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CONVERGENCE AND STABILITY OF BALANCED IMPLICIT METHODS FOR SYSTEMS OF SDES

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This paper is dedicated to academic fairness and honesty.

Abstract. Several convergence and stability issues of the balanced implicit methods (BIMs) for systems of real-valued ordinary stochastic differential equations are thoroughly discussed. These methods are linear-implicit ones, hence easily implementable and computationally more efficient than commonly known nonlinear-implicit methods. In particular, we relax the so far known convergence condition on its weight matrices c^j. The presented convergence proofs extend to the case of nonrandom variable step sizes and show a dependence on certain Lyapunov-functionals $V : \mathbb{R}^d \to \mathbb{R}^1_+$. The proof of L^2-convergence with global rate 0.5 is based on the stochastic Kantorovich-Lax-Richtmeyer principle proved by the author (2002). Eventually, p-th mean stability and almost sure stability results for martingale-type test equations document some advantage of BIMs. The problem of weak convergence with respect to the test class $C^2_{b(\kappa)}(\mathbb{R}^d, \mathbb{R}^1)$ and with global rate 1.0 is tackled too.

Key Words. Balanced implicit methods, linear-implicit methods, conditional mean consistency, conditional mean square consistency, weak V-stability, stochastic Kantorovich-Lax-Richtmeyer principle, L^2-convergence, weak convergence, almost sure stability, p-th mean stability.

1. Introduction

There are plenty of numerical methods for systems of ordinary stochastic differential equations (SDEs)

$$dX_t = a(t, X_t)\, dt + \sum_{j=1}^{m} b^j(t, X_t) dW_t^j \tag{1}$$

driven by standard one-dimensional Wiener processes $W^j = (W_t^j)_{0 \le t \le T}$ and interpreted in Itô sense (for the sake of simplicity of this representation), where $a, b^j \in C^0([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$. For an overview, e.g. see Kloeden, Platen and Schurz [8], Milstein [10], Talay [18] or Schurz [13]. However, only a few of them can tackle the problem of almost sure stochastic stability (as seen section 3) or of invariances with respect to certain subsets of $\mathbb{R}^d$ as commonly met in mathematical finance or biology. One of the successful approximation techniques in this respect is given

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by the class of *balanced implicit methods* (BIMs) as introduced by Milstein, Platen and Schurz [11]. They follow the iteration scheme

$$(2) \qquad Y_{k+1} \;=\; Y_k + \sum_{j=0}^{m} b^j(t_k, Y_k)\Delta W_k^j + \sum_{j=0}^{m} c^j(t_k, Y_k)|\Delta W_k^j|(Y_k - Y_{k+1})$$

where $\Delta W_k^j = W_{t_{k+1}}^j - W_{t_k}^j$, $c^j \in C^0([0,T]\times \mathbb{R}^d, \mathbb{R}^{d\times d})$ with the convention $W_t^0 = t$ and $b^0(t,x) = a(t,x)$ along discretizations

$$(3) \qquad 0 \le t_0 < t_1 < ... < t_k < ... < t_{n_T} \le T$$

with both variable or constant step sizes $\Delta_k = t_{k+1} - t_k$, finite, nonrandom (fixed) terminal time $T > 0$ and maximum step size

$$(4) \qquad \Delta \;=\; \Delta_{max} \;=\; \max_{k=0,1,...,n_T-1} |t_{k+1} - t_k|.$$

For the sake of abbreviation, we use the identities $b^0(t,x) = a(t,x)$ and $W_t^0 = t$ throughout this paper. In fact, these numerical methods (2) can be implemented in explicit form thanks to their linear-implicit structure. Therefore, they are easily and efficiently implementable. They can guarantee enlarged stability regions compared to the forward Euler methods with the matrix-valued weights $c^j \equiv \mathcal{O}, j = 1,2,...,m$ ($\mathcal{O}$ denotes the $d \times d$-zero matrix) contained in the family of BIMs (2). BIMs (2) possess the *one-step representations*

$$(5) \qquad Y_{s,y}(t) \;=\; y + M_{s,y}^{-1}(t)\sum_{j=0}^{m} b^j(s,y)(W_t^j - W_s^j) \quad \text{with}$$

$$(6) \qquad M_{s,y}(t) \;=\; I_d + \sum_{j=0}^{m} c^j(s,y)|W_t^j - W_s^j|$$

while assuming the existence of $M_{s,y}^{-1}(t)$ for all $0 \le t-s \le \delta_0 \le T$ and all $y \in \mathbb{R}^d$ and all $s,t \in [0,T]$, where I_d denotes the $d \times d$ unit matrix of $\mathbb{R}^{d\times d}$. Using the one-step representation (5), the *continuous polygonal representation* of the scheme (2) can recursively be written as

$$(7) \quad Y_{0,y_0}(t) \;=\; Y_k + M_{t_k,Y_k}^{-1}(t)\sum_{j=0}^{m} b^j(t_k,Y_k)(W_t^j - W_{t_k}^j) \quad \text{if} \quad t_k \le t \le t_{k+1}$$

for all times $t \in [0,T]$, started at $Y_0 = Y_{0,y_0}(t_0) = y_0 \in \mathbb{R}^d$, where we have the identity $Y_{0,y_0}(t_{k+1}) = Y_{t_k,Y_k}(t_{k+1}) = Y_{k+1}$ for all $k = 0,1,...,n_T - 1$.

The main interest of this paper is to prove rigorously convergence and stability of BIMs (2) applied to systems of SDEs (1). In detail we are going to discuss the issues of almost sure stability, exponential p-th mean and weak V-stability, conditional mean consistency with rate $r_0 \ge 1.5$, conditional mean square consistency with rate $r_2 \ge 1.0$, global L^2-convergence with rate $r_g \ge 0.5$ and weak convergence of these methods for the test class $C^2_{b(\kappa)}$ with coefficients $b^j \in C^0_{b(\kappa)} \cap C^0_{Lip}$ along nonrandom partitions of time-intervals $[0,T]$ with both variable and constant step sizes with maximum step size $\Delta_{max} \le \delta_0 \le \min(1,T)$. Due to the necessarily immense volume, we refrain from a systematic comparison study comparing with the pool of other, commonly known numerical methods in this paper. Such a more laborious work is left to the future and needs extensive simulation studies.

The paper is organized as follows. After this introduction, Section 2 investigates the class of BIMs (2) with respect to conditional mean and mean square consistency. Thereafter, we study weak V-stability, exponential p-th mean and almost sure stability of them in Section 3. Thereafter, we deal with a p-th mean boundedness of BIMs (2) which is needed to prove its maximum rate $r_w = 1.0$ of weak convergence later. Global convergence issues are the main topic of the closing Section 5. First, we present estimations of their L^2-convergence rates using the axiomatic approach by the stochastic Kantorovich-Lax-Richtmeyer principle as presented in Schurz [13, 14, 15, 16]. We close this paper with some remarks on weak convergence with global rate $r_w = 1.0$ and implementation issues (i.e. how to choose the weights c^j).

2. Conditional Mean and Conditional Mean Square Consistency

Consider the following definitions. Throughout the paper, fix the time interval $[0,T]$ with finite and nonrandom terminal time T. Let $\|.\|_d$ be the Euclidean vector norm on $\mathbb{R}^d$ and $\mathcal{M}_2([s,t])$ the Banach space of $(\mathcal{F}_u)_{s\le u\le t}$-adapted and continuous stochastic processes X with finite norm $\|X\|_{\mathcal{M}_2} = \sup_{s\le u\le t} \mathbb{E}\,||X(s)||_d^2 < +\infty$.

A numerical method Y with one-step representation $Y_{s,y}(t)$ is said to be *mean consistent with rate* r_0 on $[0,T]$ if $\exists$ Borel-measurable function $V : \mathbb{R}^d \to \mathbb{R}^1_+$ and $\exists$ real constants $K_0 \ge 0, \delta_0 > 0$ such that $\forall(\mathcal{F}_s, \mathcal{B}(\mathbb{R}^d))$-measurable random variables $Z(s)$ with $Z \in \mathcal{M}_2([0,s])$ and $\forall s,t : 0 \le t-s \le \delta_0$

$$(8)\qquad ||\mathbb{E}\,[X_{s,Z(s)}(t) - Y_{s,Z(s)}(t)|\mathcal{F}_s]||_d \quad\le\; K_0\,\sqrt{V(Z(s))}\,(t-s)^{r_0}.$$

Remark. It is well-known from Milstein [10] that the Euler methods are mean consistent with rate $r_0 \ge 1.5$ and moment control function $V(x) = 1 + ||x||_d^2$ for SDEs (1) with global Lipschitz-continuous and linear growth-bounded coefficients b^j.

A numerical method Y with one-step representation $Y_{s,y}(t)$ is said to be *mean square consistent with rate* r_2 on $[0,T]$ if $\exists$ Borel-measurable function $V : \mathbb{R}^d \to \mathbb{R}^1_+$ and $\exists$ real constants $K_0 \ge 0, \delta_0 > 0$ such that $\forall(\mathcal{F}_s, \mathcal{B}(\mathbb{R}^d))$-measurable random variables $Z(s)$ with $Z \in \mathcal{M}_2([0,s])$ and $\forall s,t : 0 \le t-s \le \delta_0$

$$(9)\qquad \Big(\mathbb{E}\,[||X_{s,Z(s)}(t) - Y_{s,Z(s)}(t)||_d^2|\mathcal{F}_s]\Big)^{1/2} \quad\le\; K_2\,\sqrt{V(Z(s))}\,(t-s)^{r_2}.$$

Remark. It is well-known from Milstein [10] that the Euler methods are mean square consistent with rate $r_2 \ge 1.0$ and moment control function $V(x) = 1 + ||x||_d^2$ for SDEs (1) with global Lipschitz-continuous and linear growth-bounded coefficients b^j.

2.1. The main assumptions. The following list of assumptions is needed for a thorough and rigorous analysis. Let all expressions K with subscripts below be non-random real constants, and $||.||_{d\times d}$ represents a matrix norm on $\mathbb{R}^{d\times d}$ which is compatible with the Euclidean vector norm $||.||_d$ on $\mathbb{R}^d$. Assume that the coefficients a and b^j of SDEs (1) are Caratheodory functions such that a strong, unique solution $X = (X_t)_{0\le t\le T}$ of related initial value problems for (1) with $X \in \mathcal{M}_2([0,T])$ exists and, in particular, we have

(A1) $\exists$ constants $K_B = K_B(T), K_V = K_V(T) \ge 0$

$$(10)\qquad \forall t \in [0,T]\;\forall x \in \mathbb{R}^d\;:\; \sum_{j=0}^{m} ||b^j(t,x)||_d^2 \quad\le\quad (K_B)^2 V(x),$$

$$(11)\qquad \sup_{0\le t\le T} \mathbb{E}\,V(X_t) \;\le\; K_V \mathbb{E}\,V(X_0) \quad<\quad +\infty$$

with appropriate Borel-measurable function $V : \mathbb{R}^d \to \mathbb{R}^1_+$.

(A2) The forward Euler method Y^E applied to Itô SDE (1) is assumed to be mean consistent with rate $r_0^E \geq 1.5$ and mean square consistent with rate $r_2^E = 1.0$ with respect to V with real constants $K_0^E, K_2^E, \delta_0 > 0$.

(A3) $\exists$ real constants $K_M = K_M(T) \geq 0, K_C = K_C(T) \geq 0$ such that, for the chosen weight matrices $c^j \in \mathbb{R}^{d\times d}$ of BIMs (2), we have

$$\forall t \in [0,T],\ \forall x \in \mathbb{R}^d \quad : \quad \sum_{j,k=0}^{m} ||c^k(t,x)b^j(t,x)||_d^2 \ \leq\ (K_C)^2 V(x), \tag{12}$$

$$\forall s,t : 0 \leq t-s \leq \delta_0,\ \forall x \in \mathbb{R}^d \qquad \exists M_{s,x}^{-1}(t) \text{ with } ||M_{s,x}^{-1}(t)||_{d\times d} \ \leq\ K_M. \tag{13}$$

Remark. $(A1)$ guarantees the existence of unique and continuous solutions to systems (1) with boundedness of moments along the function V. $(A2)$ is needed to simplify the proof-steps for mean, mean square consistency and global L^2-convergence by comparison with the behavior of related standard Euler methods. $(A3)$ ensures that the BIMs (2) are well-defined (nonexploding) for maximum step sizes $\Delta_{max} \leq \delta_0$. The existence and boundedness of matrices $M_{s,x}^{-1}(t)$ is guaranteed with the choice of positive semidefinite weights c^j. For example, one is tempted to take nonnegative multiples of the positive semidefinite parts of the Jacobian matrices $\nabla b^j(t,x)$ or negative multiples of the negative semidefinite parts of $\nabla a(t,x)$ in case of c^0. The condition (12) is new compared to that in [11]. This allows more flexibility. For example, one may numerically treat SDEs with vanishing drift $a(t,x) = 0$ and diffusion term $b^j(t,x) = \sigma_j(t)|x|^{\alpha_j}$ by BIMs with any bounded c^0 and $c^j(t,x) = |\sigma_j(t)| \cdot |x|^{\alpha_j - 1}$ while $\alpha_j \in [0.5, 1]$. Another interesting example is the Bessel-type diffusion

$$dX_t \ =\ \sum_{j=1}^{m} \sigma_j(t)\sqrt{X_t} \circ dW_t^j \ :=\ \frac{1}{4}\sum_{j=1}^{m} \sigma_j^2(t)\, dt + \sum_{j=1}^{m} \sigma_j(t)\sqrt{X_t}\, dW_t^j$$

with explicit solution $X_t = (\sqrt{X_0} + W_t)^2$ if $X_0 \geq 0$, $m = 1$ and $\sigma_1 \equiv 2$. Such an equation could successfully be treated by BIMs (2) with weights $c^0(t,x) = 0$ and $c^j(t,x) = |\sigma_j(t)|\sqrt{|x|}$ which are unbounded (assuming $\sigma_j \in L^2([0,T], \mathcal{B}([0,T]), \mu)$ - the Banach space of Borel-measurable and square μ-integrable functions f on $[0,T]$).

