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A PRIORI AND A POSTERIORI ERROR ESTIMATES FOR BOUSSINESQ EQUATIONS

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Abstract. This paper deals with an incompressible viscous flow problem, where the Navier-Stokes equations are coupled with a nonlinear heat equation. Existence and uniqueness results are established. Next, a finite element approximation of the problem is presented and analyzed. Error estimates are obtained and a posteriori error estimate is given.

Key Words. Boussinesq equations, a posteriori error estimates, finite element methods.

1. Introduction

In this paper, we are interested in an incompressible viscous fluid governed by Navier-Stokes equations, when they are coupled with a nonlinear heat equation by the intermediary of the reaction source term. The considered model is the system formed by the equations describing the flow, under the approximation of *Boussinesq*. Within the framework of this approximation, we do not take account of the variation of density. Therefore the density is regarded as constant in the equation of mass conservation. The *Boussinesq* approximation was justified and used to study some chemical phenomena as in [10, 11]. Numerical analysis and finite element approximation of this model, in non stationary form, is studied in [1, 9]. In this work, we are interested in a similar model, but in a stationary form.

Let Ω an open bounded convex domain of $\mathbb{R}^d$ (d=2,3), with Lipschitz continuous boundary Γ . In Ω, we consider the following stationary model:

$$(P)\begin{cases} -\Delta T + u.\nabla T + f(T) = 0, \quad \text{in } \Omega, \\ -\mu\Delta u + (u.\nabla)u + \nabla p = F(T), \quad \text{in } \Omega, \\ div\ u = 0, \\ u = 0 \ \text{ and } T = 0, \quad \text{on } \Gamma, \end{cases}$$

where the unknown factors are speed u, the pressure p and the temperature T; the coefficient μ (the viscosity of the fluid) is assumed to be positive. The data are a regular function F of $\mathbb{R}$ to $\mathbb{R}^d$ (typically, the function F is a gravity force proportional to the variations of density, therefore dependents on the temperature) and an other regular function f of $\mathbb{R}$ to $\mathbb{R}_+^*$ (typically, the function f is the source term of the reaction depending on the temperature and also on energy; usually this

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function is obtained by the *Arrhenius* law). On datas, we assume that the first and the second derivatives are bounded.

This model has been studied by using topological degree theory to prove the existence results in [2] and by using mixed-dual variational formulation in two dimensions in [6, 7], the authors of these last works introduced the gradient of velocity and the gradient of temperature as unknowns, on which, they give some a priori error estimates.

In the next section, we prove a result of existence and uniqueness of the continuous problem. In the third section, Some usual finite element spaces are introduced, for speed, for the pressure and for the temperature. A discrete problem is given, we prove some error estimates on the speed, on the pressure and on the temperature. Finally in the last section, a posteriori error estimate is given.

2. Existence and uniqueness

The variational form of the problem (P) can be written as following:

$$(P0)\begin{cases} \text{Find } (u,p,T) \in (H_0^1(\Omega))^d \times L_0^2(\Omega) \times H_0^1(\Omega) \text{ such that} \\ \forall v \in (H_0^1(\Omega))^d, \mu \int_\Omega \nabla u.\nabla v dx + \int_\Omega [(u.\nabla)u] v dx - \int_\Omega p div\ v dx \\ \qquad = \int_\Omega F(T) v dx, \\ \forall q \in L_0^2(\Omega), \quad \int_\Omega q div\ u dx = 0, \\ \forall s \in H_0^1(\Omega), \int_\Omega (\nabla s \nabla T + u.\nabla T) dx + \int_\Omega f(T) s dx = 0. \end{cases}$$

First of all, we will rewrite the problem in an equivalent form, allowing us to prove the existence of the weak solution. For that, we introduce the spaces:

$$V = \{v \in (H_0^1(\Omega))^d, \quad div = 0\} \quad \text{and} \quad Y = V \times H_0^1(\Omega).$$

Let $A(.,.)$ the map defined by:

$$\forall ((u,T),(v,s)) \in Y^2,$$

$$A((u,T),(v,s)) = \int_\Omega (\mu \nabla u \nabla v + (u.\nabla)uv) dx - \int_\Omega F(T) v dx$$
$$+ \int_\Omega \nabla T.\nabla s dx + \int_\Omega (u.\nabla T) s dx + \int_\Omega f(T) s dx.$$

