INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 2, Number 2, Pages 221–239

A POSTERIORI ERROR ESTIMATOR FOR FINITE ELEMENT DISCRETIZATIONS OF QUASI-NEWTONIAN STOKES FLOWS

ABDELLATIF AGOUZAL

Abstract. In this paper, we consider mixed finite elements discretizations of a class of Quasi-Newtonian Stokes flow problem. Unified a posteriori error estimator for conforming, nonconforming, with or without stabilization is obtained. We prove, without Helmholtz decomposition of the error, nor regularity and saturation assumptions, the reliability and the efficiency of our estimator.

Key Words. Quasi-Newtonian flow, conforming, nonconforming and mixed finite element, a posteriori error estimator.

1. Introduction

Adaptive finite element method is justified by using a posteriori error estimate which provides computable upper and lower error bounds, it serves then, as error indicators. The aim of the work is to unify, generalize and refine the derivation of residual error estimator for a class of Quasi-Newtonian Stokes flow problem. Indeed, the present work take on unifying proof for conforming, nonconforming, and even conforming-nonconforming scheme, with or without stabilization [4], and also mixed formulation, in two and three dimensional cases [9]. We generalize, simplify and refine the works of Verfürth [12] , Dari, Durán a nd Padra [8], Carstensen and Funcken [6] and Gatica et al [9]. We prove, without Helmholtz decomposition of the error, nor regularity of the solution or the domain, nor saturation assumption, the efficiency and the reliability of our estimator.
Let $\Omega \subset I\!R^d$ (d=2,3), be a bounded open connected and polyhedral set. In Ω, we consider the following model problem:

$$\begin{cases} \text{Find} & (u,p) \text{ such that} \\ -div(\mathcal{A}(\nabla u)) + \nabla p & = f, \ \text{ in } \Omega, \\ divu & = 0, \ \text{in } \Omega, \\ u & = 0, \ \text{on } \Gamma = \partial\Omega, \end{cases}$$

where u the velocity, p the pressure, f a regular function in the sapce $(L^2(\Omega))^d$ and $\mathcal{A} : I\!R^{d\times d} \longrightarrow I\!R^{d\times d}$ is Lipschitz continuous function satisfying, there are positives constants c_1 and c_2 such that: for all $\alpha, \beta \in I\!R^{d\times d}$,

$$c_1\|\alpha - \beta\|^2 \leq (\mathcal{A}(\alpha) - \mathcal{A}(\beta)) : (\alpha - \beta), \tag{1.1}$$

and

$$\|\mathcal{A}(\alpha) - \mathcal{A}(\beta)\| \leq c_2\|\alpha - \beta\|, \tag{1.2}$$

(Colon denotes the scalar product in $I\!R^{d\times d}$).

Received by the editors May 13, 2004 and, in revised form, August 24, 2004.
2000 *Mathematics Subject Classification.* 65N30.

This kind of nonlinear Stokes problem appears in the modeling of a large class of non-Newtonian fluids. In the particular case of Carreau law for viscoelastic flows (see, e.g. [11]), we have

$$\forall \alpha \in I\!R^{d\times d}, \quad \mathcal{A}(\alpha) = (k_0 + k_1(1+\|\alpha\|^2)^{\frac{\beta-2}{2}})\alpha,$$

with $k_0 \geq 0, k_1 > 0$ and $\beta \geq 1$. It is easy to verify that the Carreau law satisfies (1.1) and (1.2) for all $k_0 > 0$ and $\beta \in [1,2]$. In particular, with $\beta = 2$ we find the usual linear Stokes model.

In the sequel, we denote by $W^{s,p}(\Omega)$ and $W^{s,p}(\Gamma), 0 \leq s$ and $1 \leq p \leq +\infty$, the usual Sobolev spaces (see e.g [1]), endowed with the norms $||.||_{s,p,\Omega}$ and $||.||_{s,p,\Gamma}$ respectively. For a non integer s, we use the notations $|.|_{s,p,\Omega}$ and $|.|_{s,p,\Gamma}$, given explicitely, as following:

$$\text{if } p < +\infty, \qquad |v|^p_{s,p,\Omega} = \int\int_{\Omega\times\Omega} \frac{||D^{[s]}v(x) - D^{[s]}v(y)||^p}{|x-y|^{d+p\sigma}} dxdy,$$

$$\text{if } p = \infty, \qquad |v|_{s,+\infty,\Omega} = \sup_{\Omega\times\Omega} \frac{||D^{[s]}v(x) - D^{[s]}v(y)||^p}{|x-y|^{\sigma}}$$

and

$$|v|^p_{s,p,\Gamma} = \int\int_{\Gamma\times\Gamma} \frac{||D^{[s]}v(x) - D^{[s]}v(y)||^p}{|x-y|^{d-1+p\sigma}} dxdy,$$

where $[s]$ is the integer part of s and $\sigma = s - [s]$. $H^s(\Omega)$ is the usual space $W^{s,2}$ and $H^s_0(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H^s(\Omega)$.

In order to state the precise form of our estimator, we specify the hypothesis on the class of finite elements spaces under questions. Let $\mathcal{T}_h$ be a family of regular triangulations by triangles or tetrahedron of Ω in the sens of Ciarlet [7], We denote by $\mathcal{N}$ the set of all nodes in $\mathcal{T}_h$, and by $\mathcal{K} := \mathcal{N}/\Gamma$ the set of free nodes. Let ϕ_a denotes a hat function for $a \in \mathcal{N}$ which is piecewise linear function such that $\forall b \in \mathcal{N} \;\; \phi_a(b) = \delta^b_a$, by $\omega_a := \{x \in \Omega, \phi_a(x) > 0\}$ we denote the patch of $a \in \mathcal{N}$ and we set $h_a := diam(\omega_a)$. Finally, we denoted by $\mathcal{E}$ the set of all edges (faces) of $\mathcal{T}_h$ and by $\mathcal{E}_I$ the set of all interior edges (faces) of $\mathcal{T}_h$.

We introduce the following spaces:

$$\begin{aligned} V_h = \quad & \{v_h \in L^2(\Omega),\ \forall T \in \mathcal{T}_h,\ v_{h|T} \in (P_1(T))^d, \forall e \in \mathcal{E}_I,\ \int_e [v_h] d\sigma = 0, \\ & \forall e \text{ edge (face) } \subset \Gamma \quad \int_e v_h d\sigma = 0\}, \end{aligned}$$

and

$$M_h = \{q_h \in L^2_0(\Omega), \forall T \in \mathcal{T}_h,\ q_{h|T} \in P_1(T)\},$$

In the sequel, we consider $(u_h, p_h) \in (V_h)^d \times M_h$ verifying: $\forall v_h \in (V_h \cap H^1_0(\Omega))^d$,

$$\sum_{T\in\mathcal{T}_h} \int_T \mathcal{A}(\nabla u_h).\nabla v_h - \sum_{T\in\mathcal{T}_h} \int_T p_h div v_h dx = \int_\Omega f.v_h dx, \tag{1.3}$$

For abbreviation, we frequently write $\|.\|_{1,h,\omega} = \{\sum_{T\in\mathcal{T}_h, T\subset\omega} \|.\|^2_{1,T}\}^{\frac{1}{2}}$ and neglect the domain when $\omega := \Omega$ if there is no risk of confusion, and we denote by div_h the operator defined from

$$H(div; \mathcal{T}_h) := \{\sigma \in (L^2(\Omega))^{d\times d};\ \ \forall T \in \mathcal{T}_h, \sigma_{|T} \in H(div;T)\}$$

onto $L^2(\Omega)^d$ by:

$$\forall T \in \mathcal{T}_h, \quad div_h \sigma = div \sigma \qquad \text{on } T.$$

Furthermore, we define, using classical notations, the following residuals:

$$\left\{\begin{array}{ll} \eta_1^2: & = \sum_{T\in\mathcal{T}_h} \|divu_h\|_{0,T}^2, \\ \eta_2^2: & = \sum_{z\in\mathcal{K}} h_z^2 \|div_h(\mathcal{A}(\nabla u_h) - p_h Id_d) + f \\ & -\frac{1}{meas(\omega_z)} \sum_{T\in\mathcal{T}_h, T\subset\omega_z} \int_T (div(\mathcal{A}(\nabla u_h) - p_h Id_d) + f)dx\|_{0,\omega_z}^2, \\ \eta_3^2: & = \sum_{E\in\mathcal{E}} h_E \|[(\mathcal{A}(\nabla u_h) - p_h Id_d).n_E]\|_{0,E}^2, \\ \eta_4^2: & = \sum_{E\in\mathcal{E}} h_E^{-1} \|[u_h]_e\|_{0,E}^2, \end{array}\right.$$

where $[r.n_E]$ is the jump of $r.n_E$ across an interior element boundary of $E \in \mathcal{E}_I$, and is defined by $[r.n_E] = 0$ on Γ, and $[v_h]_E$ is the jump of v_h across an interior element boundary of $E \in \mathcal{E}_I$, and is defined by $(v_h)_{|E}$ on Γ.

In the sequel, we denoted by $C, C_0, C_1, ..$ various positive generic constants not dependent of $\{h_T\}_{T\in\mathcal{T}_h}$ and not necessarily the same.