2.2. Mean consistency of BIMs (2). Using the mean consistency of forward Euler methods, we may establish the mean consistency parameters for the BIMs (2).

Theorem 2.1. *Assume that* $(A1)$ - $(A3)$ *hold with a worst case rate* $r_0^E \geq 1.5$, *control functional* V *and consistency constants* K_0^E *and* δ_0. *Then the BIMs (2) are also mean consistent with worst case rate* $r_0 \geq 1.5$, *control functional* V *and consistency constants* δ_0 *and*

$$K_0 \ \leq\ K_0^E + \sqrt{m+1} \cdot K_M \cdot K_C. \tag{14}$$

Proof. Suppose that $(A1)$ - $(A3)$ hold. Let $Z(s) \in \mathcal{M}_2([0,s])$. Recall that $Y_{s,z}^E(t)$ denotes the one-step representation of the standard Euler method and $Y_{s,z}(t)$ that of BIMs (2). Then, $\forall s,t : 0 \le t-s \le \delta_0$, we have

$$\begin{aligned}
&\|\mathbb{E}\,[X_{s,Z(s)}(t) - Y_{s,Z(s)}(t)|\mathcal{F}_s]\|_d \\
&\le \|\mathbb{E}\,[X_{s,Z(s)}(t) - Y^E_{s,Z(s)}(t)|\mathcal{F}_s]\|_d + \|\mathbb{E}\,[Y^E_{s,Z(s)}(t) - Y_{s,Z(s)}(t)|\mathcal{F}_s]\|_d \\
&\le K_0^E \sqrt{V(Z(s))}\,(t-s)^{3/2} + \\
&\quad + \|\mathbb{E}\,[M^{-1}_{s,Z(s)}(t)(M_{s,Z(s)}(t) - I_d)\sum_{j=0}^m b^j(s,Z(s))(W_t^j - W_s^j)|\mathcal{F}_s]\|_d \\
&= K_0^E \sqrt{V(Z(s))}\,(t-s)^{3/2} + \\
&\quad + \|\sum_{j,k=0}^m \mathbb{E}\,[M^{-1}_{s,Z(s)}(t)c^k(s,Z(s))b^j(s,Z(s))|W_t^k - W_s^k|(W_t^j - W_s^j)|\mathcal{F}_s]\|_d \\
&= K_0^E \sqrt{V(Z(s))}\,(t-s)^{3/2} + \\
&\quad + \|\sum_{j,k=0}^m \mathbb{E}\,[M^{-1}_{s,z}(t)c^k(s,z)b^j(s,z)|W_t^k - W_s^k|(W_t^j - W_s^j)]\Big|_{z=Z(s)}\|_d \\
&= K_0^E \sqrt{V(Z(s))}\,(t-s)^{3/2} + \Big\|\sum_{k=0}^m \mathbb{E}\,[M^{-1}_{s,z}(t)c^k(s,z)a(s,z)|W_t^k - W_s^k|]\Big|_{z=Z(s)}(t-s)\Big\|_d \\
&\le K_0^E \sqrt{V(Z(s))}(t-s)^{3/2} + K_M \sum_{k=0}^m \|c^k(s,Z(s))a(s,Z(s))\|_d (t-s)^{3/2} \\
&\le \left[K_0^E \sqrt{V(Z(s))} + K_M (m+1)^{1/2}\Big(\sum_{k=0}^m \|c^k(s,Z(s))a(s,Z(s))\|_d^2\Big)^{1/2}\right](t-s)^{3/2} \\
&\le \Big(K_0^E + K_M K_C \sqrt{m+1}\Big)\sqrt{V(Z(s))}\,(t-s)^{3/2}
\end{aligned}$$

thanks to triangle and Hölder inequalities. Consequently, the BIMs (2) are mean consistent with worst case rate $r_0 \ge 1.5$ along V, hence the proof is complete. □

Remark. There is also a proof of mean consistency rate $r_0 = 1.5$ possible without using the knowledge on the mean consistency rate $r_0^E \ge 1.5$ of the related Euler method. For details, see a forthcoming paper of the author. In fact, the standard Euler method can have mean consistency rate $r_0^E = 2.0$ as best achievable rate of mean convergence (local weak convergence) under more restrictive conditions on b^j. However, in view of mean square convergence, this fact would not improve the global mean square rate $r_g = 0.5$. For further details, see Section 4.

2.3. Mean square consistency of BIMs (2). Similarly as before, we verify the mean square consistency parameters of the BIMs (2).

Theorem 2.2. *Assume that* $(A1)$ - $(A3)$ *hold with a worst case rate* $r_2^E \ge 1.0$*, control functional* V *and consistency constants* K_2^E *and* δ_0*. Then the BIMs (2) are mean square consistent with worst case rate* $r_2 \ge 1.0$*, control functional* V *and consistency constants* δ_0 *and*

$$K_2 \le K_2^E + \sqrt{3}\cdot(m+1)\cdot K_M \cdot K_C. \tag{15}$$

Proof. Suppose that $(A1)$ - $(A3)$ holds. Let $Z(s)$ be any $(\mathcal{F}_s, \mathcal{B}(\mathbb{R}^d))$-measurable random variable with $Z \in \mathcal{M}_2([0,s])$. Recall that $Y^E_{s,z}(t)$ denotes the one-step representation of the standard Euler method and $Y_{s,z}(t)$ that of BIMs (2). Then,

$\forall s,t : 0 \le t-s \le \delta_0$, we have

$$
\begin{aligned}
&\Big(\mathbb{E}\,[\|X_{s,Z(s)}(t) - Y_{s,Z(s)}(t)\|_d^2|\mathcal{F}_s]\Big)^{1/2}\\
&\le \Big(\mathbb{E}\,[\|X_{s,Z(s)}(t) - Y^E_{s,Z(s)}(t)\|_d^2|\mathcal{F}_s]\Big)^{1/2} + \Big(\mathbb{E}\,[\|Y^E_{s,Z(s)}(t) - Y_{s,Z(s)}(t)\|_d^2|\mathcal{F}_s]\Big)^{1/2}\\
&\le K_2^E\sqrt{V(Z(s))}\,(t-s) +\\
&\quad + \Big(\mathbb{E}\,[\|M^{-1}_{s,Z(s)}(t)(M_{s,Z(s)}(t)-I_d)\sum_{j=0}^m b^j(s,Z(s))(W_t^j - W_s^j)\|_d^2|\mathcal{F}_s]\Big)^{1/2}\\
&= K_2^E\sqrt{V(Z(s))}\,(t-s) +\\
&\quad + \Big(\mathbb{E}\,[\|\sum_{j,k=0}^m M^{-1}_{s,Z(s)}(t)c^k(s,Z(s))b^j(s,Z(s))|W_t^k - W_s^k|(W_t^j - W_s^j)\|_d^2|\mathcal{F}_s]\Big)^{1/2}\\
&= K_2^E\sqrt{V(Z(s))}\,(t-s) +\\
&\quad + \Big(\mathbb{E}\,[\|\sum_{j,k=0}^m M^{-1}_{s,z}(t)c^k(s,z)b^j(s,z)|W_t^k - W_s^k|(W_t^j - W_s^j)\|_d^2]\Big|_{z=Z(s)}\Big)^{1/2}\\
&\le K_2^E\sqrt{V(Z(s))}\,(t-s) +\\
&\quad + (m+1)\Big(\sum_{j,k=0}^m \mathbb{E}\,[\|M^{-1}_{s,z}(t)c^k(s,z)b^j(s,z)\|_d^2(W_t^k - W_s^k)^2(W_t^j - W_s^j)^2]\Big|_{z=Z(s)}\Big)^{1/2}\\
&\le K_2^E\sqrt{V(Z(s))}\,(t-s) +\\
&\quad + (m+1)K_M\Big(\sum_{j,k=0}^m \|c^k(s,z)b^j(s,z)\|_d^2\mathbb{E}\,[(W_t^k - W_s^k)^2(W_t^j - W_s^j)^2]\Big|_{z=Z(s)}\Big)^{1/2}\\
&\le \left[K_2^E\sqrt{V(Z(s))} + (m+1)K_M\Big(3\sum_{j,k=0}^m \|c^k(s,Z(s))b^j(s,Z(s))\|_d^2\Big)^{1/2}\right](t-s)\\
&\le \Big(K_2^E + (m+1)\sqrt{3}K_M K_C\Big)\sqrt{V(Z(s))}\,(t-s)
\end{aligned}
$$

thanks to Minkowski and Hölder inequalities, and the orthogonality of the Wiener process components with 4th moments bounded by $3(t-s)^2$ on intervals $[s,t] \subseteq [0,T]$. Consequently, the BIMs (2) are mean square consistent with worst case rate $r_2 \ge 1.0$ along V, hence the proof is complete. $\square$

Remark. There is also a proof of mean square consistency rate $r_2 = 1.0$ possible without using the knowledge on the mean square consistency rate $r_2^E \ge 1.0$ of the related Euler method. For details, see a forthcoming paper of the author.

3. Stability of Balanced Implicit Methods

This section deals with the problem of numerical almost sure stability for certain test equations and weak V-stability along Lyapunov-type functions.

3.1. Numerical weak V-stability. Introduce the following new definition.

A numerical method Y with one-step representation $Y_{s,y}(t)$ is said to be *weakly V-stable* with real constant $K_S = K_S(T)$ on $[0,T]$ if $V : \mathbb{R}^d \to \mathbb{R}^1_+$ is Borel-measurable and $\exists$ real constant $\delta_0 > 0$ such that $\forall(\mathcal{F}_s, \mathcal{B}(\mathbb{R}^d))$-measurable random variables $Z(s)$ and $\forall s,t : 0 \le t-s \le \delta_0 \le 1$

$$\mathbb{E}\,[V(Y_{s,Z(s)}(t))|\mathcal{F}_s] \le \exp(K_S(t-s))\,V(Z(s)). \tag{16}$$

Theorem 3.1. *Assume that the numerical method Y started at a $(\mathcal{F}_0, \mathcal{B}(\mathbb{R}^d))$-measurable Y_0 and constructed along any $(\mathcal{F}_t)$-adapted time-discretization of $[0,T]$ with maximum step size $\Delta_{max} \leq \delta_0$ is weakly V-stable with δ_0 and stability constant K_S on $[0,T]$ Then*

$$\mathbb{E}\, V(Y_{0,Y_0}(t)) \leq \exp(K_S T)\mathbb{E}\, V(Y_0), \tag{17}$$

$$\sup_{0\leq t\leq T} \mathbb{E}\, V(Y_{0,Y_0}(t)) \leq \exp([K_S]_+ T)\mathbb{E}\, V(Y_0) \tag{18}$$

where $[.]_+$ denotes the positive part of the inscribed expression.

Proof. Suppose that $t_k \leq t \leq t_{k+1}$ with $\Delta_k \leq \delta_0$. If $\mathbb{E}\ V(Y_0) = +\infty$ then nothing is to prove. Now, suppose that $\mathbb{E}\ V(Y_0) < +\infty$. Using elementary properties of conditional expectations, we estimate

$$\begin{aligned}
\mathbb{E}\, V(Y_{0,Y_0}(t)) &= \mathbb{E}\,\mathbb{E}\,[V(Y_{t_k,Y_k}(t))|\mathcal{F}_s] \\
&\leq \exp(K_S(t-t_k))\cdot \mathbb{E}\, V(Y_k) = \exp(K_S(t-t_k))\cdot \mathbb{E}\, V(Y_{t_{k-1},Y_{k-1}}(t_k)) \leq \ldots \\
&\leq \exp(K_S t)\cdot \mathbb{E}\, V(Y_0) \leq \exp([K_S]_+ t)\cdot \mathbb{E}\, V(Y_0) \leq \exp([K_S]_+ T)\cdot \mathbb{E}\, V(Y_0)
\end{aligned}$$

by induction. Hence, taking the supremum confirms the claim of Theorem 3.1. □

Remark. Usually V plays the role of a Lyapunov functional for controlling the stability of the numerical method Y.