We consider the problem

$$(P1)\begin{cases} \text{Find}(u,T) \in V \times H_0^1(\Omega), \text{such that} \\ \forall (v,s) \in V \times H_0^1(\Omega), \quad A((u,T),(v,s)) = 0. \end{cases}$$

It is easy to see that, if the triplet $(u,p,T) \in (H_0^1(\Omega))^d \times L_0^2(\Omega) \times H_0^1(\Omega))$ is solution of $(P0)$, then (u,T) is solution of $(P1)$. Reciprocally, for any solution $(u,T) \in V \times H_0^1(\Omega)$ of $(P1)$, there exist a unique element p of $L_0^2(\Omega)$ such that the triplet (u,p,T) is solution of $(P0)$. To prove the existence of the solution for the problem $(P1)$, we need the following theorem:

Theorem 2.1. *Let X a separable Hilbert space, and $A(.,.)$ a map defined from $X \times X$ onto $I\!R$ such that $v \longrightarrow A(u,v)$ is a linear continuous mapping. Under the following assumptions:*

(1) *There exist $(\gamma, \beta) \in I\!R_+^* \times I\!R$ such that*

$$\forall v \in X, \quad A(v,v) \geq \gamma \|v\|_X^2 - \beta \|v\|_X. \tag{2.1}$$

(2) *For any set (v_n) of X converge weakly toward v, we have*

$$\forall w \in X; \quad \lim_{n \longrightarrow \infty} A(v_n, w) = A(v,w), \tag{2.2}$$

the following problem:

$$\begin{cases} \text{Find } \ u \in X \text{ such that} \\ \forall v \in X, A(u,v) = 0 \end{cases} \tag{2.3}$$

has at least a solution.

Proof. Let us mention that without the term $\beta\|v\|_X$ in the inequality (2.1), this theorem is the same as ([8] Th 1.2, page 280). For completeness, we give only the main idea of the proof. Let a sequence $(w_m)_{m\geq 1}$ a "basis" of X and X_m the subspace of X spanned by $(w_i)_{1\leq i\leq m}$. We set Φ_m a mapping defined from X_m onto X_m by $(\Phi_m(u), w_i) = a(u, w_i), \ 1 \leq i \leq m$.

For any $u \in H_m$, we have

$$(\Phi(u), u) \geq (\gamma \|u\|_X - \beta) \|u\|_X.$$

If we set $\|u\|_X = \nu$; it is sufficient to assume that $\gamma\nu - \beta \geq 0$ to have $(\Phi_m(u), u) \geq 0$, and then to be able to apply *Brouwer*'s Theorem , i.e.

$$\exists u_m \in H_m \text{ such that } \Phi_m(u_m) = 0 \text{ and } \|u_m\|_X \leq \nu.$$

Therefore $\exists u^\star \in H$, such that u_{m_p} (a subsequence of u_m) weakly converge to $u^\star$. It is enough to use (2.2) and the fact that the finite linear combinations of w_i are dense in X, to prove the existence of the solution for the problem (2.3). □

Let the following assumptions:

(1) There exist $(a,b) \in (I\!R_+)^2$ such that

$$\forall (T,s) \in (H_0^1(\Omega))^2; \quad |\int_\Omega f(T) s dx| \leq a|T|_{1,\Omega}|s|_{1,\Omega} + b|s|_{1,\Omega}.$$

(2) There exist $c \in I\!R_+$ such that

$$\forall (T,v) \in H_0^1(\Omega) \times (H_0^1(\Omega))^d; \quad |\int_\Omega F(T) v dx| \leq c|v|_{1,\Omega} + d|v|_{1,\Omega}|T|_{1,\Omega}.$$

Theorem 2.2. *Assume that $a \in [0, 1[$ and $4\mu(1-a) > d^2$, then the problem $(P1)$ admits a solution (u,T) in $V \times H_0^1(\Omega)$.*

Proof. We will apply the Theorem 2.1. For that, we set $X = V \times H_0^1(\Omega)$. By using the compact embedding from $H_0^1(\Omega)$ onto $L^4(\Omega)$, we prove that the mapping $A(.,.)$ is continuous on X and it verifies the second assumption of Theorem 2.1 (same arguments as [8], page 286). To prove the existence of the solution, it is enough to prove that there exist two reals $\gamma \in I\!R_+^*$ and $\beta \in I\!R$, such that

$$\forall (u,T) \in X; \quad A((u,T),(u,T)) \geq \gamma(|T|_{1,\Omega}^2 + |u|_{1,\Omega}^2) - \beta(|T|_{1,\Omega}^2 + |u|_{1,\Omega}^2)^{1/2}.$$