Our main results is

Theorem 1.1. *Let $(u_h, p_h) \in (V_h)^d \times M_h$ verifying* (1.3)*, we have*

$$|u-u_h|_{1,h} + \|p-p_h\|_{0,\Omega} \leq C\{\sum_{i=1}^{4} \eta_i^2\}^{\frac{1}{2}}.$$

Moreover, for all $T \in \mathcal{T}_h$, for all $E \in \mathcal{E}$, we have

$$\|divu_h\|_{0,T} \leq C|u-u_h|_{1,T},$$

$$\begin{array}{l} h_z\|div(\mathcal{A}(\nabla u_h) - p_h Id_d) + f \\ \quad -\frac{1}{meas(\omega_z)} \sum_{T\in\mathcal{T}_h, T\subset\omega_z} \int_T (div(\mathcal{A}(\nabla u_h) - p_h Id_d) + f)dx\|_{0,\omega_z}^2 \\ \leq C\{|u-u_h|_{1,h,\omega_z} + \|p-p_h\|_{0,\omega_z} + h_z\|f - \frac{1}{meas(\omega_z)} \int_{\omega_z} fdx\|_{0,\omega_z}\}, \end{array}$$

$$\begin{array}{rll} & \forall E: & = \partial T \cap \partial K \in \mathcal{E}_I, \\ & h_E^{\frac{1}{2}} \|[(\mathcal{A}(\nabla u_h) - p_h Id_d).n_E]\|_{0,E} & \leq C\{|u-u_h|_{1,h,T\cup K} + \|p-p_h\|_{0,T\cup K}\} \\ +C_1\{h_T^2\|f - \frac{1}{meas_d(T)} \int_T fdx\|_{0,T}^2 & + \|f - \frac{1}{meas_d(K)} \int_T fdx\|_{0,K}^2\}^{\frac{1}{2}}. & \end{array}$$

$$\forall E := \partial T \cap \partial K \in \mathcal{E}_I, \quad h_E^{-\frac{1}{2}} \|[u_h]\|_{0,E} \leq C|u-u_h|_{1,h,T\cup K},$$

and

$$\forall E := \partial T \cap \Gamma, \quad h_E^{-\frac{1}{2}} \|[u_h]\|_{0,E} \leq C|u-u_h|_{1,T}.$$

2. Efficiency of the estimator

This section is devoted to the estimator efficiency. For this, we give the two following lemmas,the proof of the first lemma follows the ideas developed by Verfurth [13]:

Lemma 2.1. *Let $(u_h, p_h) \in (V_h)^d \times M_h$, for all $T \in \mathcal{T}_h$, for all $E := \partial T \cap \partial K \in \mathcal{E}_I$, we have*

$$\begin{aligned} & h_z \| div_h(\mathcal{A}(\nabla u_h) - p_h Id_d) + f \\ & -\frac{1}{meas(\omega_z)} \sum_{T \in \mathcal{T}_h, T \subset \omega_z} \int_T (div(\mathcal{A}(\nabla u_h) - p_h Id_d) + f) dx \|_{0,\omega_z}^2 \\ & \leq C\{|u - u_h|_{1,h,\omega_z} + \|p - p_h\|_{0,\omega_z} + h_z \|f - \frac{1}{meas(\omega_z)} \int_{\omega_z} f dx\|_{0,\omega_z}\} \end{aligned}$$

and

$$\begin{aligned} & h_E^{\frac{1}{2}} \|[(\mathcal{A}(\nabla u_h) - p_h Id_d).n_E]\|_{0,E} \\ & \leq C\{|u - u_h|_{1,h,T\cup K} + \|p - p_h\|_{0,T\cup K}\} \\ & + C_1\{h_T^2 \|f - \frac{1}{meas_d(T)} \int_T f dx\|_{0,T}^2 + \|f - \frac{1}{meas_d(K)} \int_T f dx\|_{0,K}^2\}^{\frac{1}{2}}. \end{aligned}$$

We have also

Lemma 2.2. *Let $(u_h, p_h) \in (V_h)^d \times M_h$, for all $T \in \mathcal{T}_h$, for all $E \in \mathcal{E}$, we have*

$$\|divu_h\|_{0,T} \leq C|u - u_h|_{1,T},$$

$$\text{If } E := \partial T \cap \partial K \in \mathcal{E}_I, \quad h_E^{-\frac{1}{2}} \|[u_h]\|_{0,E} \leq C|u - u_h|_{1,h,T\cup K},$$

and

$$\text{If } E := \partial T \cap \Gamma, \quad h_E^{-\frac{1}{2}} \|[u_h]\|_{0,E} \leq C|u - u_h|_{1,T}.$$

Proof. First, it is clear that

$$\|divu_h\|_{0,T} \leq C|u - u_h|_{1,T}.$$

Let us prove the second estimation. Let $E := \partial T \cap \partial K \in \mathcal{E}_I$, since $u_h \in (V_h)^d$ and $u \in (H_0^1(\Omega))^d$, we have

$$\int_E \{(u_h)_{|T} - u\} d\sigma = \int_E \{(u_h)_{|K} - u\} d\sigma,$$

then

$$\begin{aligned} h_E^{-\frac{1}{2}} \|[u_h]\|_{0,E} &= h_E^{-\frac{1}{2}} \|[u_h - u - c_e]\|_{0,E} \\ &\leq h_E^{-\frac{1}{2}} (\|(u_h)_{|T} - u - c_e\|_{0,E} + \|(u_h)_{|K} - u - c_e\|_{0,E}), \end{aligned}$$

where

$$c_e := \frac{1}{meas_{d-1}(E)} \int_E \{(u_h)_{|T} - u\} d\sigma = \frac{1}{meas_{d-1}(E)} \int_E \{(u_h)_{|K} - u\} d\sigma.$$

Using trace lemma, and the fact that

$$\int_E \{(u_h)_{|T} - u - c_e\} d\sigma = \int_E \{(u_h)_{|K} - u - c_e\} d\sigma = 0,$$

we obtain

$$\begin{aligned} h_E^{-\frac{1}{2}} \|[u_h]\|_{0,E} &\leq h_E^{-\frac{1}{2}} \{\|(u_h)_{|T} - u - c_e\|_{0,E} + \|(u_h)_{|K} - u - c_e\|_{0,E}\} \\ &\leq C|u - u_h|_{1,h,T\cup K}. \end{aligned}$$

If $E \in \mathcal{E} \cap \partial T$ with $T \in \mathcal{T}_h$, and $E \subset \Gamma$, since $u = 0$ on Γ and $\int_E u_h d\sigma = 0$, using the same arguments, we have

$$h_E^{-\frac{1}{2}} \|[u_h]\|_{0,E} := h_E^{-\frac{1}{2}} \|u_h\|_{0,E} = h_E^{-\frac{1}{2}} \|u_h - u\|_{0,E} \leq C|u - u_h|_{1,h,T}.$$

□

3. Reliability of the estimator

Before proving the reliability of the estimator, let us recall the following [5]:

Theorem 3.1. *There exist a linear mapping $\mathcal{I} : (H_0^1(\Omega))^d \longrightarrow (V_h)^d \cap (H_0^1(\Omega))^d$, bounded if domain and space range are endowed with H^1-semi norms, which satisfies, for all $\phi \in (H_0^1(\Omega))^d$:*

$$\{\sum_{T\in\mathcal{T}_h}\} h_T^2 \|\phi - \mathcal{I}\phi\|_{0,T}^2\}^{\frac{1}{2}} \leq C|\phi|_{1,\Omega}$$

and

$$\{\sum_{E\in\mathcal{E}} h_E^{-1} \|\phi - \mathcal{I}\phi\|_{0,E}^2\}^{\frac{1}{2}} \leq C|\phi|_{1,\Omega}.$$

In addition, there holds for all $R \in (L^2(\Omega))^d$

$$\int_\Omega R(\phi - \mathcal{I}\phi)dx \leq C|\phi|_{1,\Omega}\{\sum_{z\in\mathcal{K}} h_z^2 \|R - \frac{1}{meas(\omega_z)}\int_{\omega_z} Rdx\|_{0,\omega_z}^2\}^{\frac{1}{2}}.$$

We need also the following technical lemma.

Lemma 3.1. *There exists a linear mapping $\mathcal{R} : (V_h)^d \longrightarrow (V_h)^d \cap (H_0^1(\Omega))^d$, satisfies the following estimate:*

$$\forall u_h \in (V_h)^d, \quad \forall T \in \mathcal{T}_h, l = 0, 1, \quad |u_h - \mathcal{R}u_h|_{l,T} \leq C \sum_{E\in\mathcal{E}, \overline{E}\cap\overline{T}\neq\emptyset} h_E^{\frac{1}{2}-l} \|[u_h]_E\|_{0,E}.$$

Proof. Let $\mathcal{R}$ the operator, defined by: $\forall u_h \in (V_h)^d$, $\mathcal{R}u_h$ is the unique element of $(V_h)^d \cap (H_0^1(\Omega))^d$ where

$$\forall z \in \mathcal{K}, \mathcal{R}u_h(z) = \frac{1}{\#M_z} \sum_{K\in M_z} (u_h)_{|K}(z),$$

and $M_z := \{T \in \mathcal{T}_h,\ z \in \overline{T}\}$. Let $T \in \mathcal{T}_h$, we denoted by V_T the set of the vertex of T. On one hand, we have

$$\begin{aligned}\forall z \in V_T \cap \mathcal{E}_I, |(u_h)_{|T}(z) - \mathcal{R}u_h(z)| &= |\frac{1}{\#M_z} \sum_{K\in M_z} ((u_h)_{|T}(z) - (u_h)_{|K}(z))| \\ &\leq \sup_{K\in M_z} |(u_h)_{|T}(z) - (u_h)_{|K}(z)|\end{aligned}$$

and

$$\forall z \in E \cap V_T \quad \text{with } E \subset \Gamma, |(u_h)_{|T}(z) - \mathcal{R}u_h(z)| = |(u_h)_{|T}(z)| \leq \|[u_h]_E\|_{0,\infty,E}.$$