Theorem 3.2. *Assume that $(A1)$ and $(A3)$ with $V(x) = \rho^2 + ||x||_d^2$ ($\rho \in \mathbb{R}^1$ some real constant) hold. Then the BIMs (2) with $\Delta_{max} \leq \delta_0 \leq \min(1,T)$ are weakly V-stable with stability constant*

$$K_S \leq K_M \cdot K_B \cdot (2 + K_M \cdot K_B) \tag{19}$$

and they satisfy global weak V-stability estimates (17) and (18).

Proof. Suppose that $(A1)$ and $(A3)$ hold with $V(x) = \rho^2 + ||x||^2$. Recall that $0 \leq t-s \leq \delta_0 \leq 1$. Let $Z(s)$ be any $(\mathcal{F}_s, \mathcal{B}(\mathbb{R}^d))$-measurable random variable. Then

$$\begin{aligned}
\mathbb{E}\,[\rho^2+||Y_{s,Z(s)}||_d^2|\mathcal{F}_s] &= \mathbb{E}\,[\rho^2+||Z(s)+M^{-1}_{s,Z(s)}(t)\sum_{j=0}^{m} b^j(s,Z(s))(W_t^j-W_s^j)||_d^2|\mathcal{F}_s] \\
&= \mathbb{E}\,[\rho^2 + ||Z(s) + M^{-1}_{s,Z(s)}(t)a(s,Z(s))(t-s) + \\
&\quad +M^{-1}_{s,Z(s)}(t)\sum_{j=1}^{m} b^j(s,Z(s))(W_t^j - W_s^j)||_d^2|\mathcal{F}_s] \\
&= \rho^2 + \frac{1}{2}\mathbb{E}\,[||z + M^{-1}_{s,z}(t)a(s,z)(t-s) + M^{-1}_{s,z}(t)\sum_{j=1}^{m} b^j(s,z)(W_t^j-W_s^j)||_d^2]\Big|_{z=Z(s)} \\
&\quad +\frac{1}{2}\mathbb{E}\,[||z + M^{-1}_{s,z}(t)a(s,z)(t-s) - M^{-1}_{s,z}(t)\sum_{j=1}^{m} b^j(s,z)(W_t^j - W_s^j)||_d^2]\Big|_{z=Z(s)}
\end{aligned}$$

$$
\begin{aligned}
&= \rho^2 + \mathbb{E}\,[||z + M_{s,z}^{-1}(t)a(s,z)(t-s)||_d^2]\Big|_{z=Z(s)} + \\
&\quad + \mathbb{E}\,[||M_{s,z}^{-1}(t)\sum_{j=1}^{m} b^j(s,z)(W_t^j - W_s^j)||_d^2]\Big|_{z=Z(s)} \\
&= \rho^2 + ||Z(s)||_d^2 + 2\,[\mathbb{E}\,< z, M_{s,z}^{-1}(t)a(s,z) >_d]\Big|_{z=Z(s)}(t-s) + \\
&\quad + \mathbb{E}\,[||M_{s,z}^{-1}(t)a(s,z)||_d^2]\Big|_{z=Z(s)}(t-s)^2 + \\
&\quad + \sum_{j=1}^{m} \mathbb{E}\,[||M_{s,z}^{-1}(t)b^j(s,z)||_d^2(W_t^j - W_s^j)^2]\Big|_{z=Z(s)} \\
&\leq (1 + [2K_M K_B + K_M^2 K_B^2](t-s)) \cdot (\rho^2 + ||Z(s)||_d^2) \\
&\leq \exp([2K_M K_B + K_M^2 K_B^2](t-s)) \cdot (\rho^2 + ||Z(s)||_d^2),
\end{aligned}
$$

hence the BIMs (2) are weakly V-stable with $V(x) = \rho^2 + ||x||_d^2$. It obviously remains to apply Theorem 3.1 in order to complete the proof. □

Remark. Interestingly, by setting $\rho = 0$, we gain also a result on numerical mean square stability. However, for results on asymptotic mean square stability of BIMs, see [12].

3.2. Exponential p-th mean stability. BIMs (2) offer a way to control the numerical p-th mean stability behavior. This can be seen as follows. Let $p \neq 0$ be a real number.

A numerical method Y with one-step representation $Y_{s,y}(t)$ is said to be *(globally) exponentially p-th mean stable* with real constant $K_p = K_p(T)$ if $\exists$ real constant $\delta_0 > 0$ such that $\forall(\mathcal{F}_s, \mathcal{B}(\mathbb{R}^d))$-measurable random variables $Z(s)$ and $\forall s, t : 0 \leq t - s \leq \delta_0 \leq 1$

$$
\mathbb{E}\,[\|Y_{s,Z(s)}(t)\|^p|\mathcal{F}_s] \quad \leq \; \exp(K_p(t-s))\,\|Z(s)\|^p. \tag{20}
$$

Theorem 3.3. *Assume that the numerical method Y started at a $(\mathcal{F}_0, \mathcal{B}(\mathbb{R}^d))$-measurable Y_0 and constructed along any $(\mathcal{F}_t)$-adapted time-discretization of $[0,T]$ with maximum step size $\Delta_{max} \leq \delta_0$ is exponentially p-th mean stable with δ_0 and stability constant K_p on $[0,T]$. Then*

$$
\mathbb{E}\,\|Y_{0,Y_0}(t)\|_d^p \quad \leq \quad \exp(K_p T)\mathbb{E}\,\|Y_0\|_d^p, \tag{21}
$$

$$
\sup_{0\leq t\leq T} \mathbb{E}\,\|Y_{0,Y_0}(t)\|_d^p \quad \leq \quad \exp([K_p]_+ T)\mathbb{E}\,\|Y_0\|_d^p \tag{22}
$$

where $[.]_+$ denotes the positive part of the inscribed expression.

Proof. Suppose that $t_k \leq t \leq t_{k+1}$ with $\Delta_k \leq \delta_0$. If $\mathbb{E}\,\|Y_0\|_d^p = +\infty$ then nothing is to prove. Now, suppose that $\mathbb{E}\,\|Y_0\|_d^p < +\infty$. Using elementary properties of conditional expectations, we estimate

$$
\begin{aligned}
\mathbb{E}\,\|Y_{0,Y_0}(t)\|_d^p &= \mathbb{E}\,\mathbb{E}\,[\|Y_{t_k,Y_k}(t)\|_d^p|\mathcal{F}_s] \\
&\leq \exp(K_p(t-t_k)) \cdot \mathbb{E}\,\|Y_k\|_d^p = \exp(K_p(t-t_k)) \cdot \mathbb{E}\,\|Y_{t_{k-1},Y_{k-1}}(t_k)\|_d^p \leq \ldots \\
&\leq \exp(K_p t) \cdot \mathbb{E}\,\|Y_0\|_d^p \leq \exp([K_p]_+ t) \cdot \mathbb{E}\,\|Y_0\|_d^p \leq \exp([K_p]_+ T) \cdot \mathbb{E}\,\|Y_0\|_d^p
\end{aligned}
$$

by induction. Hence, taking the supremum confirms the claim of Theorem 3.3. □

Theorem 3.4. *Assume that (A1) and (A3) with $V(x) = ||x||_d^2$, $\delta_0 \leq \min(1,T)$, $b^j(t,x) = A^j(t,x)x$ hold and that $X = (X_t)_{0\leq t\leq T}$ satisfies (a.s.) the Itô SDE*

$$
dX_t \quad = \quad A^0(t,X_t)X_t dt + \sum_{j=1}^{m} A^j(t,X_t)X_t\, dW_t^j, \tag{23}
$$

with nonrandom $\mathbb{R}^{d\times d}$-valued matrix coefficients A^j of Caratheodory-type and there are real constants K_M^S and K_B^j satisfying $\forall t,s \in [0,T] \,:\, 0 \le t-s \le \delta_0 \; \forall x \in \mathbb{R}^d$

$$(24)\quad \|A^j(t,x)\|_{d\times d} \;\le\; K_B^j, \quad \|(I_d - A^0(t,x)(t-s))^{-1}\|_{d\times d} \;\le\; \exp(K_M^S(t-s))$$

Then the BIMs (2) applied to SDE (23) with weights $c^0(t,x) = -A^0(t,x)$ and $c^j(t,x) = \mathcal{O}$ $(j=1,2,...,m)$, and step sizes

$$(25)\qquad \Delta_k \;\le\; \Delta_{max} \le \delta_0 \le \min\Big\{1, T, \frac{1}{mp(p-1)(K_B^j)^2} : j=1,2,...,m\Big\}$$

are exponentially p-th mean stable with $p \ge 2$ and stability constant

$$(26)\qquad K_p \;\le\; p\cdot\left(m\frac{p-1}{2}\sum_{j=1}^{m}\frac{(K_B^j)^2}{1-mp(p-1)(K_B^j)^2\Delta_{max}} + K_M^S\right)$$

and they satisfy global p-th mean stability estimates (21) and (22) for $p \ge 2$.

Proof. Suppose that $(A1)$ and $(A3)$ with $V(x) = \|x\|_d^2$ and $\delta_0 \le \min(1,T)$ hold. Recall that $0 \le t-s \le \delta_0 \le \min(1,T)$. Let $Z(s)$ be any $(\mathcal{F}_s, \mathcal{B}(\mathbb{R}^d))$-measurable random variable. Define $M_{s,x}(t) = I_d - A^0(t,x)(t-s)$ and $\gamma = \sqrt{1/(p-1)}$. Then

$$\begin{aligned}
\mathbb{E}\,[\|Y_{s,Z(s)}(t)\|_d^p|\mathcal{F}_s] \;&=\; \mathbb{E}\,[\|Z(s)+M_{s,Z(s)}^{-1}(t)\sum_{j=0}^{m} b^j(s,Z(s))(W_t^j - W_s^j)\|_d^p|\mathcal{F}_s]\\
&=\; \mathbb{E}\,[\|M_{s,Z(s)}^{-1}(t)\Big(I_d + \sum_{j=1}^{m} A^j(s,Z(s))(W_t^j - W_s^j)\Big)Z(s)\|_d^p|\mathcal{F}_s]\\
&\le\; \exp(pK_M^S(t-s))\|Z(s)\|_d^p\mathbb{E}\,[\|I_d + \sum_{j=1}^{m} A^j(s,Z(s))(W_t^j - W_s^j)\|_{d\times d}^p|\mathcal{F}_s]\\
&=\; \exp(pK_M^S(t-s))\|Z(s)\|_d^p\mathbb{E}\,[\|I_d + \sum_{j=1}^{m} A^j(s,z)(W_t^j - W_s^j)\|_{d\times d}^p]\Big|_{z=Z(s)}.
\end{aligned}$$

Now, the expectation part at the right hand side is treated as follows. By using an elementary inequality originating from Clarkson [4] and Beckner [3] applied to the Banach space of random matrices with uniformly L^p-integrable coefficients (cf.

Section 4) one finds

$$\begin{aligned}
&\mathbb{E}\,[\|I_d+\sum_{j=1}^{m}A^j(s,z)(W_t^j-W_s^j)\|_{d\times d}^p]\Big|_{z=Z(s)}\\
&= \frac{1}{2}\mathbb{E}\,[\|I_d+\gamma\frac{1}{\gamma}\sum_{j=1}^{m}A^j(s,z)(W_t^j-W_s^j)\|_{d\times d}^p]\Big|_{z=Z(s)}+\\
&\quad +\frac{1}{2}\mathbb{E}\,[\|I_d-\gamma\frac{1}{\gamma}\sum_{j=1}^{m}A^j(s,z)(W_t^j-W_s^j)\|_{d\times d}^p]\Big|_{z=Z(s)}\\
&\le \mathbb{E}\left(1+m\frac{1}{\gamma^2}\sum_{j=1}^{m}\|A^j(s,Z(s))\|_{d\times d}^2(W_t^j-W_s^j)^2\right)^{p/2}\\
&\le \mathbb{E}\left(1+m(p-1)\sum_{j=1}^{m}(K_B^j)^2(W_t^j-W_s^j)^2\right)^{p/2}\\
&\le \prod_{j=1}^{m}\mathbb{E}\,\exp\left(\frac{1}{2}mp(p-1)(K_B^j)^2(W_t^j-W_s^j)^2\right)\\
&\le \exp\left(m\frac{p(p-1)}{2}\sum_{j=1}^{m}\frac{(K_B^j)^2}{1-mp(p-1)(K_B^j)^2\Delta_{max}}(t-s)\right)
\end{aligned}$$

for $0\le t-s\le\Delta_{max}\le\delta_0\le\min(1,T,1/[mp(p-1)(K_B^j)^2])$. Exploiting this fact after returning to the original estimation yields

$$\mathbb{E}\,[\|Y_{s,Z(s)}(t)\|_d^p|\mathcal{F}_s]\le\exp\left(p[m\frac{p-1}{2}\sum_{j=1}^{m}\frac{(K_B^j)^2}{1-mp(p-1)(K_B^j)^2\Delta_{max}}+K_M^S](t-s)\right)\cdot\|Z(s)\|_d^p.$$

Therefore, the BIMs (2) are exponentially p-th mean stable for $p\ge 2$. It obviously remains to apply Theorem 3.3 in order to complete the proof with K_p as in (26). □

Remark. Interestingly, we also gain asymptotic p-th mean stability of BIMs provided that $K_M^S<-\frac{p-1}{2}\sum_{j=1}^{m}(K_B^j)^2$ (compare with the simple onedimensional case $dX_t=\alpha X_t dt+\sigma X_t dW_t$ when $\alpha+(p-1)\sigma^2/2<0$). Conditions (24) can be guaranteed for negative semidefinite matrices A^0 and uniformly bounded A^j for $j=1,2,...,m$. For practical implementation, one may also take the stabilizing, negative semidefinite part of A^0 as weight matrix c^0 instead of the entire structure of A^0.