Since $u \in V$, we have

$$\forall v \in V, \forall T \in H_0^1(\Omega), \quad \int_\Omega (u.\nabla)u.v dx = \int_\Omega (u.\nabla T) T dx = 0.$$

Then

$$\forall (u,T) \in V \times H_0^1(\Omega),$$

$$A((u,T),(u,T)) \geq \mu |u|_{1,\Omega}^2 - c|u|_{1,\Omega} + (1-a)|T|_{1,\Omega}^2 - b|T|_{1,\Omega} - d|u|_{1,\Omega}|T|_{1,\Omega}$$

$$\geq min(\mu, 1-a)(|T|_{1,\Omega}^2 + |u|_{1,\Omega}^2) - 2^{\frac{1}{2}} max(c,b)(|T|_{1,\Omega}^2 + |u|_{1,\Omega}^2)^{\frac{1}{2}}$$

$$-d|u|_{1,\Omega}|T|_{1,\Omega}.$$

From the inequality $4\mu(1-a) > d^2$, we deduce that there exist a strictly positive constant γ, such that

$$\forall (u,T) \in V \times H_0^1(\Omega) A((u,T),(u,T)) \geq \gamma(|T|_{1,\Omega}^2 + |u|_{1,\Omega}^2)$$

$$-2^{\frac{1}{2}} max(c,b)(|T|_{1,\Omega}^2 + |u|_{1,\Omega}^2)^{\frac{1}{2}}.$$

By using the Theorem 2.1, we deduce that the problem $(P1)$ admits at least one solution in $X = V \times H_0^1(\Omega)$. □

We set

$$N_1 := \sup_{u \in (H_0^1(\Omega))^d, (T,s) \in (H_0^1(\Omega))^2} \frac{\displaystyle\int_\Omega (u.\nabla T) s dx}{|u|_{1,\Omega}|T|_{1,\Omega}|s|_{1,\Omega}},$$

$$N_2 := \sup_{(u,v,w) \in ((H_0^1(\Omega))^d)^3} \frac{\displaystyle\int_\Omega (u.\nabla) v.w dx}{|u|_{1,\Omega}|v|_{1,\Omega}|w|_{1,\Omega}},$$

$$\overline{\beta} := \begin{cases} 0, \text{ if } f \text{ is an increasing function} \\ \displaystyle\sup_{(T,s) \in (H_0^1(\Omega))^2} \frac{\displaystyle\int_\Omega (f(T) - f(s))(T-s) dx}{|T-s|_{1,\Omega}^2} \text{ otherwise} \end{cases}$$

and

$$\overline{\gamma} := \sup_{(T,s) \in (H_0^1(\Omega))^2, v \in (H_0^1(\Omega))^d} \frac{\displaystyle\int_\Omega (F(T) - F(s)) v dx}{|T-s|_{1,\Omega}|v|_{1,\Omega}}.$$

Thanks to the compact embedding of $H_0^1(\Omega)$ onto $L^4(\Omega)$, we prove easily that $N_1, N_2, \overline{\beta}$ and $\overline{\gamma}$ are positive reals. Concerning the existence and the uniqueness, we have the following Theorem:

Theorem 2.3. *Under the following assumptions:*

$$a \in [0,1[, \qquad 4\mu(1-a) > d^2 \qquad \overline{\beta} \in [0,1[\qquad and \qquad N_2 \frac{c}{\mu} + \overline{\gamma} \frac{N_1}{1-\overline{\beta}} \frac{b}{1-a} < \mu,$$

the problem $(P1)$ *has a unique solution* $(u,T) \in V \times H_0^1(\Omega)$.

Proof. Recall that for $a \in [0,1[$ and $4\mu(1-a) > d^2$, the problem has a solution. We will prove then the uniqueness.