Since $u_h - \mathcal{R}u_h \in P_1(T)$, we have

$$\|(u_h)_{|T} - \mathcal{R}u_h\|_{0,\infty,T} = \sup_{z\in V_T} |(u_h)_{|T}(z) - \mathcal{R}u_h(z)| \leq \sum_{E\in\mathcal{E}, \overline{E}\cap\overline{T}\neq\emptyset} \|[u_h]_E\|_{0,\infty,E},$$

using the inverse inequality, we obtain

$$\|(u_h)_{|T} - \mathcal{R}u_h\|_{0,\infty,T} \leq C \sum_{E\in\mathcal{E}, \overline{E}\cap\overline{T}\neq\emptyset} h_E^{\frac{1-d}{2}} \|[u_h]_e\|_{0,E}. \tag{3.1}$$

On the other hand, using again the inverse inequality, we have

$$\|(u_h)_{|T} - \mathcal{R}u_h\|_{l,T} \leq C h_T^{\frac{d}{2}-l} \|(u_h)_{|T} - \mathcal{R}u_h\|_{0,\infty,T}. \tag{3.2}$$

Finally, by using (3.1)-(3.2), we obtain the result. ∎

Lemma 3.2. *We have the generalized inf-sup condition:*
for all $((u,p),(v,q)) \in ((H_0^1(\Omega))^d \times L_0^2(\Omega))^2$,

$$|u-v|_{1,\Omega} + \|p-q\|_{0,\Omega} \leq C \sup_{(w,s)\in(H_0^1(\Omega))^d\times(L^2(\Omega))^d} \frac{APS}{\|q\|_{0,\Omega} + |v|_{1,\Omega}},$$

where

$$APS = \int_\Omega ((\mathcal{A}(\nabla u) - \mathcal{A}(\nabla v)).\nabla w dx - \int_\Omega (p-q) div w dx + \int_\Omega s\ div(u-v)dx.$$

Proof: Let $(\hat{w},\hat{q}) \in (H_0^1(\Omega))^d \times L_0^2(\Omega)$ the unique weak solution of Stokes problem

$$\Delta\hat{w} - \nabla\hat{q} = 0 \quad \text{and} \quad div\hat{w} = p-q \quad \text{on } \Omega.$$

we set $w = \gamma(u-v) - \hat{w}$, with $\gamma > 0$. On one hand, we have

$$\begin{aligned}
&\int_\Omega ((\mathcal{A}(\nabla u) - \mathcal{A}(\nabla v)).\nabla w dx - \int_\Omega (p-q) div w dx \\
&= \gamma \int_\Omega ((\mathcal{A}(\nabla u) - \mathcal{A}(\nabla v)).\nabla(u-v) dx - \gamma \int_\Omega (p-q) div(u-v) dx \\
&- \int_\Omega \int_\Omega ((\mathcal{A}(\nabla u) - \mathcal{A}(\nabla v)).\nabla\hat{w} dx + \|p-q\|_{0,\Omega}^2.
\end{aligned}$$

Since

$$\gamma \int_\Omega ((\mathcal{A}(\nabla u) - \mathcal{A}(\nabla v)).\nabla(u-v) dx \geq c\gamma|u-v|_{1,\Omega}^2,$$

$$\begin{aligned}
\int_\Omega ((\mathcal{A}(\nabla u) - \mathcal{A}(\nabla v)).\nabla\hat{w} dx &\leq c_2|\hat{w}|_{1,\Omega}|u-v|_{1,\Omega} \\
&\leq \frac{c_2c_3}{2}|u-v|_{1,\Omega}^2 + \frac{1}{2}\|p-q\|_{0,\Omega}^2
\end{aligned}$$

and

$$\begin{aligned}
\gamma \int_\Omega (p-q) div(u-v) dx &\leq \gamma\|p-q\|_{0,\Omega}\|div(u-v)\|_{0,\Omega} \\
&\leq \gamma^2\|div(u-v)\|_{0,\Omega}^2 + \frac{1}{4}\|p-q\|_{0,\Omega}^2,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\int_\Omega ((\mathcal{A}(\nabla u) - \mathcal{A}(\nabla v)).\nabla w dx - \int_\Omega (p-q) div w dx \\
&\geq c\gamma|u-v|_{1,\Omega}^2 - \frac{1}{4}\|p-q\|_{0,\Omega}^2 - \gamma^2\|div(u-v)\|_{0,\Omega} \\
&\quad -\frac{c_2c_3}{2}|u-v|_{1,\Omega}^2 - \frac{1}{2}\|p-q\|_{0,\Omega}^2 + \|p-q\|_{0,\Omega}^2.
\end{aligned}$$

which implie

$$\begin{aligned}
&(c\gamma - \frac{c_2c_3}{2})\|u-v\|_{1,\Omega}^2 + \frac{1}{4}\|p-q\|_{0,\Omega}^2 \\
&\leq \int_\Omega ((\mathcal{A}(\nabla u) - \mathcal{A}(\nabla v)).\nabla w dx - \int_\Omega (p-q) div w dx + \gamma^2\|div(u-v)\|_{0,\Omega}^2.
\end{aligned}$$

On the other hand, since

$$|w|_{1,\Omega} \leq \gamma|u-v|_{1,\Omega} + |\hat{w}|_{1,\Omega} \leq C(|u-v|_{1,\Omega} + |p-q|_{0,\Omega}),$$

and

$$s = div(u-v) \in L_0^2(\Omega),$$

we deduce for $c\gamma = c_2c_3$ that

$$\int_\Omega ((\mathcal{A}(\nabla u) - \mathcal{A}(\nabla v)).\nabla w dx - \int_\Omega (p-q) div w dx + \gamma^2 \int_\Omega s div(u-v) dx$$

$$\geq C(|w|_{1,\Omega} + \|s\|_{0,\Omega})(|u-v|_{1,\Omega} + \|p-q\|_{0,\Omega}).$$

which implies

$$|u-v|_{1,\Omega} + \|p-q\|_{0,\Omega} \leq C \sup_{(w,s)\in(H_0^1(\Omega))^d\times(L^2(\Omega))^d} \frac{APS}{\|q\|_{0,\Omega} + |v|_{1,\Omega}}.$$

□

With slightly modification of last argument we can prove more general results, more precisely, let $\{\Omega_i\}_{i=1}^I$ a domain decomposition without overlapping of Ω. We introduce the bilinear form defined on $(\prod_{i=1}^I (H^1(\Omega_i))^d) \times L^2(\Omega)$ by

$$\forall((u,p);(v,q)) \in ((\prod_{i=1}^I (H^1(\Omega_i))^d) \times L^2(\Omega))^2,$$

$$a((u,p);(v,q)) = \sum_{i=1}^I \int_{\Omega_i} \nabla u.\nabla v dx - \sum_{i=1}^I \int_{\Omega_i} p div v dx + \sum_{i=1}^I \int_{\Omega_i} q div u dx.$$

We have the following

Theorem 3.2. *Let (X, M) two subspaces of $\prod_{i=1}^I (H^1(\Omega))^d$ and $L^2(\Omega)$. We assume that there exist two spaces $(\overline{X}, \overline{M})$ such that:*

1. *$X \subset \overline{X} \subset \prod_{i=1}^I (H^1(\Omega))^d$ and $M \subset \overline{M} \subset L^2(\Omega)$.*
2. *There exist $C > 0$ such that*

$$\inf_{q\in\overline{M}} \sup_{v\in\overline{M}} \frac{\sum_{i=1}^I \int_{\Omega_i} q div v dx}{\|q\|_{0,\Omega}\{\sum_{i=1}^I |v|_{\Omega_i}^2\}^{\frac{1}{2}}} \geq C$$

3.

$$\textit{If } v \in \overline{X} \quad \textit{and} \quad \{\sum_{i=1}^I |v|_{\Omega_i}^2\}^{\frac{1}{2}} = 0 \quad \textit{then} \quad v = 0.$$

Then, there exist a constant C_2 such that, for all $((u,p);(v,q)) \in X \times M$, we have

$$\{\sum_{i=1}^I |u-v|_{\Omega_i}^2\}^{\frac{1}{2}} + \|p-q\|_{0,\Omega} \leq C_2 \sup_{(w,s)\overline{X}\times R} \frac{a((u,p);(w,s)) - a((v,q);(w,s))}{\|s\|_{0,\Omega} + \{\sum_{i=1}^I |w|_{\Omega_i}^2\}^{\frac{1}{2}}},$$

where

$$R := \{q \in L^2(\Omega);\ \textit{there exist } u \in X \ \textit{ such that } q = div u \ \textit{ on } \Omega_i, i = 1,..,I\}.$$

Now we are able to prove the reliability of our estimator, more precisely, we have

Theorem 3.3. *Let* $(u_h, p_h) \in (V_h)^d \times M_h$ *verifying* (2.1), *we have*

$$|u - u_h|_{1,h} + \|p - p_h\|_{0,\Omega} \leq C\{\sum_{i=1}^{4} \eta_i^2\}^{\frac{1}{2}}.$$