3.3. Numerical almost sure stability. In the following we discuss the almost sure stability behavior of BIMs with both constant and variable step sizes with respect to the trivial equilibrium $0\in\mathbb{R}^d$. For this purpose, consider the following definition.

A sequence $Y=(Y_n)_{n\in\mathbb{N}}$ with $Y_n:(\Omega,\mathcal{F}_n,\mathbb{P})\to\mathbb{R}^d$ is called *(globally) asymptotically stable with probability one* (or *(globally) asymptotically a.s. stable*) if

$$\lim_{n\to+\infty}\|Y_n\|_d = 0\ (a.s.)$$

for all $Y_0=y_0\in\mathbb{R}^d\setminus\{0\}$, where $y_0\in\mathbb{R}^d$ is nonrandom, otherwise *asymptotically a.s. unstable.*

Lemma 3.1. *Let $V = (V(n))_{n\in I\!N}$ be a sequence of nonnegative random variables $V(n) : (\Omega, \mathcal{F}_n, \mathbb{P}) \to I\!R^1_+$ with $V(0) > 0$ satisfying the recursive scheme*

$$V(n+1) \;=\; V(n)G(n) \tag{27}$$

where $G(n) : (\Omega, \mathcal{F}_n, \mathbb{P}) \to I\!R^1_+$ are i.i.d. random variables with $I\!E\,|\ln[G(n)]| < +\infty$. Then

$$V \text{ (globally) asymptotically a.s. stable} \qquad \text{iff} \qquad I\!E\,\ln[G(n)] < 0\ .$$

Proof. The main idea is to use the strong law of large numbers (SLLN) in conjunction with the law of iterated logarithm (LIL). Note that V possesses the explicit representation

$$V(n+1) \;=\; \left(\prod_{k=0}^{n} G(k)\right) V(0) \tag{28}$$

for all $n \in \mathbb{N}$. Now, define

$$\mu := \mathbb{E}\,[\ln(G(n))], \quad S_n := \sum_{k=0}^{n-1} \ln G(k),$$

hence $V(n+1) = \exp(S_{n+1})V(0)$ and $\mathbb{E}\,[S_n] = n\mu$ for $n \in \mathbb{N}$. By SLLN we may conclude that

$$\lim_{n\to+\infty} \frac{S_n}{n} \;=\; \mu\ (a.s.)$$

thanks to the $\mathbb{P}$-integrability of $G(k)$. This fact implies that if $\mu < 0$ then $S_n \to -\infty$, i.e. $V(n) \to 0$ as n tends to $+\infty$ and if $\mu > 0$ then $S_n \to +\infty$, i.e. $V(n) \to +\infty$ as n tends to $+\infty$. Moreover, in the case $\mu = 0$, we may use LIL (at first, under $\sigma^2 = Var(\ln G(k)) < +\infty$, later we may drop $\sigma^2 < +\infty$ by localization procedures) to get

$$\liminf_{n\to+\infty} \frac{S_n}{\sqrt{2n\ln\ln n}} = -|\sigma|, \quad \limsup_{n\to+\infty} \frac{S_n}{\sqrt{2n\ln\ln n}} = |\sigma|,$$

hence $\lim_{n\to+\infty} S_n$ does not exist, and therefore

$$\lim_{n\to+\infty} V(n) \;=\; \lim_{n\to+\infty} \exp(S_n)V(0)$$

does not exist either (a.s.). Thus, $\lim_{n\to+\infty} V(n) \neq 0$ and the proof is complete. □

Now, consider the onedimensional test class of pure diffusion equations

$$dX_t \quad = \quad \sigma X_t\, dW_t \tag{29}$$

as suggested by Milstein, Platen and Schurz [11]. Then, the following result provides a mathematical evidence that their numerical experiments for BIMs (2) led to the correct observation of numerical stability due to its asymptotic a.s. stability.

Theorem 3.5. *The BIMs (2) with scalar weights $c^0 = 0$ and $c^1 = |\sigma|$ applied to martingale test equations (29) for any parameter $\sigma \in I\!R^1 \setminus \{0\}$ with any equidistant step size Δ provide (globally) asymptotically a.s. stable sequences $Y = (Y_n)_{n\in I\!N}$.*

Proof. Suppose $|\sigma| > 0$. Then, the proof is an application of Lemma 3.1. For this purpose, consider the sequence $V = (V(n))_{n\in\mathbb{N}} = (|Y_n|)_{n\in\mathbb{N}}$. Note that $V(n+1) = G(n)V(n)$, $\mathbb{E}\,|\ln G(n)| < +\infty$ and $\mathbb{E}\,[\ln G(n)] < 0$ since

$$\mathbb{E}\,[|\ln G(n)|] \;\leq\; (\mathbb{E}\,[\ln G(n)]^2)^{1/2} \;\leq\; \ln(2) + |\sigma|\sqrt{\Delta} \text{ and}$$

$$\begin{aligned}
\mathbb{E}\,[\ln G(n)] &= \mathbb{E}\,\left[\ln\left|\frac{1+|\sigma\Delta W_n|+\sigma\Delta W_n}{1+|\sigma\Delta W_n|}\right|\right] \;=\; \mathbb{E}\,\left[\ln\left|1+\frac{\sigma\Delta W_n}{1+|\sigma\Delta W_n|}\right|\right]\\
&= \frac{1}{2}\mathbb{E}\,\left[\ln\left|1+\frac{\sigma\Delta W_n}{1+|\sigma\Delta W_n|}\right|\right]+\frac{1}{2}\mathbb{E}\,\left[\ln\left|1-\frac{\sigma\Delta W_n}{1+|\sigma\Delta W_n|}\right|\right]\\
&= \frac{1}{2}\mathbb{E}\,\left[\ln\left|1-\left(\frac{\sigma\Delta W_n}{1+|\sigma\Delta W_n|}\right)^2\right|\right] \;<\; -\frac{1}{2}\mathbb{E}\,\left[\left(\frac{|\sigma\Delta W_n|}{1+|\sigma\Delta W_n|}\right)^2\right] \;<\; 0
\end{aligned}$$

with independently identically Gaussian distributed increments $\Delta W_n \in \mathcal{N}(0,\Delta)$ (In fact, note that, for all $\sigma \neq 0$ and Gaussian ΔW_n, we have

$$0 \;<\; 1-\left(\frac{\sigma\Delta W_n}{1+|\sigma\Delta W_n|}\right)^2 \;<\; 1$$

with probability one, hence, that ΔW_n has a nondegenerate probability distribution with nontrivial support is essential here!). Therefore, the assumptions of Lemma 3.1 are satisfied and an application of Lemma 3.1 yields the claim of Theorem 3.5. Thus, the proof is complete. □

Remark. The increments $\Delta W_n \in \mathcal{N}(0,\Delta_n)$ can also be replaced by multi-point discrete probability distributions such as

$$\mathbb{P}\,\{\Delta W_n = \pm\sqrt{\Delta_n}\} \;=\; \frac{1}{2}$$

$$\text{or}\quad \mathbb{P}\,\{\Delta W_n = 0\} \;=\; \frac{2}{3},\;\; \mathbb{P}\,\{\Delta W_n = \pm\sqrt{3\Delta_n}\} \;=\; \frac{1}{6}$$

as commonly met in weak approximations. In this case, the almost sure stability of the BIMs as chosen by Theorem 3.5 is still guaranteed, as seen by our proof above (due to the inherent symmetry of ΔW_n with respect to 0).

For variable step sizes, we can also formulate and prove a general assertion with respect to asymptotic a.s. stability. Let $Var(Z)$ denote the variance of the inscribed random variable Z.

Lemma 3.2. *Let $V = (V(n))_{n\in\mathbb{N}}$ be a sequence of nonnegative random variables $V(n) : (\Omega,\mathcal{F}_n,\mathbb{P}\,) \to \mathbb{R}^1_+$ with $V(0) > 0$ satisfying the recursive scheme*

$$V(n+1) \;=\; V(n)G(n), \tag{30}$$

where $G(n) : (\Omega,\mathcal{F}_n,\mathbb{P}\,) \to \mathbb{R}^1_+$ are independent random variables such that $\exists$ nonrandom sequence $b = (b_n)_{n\in\mathbb{N}}$ with $b_n \to +\infty$ as $n \to +\infty$

$$\sum_{k=0}^{+\infty}\frac{Var(\ln(G(k)))}{b_k^2} \;<\; +\infty,\quad \exists \lim_{n\to+\infty}\frac{\sum_{k=0}^{n-1}\mathbb{E}\,\ln(G(k))}{b_n} \;<\; 0. \tag{31}$$

Then $V = (V(n))_{n\to+\infty}$ is (globally) asymptotically a.s. stable sequence, i.e. we have $\lim_{n\to+\infty} V(n) \;=\; 0$ (a.s.).
Moreover, if

$$\sum_{k=0}^{+\infty}\frac{Var(\ln(G(k)))}{b_k^2} \;<\; +\infty,\quad \exists \lim_{n\to+\infty}\frac{\sum_{k=0}^{n-1}\mathbb{E}\,\ln(G(k))}{b_n} \;>\; 0 \tag{32}$$

then $V = (V(n))_{n\to+\infty}$ is (globally) asymptotically a.s. unstable sequence, i.e. we have $\lim_{n\to+\infty} V(n) \;=\; +\infty$ (a.s.) for all nonrandom $y_0 \neq 0$.

Proof. The main idea is to apply Kolmogorov's SLLN, see Shiryaev [17], p. 389. Recall that V possesses the explicit representation (28). Now, define

$$S_n := \sum_{k=0}^{n-1} \ln G(k),$$

hence $V(n+1) = \exp(S_{n+1})V(0)$ for $n \in \mathbb{N}$. By Kolmogorov's SLLN we may conclude that

$$\lim_{n\to+\infty} \frac{S_n}{b_n} = \lim_{n\to+\infty} \frac{\mathbb{E}\, S_n}{b_n} = \lim_{n\to+\infty} \frac{\sum_{k=0}^{n-1} \mathbb{E}\, \ln(G(k))}{b_n} < 0 \ (a.s.)$$

thanks to the assumptions (31) of $\mathbb{P}$ -integrability of $G(k)$. This fact together with $b_n \to +\infty$ implies that $S_n \to -\infty$ (a.s.), i.e. $V(n) \to 0$ as n tends to $+\infty$. The reverse direction under (32) is proved analogously to previous proof-steps. Thus, the proof is complete. □

Now, let us apply this result to BIMs (2) applied to test equation (29). For $k = 0, 1, ..., n_T$, define

$$G(k) := \left| \frac{1 + |\sigma\Delta W_k| + \sigma\Delta W_k}{1 + |\sigma\Delta W_k|} \right|. \tag{33}$$

Theorem 3.6. *Assume that $\exists$ nonrandom sequence $b = (b_n)_{n\in\mathbb{N}}$ with $b_n \to +\infty$ as $n \to +\infty$ for a fixed choice of step sizes $\Delta_n > 0$ such that*

$$\sum_{k=0}^{+\infty} \frac{Var(\ln(G(k)))}{b_k^2} < +\infty, \quad \exists \lim_{n\to+\infty} \frac{\sum_{k=0}^{n-1} \mathbb{E}\, \ln(G(k))}{b_n} < 0.$$

Then the BIMs (2) with scalar weights $c^0 = 0$ and $c^1 = |\sigma|$ applied to martingale test equations (29) with parameter $\sigma \in \mathbb{R}^1 \setminus \{0\}$ with the fixed sequence of variable step sizes Δ_n provide (globally) asymptotically a.s. stable sequences $Y = (Y_n)_{n\in\mathbb{N}}$.