Let (u_1, T_1) and (u_2, T_2) two solutions of the problem $(P1)$.
we have

$$|T_1 - T_2|^2_{1,\Omega}$$

$$= -\int_\Omega (u_1.\nabla T_1 - u_2.\nabla T_2).\nabla(T_1 - T_2)dx - \int_\Omega (f(T_1) - f(T_2))(T_1 - T_2)dx$$

$$= -\int_\Omega (u_1.\nabla(T_1 - T_2))(T_1 - T_2)dx - \int_\Omega ((u_1 - u_2).\nabla T_2)(T_1 - T_2)dx$$

$$- \int_\Omega (f(T_1) - f(T_2))(T_1 - T_2)dx$$

$$\leq N_1 |u_1 - u_2|_{1,\Omega} |T_2|_{1,\Omega} |T_1 - T_2|_{1,\Omega} + \overline{\beta} |T_1 - T_2|^2_{1,\Omega}.$$

Then

$$|T_1 - T_2|_{1,\Omega} \leq \frac{N_1}{1 - \overline{\beta}} |T_2|_{1,\Omega} |u_1 - u_2|_{1,\Omega}. \tag{2.4}$$

We have also

$$\mu |u_1 - u_2|^2_{1,\Omega} + \int_\Omega [(u_1.\nabla)u_1 - (u_2.\nabla)u_2](u_1 - u_2)dx$$

$$= \int_\Omega (F(T_1) - F(T_2)).(u_1 - u_2)dx.$$

By noticing that

$$\int_\Omega (u_1.\nabla)(u_1 - u_2)(u_1 - u_2)dx = 0,$$

we obtain

$$\mu |u_1 - u_2|^2_{1,\Omega} \leq N_2 |u_2|_{1,\Omega} |u_1 - u_2|^2_{1,\Omega} + \overline{\gamma} |T_1 - T_2|_{1,\Omega} |u_1 - u_2|_{1,\Omega}. \tag{2.5}$$

However (u_2, T_2) is solution of the problem $(P1)$; we have the following estimates:

$$|u_2|_{1,\Omega} \leq \frac{c}{\mu} \qquad \text{and } |T_2|_{1,\Omega} \leq \frac{b}{1 - a}.$$

By using (2.4)-(2.5), we deduce

$$\mu |u_1 - u_2|^2_{1,\Omega} \leq N_2 \frac{c}{\mu} |u_1 - u_2|^2_{1,\Omega} + \overline{\gamma} \frac{N_1}{1 - \overline{\beta}} \frac{b}{1 - a} |u_1 - u_2|^2_{1,\Omega}.$$

Then, with the assumption

$$N_2 \frac{c}{\mu} + \overline{\gamma} \frac{N_1}{1 - \overline{\beta}} \frac{b}{1 - a} < \mu,$$

we obtain: $u_1 = u_2$, and by the inequality (2.4), we get: $T_1 = T_2$. □

3. Presentation of the discrete problem

Let S an operator defined by:

$$S : (H^{-1}(\Omega))^d \longrightarrow (H^1_0(\Omega))^d, \quad g \longrightarrow w,$$

where the couple (w, q) is the solution in $(H^1_0(\Omega))^d \times L^2_0(\Omega)$ of Stokes problem

$$\begin{cases} -\mu \Delta w + \nabla q = g \quad \text{in } \Omega, \\ divw = 0 \quad \text{in } \Omega, \\ w = 0 \quad \text{on } \Gamma. \end{cases}$$

Let $\overline{S}g$ the function q in the couple (w,q). Let L an operator defined by:

$$\begin{array}{c} L:(H^{-1}(\Omega))\longrightarrow H_0^1(\Omega),\\ h\longrightarrow Q, \end{array}$$

where Q is the solution in $H_0^1(\Omega)$ of the problem

$$\left\{\begin{array}{l} -\Delta Q=h \quad \text{in } \Omega,\\ Q=0 \quad \text{on } \Gamma. \end{array}\right.$$

Finally, we introduce the operator H defined by:

$$H:Y=(H_0^1(\Omega))^d\times H_0^1(\Omega)\longrightarrow Y=(H_0^1(\Omega))^d\times H_0^1(\Omega),$$

$$V=(v,s)\longrightarrow H(V)=V-(S(\ F(s)-(v.\nabla)v),-L(f(s)+v.\nabla s)).$$

The continuity and the differentiability of this operator are easy to be verified thanks to the *Sobolev* embeddings. It is easy to verify that if the triplet (u,p,T) of $(H_0^1(\Omega))^d\times L_0^2(\Omega)\times H_0^1(\Omega)$ is a weak solution of the problem (P) then $H(U)=0$ with $U=(u,T)$. Reciprocally, for any solution $U=(u,T)$ of equation $H(U)=0$, there exist a unique $p\in L_0^2(\Omega)$ such that the triplet (u,p,T) is a weak solution of the problem (P).
We assume that the couple $U=(u,T)$ is a nonsingular solution of the equation $H(U)=0$, in such way that $DH(U)\in Isom(Y,Y)$.
By writing the operator $DH(U)$ explicitly, for all $V=(v,s)\in Y$,