Proof . Since $(\mathcal{R}u_h, p_h) \in (H_0^1(\Omega))^d \times L_0^2(\Omega)$, let us remark that

$$(3.1)\ \left\{ \begin{array}{l} |u - \mathcal{R}u_h|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \leq \\ C \sup\limits_{(v,q) \in (H_0^1(\Omega))^d \times L_0^2(\Omega)} \dfrac{a((u,p);(v,q)) - a((\mathcal{R}u_h, p_h);(v,q))}{|v|_{1,\Omega} + \|q\|_{0,\Omega}}, \end{array} \right.$$

where $a(.;.)$ is defined by

$$\begin{aligned} & \forall((u,p);(v,q)) \in ((H_0^1(\Omega))^d \times L_0^2(\Omega))^2, \\ a((u,p);(v,q)) \quad &= \int_\Omega \mathcal{A}(\nabla u).\nabla v dx - \int_\Omega p div v dx + \int_\Omega q div u dx. \end{aligned}$$

On one hand, since $divu = 0$ on Ω, we have

$$|\int_\Omega q div(u - \mathcal{R}u_h)dx| \leq C \sum_{T \in \mathcal{T}_h} \|q\|_{0,T} \|div \mathcal{R}u_h\|_{0,T},$$

using Lemma 3.1, we obtain

$$(3.2)\quad \begin{aligned} \forall T \in \mathcal{T}_h,\ \|div\mathcal{R}u_h\|_{0,T} \quad &\leq \|divu_h\|_{0,T} + \|divu_h - div\mathcal{R}u_h\|_{0,T} \\ &\leq \|divu_h\|_{0,T} + C \sum_{e \in \mathcal{E}, \overline{e} \cap \overline{T} \neq \emptyset} h_E^{-\frac{1}{2}} \|[u_h]_e\|_{0,e}, \end{aligned}$$

then

$$|\int_\Omega q div(u - \mathcal{R}u_h)dx| \leq C\|q\|_{0,\Omega}\{\eta_1^2 + \eta_4^2\}^{\frac{1}{2}}.$$

On the other hand, we have

$$\begin{aligned} & \int_\Omega (\mathcal{A}(\nabla u) - \mathcal{A}(\nabla \mathcal{R}u_h)).\nabla v dx - \int_\Omega (p - p_h) div v dx \\ = \quad & \sum_{T \in \mathcal{T}_h} \int_T (\mathcal{A}(\nabla u) - \mathcal{A}(\nabla u_h)).\nabla v dx \\ & + \sum_{T \in \mathcal{T}_h} \int_T (\mathcal{A}(\nabla u_h) - \mathcal{A}(\nabla \mathcal{R}u_h)).\nabla v dx - \int_\Omega (p - p_h) div v dx \\ = \quad & - \sum_{T \in \mathcal{T}_h} \int_T \mathcal{A}(\nabla u_h).\nabla v dx + \int_\Omega p_h div v dx \\ & + \int_\Omega f v dx + \sum_{T \in \mathcal{T}_h} \int_T (\mathcal{A}(\nabla u_h) - \mathcal{A}(\nabla \mathcal{R}u_h)) \nabla v dx. \end{aligned}$$

First, Using lemma 3.1, it is clear that

$$(3.3)\ |\sum_{T \in \mathcal{T}_h} \int_T (\mathcal{A}(\nabla u_h) - \mathcal{A}(\mathcal{R}u_h)) \nabla v dx| \leq \sum_{T \in \mathcal{T}_h} |u_h - \mathcal{R}u_h|_{1,T} |v|_{1,T} \leq C\eta_4.|v|_{1,\Omega}.$$

Since (u_h, p_h) satisfies (2.1), by elementwise integration by parts, we infer

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h}\int_T \mathcal{A}(\nabla u_h).\nabla v dx - \int_\Omega p_h div v dx - \int_\Omega f v dx \\
&= \sum_{T\in\mathcal{T}_h}\int_T \mathcal{A}(\nabla u_h).\nabla (v-\mathcal{I}v) dx - \int_\Omega p_h div (v-\mathcal{I}v) dx - \int_\Omega f (v-\mathcal{I}v) dx \\
&= \sum_{T\in\mathcal{T}_h}\int_T (-div(\mathcal{A}(\nabla u_h) - p_h Id_d) + f)(v-\mathcal{I}v) dx \\
&\quad + \sum_{E\in\mathcal{E}}\int_E [(\mathcal{A}(\nabla u_h) - p_h Id_d).n_E](v-\mathcal{I}v) d\sigma,
\end{aligned}$$

Recall that $[r.n_E]$ is the jump of $r.n_E$ across an interior element boundary of $E \in \mathcal{E}$, and is defined by $[r.n_E] = 0$ on Γ. From Cauchy inequality and using theorem 3.1, we conclude

$$(3.4)\qquad |\sum_{T\in\mathcal{T}_h}\int_T \mathcal{A}(\nabla u_h).\nabla v dx - \int_\Omega p_h div v dx - \int_\Omega f v dx| \le C(\eta_2^2 + \eta_3^2)^{\frac{1}{2}} |v|_{1,\Omega}.$$

Finally, since

$$|u-u_h|_{1,h} \le |u - \mathcal{R}u_h|_{1,\Omega} + |u_h - \mathcal{R}u_h|_{1,h} \le |u-\mathcal{R}u_h|_{1,\Omega} + C\eta_4,$$

using (3.1)-(3.4), we obtain the result. □

4. Applications.

4.1. Nonconforming Approximations. We consider the discrete problem:

$$\left\{\begin{array}{lll}
\text{Find } (u_h, p_h) & \in (V_h)^d \times \dot{M}_h \quad \text{such that:} & \\
\forall v_h \in (V_h)^d, & \sum_{T\in\mathcal{T}_h}\int_T \mathcal{A}(\nabla u_h).\nabla v_h dx - \sum_{T\in\mathcal{T}_h}\int_T p_h div v_h dx = \int_\Omega f v_h dx, & \\
\forall q_h \in \dot{M}_h, & \sum_{T\in\mathcal{T}_h}\int_T q_h div v_h dx = 0, &
\end{array}\right.$$

where

$$\dot{M}_h := \{q_h \in L_0^2(\Omega); \quad \forall T \in \mathcal{T}_h,\ \{v_h\}_{|T} \in P_0(T)\}.$$

We assume that, for all $v_h \in (V_h)^d$ and $T \in \mathcal{T}_h$, the matrix $\mathcal{A}(\nabla v_h)_{|T}$ is matrix with constant components. It is clear that this problem has unique solution (u_h, p_h) ([4],[10]), moreover (u_h, p_h) satisfies (1.3) and

$$\forall T \in \mathcal{T}_h, \quad div u_h = 0 \quad \text{on } T.$$

Let $\Pi : (H_0^1(\Omega))^d \longrightarrow (V_h)^d$ the linear operator defined by:

$$\forall v \in (H_0^1(\Omega))^d, \forall E \in \mathcal{E}, \quad \int_E (\Pi v - v) d\sigma = 0.$$

First, we have

Lemma 4.1. *The linear operator* Π *satisfies, for all* $v \in (H_0^1(\Omega))^d$ *and for all* $T \in \mathcal{T}_h$*:*

$$\forall s_h \in (P_0(T))^{d\times d}, \quad \int_T s_h.\nabla(v - \Pi v) dx = 0,$$

$$\forall q_h \in P_0(T), \quad \int_T q_h div(v - \Pi v) dx = 0,$$

and

$$\forall R \in (L^2(\Omega))^d, \quad |\sum_{T\in\mathcal{T}_h}\int_T R(v-\Pi v)dx| \leq C|v|_{1,\Omega}\{\sum_{T\in\mathcal{T}_h} h_T^2\|R\|_{0,T}^2\}^{\frac{1}{2}}.$$

Proof First, let $v \in (H_0^1(\Omega))^d$ and $s_h \in (P_0(T))^{d\times d}$, by elementwise integration by part, we infer

$$\begin{aligned}\int_T s_h.\nabla(v-\Pi v)dx &= -\int_T (v-\Pi v)divs_h dx + \int_{\partial T}(v-\Pi v)s_h.nd\sigma \\ &= \int_{\partial T}(v-\Pi v)s_h.nd\sigma = 0.\end{aligned}$$

Again, by elementwise integration by part, we infer

$$\forall q_h \in P_0(T), \int_T q_h div(v-\Pi v)dx = -\int_T \nabla q_h.(v-\Pi v)dx + \int_{\partial T} q_h(v-\Pi v).n_T d\sigma = 0.$$

Let $R \in (L^2(\Omega))^d$, since

$$\|v-\Pi v\|_{0,T} \leq Ch_T|v|_{1,T},$$

we have

$$|\sum_{T\in\mathcal{T}_h}\int_T R(v-\Pi v)dx| \leq C|v|_{1,\Omega}\{\sum_{T\in\mathcal{T}_h} h_T^2\|R\|_{0,T}^2\}^{\frac{1}{2}}.$$

Modification of the last arguments give to us the following

Theorem 4.1. *Let $(u_h,p_h) \in (V_h)^d \times \dot{M}_h$ the solution of the the problem (P_h) , we have*

$$|u-u_h|_{1,h} + \|p-p_h\|_{0,\Omega} \leq C\{\eta_4^2 + \sum_{T\in\mathcal{T}_h} h_T^2\|f\|_{0,T}\}^{\frac{1}{2}}.$$