Proof. We may apply Lemma 3.2 since the assumptions are satisfied for the BIMs (2) with scalar weights $c^0 = 0$ and $c^1 = |\sigma|$ applied to martingale test equations (29). Hence, the proof is complete. □

Theorem 3.7. *The BIMs (2) with scalar weights $c^0 = 0$ and $c^1 = |\sigma|$ applied to martingale test equations (29) with parameter $\sigma \in \mathbb{R}^1 \setminus \{0\}$ with any variable step sizes Δ_k satisfying $0 < \Delta_{min} \le \Delta_k \le \Delta_{max}$ provide (globally) asymptotically a.s. stable sequences $Y = (Y_n)_{n\in\mathbb{N}}$.*

Proof. We may again apply Lemma 3.2. For this purpose, we check the assumptions. Define $b_n := n$. Note that the variance $Var(\ln(G(k)))$ is uniformly bounded since $\Delta W_n \in \mathcal{N}(0, \Delta_n)$ and $0 < \Delta_{min} \le \Delta_k \le \Delta_{max}$. More precisely, we have

$$\begin{aligned}
&Var(\ln(G(k))) \le \mathbb{E}\, [\ln(G(k))]^2 \\
&= \mathbb{E}\, [I_{\{\Delta W_n > 0\}} \ln(G(k))]^2 + \mathbb{E}\, [I_{\{\Delta W_n < 0\}} \ln(G(k))]^2 \\
&< p_n[\ln(2)]^2 + \mathbb{E}\, [\ln(1 + |\sigma\Delta W_n|)]^2 \le p_n[\ln(2)]^2 + \mathbb{E}\, [\ln(\exp(|\sigma\Delta W_n|))]^2 \\
&\le p_n[\ln(2)]^2 + \mathbb{E}\, [\sigma\Delta W_n]^2 = p_n[\ln(2)]^2 + \sigma^2\Delta_n \le p_n[\ln(2)]^2 + \sigma^2\Delta_{max}
\end{aligned}$$

for $G(k)$ as defined in (33), where $I_{\{Q\}}$ denotes the indicator function of the inscribed set Q and $p_n = \sqrt{\mathbb{P}\, \{\Delta W_n > 0\}}$. Note that $0 < p_n = \sqrt{2}/2 < 1$ if

ΔW_n is Gaussian distributed. Therefore, there is a finite real constant $K_2^G < (\ln(2))^2 + \sigma^2 \Delta_{max}$ such that

$$\sum_{k=0}^{+\infty} \frac{Var(\ln(G(k)))}{k^2} \leq \sum_{k=0}^{+\infty} \frac{K_2^G}{k^2} = K_2^G \frac{\pi^2}{6} < +\infty.$$

It remains to check whether

$$\lim_{n \to +\infty} \frac{\sum_{k=0}^{n-1} \mathbb{E} \ \ln(G(k))}{n} < 0.$$

For this purpose, we only note that $\mathbb{E} \ \ln(G(k))$ is decreasing for increasing $\sqrt{\Delta_k}$ for all $k \in \mathbb{N}$ (see the proof of Theorem 3.5). Therefore, we can estimate this expression by

$$\mathbb{E} \ \ln(G(k)) \leq \frac{1}{2} \mathbb{E} \left[\ln \left| 1 - \left(\frac{\sigma \sqrt{\Delta_{min}} \xi}{1 + |\sigma \sqrt{\Delta_{min}} \xi|} \right)^2 \right| \right] := K_1^G < 0$$

where $\xi \in \mathcal{N}(0,1)$ is a standard Gaussian distributed random variable and K_1^G the negative real constant as defined above. Thus,

$$\lim_{n \to +\infty} \frac{\sum_{k=0}^{n-1} \mathbb{E} \ \ln(G(k))}{n} \leq K_1^G < 0.$$

Hence, thanks to Lemma 3.2 (or 3.6), the proof is completed. □

4. Boundedness of p-th Moments of Balanced Implicit Methods

It is neccessary to verify the uniform boundedness of p-th moments of the outcomes of BIMs (2) in order to prove the maximum possible rate $r_w = 1.0$ of global weak convergence.

4.1. Three auxiliary lemmas. We begin with a random version of *Clarkson-Beckner inequality.*

Lemma 4.1. *Let X, Y be two elements of a Hilbert space $(H, < .,. >_H)$ equipped with its scalar product $< .,. >_H$, $\mathbb{R}^1$ as its set of scalars and naturally induced norm $\|Z\|_H = (< Z, Z >_H)^{1/2}$. Assume that*

$$\mathbb{E} \left[\|X\|_H^p + \|Y\|_H^p \right] < +\infty$$

for a $p \geq 2$. Then, we have

$$\frac{\mathbb{E} \|X + Y\|_H^p + \mathbb{E} \|X - Y\|_H^p}{2} \leq \mathbb{E} \left(\|X\|_H^2 + (p-1) \|Y\|_H^2 \right)^{p/2}. \tag{34}$$

Proof. Define $B := \{X \in (H, < .,. >_H) : \|X\|_B^p = \mathbb{E} \ (\|X\|_H^p) < +\infty\}$. Then $(B, \|.\|_B)$ forms a Banach space as a closed subset of H. Suppose that $X, Y \in B$. Set $\gamma = 1/\sqrt{p-1}$, $z_1 = X + \sqrt{p-1} Y$, $z_2 = X - \sqrt{p-1} Y$, $u_1 = (\|z_1\|_H + \|z_2\|_H)/2$ and $u_2 = |\|z_1\| - \|z_2\||/2$. Then, Clarkson-Beckner inequality (see [4] and [3]) which says that

$$\Big(\frac{|1+u|^q + |1-u|^q}{2} \Big)^{1/q} \leq \Big(\frac{|1 + \sqrt{(q-1)/(p-1)} u|^p + |1 - \sqrt{(q-1)/(p-1)} u|^p}{2} \Big)^{1/p}$$

for all numbers $u \geq 0, 1 < p \leq q$ and parallelogram identity on Hilbert spaces imply that

$$\begin{aligned}
\Big(\frac{\|X+Y\|_H^p + \|X-Y\|_H^p}{2}\Big)^{1/p} &= \Big(\frac{\|X+\gamma\frac{1}{\gamma}Y\|_H^p + \|X-\gamma\frac{1}{\gamma}Y\|_H^p}{2}\Big)^{1/p} \\
\leq \Big(&\frac{((1+\gamma)\|z_1\|_H/2+(1-\gamma)\|z_2\|_H/2)^p+((1-\gamma)\|z_1\|_H/2+(1+\gamma)\|z_2\|_H/2)^p}{2}\Big)^{1/p} \\
= \Big(&\frac{|u_1+\gamma u_2|^p + |u_1-\gamma u_2|^p}{2}\Big)^{1/p} \leq \Big(\frac{|u_1+u_2|^2+|u_1-u_2|^2}{2}\Big)^{1/2} \\
= \Big(&\frac{\|z_1\|_H^2 + \|z_2\|_H^2}{2}\Big)^{1/2} = \Big(\|X\|_H^2 + (p-1)\|Y\|_H^2\Big)^{1/2}.
\end{aligned}$$

Now, it remains to take the p-th power and expectation in order to arrive at (34). Thus, the proof is complete. □

Observe the following property of moments of Gaussian exponentials.

Lemma 4.2. *Assume that $X \in \mathcal{N}(0,\Delta)$. Then*

$$\forall \sigma \in \Big(-\frac{1}{\sqrt{2\Delta}}, \frac{1}{\sqrt{2\Delta}}\Big) \quad \mathbb{E}\exp(\sigma^2 X^2) \leq \frac{1}{\sqrt{1-2\sigma^2\Delta}} \leq \exp\Big(\frac{\sigma^2\Delta}{1-2\sigma^2\Delta}\Big). \tag{35}$$

Proof. Define $\xi = X/\sqrt{\Delta}$. Note that $\xi \in \mathcal{N}(0,1)$. Calculate

$$\begin{aligned}
\mathbb{E}\exp(\sigma^2X^2) &= \mathbb{E}\exp(\sigma^2\Delta\,\xi^2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\Big(\sigma^2\Delta x^2 - \frac{x^2}{2}\Big)\,dx \\
= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}&\exp\Big(-(1-2\sigma^2\Delta)\frac{x^2}{2}\Big)\,dx = \frac{1}{\sqrt{1-2\sigma^2\Delta}} \leq \exp\Big(\frac{\sigma^2\Delta}{1-2\sigma^2\Delta}\Big)
\end{aligned}$$

using the elementary inequality $1/(1-z) \leq \exp(z/(1-z))$ for $z = 2\sigma^2\Delta < 1$. Thus, the proof is complete. □

Linear-polynomial boundedness of Lipschitz continuous functions can be established too. Let $C^0_{b(\kappa)}([0,T]\times\mathbb{R}^d,\mathbb{R}^l)$ denote the set of all continuous functions $f:[0,T]\times\mathbb{R}^d \to \mathbb{R}^l$ which are uniformly polynomially bounded such that

$$\|f(t,x)\|_l \leq K_f\cdot(1+\|x\|_l^\kappa)$$

for all $x \in \mathbb{R}^d$, where $K_f \geq 0$ and $\kappa \geq 0$ are appropriate real constants.

Lemma 4.3. *Assume that $f \in C^0_{b(\kappa)}([0,T]\times\mathbb{R}^d,\mathbb{R}^l)$ with constants $\kappa \geq 0$ and K_f is uniformly Lipschitz continuous with constant K_L, i.e.*

$$\forall t\in[0,T],\ \forall x,y\in\mathbb{R}^d \quad \|f(t,x)-f(t,y)\|_l \leq K_L\,\|x-y\|_l. \tag{36}$$

Then, there exist constants $K_{b(p)} = K_{b(p)}(p,T,K_f,K_L)$ such that $\forall t\in[0,T]$ $\forall x\in\mathbb{R}^d$

$$\|f(t,x)\|_l \leq 2^{-(p-1)/p}K_{b(p)}\cdot(1+\|x\|_l) \leq K_{b(p)}\cdot(1+\|x\|_l^p)^{1/p} \tag{37}$$

for all $p \geq 1$, where the real constants $K_{b(p)}$ can be estimated by

$$0 \leq K_{b(p)} \leq 2^{(p-1)/p}\cdot\max\{K_f,K_L\}. \tag{38}$$

Proof. Estimate

$$\begin{aligned}
0 &\leq \|f(t,x)\|_l \leq \|f(t,0)\|_l + \|f(t,x)-f(t,0)\|_l \leq K_f + K_L\|x\|_l \\
&\leq \max\{K_f,K_L\}(1+\|x\|_l) \leq 2^{(p-1)/2}\max\{K_f,K_L\}(1+\|x\|_l^p)^{1/p}.
\end{aligned}$$

Therefore, constant $K_{b(p)}$ can be chosen as in (38). Thus, the proof is complete. □

Remark. In fact, it suffices that $\sup_{0\leq t\leq T}\|f(t,x_*)\|_l < +\infty$ for some $x_* \in \mathbb{R}^d$ and f is Lipschitz continuous in $x \in \mathbb{R}^d$ with constant $K_L(t)$ which is uniformly bounded with respect to $t \in [0,T]$. However, $K_{b(p)}$ may depend on κ too.

4.2. Uniform boundedness of p-th moments. Consider BIMs (2) with both variable or constant step sizes $\Delta_k \leq \Delta_{max}$ where Δ_{max} sufficiently small. Let $ent[p]$ be the maximum integer which is smaller than or equal to the inscribed real number p (i.e. such that $ent[p]+1 > p \geq ent[p]$, $ent[p] \in \mathbb{N}$ for all $p \in \mathbb{R}_+$). Then, uniform boundedness of p-th moments can be established as follows.