$$DH(U).V=V-(S[\ F'(T)s-(u.\nabla)v-(v.\nabla)u],-L[f'(T)s+u.\nabla s+v.\nabla T]),$$

and by using the compactness of the nonlinear terms, we verify that the assumption of non-singularity is equivalent to say that the only solution $(w,q,s)\in Y$ of the following problem:

$$\left\{\begin{array}{l} \forall v\in(H_0^1(\Omega))^d,\ \mu\displaystyle\int_\Omega\nabla w.\nabla v dx+\int_\Omega[(u.\nabla)w+(w.\nabla)u]v dx\\ \qquad\qquad\qquad\qquad -\int{}_\Omega q div\ v dx=\displaystyle\int_\Omega F'(T)sv dx,\\ \forall r\in L_0^2(\Omega),\quad \displaystyle\int_\Omega r div w dx=0,\\ \forall z\in H_0^1(\Omega),\ \displaystyle\int_\Omega(\nabla s\nabla z+[w.\nabla T+u.\nabla s]z)dx+\int_\Omega f'(T)zs dx=0, \end{array}\right.$$

is the solution zero. By the local inversion Theorem, the non-singularity assumption implies a local uniqueness of the solution U.

For all values of the real parameter $h>0$, we consider three spaces X_h, M_h and W_h such that

$$X_h\subset(H_0^1(\Omega))^d\ ,\ M_h\subset L_0^2(\Omega)\ \text{ and } W_h\subset H_0^1(\Omega),$$

we set

$$V_h:=\{v_h,\ \ \forall q_h\in M_h,\ b(q_h,v_h)=0\},$$

and we assume that it satisfies the following assumptions:

(1) For all $0 < \sigma \leq 1$, there exist a linear continuous operator P_h from $H^\sigma(\Omega)\cap L_0^2(\Omega)$ onto M_h such that

$$\forall q \in H^\sigma(\Omega)\cap L_0^2(\Omega),\ \|q - P_h q\|_{0,\Omega} \lesssim h^\sigma |q|_{\sigma,\Omega)},$$

(2) For all $\frac{d}{2} < \sigma \leq 1$, there exist a linear continuous operator $\mathcal{I}_h$ from $(H^{1+\sigma}(\Omega))^d \cap (H_0^1(\Omega))^d$ onto X_h such that

$$\forall u \in (H^{1+\sigma}(\Omega))^d \cap (H_0^1(\Omega))^d,\quad \|u - \mathcal{I}_h u\|_{1,\Omega} \lesssim h^\sigma |u|_{1+\sigma,\Omega}.$$

(3) There exist a constant β independent of h, such that

$$\forall q_h \in M_h,\ \exists v_h \in X_h,\ \text{ such that } (div\ v_h, q_h)_{0,\Omega} \geq \beta \|q_h\|_{0,\Omega}\|v_h\|_{1,\Omega}.$$

(4) For all $\frac{d}{2} < \sigma \leq 1$, there exist a linear continuous operator i_h from $H^{1+\sigma}(\Omega)\cap H_0^1(\Omega)$ onto W_h such that

$$\forall T \in H^{1+\sigma}(\Omega)\cap H_0^1(\Omega),\quad \|T - i_h T\|_{1,\Omega} \lesssim h^\sigma |T|_{1+\sigma,\Omega}.$$

We introduce now three spaces X_h, M_h and W_h such that the previous assumptions are satisfied. For that, we assume that the open ω is polyhedric and we assume a regular triangulations family $\mathcal{T}_h$ of ω (see [[5] Chapter 3, Parag. 3]), where for all h the triangulation $\mathcal{T}_h$ is d-simplexes set of diameters bounded above by h. For all K of $\mathcal{T}_h$, we define by $P_k(K)$ the polynomial space of total degree $\leq k$ on T, where k is a strictly positive real.

Let the space W_h defined as following:

$$W_h = \{T_h \in \mathcal{C}^0(\Omega)\cap H_0^1(\Omega),\quad \forall K \in \mathcal{T}_h,\ T_{h|K} \in P_1(K)\}.$$

This space verifies the assumption (4) as in [5]. We denote by I_h the operator defined as following:

$$\begin{aligned} I_h : (\mathcal{C}^0(\Omega))^d &\longrightarrow (W_h)^d \\ F = (f_1, .., f_d) &\longrightarrow I_h F = (i_h f_1, .., i_h f_d). \end{aligned}$$

where i_h is the classic Lagrange interpolation operator.