Moreover, for all $T \in \mathcal{T}_h$, for all $E \in \mathcal{E}$, we have

$$\textit{If } E := \partial T \cap \partial K \in \mathcal{E}_I, \quad h_E^{-\frac{1}{2}}\|[u_h]\|_{0,E} \leq C|u-u_h|_{1,h,T\cup K},$$

$$\textit{If } E \subset \partial T \cap \Gamma, \quad h_E^{-\frac{1}{2}}\|[u_h]\|_{0,E} \leq C|u-u_h|_{1,T},$$

and

$$h_T\|f_h\|_{0,T} \leq C\{|u-u_h\|_{1,T} + \|p-p_h\|_{0,T} + h_T\|f-f_h\|_{0,T}\}.$$

Proof . Let us recall that (see Lemma 3.2.):

$$(4.1)\left\{\begin{array}{l}|u-\mathcal{R}u_h|_{1,\Omega} + \|p-p_h\|_{0,\Omega} \leq \\ C \sup\limits_{(v,q)\in(H_0^1(\Omega))^d\times L_0^2(\Omega)} \dfrac{a((u,p);(v,q)) - a((\mathcal{R}u_h,p_h);(v,q))}{|v|_{1,\Omega} + \|q\|_{0,\Omega}},\end{array}\right.$$

As in Theorem 3.3, since $divu_h = 0$ on $T \in \mathcal{T}_h$, on one hand:

$$(4.2) \qquad |\int_\Omega q div(u-\mathcal{R}u_h)dx| \leq C\|q\|_{0,\Omega}\eta_4.$$

On the other hand, we have

$$\begin{aligned}
&\int_\Omega (\mathcal{A}(\nabla u) - \mathcal{A}(\nabla \mathcal{R}u_h)).\nabla v dx - \int_\Omega (p-p_h) divv dx \\
= &\sum_{T\in\mathcal{T}_h} \int_T (\mathcal{A}(\nabla u) - \mathcal{A}(\nabla u_h)).\nabla v dx - \int_\Omega (p-p_h) divv dx \\
&\qquad + \sum_{T\in\mathcal{T}_h} \int_T (\mathcal{A}(\nabla u_h) - \mathcal{A}(\nabla \mathcal{R}u_h)).\nabla v dx \\
= &-\sum_{T\in\mathcal{T}_h} \int_T \mathcal{A}(\nabla u_h).\nabla v dx + \int_\Omega p_h divv dx + \int_\Omega f v dx \\
&\qquad + \sum_{T\in\mathcal{T}_h} \int_T (\mathcal{A}(\nabla u_h) - \mathcal{A}(\nabla \mathcal{R}u_h))\nabla v dx.
\end{aligned}$$

First, we have

$$|\sum_{T\in\mathcal{T}_h} \int_T (\mathcal{A}(\nabla u_h) - \mathcal{A}(\mathcal{R}u_h))\nabla v dx| \leq C\eta_4.|v|_{1,\Omega}. \tag{4.3}$$

Since (u_h, p_h) is solution of discrete problem, we have

$$\left\{\begin{aligned}
&\sum_{T\in\mathcal{T}_h} \int_T \mathcal{A}(\nabla u_h).\nabla v dx - \int_\Omega p_h divv dx - \int_\Omega f v dx \\
&= \sum_{T\in\mathcal{T}_h} \int_T \mathcal{A}(\nabla u_h).\nabla \Pi v dx - \int_\Omega p_h div(\Pi v) dx - \int_\Omega f v dx \\
&= \int_\Omega f(\Pi v - v) dx \leq C|v|_{1,\Omega}\{\sum_{T\in\mathcal{T}_h} h_T^2 \|f\|_{0,T}^2\}^{\frac{1}{2}},
\end{aligned}\right. \tag{4.4}$$

Finally, since

$$|u-u_h|_{1,h} \leq |u-\mathcal{R}u_h|_{1,\Omega} + |u_h - \mathcal{R}u_h|_{1,h} \leq |u-\mathcal{R}u_h|_{1,\Omega} + C\eta_4,$$

using (4.1)-(4.4), we obtain the upper bound.

The lower bound can be proved using the same arguments as in section 2, and so the details are omitted. ∎

4.2. Dual Mixed Approximation. In this section, we assume that there exist a function $\phi : I\!R^{d\times d} \longrightarrow I\!R$ such that:

$$\forall \alpha \in I\!R^{d\times d}, \quad \mathcal{A}(\alpha) = \phi(\|\alpha\|)\alpha.$$

The mixed dual formulation of continuous problems is [9]: Find $(t, \sigma, \overline{u}, \overline{p}, \zeta) \in (L^2(\Omega))^{d\times d} \times H(div;\Omega) \times (L^2(\Omega))^d \times L^2(\Omega) \times I\!R$ such that

$$\left\{\begin{aligned}
&\forall s \in (L^2(\Omega)^{d\times d}, \quad \int_\Omega \mathcal{A}(t).s dx - \int_\Omega \sigma.s dx - \int_\Omega \overline{p} trace(s) dx = 0, \\
&\forall (\tau, q) \in H(div;\Omega) \times L^2(\Omega), \quad -\int_\Omega \tau t dx - \int_\Omega q trace(t) dx \\
&\qquad\qquad - \int_\Omega \overline{u}.div\tau dx + \int_\Omega \zeta trace(\tau) dx = 0, \\
&\forall (v, \eta) \in (L^2(\Omega))^d \times R, \quad -\int_\Omega v div\sigma dx + \int_\Omega \eta trace(\sigma) dx = \int_\Omega f.v dx.
\end{aligned}\right.$$

In this section, we give a priori and a posteriori error estimates without using the two-fold saddle point theory [9].

Lemma 4.2. *The mixed problem has unique solution*

$$(t,\sigma,u,p,\zeta)\in(L^2(\Omega))^{d\times d}\times H(div;\Omega)\times(L^2(\Omega))^d\times L^2(\Omega)\times I\!R,$$

where,

$$t=\nabla u\quad,\quad\sigma=\mathcal{A}(u)-pId_d\quad,\quad\overline{u}=u\quad,\quad\overline{p}=p\quad and\ \zeta=0,$$

and (u,p) is the weak solution of model problem.

Proof . It is clear that $(t,\sigma,\overline{u},\overline{p},\zeta)$, where

$$t=\nabla u\quad,\quad\sigma=\mathcal{A}(u)-pId_d\quad,\quad\overline{u}=u\quad,\quad\overline{p}=p\quad\text{and }\zeta=0,$$

is weak solution of mixed problem. Let us prove the uniqueness.

Let $(t_i,\sigma_i,\overline{u}_i,\overline{p}_i,\zeta_i)$, i=1,2, be two weak solutions of mixed problem, we set:

$$t=t_1-t_2\quad,\sigma=\sigma_1-\sigma_2\quad,u=\overline{u}_1-\overline{u}_2\quad,\quad p=\overline{p}_1-\overline{p}_2\quad\text{and }\zeta=\zeta_1-\zeta_2.$$

First, we have

$$\forall(v,\eta)\in(L^2(\Omega))^d\times I\!R,-\int_\Omega v div\sigma dx+\int_\Omega\eta trace(\sigma)dx=0,$$

which implies

$$div\sigma=0\quad\text{on}\quad\Omega\quad\text{and}\quad\int_\Omega trace(\sigma)dx=0.$$

Since $\forall(\tau,q)\in H(div;\Omega)\times L^2(\Omega)$,

$$-\int_\Omega\tau t dx-\int_\Omega q trace(t)dx-\int_\Omega\overline{u}.div\tau dx+\int_\Omega\zeta trace(\tau)dx=0,$$

for $\tau=\sigma$, and using the fact that

$$div\sigma=0\quad\text{and}\quad\int_\Omega trace(\sigma)dx=0,$$

we have

$$trace(t)=0\quad\text{and}\ \int_\Omega t.\sigma=0.$$

By choosing the test function $s=t$ in the first equation, we obtain

$$\int_\Omega(\mathcal{A}(t_1)-\mathcal{A}(t_2)):(t_1-t_2)dx=0,$$

and then $t=t_1-t_2=0$. Again, by choosing the test function $s=\sigma+pId_d$ in the first equation , we obtain $\sigma+pId_d=0$ on Ω. By choosing $s=Id_d$ in the first equation, we have

$$\int_\Omega trace(\sigma)dx+d\int_\Omega pdx=0,$$

which implies $p\in L^2_0(\Omega)$, and since $\sigma\in H(div;\Omega)$, then $d\nabla p=-div\sigma=0$, which implies $p=0$ and then $\sigma=0$.

Finally, Let $(w,q)\in(H^1_0(\Omega))\times L^2_0(\Omega)$ the weak solution of

$$-\Delta w+\nabla q=u\quad\text{and}\quad divw=0\quad\text{on }\Omega,$$

we set $\tau=\nabla w-qId_d$, it is clear that

$$\tau\in H(div;\Omega)\quad\int_\Omega trace(\tau)dx=0\quad\text{and }-div\tau=u\quad\text{on }\Omega.$$

since

$$\forall\beta\in H(div;\Omega)\ ;\quad\int_\Omega u div\beta dx+\zeta\int_\Omega trace(\beta)dx=0,$$

we have, using $\beta = \tau$ and $\beta = Id_d$ as test function:

$$u = 0 \quad \text{and} \quad \zeta = 0.$$

Which prove the uniqueness of the mixed problem.

