beta_2}
\leq \epsilon|\|G_1 - G_1^h\|_{\beta_2}^2 + C \left( \int_t^T |\|G_1 - G_1^h\|_{\beta_2} ds \right)^2
+ Ch^2 \log h|(1 + \phi_1(t))^2.
Under the assumptions of Theorem 3.3, we have

\[ \|G_1 - G^h_1\|_{\beta^2} \leq C h^2 \log h |1 + \phi_1(t)|^2 + C \left( \int_t^T \|G_1 - G^h_1\|_{\beta^2} ds \right)^2, \]

so that Gronwall’s lemma yields

\[ (6.11) \quad \|G_1 - G^h_1\|_{\beta^2} \leq C h \log h |1/2(1 + \phi_1(t)). \]

Hence, Lemma 6.3 follows from (6.5) and (6.11).

**Lemma 6.4.** Under the assumptions of Theorem 3.3, we have

\[ \begin{align*}
\|G_2 - G^h_2\|_{L^1(\Omega)} & \leq C \|h^{1/2(1 + \phi_2(t))}, \\
\|G_2 - G^h_2\|_{0} & \leq C h^{-1}(1 + \phi_2(t)), \\
\|\nabla \lambda_2\|_{0} & \leq C h^{-1}(1 + \phi_2(t)).
\end{align*} \]

**Proof.** We have by virtue of Schwartz inequality and (3.1) that

\[ (6.12) \quad \|G_2 - G^h_2\|_{L^1(\Omega)} \leq C \|h^{1/2}\|_{\beta^2} \|G_2 - G^h_2\|_{\beta^2}. \]

Let

\[ \Psi_2 := \beta^2(G_2 - G^h_2). \]

Then, it follows from a similar argument to that for Lemma 6.3 that

\[ (6.13) \quad \|G_2 - G^h_2\|_{\beta^2} \leq C_0(a(G_2 - G^h_2), \Psi_2 - \Pi^h_2 \Psi_2) - C_0(\lambda_2 - \lambda^h_2, \text{div} \Pi^h_2 \Psi_2) \]

\[ - C_0(M \ast * (G_2 - G^h_2), \Pi^h_2 \Psi_2) := M_1^2 + M_2^2 + M_3^2. \]

Thus, we know from Lemma 6.5 below that

\[ (6.14) \quad |M_1^2| \leq \epsilon \|G_2 - G^h_2\|_{\beta^2}^2 + C(1 + \phi_2(t))^2. \]

Moreover, we see from Remark 6.1 and the same arguments as those for (6.8) that

\[ (6.15) \quad |M_2^2| \leq C(1 + \phi_2(t)) \|\text{div} \Psi_2\|_0. \]

It follows from (3.4) that

\[ \text{div} \Psi_2 = \nabla (\beta^2 \cdot (G_2 - G^h_2)), \]

which yields by (6.15) that

\[ (6.16) \quad |M_2^2| \leq \epsilon \|G_2 - G^h_2\|_{\beta^2}^2 + C(1 + \phi_2(t))^2. \]

Also, we obtain according to the similar steps for (6.10) that

\[ (6.17) \quad |M_3^2| \leq \epsilon \|G_2 - G^h_2\|_{\beta^2}^2 + C \left( \int_t^T \|G_2 - G^h_2\|_{\beta^2} ds \right)^2 + C(1 + \phi_2(t))^2. \]

Combining (6.14), (6.16) and (6.17) with (6.13), we have via using Gronwall’s lemma that

\[ \|G_2 - G^h_2\|_{\beta^2} \leq C(1 + \phi_2(t)). \]

Hence, from (6.12) we obtain

\[ \|G_2 - G^h_2\|_{L^1(\Omega)} \leq C \|h^{1/2}\|_{\beta^2} \|G_2 - G^h_2\|_{\beta^2}. \]

By the $H^2$-regularity assumption, we have

\[ \|\nabla \lambda_2\|_0 \leq C h^{-1}(1 + \phi_2(t)). \]
Thus, from [26] we see that
\[ \|G_2 - G_2^h\|_0 \leq Ch^{-1}(1 + \phi_2(t)). \]

\[ \square \]

**Lemma 6.5.** Let \( \Psi_i \ (i = 1, 2) \) be the functions defined as before. Then, we have
\[ \|\Psi_1 - \Pi_h^i \Psi_1\|_{\beta-2} \leq Ch| \log h|^{1/2}(1 + \phi_1(t)), \]
\[ \|\Psi_2 - \Pi_h^i \Psi_2\|_{\beta-2} \leq C(1 + \phi_2(t)). \]

**Proof.** Recall
\[ \Psi_i = \beta^2(G_i - G_i^h), \quad i = 1, 2, \]
and rewrite them as
\[ \Psi_i = \beta^2(G_i - \Pi_h^i G_i) + \beta^2(\Pi_h^i G_i - G_i^h) := \Psi_{i1} + \Psi_{i2}. \]

Thus,
\[ \|\Psi_i - \Pi_h^i \Psi_i\|_{\beta-2} \leq \|\Psi_{i1} - \Pi_h^i \Psi_{i1}\|_{\beta-2} + \|\Psi_{i2} - \Pi_h^i \Psi_{i2}\|_{\beta-2}. \]