Example 1 In dimension d=2, we set

$$X_h = \{v_h \in (\mathcal{C}^0(\Omega))^2 \cap (H_0^1(\Omega))^2,\quad \forall K \in \mathcal{T}_h,\ v_{h|K} \in (P_2(K))^2\},$$

$$M_h = \{q_h \in L_0^2(\Omega),\quad \forall K \in \mathcal{T}_h,\ q_{h|K} \in P_0(K)\}.$$

Example 2 In dimension d=2, we set

$$X_h = \{v_h \in (\mathcal{C}^0(\Omega))^2 \cap (H_0^1(\Omega))^2,\quad \forall K \in \mathcal{T}_h,\ v_{h|K} \in (P_2(K))^2\},$$

$$M_h = \{q_h \in L_0^2(\Omega) \cup \mathcal{C}^0(\Omega),\quad \forall K \in \mathcal{T}_h,\ q_{h|K} \in P_1(K)\}.$$

Example 3 For all T of $\mathcal{T}_h$ of vertices a_i, $1 \leq i \leq d+1$, we note by λ_i the barycentric coordinate associated to the vertices a_i and by $\mathbf{n}_i$ the normal vector on the face not containing a_i. We set P_T the space engendered by the polynoms of $(P_1(T))^d$ and by the functions

$$\mathbf{p}_i = \{ \prod_{j=1, j\neq i}^{d+1} \lambda_j \mathbf{n}_i, \quad 1 \leq i \leq d+1 \}.$$

We set then

$$X_h = \{v_h \in (\mathcal{C}^0(\Omega))^d \cap (H_0^1(\Omega))^d, \quad \forall K \in \mathcal{T}_h, \ v_{h|K} \in P_K\},$$
$$M_h = \{q_h \in L_0^2(\Omega) \cup \mathcal{C}^0(\Omega), \quad \forall K \in \mathcal{T}_h, \ q_{h|K} \in P_1(K)\}.$$

In the three examples above, the assumptions (1)-(3) are satisfied and the constant β is independent of h ([3],[8]).

We will specify the approximation of the nonlinear terms F and f. A continuous function T being known on the nodes of the interpolation operators I_h and i_h, we can calculate the quantities $F(T)$ and $f(T)$ in the nodes, and then to construct the interpolates of $F(T)$ and of $f(T)$. More precisely, we define F_h and f_h by:

$$F_h : (\mathcal{C}^0(\Omega))^d \longrightarrow X_h$$
$$s \longrightarrow F_h(s) = I_h(F(s)) = I_h(F(i_h s))$$

and

$$f_h : \mathcal{C}^0(\Omega) \longrightarrow W_h$$
$$s \longrightarrow f_h(s) = i_h(f(s)) = i_h(f(i_h s)).$$

It is easy to verify that for all continuous functions s and q on Ω, the differentials $DF_h(s)q$ and $Df_h(s)q$ are written as following:

$$DF_h(s)q = I_h[F'(s)q] = I_h[F'(i_h s) i_h q],$$
$$Df_h(s)q = i_h[F'(s)q] = i_h[F'(i_h s) i_h q].$$

Moreover, if $F(s) \in (H^\sigma(\Omega))^d$ and $f(s) \in H^\sigma(\Omega)$, with $\frac{d}{2} < \sigma \leq 2$, we have

$$\|F(s) - F_h(s)\|_{0,\Omega} \leq Ch^\sigma \|F(s)\|_{\sigma,\Omega} \text{ and } \|f(s) - f_h(s)\|_{0,\Omega} \leq Ch^\sigma \|f(s)\|_{\sigma,\Omega}.$$

We can write the discrete problem in the following variational form:

$$(P_h) \begin{cases} \text{Find a triplet } (u_h, p_h, T_h) \in X_h \times M_h \times W_h \text{ such that} \\ \forall v_h \in X_h, \ \mu \int_\Omega \nabla u_h . \nabla v_h dx + \int_\Omega (u_h . \nabla) u_h . v_h dx \\ \qquad\qquad - \int_\Omega p_h \text{ div } v_h \ dx = \int_\Omega F_h(T_h) v_h dx, \\ \forall q_h \in M_h, \quad \int_\Omega q_h \text{ div } u_h dx = 0, \\ \forall s_h \in W_h, \ \int_\Omega (\nabla T_h \nabla s_h + u_h . \nabla T_h s_h) dx + \int_\Omega f_h(T_h) s_h dx = 0. \end{cases}$$