To be able to state the discrete mixed formulation, we introduce the finite elements spaces:

$$X_{1,h} = \{s \in (L^2(\Omega))^{d\times d} \ : \ s_{|T} \in (P_0(T))^{d\times d} \ \ \forall T \in \mathcal{T}_h\},$$

$$M^{\sigma}_{1,h} := \{\tau = (\tau_{ij}) \in H(div,\Omega) \ : \ (\tau_{i,1}.\tau_{i,d})_{|T} \in RT_0(T) = (P_0(T))^d + xP_0(T) \\ i = 1,..,d, \ \forall T \in \mathcal{T}_h\},$$

$$\overline{M}_h = \{q_h \in L^2(\Omega), \ q_{h|T} \in P_0(T), \ \forall T \in \mathcal{T}_h, \},$$

and

$$M^u_h := \{v \in (L^2(\Omega))^d \ \ , v_{|T} \in (P_0(T))^d, \ \forall T \in \mathcal{T}_h\}.$$

The discrete mixed problem is

$$\begin{cases} \text{Find } (t_h, \sigma_h, p_h, \overline{u}_h, \zeta) & \in X_{1,h} \times M^{\sigma}_{1,h} \times \overline{M}_h \times M^u_h \times I\!R \quad \text{such that} \\ \forall s_h \in X_{1,h}, & \displaystyle\int_\Omega \mathcal{A}(t_h).s_h dx - \int_\Omega \sigma_h.s_h dx - \int_\Omega p_h trace(s_h) dx = 0, \\ \forall (\tau_h, q_h) \in M^{\sigma}_h \times \overline{M}_h, & \displaystyle -\int_\Omega \tau_h t_h dx - \int_\Omega q_h trace(t_h) dx \\ & \displaystyle - \int_\Omega \overline{u}_h.div\tau_h dx + \int_\Omega \zeta_h trace(\tau_h) dx = 0, \\ \forall (v_h, \eta) \in M^u_h \times I\!R, & \displaystyle -\int_\Omega v_h div\sigma_h dx + \int_\Omega \eta trace(\sigma_h) dx = \int_\Omega f.v_h dx, \end{cases}$$

To prove the existence, uniqueness of discret solution and to obtain a priori and a posteriori error estimate. We consider first, the discrete problem:

$$(P_h) \quad \begin{cases} \text{Find } (u_h, p_h) & \in (V_h)^d \times \dot{M}_h \quad \text{such that:} \\ \forall v_h \in (V_h)^d, & \displaystyle\sum_{T\in\mathcal{T}_h} \int_T \mathcal{A}(\nabla u_h).\nabla v_h dx - \sum_{T\in\mathcal{T}_h} \int_T p_h div v_h dx \\ & \displaystyle = \int_\Omega f_h v_h dx, \\ \forall q_h \in \dot{M}_h, & \displaystyle\sum_{T\in\mathcal{T}_h} \int_T q_h div v_h dx = 0, \end{cases}$$

where

$$\dot{M}_h := \{q_h \in L^2_0(\Omega); \quad \forall T \in \mathcal{T}_h, \ v_{h|T} \in P_0(T)\},$$

and

$$\forall T \in \mathcal{T}_h, \quad f_h = \frac{1}{mes(T)} \int_T f dx \quad \text{on } T.$$

Theorem 4.2. *The discret problem (P_h) has unique solution $(u_h, p_h) \in (V_h)^d \times \dot{M}_h$. Moreover, if the weak solution (u, p) of continuous problem satisfies*

$$\sigma := \mathcal{A}(u) - pId_d \in (H^s(\Omega))^{d\times d} \cap H(div;\Omega),$$

with $s \in]0, 1]$, we have

$$|u - u_h|_{1,h} + \|p - p_h\|_{0,\Omega} \le C(h^s\|\sigma\|_{s,\Omega} + \{\sum_{T\in\mathcal{T}_h} h_T^2 \|f\|^2_{0,T}\}^{\frac{1}{2}}).$$

Let us remark that the solution (u_h, p_h) of (P_h) does not satisfies (1.3), but the modification of the last arguments give the following

Theorem 4.3. *Let $(u_h,p_h)\in (V_h)^d\times \dot{M}_h$ the solution of the the problem (P_h) , we have*

$$|u-u_h|_{1,h}+\|p-p_h\|_{0,\Omega}\leq C\{\eta_4^2+\sum_{T\in\mathcal{T}_h}h_T^2\|f_h\|_T^2\}^{\frac{1}{2}}+\{\sum_{T\in\mathcal{T}_h}h_T^2\|f-f_h\|_{0,T}^2\}^{\frac{1}{2}}.$$

Moreover, for all $T\in\mathcal{T}_h$, for all $E\in\mathcal{E}$, we have

$$\text{If } E:=\partial T\cap\partial K\in\mathcal{E}_I,\quad h_E^{-\frac{1}{2}}\|[u_h]\|_{0,E}\leq C|u-u_h|_{1,h,T\cup K},$$

$$\text{If } E:\subset\partial T\cap\Gamma,\quad h_E^{-\frac{1}{2}}\|[u_h]\|_{0,E}\leq C|u-u_h|_{1,T},$$

and

$$h_T\|f_h\|_{0,T}\leq C\{|u-u_h\|_{1,T}+\|p-p_h\|_{0,T}+h_T\|f-f_h\|_{0,T}\}.$$

To study the mixed formulation and to adapt the a posteriori error estimator to it, we set

$$\forall T\in\mathcal{T}_h,\quad \sigma_h=\mathcal{A}(\nabla u_h)-p_hId_d-\frac{f_h\times(x-x_g)}{d},\ \text{on } T,$$

where x_g is the barycenter of T and by using the notations:

$$\forall f,g\in R^d,\ (f\times g)_{ij}=f_ig_j,\quad i,j=1,..,d.$$

We have

Lemma 4.3. *The tensor σ_h satisfies*

$$\sigma_h\in H(div;\Omega)\quad,\quad -div\sigma_h=f_h\ \text{on } \Omega\quad \text{and} \int_\Omega trace(\sigma_h)dx=0.$$

Proof: Remark that

$$\forall T\in\mathcal{T}_h,\quad \sigma_h\in(RT_0(T))^d,$$

Let $e=\partial T_1\cap\partial T_2\in\mathcal{E}_I$ and $v_h\in(V_h)^d$ such that:

$$\forall f\in\mathcal{E},\quad \int_f v_hd\sigma=\delta_e^f.$$

Since

$$\forall T\in\mathcal{T}_h,\forall v_h\in V_h,\quad \int_T\frac{f_h\times(x-x_g)}{d}.\nabla v_hdx=0,$$

and using Green formula, we have:

$$\begin{aligned}[\sigma_h.n]_e &=\int_e[\sigma_h.n]v_hd\sigma=\sum_{i=1}^2\{\int_{T_i}(\sigma_h.\nabla v_h+v_hdiv\sigma_h)dx\}\\ &=\sum_{i=1}^2\{\int_{T_i}(\mathcal{A}(\nabla u_h)-p_hId_d-\frac{f_h\times(x-x_g)}{d}).\nabla v_h-\int_{T_i}f_hv_hdx\\ &=\sum_{i=1}^2\{\int_{T_i}\mathcal{A}(\nabla u_h).\nabla v_h-\int_{T_i}p_hdivv_hdx-\int_{T_i}f_hv_hdx\}=0,\end{aligned}$$

then $\sigma_h\in H(div,\Omega)$. Finally, it is clear that $-div\sigma_h=f_h$ on Ω, and since

$$\forall T\in\mathcal{T}_h,\qquad divu_h=0\quad \text{on } T,$$

we have

$$\int_\Omega trace(\sigma_h)dx=\sum_{T\in\mathcal{T}_h}\{\int_T\alpha((|\nabla u_h|)divu_hdx-d\int_Tp_hdx-\int_T\frac{f_h\times(x-x_g)}{d}dx=0.$$

∎

Concerning the existence and the uniqueness of mixed discrete problem solution, we have:

Theorem 4.4. *Let* $(u_h, p_h) \in (V_h)^d \times \dot{M}_h$ *the unique solution of the discrete problem* (P_h). *We set*

$$\forall T \in \mathcal{T}_h, \quad t_h = \nabla u_h, \quad \text{on } T,$$

$$\forall T \in \mathcal{T}_h, \quad \overline{u}_h = \frac{1}{mes_d(T)} \int_T u_h dx, \quad \text{on } T.$$

Then $(t_h, \sigma_h, p_h, \overline{u}_h, \zeta = 0) \in X_{1,h} \times M^\sigma_{1,h} \times \overline{M}_h \times M^u_h \times I\!R$ *is the unique solution of the following mixed problem:*

$$\left\{\begin{array}{rl} \text{Find } (t_h, \sigma_h, p_h, \overline{u}_h, \zeta) & \in X_{1,h} \times M^\sigma_{1,h} \times M_h \times M^u_h \times I\!R \quad \text{such that} \\ \forall s_h \in X_{1,h}, & \displaystyle\int_\Omega \mathcal{A}(t_h).s_h dx - \int_\Omega \sigma_h.s_h dx - \int_\Omega p_h trace(s_h) dx = 0, \\ \forall (\tau_h, q_h) \in M^\sigma_h \times \overline{M}_h, & \displaystyle -\int_\Omega \tau_h t_h dx - \int_\Omega q_h trace(t_h) dx \\ & \displaystyle \qquad - \int_\Omega \overline{u}_h.div\tau_h dx + \int_\Omega \zeta_h trace(\tau_h) dx = 0, \\ \forall (v_h, \eta) \in M^u_h \times I\!R, & \displaystyle -\int_\Omega v_h div\sigma_h dx + \int_\Omega \eta trace(\sigma_h) dx = \int_\Omega f.v_h dx, \end{array}\right.$$