Theorem 4.1. *Assume that BIMs (2) with step sizes $\Delta_k \leq \Delta_{max} \leq 1$ and*

$$ent[\frac{p}{2}](2ent[\frac{p}{2}]-1)mK_M^2(K_{b(2)}^j)^2\Delta_{max} < 1 \tag{39}$$

satisfy (A3), $\mathbb{E}\,\|Y_0\|_d^p < +\infty$ for a $p \geq 2\kappa \geq 2$, and

$$\forall t \in [0,T],\ \forall x \in \mathbb{R}^d \quad \|b^j(t,x)\|_d^2 \ \leq\ (K_{b(2)}^j)^2(1+\|x\|_d^2). \tag{40}$$

Then, all $2ent[p/2]$-moments of BIMs (2) are uniformly bounded and, more precisely, for all $k = 0,1,...,n_T$ and all $\kappa \in \mathbb{N}$ with $2\kappa \leq ent[p]$, we have

$$\begin{aligned}\mathbb{E}\,\|Y_k\|_d^{2\kappa} &\leq \mathbb{E}\,[1+\|Y_k\|_d^2]^\kappa \ \leq\ \exp(K_{2\kappa}t_k)\mathbb{E}\,[1+\|Y_0\|_d^2]^\kappa \\ &\leq \exp(K_{2\kappa}T)\mathbb{E}\,[1+\|Y_0\|_d^2]^\kappa\end{aligned} \tag{41}$$

with appropriate real constant

$$K_{2\kappa} \quad \leq \quad \kappa K_M\Big[2K_{b(2)}^0 + (2\kappa-1)mK_M\sum_{j=1}^m \frac{(K_{b(2)}^j)^2}{1-2\kappa(2\kappa-1)mK_M^2(K_{b(2)}^j)^2\Delta_k}\Big]. \tag{42}$$

Proof. Define $v_0(k) := \mathbb{E}\,[\|Y_k\|_d^p]$ for all $k = 0,1,...,n_T$. First, note that

$$\begin{aligned} v_0(k+1) \ &=\ \mathbb{E}\,[\|Y_k + M_{t_k,Y_k}^{-1}(t_{k+1})\sum_{j=0}^m b^j(t_k,Y_k)\Delta W_k^j\|_d^p] \\ =\ & \frac{1}{2}\mathbb{E}\,[\|Y_k + M_{t_k,Y_k}^{-1}(t_{k+1})a(t_k,Y_k)\Delta_k + \sum_{j=1}^m M_{t_k,Y_k}^{-1}(t_{k+1})b^j(t_k,Y_k)\Delta W_k^j\|_d^p] + \\ & +\frac{1}{2}\mathbb{E}\,[\|Y_k + M_{t_k,Y_k}^{-1}(t_{k+1})a(t_k,Y_k)\Delta_k - \sum_{j=1}^m M_{t_k,Y_k}^{-1}(t_{k+1})b^j(t_k,Y_k)\Delta W_k^j\|_d^p]. \end{aligned}$$

Second, apply the random version of Clarkson-Beckner inequality as stated by (34) in Lemma 4.1 and obtain

$$\begin{aligned} & v_0(k+1) \leq \\ & \mathbb{E}\Big(\|Y_k + M_{t_k,Y_k}^{-1}(t_{k+1})a(t_k,Y_k)\Delta_k\|_d^2 + (p-1)\|\sum_{j=1}^m M_{t_k,Y_k}^{-1}(t_{k+1})b^j(t_k,Y_k)\Delta W_k^j\|_d^2\Big)^{p/2}. \end{aligned}$$

Under $(A3)$ this implies

$$\begin{aligned} v_0(k) &= \mathbb{E}\,[\|Y_k\|_d^2]^{p/2} \\ &\leq \mathbb{E}\,\Big(\|Y_k\|_d^2 + 2K_M\|Y_k\|_d\|a(t_k,Y_k)\|_d\Delta_k + K_M^2\|a(t_k,Y_k)\|_d^2\Delta_k^2 + \\ &\qquad +(p-1)mK_M^2\sum_{j=1}^m \|b^j(t_k,Y_k)\|_d^2(\Delta W_k^j)^2\Big)^{p/2} \qquad (43)\\ &\leq \mathbb{E}\,\Big(\|Y_k\|_d^2 + (1+\|Y_k\|_d^2)\Big[2K_MK_{b(2)}^0\Delta_k + K_M^2(K_{b(2)}^0)^2\Delta_k^2 + \\ &\qquad +(p-1)mK_M^2\sum_{j=1}^m (K_{b(2)}^j)^2(\Delta W_k^j)^2\Big]\Big)^{p/2}. \qquad (44)\end{aligned}$$

Third, repeat the previous estimation for all exponents 2κ with $0 < 2\kappa \leq ent[p]$ instead of p. This leads to inequalities (44) for $2\kappa \leq ent[p]$ instead of p. Define $v_p(k) := \mathbb{E}\,[1+\|Y_k\|^2]_d^{p/2}$ for all $k = 0,1,...,n_T$. In particular, we are interested in $v_{2\kappa}(k) = \mathbb{E}\,[1+\|Y_k\|^2]_d^{\kappa}$ for all $k = 0,1,...,n_T$ and all $\kappa \in [0, ent[p/2]]$. For simplicity, suppose that $\kappa \in \mathbb{N}$. Apply the binomial theorem in order to estimate

$$v_{2\kappa}(k) \quad = \quad \mathbb{E}\,[1+\|Y_k\|_d^2]^\kappa \;=\; \sum_{n=0}^{\kappa}\binom{ent[p/2]}{n}\mathbb{E}\,[\|Y_k\|_d^2]^n$$

for all $\kappa \in [0, ent[p/2]] \cap \mathbb{N}$. Set $\zeta_k^j = (K_{b(2)}^j)^2(\Delta W_k^j)^2$. Adding the inequalities (44) for all $2n \leq 2\kappa \leq ent[p]$ instead of p, multiplied by the related binomial coefficients, leads to

$$\begin{aligned} &v_{2\kappa}(k+1) \\ \leq\;& \mathbb{E}\Big((1+\|Y_k\|_d^2)\Big[1+2K_MK_{b(2)}^0\Delta_k+K_M^2(K_{b(2)}^0)^2\Delta_k^2+(2\kappa-1)mK_M^2\sum_{j=1}^m\zeta_k^j\Big]\Big)^\kappa \\ \leq\;& \mathbb{E}\,\Big((1+\|Y_k\|_d^2)\exp\Big(2K_MK_{b(2)}^0\Delta_k+(2\kappa-1)mK_M^2\sum_{j=1}^m\zeta_k^j\Big)\Big)^\kappa \\ \leq\;& \mathbb{E}\Big((1+\|Y_k\|_d^2)^\kappa\exp\Big(2\kappa K_MK_{b(2)}^0\Delta_k\Big)\prod_{j=1}^m\mathbb{E}\Big[\exp\Big(\kappa(2\kappa-1)mK_M^2\zeta_k^j\Big)\Big|\mathcal{F}_k\Big]\Big) \\ \leq\;& \mathbb{E}\,\Big(1+\|Y_k\|_d^2\Big)^\kappa\exp\Big(2\kappa K_MK_{b(2)}^0\Delta_k\Big)\prod_{j=1}^m\mathbb{E}\,\Big[\exp\Big(\kappa(2\kappa-1)mK_M^2(K_{b(2)}^j)^2\Delta_k(\xi_k^j)^2\Big)\Big]\end{aligned}$$

with i.i.d. $\xi_k^j \in \mathcal{N}(0,1)$, thanks to monotonicity of expectations, tower property of conditional expectations and independence of increments $\Delta W_k^j = \sqrt{\Delta_k}\,\xi_k^j$. Fourth, suppose that the constants $\sigma^2 := \kappa(2\kappa-1)mK_M^2(K_{b(2)}^j)^2$ satisfy $2\sigma^2\Delta_k < 1$. Apply Lemma 4.2 with σ^2 to treat the latter estimate. This implies that

$$0 \;\leq\; \mathbb{E}\,\|Y_k\|_d^{2\kappa} \;<\; v_{2\kappa}(k+1) \quad\leq\quad v_{2\kappa}(k)\exp\Big(c_H(k)\Big) \qquad (45)$$

where the coefficients c_H are given by

$$c_H(k) \quad = \quad \kappa K_M\Big(2K_{b(2)}^0+(2\kappa-1)mK_M\sum_{j=1}^m\frac{(K_{b(2)}^j)^2}{1-2\kappa(2\kappa-1)mK_M^2(K_{b(2)}^j)^2\Delta_k}\Big)\Delta_k.$$

Therefore, $(v_k)_{k=0,1,...,n_T}$ is governed by a linear homogeneous inequality (45) whose maximum solution can be estimated by the discrete variation-of-constants formula (i.e. discrete Gronwall-Bellman Lemma) as proven in [12] and applied in [14, 15, 16]. Thus, we arrive at

$$0 \le \mathbb{E}\, \|Y_k\|_d^{2\kappa} < \quad v_{2\kappa}(k+1) \quad \le v_{2\kappa}(k) \exp\Big(K_{2\kappa}\Delta_k\Big) \le v_{2\kappa}(0) \exp\Big(K_{2\kappa} t_{k+1}\Big).$$

This gives the estimates (41) with constants $K_{2\kappa}$ estimated as in (42). Note that $K_{2\kappa}$ is increasing for increasing κ, hence $K_{2\kappa} \le K_p$ and the uniform boundedness of all 2κ-moments of BIMs (2) is obtained for all $\kappa \in [0, ent[p/2]]$ provided that $\mathbb{E}\, \|Y_0\|_d^{ent[p]} < +\infty$. Thus, the proof is complete. □

5. Convergence of Balanced Implicit Methods

This section presents results on the convergence of BIMs (2) applied to SDEs (1) with variable step sizes on fixed time-intervals $[0, T]$.

5.1. L^2-convergence of balanced implicit methods. Define the *pointwise L^2-error* for the numerical method Y approximating the stochastic process X by

$$(46) \qquad \varepsilon_2(t) \quad := \quad \Big(\mathbb{E}\, \|X_{0,x_0}(t) - Y_{0,y_0}(t)\|_d^2\Big)^{1/2}$$

for all $t \in [0, T]$, and the *uniform (weak) L^2-error* by

$$(47) \qquad u_2(t) \quad := \quad \Big(\sup_{0 \le s \le t} \mathbb{E}\, \|X_{0,x_0}(s) - Y_{0,y_0}(s)\|_d^2\Big)^{1/2}$$

for all $t \in [0, T]$. Let $[K]_+$ be the positive part of inscribed expression K and $[K]_-$ the negative part of K such that $K = [K]_+ - [K]_-$.

Theorem 5.1. *Assume that (A1) - (A3) with control function V, constants K_0^E, K_2^E, K_M, K_C, K_B and $\delta_0 \le \min(1, T)$ hold, the coefficients $b^j (j = 1, 2, ..., m)$ are uniform Lipschitz-continuous with Lipschitz constant K_{SM} such that*

$$(48) \ \forall t \in [0, T],\ \forall x, y \in \mathbb{R}^d \quad \sum_{j=1}^{m} \|b^j(t, x) - b^j(t, y)\|_d^2 \le K_{SM}^2 \|x - y\|_d^2.$$

Furthermore, let X be conditionally mean square contractive, i.e. $\exists$ real constant K_C^X such that, for all $0 \le t - s \le \delta_0 \le \min(1, T)$ and all $(\mathcal{F}_s, \mathcal{B}(\mathbb{R}^d))$-measurable random variables $Y(s), Z(s)$ with $Y, Z \in \mathcal{M}_2([0, s])$, we have

$$(49) \ \Big(\mathbb{E}\, [\|X_{s,Y(s)}(t) - X_{s,Z(s)}(t)\|_d^2 | \mathcal{F}_s]\Big)^{1/2} \quad \le \quad \exp(K_C^X (t - s)) \|Y(s) - Z(s)\|_d.$$

Then the BIMs (2) applied to SDEs (1) with (nonrandom) variable step sizes $\Delta_n \le \delta_0$ are globally mean square converging with worst case rate $r_2 = 0.5$ on time-intervals $[0, T]$. Moreover, their pointwise L^2-error ε_2 and uniform (weak) L^2-error u_2 satisfy the universal estimates

$$(50) \ \varepsilon_2(t) \quad \le \quad \begin{cases} \exp((K_C^X + \rho^2)t)\varepsilon_2(0) + \\ \\ +K_g \exp(K_S t) \sqrt{\dfrac{\exp(2(K_C^X + \rho^2 - K_S)t) - 1}{2(K_C^X + \rho^2 - K_S)}} \sqrt{\Delta_{max}} \end{cases},$$

$$(51) \ u_2(t) \quad \le \quad \begin{cases} \exp([K_C^X + \rho^2]_+ t) u_2(0) + \\ \\ +K_g \exp([K_S]_+ t) \sqrt{\dfrac{\exp(2(K_C^X + \rho^2 - K_S)t) - 1}{2(K_C^X + \rho^2 - K_S)}} \sqrt{\Delta_{max}} \end{cases}$$

on $[0, T]$, *where* $\rho > 0$ *is any real constant and*

$$\begin{aligned} K_g &= \frac{1}{\rho}\sqrt{K_0^2 + K_2^2(\rho^2 + K_{SM}^2)} \cdot \sqrt{\mathbb{E}\,V(y_0)} \cdot \exp(([K_C^X]_- + [K_S]_-)\Delta_{max}), \\ K_S &= K_M \cdot K_B \cdot (2 + K_M \cdot K_B), \\ K_0 &= K_0^E + \sqrt{m+1} \cdot K_M \cdot K_C, \\ K_2 &= K_2^E + \sqrt{3} \cdot (m+1) \cdot K_M \cdot K_C. \end{aligned}$$

Proof. It only remains to apply the axiomatic approach as presented and proven in [14, 15, 16]. We know about V-stability with constant K_S from Theorem 3.2 (or Theorem 4.1 with $V(x) = (1 + \|x\|_d^2)^\kappa$, $K_S \le K_{2\kappa}$ for $\kappa \ge 1$, mean consistency with constant K_0 and worst case rate $r_0 = 1.5$ from Theorem 2.1 and mean square consistency with constant K_2 and worst case rate $r_2 = 1.0$ from Theorem 2.2. Furthermore, the diffusion part of SDEs (1) is mean square Hölder-continuous with Hölder exponent $r_{sm} = 0.5$ due to assumption $(A1)$. Therefore, all conditions of the stochastic Kantorovich-Lax-Richtmeyer principle proven by Schurz [14, 15, 16] are met. Hence, the global mean square rate $r_g = r_0 + r_{sm} - 1.0 = 0.5$ is established together with the universal estimates (50) and (51). For example, see Theorem 3.1 in [15] or Theorem 2.1 in [14]. Thus, the proof is complete. □

Remark. In Milstein, Platen and Schurz [11] one finds a proof for L^2-convergence of BIMs (2) with equidistant step sizes $\Delta_k = T/N$ and control functions $V(x) = 1 + \|x\|_d^2$. In contrast to that paper, here we allow variable step sizes $\Delta_k \le \Delta_{max} \le \delta_0$ and other functions $V(x)$ (different from $1 + \|x\|_d^2$) by our proof. Moreover, we show the dependence of the error estimates on all constants K and functions V as well as on the length of the integration interval $[0, T]$.