Since \( \Pi_h^i \) is a local projection operator, it follows from [26] that
\[ \|\Psi_{i1} - \Pi_h^i \Psi_{i1}\|_{\beta-2} \leq C\|\Psi_{i1}\|_{\beta-2} \leq C\|G_i - \Pi_h^i G_i\|_{\beta^2} \leq Ch\|\nabla^2 \lambda_i\|_{\beta^2}. \]

Then, Theorem 3.1 and (6.28) below lead to
\[ \|\Psi_{i1} - \Pi_h^i \Psi_{i1}\|_{\beta-2} \leq \begin{cases} Ch| \log h|^{1/2}(1 + \phi_1(t)), & \text{for } i = 1, \\ C(1 + \phi_2(t)), & \text{for } i = 2. \end{cases} \]

Following [26] we obtain from Lemmas 6.1 and 6.4 that
\[ \|\Psi_{i2} - \Pi_h^i \Psi_{i2}\|_{\beta-2} \leq \begin{cases} Ch(1 + \phi_1(t)), & \text{for } i = 1, \\ C(1 + \phi_2(t)), & \text{for } i = 2. \end{cases} \]

Now, (6.19) and (6.20) lead (6.18) to
\[ \|\Psi_i - \Pi_h^i \Psi_i\|_{\beta-2} \leq \begin{cases} Ch| \log h|^{1/2}(1 + \phi_1(t)), & \text{for } i = 1, \\ C(1 + \phi_2(t)), & \text{for } i = 2. \end{cases} \]

which verifies the conclusions of Lemma 6.5. \[ \square \]

**Lemma 6.6.** Under the assumptions of Theorem 3.3 there hold
\[ \|\lambda_2\|_0 \leq C| \log h|^{1/2}(1 + \phi_2(t)), \]
\[ \|\nabla \lambda_2\|_{L^1(\Omega)} \leq C| \log h|^{1}(1 + \phi_2(t)), \]
\[ \|\nabla^2 \lambda_2\|_{L^1(\Omega)} \leq Ch^{-1}| \log h|^{1/2}(1 + \phi_2(t)). \]

**Proof.** From Schwarz’s inequality and (3.1) we have
\[ \|\nabla \lambda_2\|_{L^1(\Omega)} \leq C| \log h|^{1/2}|\nabla \lambda_2\|_{\beta^2}. \]

Furthermore, it follows from (3.4) and Green’s formula that
\[ \|\nabla \lambda_2\|_{L^2}^2 = (\nabla \lambda_2, \beta^2 \nabla \lambda_2) = -(\Delta \lambda_2, \beta^2 \lambda_2) + \frac{1}{2}(\lambda_2, \Delta(\beta^2)\lambda_2) \leq |(\text{div}\delta^2 \phi_2(t), \beta^2 \lambda_2)| + C|\lambda_2|_0^2 \leq C(\phi_2^2(t) + |\lambda_2|_0^2). \]

Now, let us consider the following auxiliary Dirichlet problem to bound \( \|\lambda_2\|_0 \):
\[ -\Delta r = \lambda_2 \quad \text{in } \Omega, \]
\[ r = 0 \quad \text{on } \partial \Omega. \]
From the regularity assumption on the domain $\Omega$ we have
\[ (6.23) \quad \| \nabla^2 r \|_0 \leq C \| \lambda_2 \|_0. \]
In addition, it follows from (3.4) and Green's formula that
\[ (6.24) \quad \| \lambda_2 \|_0^2 = (\nabla \lambda_2, \nabla r) = - (\nabla^2 \lambda_2, r) = (\text{div} \delta^h_2, r) + (\delta^h_2, \nabla r) \phi_2(t) := N_. \]

Following the procedure in [26], we have, according to (3.5), (6.23) and the standard inverse estimate, that
\[ (6.25) \quad \| (\nabla r)^j \|_\infty \leq C \log h^{1/2} \| (\nabla r)^j \|_1 \leq C \log h^{1/2} \| \nabla^2 r \|_0, \]
and
\[ (6.26) \quad \| \lambda_2 \|_0 \leq C \left( 1 + \log h^{1/2} \right) \phi_2(t). \]

Now, (6.26) and (6.22) lead (6.21) to
\[ (6.27) \quad \| \nabla \lambda_2 \|_{L^1(\Omega)} \leq C \log h \left( 1 + \phi_2(t) \right). \]

Again, we use Schwarz's inequality and (3.1) to obtain
\[ (6.28) \quad \| \nabla^2 \lambda_2 \|_{L^1(\Omega)} \leq C \log h^{1/2} \| \nabla^2 \lambda_2 \|_{\beta^2}. \]
Following [26] we further have
\[ (6.29) \quad \| \nabla^2 \lambda_2 \|_{\beta^2} \leq C h^{-1} (1 + \phi_2(t)). \]
Thus,
\[ (6.30) \quad \| \nabla^2 \lambda_2 \|_{L^1(\Omega)} \leq C h^{-1} \log h^{1/2} (1 + \phi_2(t)). \]

References