Proof . First, recall that $\sigma_h \in M^\sigma_h$. Using lemma 4.1, we have:

$$\forall (v_h, \eta) \in M^u_h \times I\!R, -\int_\Omega v_h div\sigma_h dx + \int_\Omega \eta trace(\sigma_h) dx = \int_\Omega f.v_h dx.$$

On the one hand, since $\forall T \in \mathcal{T}_h,\ trace(t_h) := divu_h = 0$ on T, we have

$$\forall q_h \in M_h \ , \quad \int_\Omega q_h trace(t_h) dx = 0,$$

and, using Green formula, we have:

$$\begin{aligned} \forall \tau_h \in M^\sigma_h, \int_\Omega \tau_h.t_h + \int_\Omega \overline{u}_h div\tau_h dx &= \sum_{T\in\mathcal{T}_h} \{\int_T \tau_h.\nabla u_h dx + \int_T u_h div\tau_h dx\} \\ &= \sum_{T\in\mathcal{T}_h} \int_{\partial T} \tau_h.n \ . \ u_h d\gamma \\ &= \tfrac{1}{2} \sum_{T\in\mathcal{T}_h} \int_{\partial T} \tau_h.n \ . \ [u_h] d\gamma = 0. \end{aligned}$$

On the other hand $\forall s_h \in X_h$,

$$\begin{aligned} & \int_\Omega \phi(\|t_h\|) t_h.s_h dx - \int_\Omega \sigma_h.s_h dx - \int_\Omega p_h trace(s_h) dx \\ = & \int_\Omega (\phi(\|t_h\|) t_h - \sigma_h - p_h Id_d) s_h dx \\ = & \int_\Omega s_h. \frac{f_h \times (x - x_g)}{d} dx = 0. \end{aligned}$$

Then $(t_h, \sigma_h, p_h, \overline{u}_h, \zeta = 0) \in X_{1,h} \times M^\sigma_{1,h} \times M_h \times M^u_h \times I\!R$ is solution of the following problem:

$$\left\{\begin{array}{rl} & \text{Find} \quad (t_h, \sigma_h, p_h, \overline{u}_h, \zeta) \in X_{1,h} \times M^\sigma_{1,h} \times \dot{M}_h \times M^u_h \times I\!R \quad \text{such that} \\ \forall s_h \in X_{1,h} & , \int_\Omega \mathcal{A}(t_h).s_h dx - \int_\Omega \sigma_h.s_h dx - \int_\Omega p_h trace(s_h) dx = 0, \\ \forall (\tau_h, q_h) \in M^\sigma_h \times M_h & , -\int_\Omega \tau_h t_h dx - \int_\Omega q_h trace(t_h) dx \\ & \quad - \int_\Omega \overline{u}_h.div\tau_h dx + \int_\Omega \zeta_h trace(\tau_h) dx = 0, \\ \forall (v_h, \eta) \in M^u_h \times R & , -\int_\Omega v_h div\sigma_h dx + \int_\Omega \eta trace(\sigma_h) dx = \int_\Omega f.v_h dx. \end{array}\right.$$

The uniqueness of discrete solution can be proved using the same ideas as in Lemma 4.2, we need only to prove that If $(u_h, \zeta) \in M^u_h \times I\!R$ satisfies:

$$\forall \beta_h \in M^\sigma_h, \qquad \int_\Omega u_h div\beta_h dx + \int_\Omega \zeta trace(\beta_h) dx = 0,$$

then $(u_h, \zeta) = (0, 0)$.

Let $(w, q) \in (H^1_0(\Omega))^d \times L^2_0(\Omega)$ the weak solution of

$$-\Delta w + \nabla q = u_h \quad \text{and} \quad divw = 0 \quad \text{on } \Omega,$$

we set $\tau = \nabla w - qId_d$, it is clear that

$$\tau \in H(div;\Omega) \quad \text{and} \quad -div\tau = u_h \quad \text{on } \Omega.$$

Since $\tau \in (H^s(\Omega))^{d\times d} \cap H(div;\Omega)$ with $s > 0$, we can define the equilibrium interpolation $\Pi_h\tau$ of τ on M^σ_h [2]. We set

$$\tau_h = \Pi_h\tau - \frac{1}{d \times meas_d(\Omega)}(\int_\Omega trace(\Pi_h\tau)dx)Id_d,$$

we have

$$div\tau_h = div\Pi_h\tau = -u_h \quad \text{and} \quad \int_\Omega trace(\tau_h)dx = 0,$$

By choosing $\beta_h = \tau_h$ and $\beta_h = Id_d$ as test function , we obtain $(u_h, \zeta) = (0, 0)$. □

Lemma 4.4. *Let $(u_h, p_h) \in (V_h)^d \times \dot{M}_h$ the solution of nonconforming discrete problem and $(t_h, \sigma_h, \overline{p}_h, \overline{u}_h, \zeta) \in X_{1,h} \times M^\sigma_{1,h} \times \overline{M}_h \times M^u_h \times I\!R$ the solution of discrete mixed formulation, we have*

$$\begin{aligned} |u - u_h|_{1,h} + \|p - p_h\|_{0,\Omega} &\le \|\sigma - \sigma_h\|_{0,\Omega} + \|t - t_h\|_{0,\Omega} + \|p - \overline{p}_h\|_{0,\Omega} \\ &\le C\{|u - u_h|_{1,h} + \|p - p_h\|_{0,\Omega}\} + \{\sum_{T\in\mathcal{T}_h} h^2_T \|f\|^2_{0,T}\}^{\frac{1}{2}}. \end{aligned}$$

Proof . On one hand, since

$$\forall T \in \mathcal{T}_h, \quad t_h = \nabla u_h \quad \text{and } \overline{p}_h = p_h \quad \text{on } T,$$

we have

$$\|t - t_h\|_{0,\Omega} + \|p - \overline{p}_h\|_{0,\Omega} = |u - u_h|_{1,h} + \|p - p_h\|_{0,\Omega},$$

and then

$$|u-u_h|_{1,h} + \|p-p_h\|_{0,\Omega} \le \{\|\sigma-\sigma_h\|_{0,\Omega} + \|t-t_h\|_{0,\Omega} + \|p-\overline{p}_h\|_{0,\Omega}\} + \{\sum_{T\in\mathcal{T}_h} h^2_T \|f\|^2_{0,T}\}^{\frac{1}{2}}.$$

On the other hand, since

$$\forall T \in \mathcal{T}_h, \quad \sigma_h = \mathcal{A}(\nabla u_h) - p_h Id_d - \frac{f_h \times (x - x_g)}{d}, \text{ on } T,$$

we have

$$\|\sigma-\sigma_h\|_{0,\Omega} \leq C\{|u-u_h|_{1,h} + \|p-p_h\|_{0,\Omega} + \{\sum_{T\in\mathcal{T}_h} h_T^2\|f\|_{0,T}^2\}^{\frac{1}{2}},$$

Using the last inequalities, we have:

$$\|\sigma-\sigma_h\|_{0,\Omega}+\|t-t_h\|_{0,\Omega}+\|p-\overline{p}_h\|_{0,\Omega} \leq C\{|u-u_h|_{1,h}+\|p-p_h\|_{0,\Omega}\}+\{\sum_{T\in\mathcal{T}_h} h_T^2\|f\|_{0,T}^2\}^{\frac{1}{2}}.$$

□

In the sequel, we set

$$\forall T\in\mathcal{T}_h, \quad t_h = \nabla u_h, \quad \text{on } T,$$

$$\forall T\in\mathcal{T}_h, \quad \overline{u}_h = \frac{1}{mes_d(T)}\int_T u_h dx,, \quad \text{on } T,$$

where $(u_h, p_h) \in (V_h)^d \times \dot{M}_h$ is the unique solution of the discrete problem (P_h).

Lemma 4.5. *Let $(u_h, p_h) \in (V_h)^d \times \dot{M}_h$ the solution of nonconforming discrete problem and*
$(t_h, \sigma_h, \overline{p}_h, \overline{u}_h, \zeta) \in X_{1,h} \times M_{1,h}^{\sigma} \times \overline{M}_h \times M_h^u \times I\!R$ be the solution of discrete mixed formulation, we have

$$\|u-\overline{u}_h\|_{0,\Omega} \leq C\{|u-u_h|_{1,h} + \{\sum_{T\in\mathcal{T}_h} h_T^2\|t_h\|_{0,T}^2\}^{\frac{1}{2}},$$

and

$$\forall T\in\mathcal{T}_h, \quad h_T\|t_h\|_{0,T} \leq C\{\|t-t_h\|_{0,T} + \|u-\overline{u}_h\|_{0,T}\}.$$

Proof . One one hand, since

$$\forall T\in\mathcal{T}_h, \quad t_h = \nabla u_h \quad \text{and } \overline{u}_h = \frac{1}{meas_d(T)}\int_T u_h dx \quad \text{on } T,$$

we have

$$\forall T\in\mathcal{T}_h, \quad u_h = \overline{u}_h + \nabla u_h \times (x-x_g) = \overline{u}_h + t_h \times (x-x_g) \quad \text{on } T,$$

then

$$\|u-\overline{u}_h\|_{0,\Omega} \leq \|u-u_h\|_{0,\Omega} + \|u_h-\overline{u}_h\|_{0,\Omega} \leq C\{|u-u_h|_{1,h} + \{\sum_{T\in\mathcal{T}_h} h_T^2\|t_h\|_{0,T}^2\}^{\frac{1}{2}}.$$