5.2. How to choose the weight matrices c^j. Suppose one is only interested in weak convergence, i.e. the convergence of BIMs with respect to appropriate test functions $F : \mathbb{R}^d \to \mathbb{R}^l$ or path-dependent functionals $F : C^0([0, T], \mathbb{R}^d) \to \mathbb{R}^1$. Then, of course the weights c^j $(j = 1, 2, ..., m)$ should be set to be the zero matrix $\mathcal{O}$ in order to not destroy the global rate of weak convergence $r_w = 1.0$ compared to the forward or backward Euler methods. In general, it is an open problem whether it is possible to construct higher order weakly converging methods (2) which exploit nonzero random weights c^j with $j \ge 1$ and still guarantee $r_w = 1.0$. It is rather obvious that we have a crude estimate $r_w \ge 0.5$ due to our previous L^2-analysis (e.g. apply Lyapunov-inequality) under the commonly met assumptions for BIMs (2). For general path-dependent functionals F, the weights c^j need to be chosen more carefully. A detailed discussion requires further research. Anyway, it is advicable that the weights should be chosen such that numerical stability (i.e. in almost sure, weak or p-th moment sense) is achieved. Suppose that $b^j \in C^0([0, T] \times \mathbb{R}^d \to \mathbb{R}^d)$ and

$$c_1^2(t)(1 + ||x||_d^2) \le ||(\nabla b^j(t, x))b^k(t, x)||_d^2 \le c_2^2(t)(1 + ||x||_d^2)$$

with $c_1, c_2 \in L^2([0, T], \mathcal{B}([0, T]), \mu)$, where ∇b^j represents the Jacobian matrix of b^j with respect to the variable $x \in \mathbb{R}^d$. If $||\nabla b^j(t, x)b^k(t, x)||_d \in C^0([0, T] \times \mathbb{R}^d, \mathbb{R}_+^1)$ and $[\nabla b^j(t, x)]_+$ is a positive semidefinite matrix part and $[\nabla b^j(t, x)]_-$ a negative semidefinite matrix part of the Jacobian $\nabla b^j(t, x)$ for all $0 \le t \le T$ and all $x \in \mathbb{R}^d$ then a recommendable choice of $c^j(t, x)$ is given by

$$c^0(t, x) = 0.5\nabla a(t, x), \quad c^j(t, x) = [\nabla b^j(t, x)]_+ + [\nabla b^j(t, x)]_- \quad (j = 1, 2, ..., m)$$

due to the stability and boundedness assertions from previous sections while maintaining the convergence in L^2-sense. However, moment-stable approximations can already be obtained by BIMs (2) with vanishing weights $c^j = \mathcal{O}$ $(j = 1, 2, ..., m)$.

5.3. Weak convergence of balanced implicit methods for $C^2_{b(\kappa)}(\mathbb{R}^d, \mathbb{R}^1)$. For approximations in the weak sense, one should rather take the weights $c^j \equiv \mathcal{O}$ for $j = 1, 2, ..., m$ to guarantee the maximum rate of weak convergence. More degree of freedom is in the choice of c^0. A preferrable choice is $c^0(t, x) = 0.5\nabla a(t, x)$ due to a reasonable replication of the p-th moment stability behavior of such BIMs compared to the underlying SDEs. This choice would also coincide with linearly drift-implicit midpoint and trapezoidal methods for bilinear SDEs. Let $C^l_{b(\kappa)}(\mathbb{R}^d, \mathbb{R}^1)$ denote the set of all l-times ($l \in \mathbb{N}$) continuously differentiable functions $f : \mathbb{R}^d \to \mathbb{R}^1$ with uniformly bounded derivatives up to l-th order such that

$$\max\{|f(x)|, ||\nabla f(x)||_d, \|\nabla^2 f(x)\|_{d\times d}, ...\} \ \leq \ K_f \cdot (1 + \|x\|^\kappa_d)$$

for all $x \in \mathbb{R}^d$, where K_f and κ are appropriate real constants.

Theorem 5.2. *Assume that* $(A1)$ *and* $(A3)$ *with* $V(x) \in C^2_{b(\kappa)}(\mathbb{R}^d, \mathbb{R}^1_+)$ *hold,* $\mathbb{E}\,\|Y_0\|^{4\kappa} < +\infty$ *for an integer* $\kappa \geq 1$*, all coefficients* $a, b^j \in C^2_{b(\kappa)}([0, T] \times \mathbb{R}^d, \mathbb{R}^d)$ *of SDE (1) are Lipschitz-continuous with Lipschitz constants* K^j_L *with respect to both variables* t, x *and*

$$\forall t \in [0, T]\, \forall x \in \mathbb{R}^d \ \sum_{j=0}^{m} \|C^0(t, x)b^j(t, x)\|^4_d \ \leq \ (K_C)^4(1 + \|x\|^{4\kappa}_d). \tag{52}$$

Then the subclass of BIMs (2) with weights $c^j(t, x) \equiv \mathcal{O}$ *for* $j = 1, 2, ..., m$ *(i.e. BIMs with nonrandom weights) is weakly converging with rate* $r_w = 1.0$ *with respect to the test class* $f \in C^2_{b(\kappa)}(\mathbb{R}^d, \mathbb{R}^1)$*. More precisely, for all test functions* $f \in C^2_{b(\kappa)}(\mathbb{R}^d, \mathbb{R}^1)$ *for which the standard Euler method weakly converges with rate* $r^E_w = 1.0$*, there is a real constant* $K_w = K_w(T, K_f, b^j)$ *such that*

$$|\mathbb{E}\, f(X_T) - \mathbb{E}\, f(Y_{n_T})| \ \leq \ K_w \cdot \Big(\max_{k=0,1,...,n_T} \mathbb{E}\,(1 + \|Y_k\|^{4\kappa}_d)\Big) \cdot \Delta_{max} \tag{53}$$

where the maximum step size Δ_{max} *satisfies the condition*

$$2\kappa(4\kappa - 1)mK^2_M(K^j_{b(2)})^2\Delta_{max} < 1 \tag{54}$$

with constants $K^j_{b(2)}$ *chosen as in (38) for all* $j = 1, 2, ..., m$ *(i.e. for* b^j *instead of* f*).*

Proof. Recall that the forward Euler methods weakly converge with worst case global rate $r^E_w = 1.0$ and error-constants $K^E_w = K^E_w(T) \geq 0$ under the given assumptions (see Milstein [10] and Talay [18]). Let $f \in C^2_{b(\kappa)}(\mathbb{R}^d, \mathbb{R}^1)$ have uniformly bounded derivatives satisfying

$$\max\Big(|f(x)|, \|\nabla f(x)\|_d, \|\nabla^2 f(x)\|_{d\times d}\Big) \ \leq \ K_f(1 + \|x\|^{4\kappa})^{1/4} \ \leq \ K_f(1 + \|x\|^\kappa)$$

with constant K_f. Moreover, for such functions f, one can find an appropriate real constant $K^E_w = K^E_w(T, f, b^j)$ such that it satisfies the conditional estimates of the local weak error

$$\Big|\mathbb{E}\,[f(X_{s,x}(t)) - f(Y^E_{s,x}(t))]\Big| \ \leq \ K^E_w \cdot (1 + \|x\|^{4\kappa}_d) \cdot (t - s)^2$$

for sufficiently small $0 \le t - s \le \Delta_{max} \le \delta_0$ and $x \in \mathbb{R}^d$, and the global weak error

$$\Big| \mathbb{E}\,[f(X_{0,x}(T)) - f(Y^E_{0,x}(T))] \Big| \;\le\; K^E_w \cdot (1 + \|x\|_d^{4\kappa}) \cdot T \cdot \Delta_{max},$$

for sufficiently small $\Delta_{max} \le \delta_0 \le \min(1,T)$. Now, define the auxiliary functions $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}^1$ by

$$u(s,x) \;=\; \mathbb{E}\, f(X_{s,x}(t_{k+1}))$$

for $0 \le s \le t_{k+1}$. Suppose that $0 \le \Delta_k \le \delta_0 \le \min(1,T)$. For simplicity, assume that X and Y are constructed on one and the same complete probability space (which does not exhibit a real restriction due to Kolmogorov's extension theorem). Then, by following similar ideas as in Milstein [10] extended to the variable step size case, we arrive at

$$\begin{aligned}
\varepsilon_0(t_{k+1}) &:= \Big| \mathbb{E}\,[f(X_{0,x_0}(t)) - f(Y_{0,y_0}(t))] \Big| \\
&= \Big| \sum_{i=0}^{k-1} \Big(\mathbb{E}\,[u(t_{i+1}, X_{t_i,Y_i}(t_{i+1}))] - \mathbb{E}\,[u(t_{i+1}, Y_{t_i,Y_i}(t_{i+1}))] \Big) + \\
&\qquad + \mathbb{E}\,[f(X_{t_k,Y_k}(t_{k+1}))] - \mathbb{E}\,[f(Y_{t_k,Y_k}(t_{k+1}))] \Big| \\
&\le \sum_{i=0}^{k-1} \mathbb{E} \Big| \mathbb{E}\,[u(t_{i+1}, X_{t_i,Y_i}(t_{i+1})) - u(t_{i+1}, Y_{t_i,Y_i}(t_{i+1})) | \mathcal{F}_{t_i}] \Big| + \\
&\qquad + \mathbb{E} \Big| \mathbb{E}\,[f(X_{t_k,Y_k}(t_{k+1})) - f(Y_{t_k,Y_k}(t_{k+1})) | \mathcal{F}_{t_k}] \Big| \\
&\le \sum_{i=0}^{k-1} \mathbb{E} \Big| \mathbb{E}\,[u(t_{i+1}, X_{t_i,Y_i}(t_{i+1})) - u(t_{i+1}, Y^E_{t_i,Y_i}(t_{i+1})) | \mathcal{F}_{t_i}] \Big| + \\
&\qquad + \sum_{i=0}^{k-1} \mathbb{E} \Big| \mathbb{E}\,[u(t_{i+1}, Y^E_{t_i,Y_i}(t_{i+1})) - u(t_{i+1}, Y_{t_i,Y_i}(t_{i+1})) | \mathcal{F}_{t_i}] \Big| + \\
&\qquad + \mathbb{E} \Big| \mathbb{E}\,[f(X_{t_k,Y_k}(t_{k+1})) - f(Y^E_{t_k,Y_k}(t_{k+1})) | \mathcal{F}_{t_k}] \Big| + \\
&\qquad + \mathbb{E} \Big| \mathbb{E}\,[f(Y^E_{t_k,Y_k}(t_{k+1})) - f(Y_{t_k,Y_k}(t_{k+1})) | \mathcal{F}_{t_k}] \Big| \\
&\le K^E_w \cdot \max_{i=0,1,\ldots,k+1} (1 + \mathbb{E}\, \|Y_i\|_d^{4\kappa}) \cdot \sum_{i=0}^{k} \Delta_i^2 + \\
&\qquad + \sum_{i=0}^{k-1} \mathbb{E} \Big| \mathbb{E}\,[u(t_{i+1}, Y^E_{t_i,Y_i}(t_{i+1})) - u(t_{i+1}, Y_{t_i,Y_i}(t_{i+1})) | \mathcal{F}_{t_i}] \Big| + \\
&\qquad + \mathbb{E} \Big| \mathbb{E}\,[f(Y^E_{t_k,Y_k}(t_{k+1})) - f(Y_{t_k,Y_k}(t_{k+1})) | \mathcal{F}_{t_k}] \Big| \\
&= K^E_w \cdot \max_{i=0,1,\ldots,k+1} (1 + \mathbb{E}\, \|Y_i\|_d^{4\kappa}) \cdot t_{k+1} \cdot \Delta_{max} + m_1(k) + m_2(k).
\end{aligned}$$

$$\begin{aligned}
\text{where} \quad m_1(k) &= \sum_{i=0}^{k-1} \mathbb{E} \Big| \mathbb{E}\,[u(t_{i+1}, Y^E_{t_i,Y_i}(t_{i+1})) - u(t_{i+1}, Y_{t_i,Y_i}(t_{i+1})) | \mathcal{F}_{t_i}] \Big|, \\
m_2(k) &= \mathbb{E} \Big| \mathbb{E}\,[f(Y^E_{t_k,Y_k}(t_{k+1})) - f(Y_{t_k,Y_k}(t_{k+1})) | \mathcal{F}_{t_k}] \Big|.
\end{aligned}$$