On the other hand, following [12], let b_T the bubble function on T with $\max_T b_T = 1$. Then the norms $\|.\|_{0,T}$ and $\|b_T.\|_{0,T}$ are equivalent on $(P_1(T))^{d\times d}!$, and so

$$\|t_h\|_{0,T}^2 \leq C\int_T t_h(b_T t_h)dx = C\{\int_T (t-t_h)(b_T t_h)dx + \int_T t(b_T t_h)dx.$$

since $\int_T div(b_T t_h)dx = 0$ and $\overline{u}_h \in (P_0(T))^d$, we have

$$\int_T t(b_T t_h)dx = \int_T \nabla u.(b_T t_h)dx = -\int_T u div(b_T t_h)dx = \int_T (\overline{u}_h - u) div(b_T t_h)dx,$$

now using the inverse inequality $\|div(b_T t_h)\|_{0,T} \leq C h_T^{-1}\|t_h\|_{0,T}$, we have

$$h_T\|t_h\|_{0,T} \leq C\{\|t-t_h\|_{0,T} + \|u-\overline{u}_h\|_{0,T}\}.$$

□

Now, we are able to give a priori and a posteriori error estimator for mixed formulation, more precisely, on one hand, using Theorem 4.1 and Lemma 4.4, we have

Theorem 4.5. *Let $(t_h, \sigma_h, \overline{p}_h, \overline{u}_h, \zeta) \in X_{1,h} \times M^\sigma_{1,h} \times \overline{M}_h \times M^u_h \times I\!R$ the solution of discrete mixed formulation, we have If the weak solution (u,p) of continuous problem satisfies*

$$\mathcal{A}(u) - pId_d \in (H^s(\Omega))^{d\times d} \cap H(div;\Omega),$$

with $s \in]0,1]$, we have

$$\|u-\overline{u}_h\|_{0,\Omega}+\|\sigma-\sigma_h\|_{0,\Omega}+\|t-t_h\|_{0,\Omega}+\|p-\overline{p}_h\|_{0,\Omega} \le C(h^s\|\sigma\|_{s,\Omega}+\{\sum_{T\in\mathcal{T}_h} h_T^2\|f\|^2_{0,T}\}^{\frac{1}{2}}).$$

On the other hand, using Theorem 4.3, Lemmas 4.4 and 4.5, we have

Theorem 4.6. *Let $(t_h, \sigma_h, \overline{p}_h, \overline{u}_h, \zeta) \in X_{1,h} \times M^\sigma_{1,h} \times \overline{M}_h \times M^u_h \times I\!R$ the solution of discrete mixed formulation, we have*

$$\begin{aligned} & \|\sigma-\sigma_h\|_{0,\Omega} + \|u-\overline{u}_h\|_{0,\Omega} + \|t-t_h\|_{0,\Omega} + \|p-\overline{p}_h\|_{0,\Omega} \\ \le\ & C\{\eta_4^2 + \sum_{T\in\mathcal{T}_h} h_T^2\|f_h\|^2_{0,T} + \sum_{T\in\mathcal{T}_h} h_T^2\|t_h\|^2_{0,T}\}^{\frac{1}{2}} + \{\sum_{T\in\mathcal{T}_h} h_T^2\|f-f_h\|^2_{0,T}\}^{\frac{1}{2}}. \end{aligned}$$

Moreover, for all $T \in \mathcal{T}_h$, for all $E \in \mathcal{E}$, we have

$$h_T\|t_h\|_{0,T} \le C\{\|t-t_h\|_{0,T} + \|u-\overline{u}_h\|_{0,T}\}.$$

$$\text{If } E := \partial T \cap \partial K \in \mathcal{E}_I, \quad h_E^{-\frac{1}{2}}\|[u_h]\|_{0,E} \le C|t-t_h|_{0,T\cup K},$$

$$\text{If } E :\subset \partial T \cap \Gamma, \quad h_E^{-\frac{1}{2}}\|[u_h]\|_{0,E} \le C|t-t_h|_{0,T},$$

and

$$h_T\|f_h\|_{0,T} \le C\{|t-t_h\|_{0,T} + \|p-\overline{p}_h\|_{0,T} + h_T\|f-f_h\|_{0,T}\},$$

where u_h is defined by:

$$\forall T \in \mathcal{T}_h, \quad u_h = \overline{u}_h + t_h \times (x-x_g) \quad on\ T.$$

4.3. Finite Element Pressure Gradient Stabilization. In this subsection, we consider the Galerkin weighted least squares stabilizations (GLS) for our model problem (see e.g [3]). First, we set

$$W_h = (V_h \cap H^1_0(\Omega))^d \quad \text{and } N_h = M_h \cap H^1(\Omega).$$

The discrete problem is

$$(P_h)\left\{\begin{array}{ll} \text{Find } (u_h,p_h) & \in W_h \times N_h \ \text{ such that:} \\ \quad \forall v_h \in W_h, & \displaystyle\int_\Omega \mathcal{A}(\nabla u_h).\nabla v_h dx - \int_\Omega p_h div v_h dx = \int_\Omega f v_h dx, \\ \quad \forall q_h \in N_h, & \displaystyle\int_\Omega q_h div v_h + \sum_{T\in\mathcal{T}_h} \delta_T \int_T \nabla p_h.\nabla q_h dx = \sum_{T\in\mathcal{T}_h} \delta_T \int_T f.\nabla q_h dx \end{array}\right.$$

where, for all $T \in \mathcal{T}_h$, $\quad C_1 h_T^2 \le \delta_T \le C_2 h_T^2$.

The discrete problem has unique solution $(u_h,p_h) \in W_h \times N_h$ which satisfies (2.1) and $u_h \in (H^1_0(\Omega))^d$. Using Theorem 1.1, we have the following

Theorem 4.7. *Let $(u_h,p_h) \in (W_h)^d \times N_h$ the unique solution of (P_h), we have*

$$|u-u_h|_{1,\Omega)} + \|p-p_h\|_{0,\Omega} \le C\{\sum_{i=1}^3 \eta_i^2\}^{\frac{1}{2}}.$$

Moreover, for all $T \in \mathcal{T}_h$, for all $E := \partial T \cap \partial K \in \mathcal{E}_I$, we have

$$\|div u_h\|_{0,T} \le C|u-u_h|_{1,T},$$

$$h_z\|div_h(\mathcal{A}(\nabla u_h) - p_h Id_d) + f - \frac{1}{meas(\omega_z)} \sum_{T\in\mathcal{T}_h, T\subset\omega_z} \int_T (div(\mathcal{A}(\nabla u_h) - p_h Id_d) + f)dx\|^2_{0,\omega_z}$$
$$\leq C\{|u-u_h|_{1,h,\omega_z} + \|p-p_h\|_{0,\omega_z} + h_z\|f - \frac{1}{meas(\omega_z)}\int_{\omega_z} f dx\|_{0,\omega_z}\},$$

and

$$h_E^{\frac{1}{2}}\|[(\mathcal{A}(\nabla u_h) - p_h Id_d).n_E]\|_{0,E} \leq C\{|u-u_h|_{1,h,T\cup K} + \|p-p_h\|_{0,T\cup K}\}$$

References

[1] R. A. Adams. *Sobolev Spaces.* Academic Press, New York, 1975.

[2] A. Agouzal and J.-M. Thomas, An extension theorem for equilibrium finite elements spaces, Japan J. Indust. Appl. Math. **13**, no. 2, pp 257-266. 1996.

[3] B. Achchab and A. Agouzal, Formulation mixte augmentée et applications , M2AN Math. Model. Numer. Anal. **33**, no. 3, pp 459-478. 1999.

[4] F. Brezzi and M. Fortin, Mixed and hybrid finite element methods , Springer, New York, 1991.

[5] C. Carstensen, Quasi-interpolation and a posteriori error analysis in finite element methods, M2AN , Vol 33, No 6, pp 1187–1202. 1999.

[6] C. Carstensen and S. A. Funken, *A posteriori error control in low-order finite element discretisations of incompressible stationary flow problems* , Math. Comp., Vol 70, No 236, pp 1353-1381. 2001.

[7] P. G. Ciarlet, The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam, New York, Oxford, 1978.

[8] E. Dari and R. Durán and C. Padra, *Error estimators for nonconforming finite element approximations of the Stokes problem*, Math. Comp., Vol 64, No 211, pp 1017–1033. 1995.

[9] G. N. Gatica, M. Gonzáles and S. Meddahi A low-order mixed finite element method for a class of casi-Newtonian Stokes flows Tech. Rep. 02-01. Dept. Ingeniería Matemática. Universidad de Concepción. Chile.

[10] V. Girault and P.-A. Raviart, Finite element methods for Navier-Stokes equations, Springer, Berlin, 1986.

[11] A. F. D. Loula and J. N. C. Guerreiro *Finite element analysis of nonlinear creeping flows*, Computer Methods in Applied Mechanics and Engineering, Vol 79, pp 87-109, 1990.

[12] R. Verfürth, *A posteriori error estimators for the Stokes equations. II. Nonconforming discretizations*, Numer. Math., Vol 60, No 2, pp 235–249, 19991

[13] R. Verfürth, *A review of a posteriori error estimation and adaptive mesh-refinement techniques.* Teubner Skripten zur Numerik, B.G. Teubner Stuttgarti, 1996.

U.M.R. 5585 - Equipe de Mathématiques appliquées de Lyon. Université de Lyon 1. Bât. 101, 69 622 Villeurbanne Cedex - FRANCE

E-mail: agouzal@univ-lyon1.fr