Next, we analyze the remaining terms m_1 and m_2. For this purpose, suppose that $g \in C^2_{b(\kappa)}(\mathbb{R}^d, \mathbb{R}^1)$. Then, the expressions m_1 and m_2 have only terms of the form

$\mathbb{E}\,|\mathbb{E}\,[g(Y^E_{t_k,Y_k}(t_{k+1}))-g(Y_{t_k,Y_k}(t_{k+1}))|\mathcal{F}_{t_k}]|$. Thus, it remains to estimate them by $K_k\Delta_k^2$ with constants K_k. Note also that $M_{s,x}(t)=I_d-c^0(s,x)(t-s)$ is nonrandom, $Y_{t_k,Y_k}(t_{k+1})=Y_{k+1}$ by definition, and

$$d_{s,x}(t)\ :=\ Y^E_{s,x}(t)-Y_{s,x}(t)\ =\ M^{-1}_{s,x}(t)\sum_{j=0}^{m}c^0(s,x)b^j(s,x)(W^j_t-W^j_s)(t-s).$$

Now, we obtain

$$\begin{aligned}
m(k) &:= \mathbb{E}\,\Big|\mathbb{E}\,[g(Y^E_{t_k,Y_k}(t_{k+1}))-g(Y_{t_k,Y_k}(t_{k+1}))|\mathcal{F}_{t_k}]\Big|\\
&= \mathbb{E}\,\Big|\mathbb{E}\,[<\nabla g(Y_k),Y^E_{t_k,Y_k}(t_{k+1})-Y_{t_k,Y_k}(t_{k+1})>_d|\mathcal{F}_{t_k}]+\\
&\quad \mathbb{E}\,[<\nabla g(\eta_1(t_{k+1}))-\nabla g(Y_k)),Y^E_{t_k,Y_k}(t_{k+1})-Y_{t_k,Y_k}(t_{k+1})>_d|\mathcal{F}_{t_k}]\Big|\\
&= \mathbb{E}\,\Big|<\nabla g(Y_k),\mathbb{E}\,[d_{t_k,Y_k}(t_{k+1})|\mathcal{F}_{t_k}]>_d+\\
&\quad +\mathbb{E}\,[<\nabla^2 g(\eta_2(t_{k+1}))(\eta_1(t_{k+1})-Y_k),d_{t_k,Y_k}(t_{k+1})>_d|\mathcal{F}_{t_k}]\Big|\\
&= \mathbb{E}\,\Big|<\nabla g(Y_k),\mathbb{E}\,[d_{t_k,Y_k}(t_{k+1})|\mathcal{F}_{t_k}]>_d+\\
&\quad +\mathbb{E}\,[\theta^1_k<\nabla^2 g(\eta_2(t_{k+1}))d_{t_k,Y_k}(t_{k+1}),d_{t_k,Y_k}(t_{k+1})>_d|\mathcal{F}_{t_k}]\Big|\\
&= \mathbb{E}\,\Big|<\nabla g(Y_k),M^{-1}_{t_k,Y_k}(t_{k+1})c^0(t_k,Y_k)a(t_k,Y_k)>_d\Delta_k^2+\\
&\quad +\mathbb{E}\,[\theta^1_k<\nabla^2 g(\eta_2(t_{k+1}))d_{t_k,Y_k}(t_{k+1}),d_{t_k,Y_k}(t_{k+1})>_d|\mathcal{F}_{t_k}]\Big|\\
&\le K_M\Big(\mathbb{E}\,[\|\nabla g(Y_k)\|^2_{d\times d}]\Big)^{1/2}\Big(\mathbb{E}\,[\|c^0(t_k,Y_k)a(t_k,Y_k)\|_d^2]\Big)^{1/2}\Delta_k^2+\\
&\quad +\Big(\mathbb{E}\,[\|\nabla^2 g(\eta_2(t_{k+1}))\|^2_{d\times d}]\Big)^{1/2}\Big(\mathbb{E}\,[\|d_{t_k,Y_k}(t_{k+1})\|_d^4]\Big)^{1/2}\\
&\le K_M\Big(\mathbb{E}\,[\|\nabla g(Y_k)\|^4_{d\times d}]\Big)^{1/4}\Big(\mathbb{E}\,[\|c^0(t_k,Y_k)a(t_k,Y_k)\|_d^4]\Big)^{1/4}\Delta_k^2+\\
&\quad +\Big(\mathbb{E}\,[\|\nabla^2 g(\eta_2(t_{k+1}))\|^4_{d\times d}]\Big)^{1/4}\Big(\mathbb{E}\,[\|d_{t_k,Y_k}(t_{k+1})\|_d^4]\Big)^{1/2}\\
&\le 2^{3/2}K_gK_MK_C\Big(\mathbb{E}\,[1+\|Y_k\|_d^{4\kappa}]\Big)^{1/2}\Delta_k^2+\\
&\qquad +\sqrt{3}\,2^{3/2}(m+1)^{3/2}K_gK_M^2K_C^2\Big(\mathbb{E}\,[1+\|Y_k\|_d^{4\kappa}]\Big)^{3/4}\Delta_k^3\\
&\le 2^{3/2}K_gK_MK_C(1+\sqrt{3}\,(m+1)^{3/2}K_MK_C)\cdot\Big(\max_{i=0,1,\ldots,k}\mathbb{E}\,[1+\|Y_{i+1}\|_d^{4\kappa}]\Big)\cdot\Delta_k^2
\end{aligned}$$

where $\eta(t)$ is an intermediate value between $Y^E_{t_k,Y_k}(t)$ and $Y_{t_k,Y_k}(t)$, i.e. $\eta(t)=Y_k+\theta_k(Y^E_{t_k,Y_k}(t)-Y_{t_k,Y_k}(t))$ with scalar $\theta_k\in[0,1]$. Therefore, we may conclude that

$$\begin{aligned}
&m_1(k)\\
&\le 2^{3/2}K_fK_MK_C(1+\sqrt{3}\,(m+1)^{3/2}K_MK_C)(\max_{i=0,1,\ldots,k}\mathbb{E}\,[1+\|Y_{i+1}\|_d^{4\kappa}])\sum_{i=0}^{k-1}\Delta_i^2\\
&\le 2^{3/2}K_fK_MK_C(1+\sqrt{3}\,(m+1)^{3/2}K_MK_C)(\max_{i=0,1,\ldots,k}\mathbb{E}\,[1+\|Y_{i+1}\|_d^{4\kappa}])t_k\Delta_{max},
\end{aligned}$$

and

$$m_2(k) \leq 2^{3/2} K_f K_M K_C (1+\sqrt{3}\,(m+1)^{3/2} K_M K_C)(\max_{i=0,1,\ldots,k} \mathbb{E}\,[1+\|Y_{i+1}\|_d^{4\kappa}])\Delta_k \Delta_{max}.$$

Consequently, for all $k = 0, 1, ..., n_T - 1$, the weak error ε_0 of $BIMs$ (2) with nonrandom weights c^0 must satisfy

$$\begin{aligned} \varepsilon_0(t_{k+1}) &\leq K_w(t_{k+1}) \cdot \max_{i=0,1,\ldots,k} \mathbb{E}\,[1 + \|Y_{i+1}\|_d^{4\kappa}] \cdot \Delta_{max} \\ &\leq K_w(T) \cdot \max_{i=0,1,\ldots,n_T-1} \mathbb{E}\,[1 + \|Y_{i+1}\|_d^{4\kappa}] \cdot \Delta_{max}. \end{aligned}$$

where $K_w(t) \leq (K_w^E + 2^{3/2} K_f K_M K_C (1+\sqrt{3}\,(m+1)^{3/2} K_M K_C))t$. The $p = 4\kappa$-moments of the BIMs (2) with vanishing weights c^j ($j = 1, 2, ..., m$) and sufficiently small step sizes $\Delta_k \leq \Delta_{max}$ are uniformly bounded, as seen by Theorem 4.1. Thus, weak convergence with worst case rate $r_w \geq 1.0$ can be established under the given assumptions of Theorem 5.2, hence the proof is complete. □

Remark. Theorem 5.2 says that the BIMs with nonrandom weights have the same rate of weak convergence as the forward Euler methods have. For further details and more general classes of functionals F, see Talay [18]. One can also find estimates of K_w which are monotonically increasing in K_f, thanks to Theorem 5.2. Therefore, we obtain uniform weak convergence with respect to all test functions $f \in C^2_{b(\kappa)}(\mathbb{R}^d, \mathbb{R}^1)$ which have boundedness constants bounded by $K_f \leq c < +\infty$.

BIMs are implementable very easily while gaining numerical stability compared to explicit methods (as that of Euler-Maruyama) and maintaining the same convergence rates as their explicit counterparts. Thus, we can justify them as a useful and remarkable alternative to the most used numerical methods for SDEs.

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References

[1] M. I. Abukhaled and E. J. Allen, Expectation stability of second order weak numerical methods for stochastic differential equations, Stoch. Anal. Applic. 20 (2002), 693-707.

[2] S. S. Artemiev and T. A. Averina, Numerical Analysis of Systems of Ordinary and Stochastic Differential Equations, VSP, Utrecht, 1997.

[3] W. Beckner, Inequalities in Fourier analysis, Ann. of Math. 102 (1975), 159-182.

[4] J. A. Clarkson, Uniformly convex spaces, Trans. Amer. Math. Soc. 40 (1936), 396-414.

[5] N. Bouleau and D. Lépingle, Numerical Methods for Stochastic Processes, Wiley & Sons, Inc., New York, 1993.

[6] K. Burrage, P. Burrage and T. Mitsui, Numerical solutions of stochastic differential equations—implementation and stability issues, J. Comput. Appl. Math. 125 (2000), 171-182.

[7] Y. Hu, S. E. A. Mohammed and F. Yan, Discrete-time approximations of stochastic delay equations: The Milstein scheme, Ann. Probab. (2004, to appear).

[8] P. E. Kloeden, E. Platen, and H. Schurz, Numerical Solution of SDEs Through Computer Experiments, Third corrected printing, Springer-Verlag, New York, NJ, 2003.

[9] T. Kurtz and P. Protter, *Wong-Zakai corrections, random evolutions and numerical schemes for S.D.E.'s*, In Stochastic Analysis: Liber Amicorum for Moshe Zakai, p. 331-346, Academic Press, Boston, 1991.

[10] G. N. Milstein, Numerical Integration of Stochastic Differential Equations, Kluwer Academic Publishers, Dordrecht, 1995.

[11] G. N. Milstein, E. Platen, and H. Schurz, Balanced implicit methods for stiff stochastic systems, SIAM J. Numer. Anal. 35 (1998), 1010–1019.
[12] H. Schurz, Stability, Stationarity, and Boundedness of Some Implicit Numerical Methods for SDEs and Applications, Logos-Verlag, Berlin, 1997.
[13] H. Schurz, *Numerical analysis of SDEs without tears*, In Handbook of Stochastic Analysis (Ed. D. Kannan and V. Lakshmikantham), p. 237-359, Marcel Dekker, basel, 2002.
[14] H. Schurz, General theorems for numerical approximation of stochastic processes on the Hilbert Space $H_2([0,T],\mu,\mathbb{R}^d)$, Electr. Trans. Numer. Anal. 16 (2003), 50-69.
[15] H. Schurz, An axiomatic approach to the approximation of some Hilbert-space-valued stochastic processes, pp. 18, submitted (also Preprint M-02-007, Department of Mathematics, Southern Illinois University, Carbondale, IL, 2002).
[16] H. Schurz, *Two general theorems for the numerical approximation of stochastic processes on separable Hilbert spaces*, In Proceedings of Neural, Parallel, and Scientific Computations 2 (Ed. M.P. Bekakos, G.S. Ladde, N.G. Medhin and M. Sambandham), p. 55-58. Dynamic Publishers, Atlanta, 2002.
[17] A. N. Shiryaev, Probability, Second edition, Springer-Verlag, New York, 1996.
[18] D. Talay, *Simulation of stochastic differential systems*, In Probabilistic Methods in Applied Physics, Springer Lecture Notes in Physics 451 (Ed. P. Krée and W. Wedig), p. 54-96, Springer-Verlag, Berlin, 1995

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