To study this system, we will rewrite it in same manner as the continuous problem form. For this goal, we introduce the operators S_h and L_h, the discrete analogues of the operators S and L. More precisely:

$$S_h : (H^{-1}(\Omega))^d \longrightarrow X_h,$$
$$g \longrightarrow w_h,$$

where the couple (w_h, q_h) is the solution in $X_h \times M_h$ of Stokes problem:

$$\begin{cases} \forall v_h \in X_h,\ \mu \displaystyle\int_\Omega \nabla w_h.\nabla v_h dx - \int_\Omega q_h \text{ div } v_h \ dx =< g, v_h >, \\ \forall r_h \in M_h, \ \displaystyle\int_\Omega r_h \text{ div } w_h dx = 0. \end{cases}$$

The operator L_h is defined by:

$$\begin{aligned} L_h : (H^{-1}(\Omega)) &\longrightarrow W_h, \\ h &\longrightarrow Q_h, \end{aligned}$$

where Q_h is the solution in W_h of the following problem:

$$\forall s_h \in W_h, \ \int_\Omega \nabla Q_h.\nabla s_h dx =< h, s_h > .$$

We have

$$\begin{aligned} \forall (g,s) \ \in \ & (H^{-1}(\Omega))^d \times H^{-1}(\Omega); \ \ \|S_h g\|_{1,\Omega} \le C\|g\|_{(H^{-1}(\Omega))^d} \\ & \text{and } \|L_h s\|_{1,\Omega} \le C\|s\|_{H^{-1}(\Omega)}. \end{aligned}$$

Moreover, if $S(g) \in (H^\sigma(\Omega))^d$ and $L(s) \in H^\sigma(\Omega)$, with $\frac{d}{2} < \sigma \le 2$, we have

$$\|(S - S_h)g\|_{1,\Omega} \le Ch^{\sigma-1}(\|Sg\|_{\sigma,\Omega} + (\|\overline{S}g\|_{\sigma-1,\Omega})$$

and

$$\|(L - L_h)s\|_{1,\Omega} \le Ch^{\sigma-1}\|Ls\|_{\sigma,\Omega}.$$

Finally, we introduce the operator H_h defined from the space Y onto Y by:

$$H_h : Y := (H_0^1(\Omega))^d \times H_0^1(\Omega) \longrightarrow Y := (H_0^1(\Omega))^d \times H_0^1(\Omega),$$

$$V = (v,s) \longrightarrow H_h(V) = V - (S_h(\ F(s) - (v.\nabla)v), -L_h(f(s) + v.\nabla s)).$$

It should be noted that this mapping is continuous differentiable from $X_h \times W_h$ onto $X_h \times W_h$. In addition, the system (P_h) is written now in equivalent form $H_h(U_h) = 0$, where $U_h = (u_h, T_h)$. The assumption of compatibility allows to calculate the pressure p_h in M_h, in a unique manner.
the formulation above makes it possible to study the problem (P_h) by using the discrete implicit function Theorem according to [8]:

Theorem 3.1. *We assume that there exist a couple $\overline{U}_h \in X_h \times W_h$ such that $DH_h(\overline{U}_h) \in Isom(X_h \times W_h, X_h \times W_h)$. We assume*

$$\epsilon_h = \|H(\overline{U}_h)\|_{((H_0^1(\Omega))^d \times H_0^1(\Omega)},$$

$$\gamma_h = \|(DH_h(\overline{U}_h))^{-1}\|_{\mathcal{L}(X_h \times W_h, X_h \times W_h)},$$

$$\Lambda_h(\nu) = \sup_{V_h \in B(\overline{U}_h, \nu)} \|DH_h(\overline{U}_h) - DH_h(V_h)\|_{\mathcal{L}(X_h \times W_h, X_h \times W_h)},$$

where

$$B(\overline{U}_h, \nu) = \{V_h \in X_h \times W_h; \ \ \|\overline{U}_h - V_h\|_Y \le \nu\}.$$

If we have

$$2\gamma_h \Lambda_h(2\gamma_h \epsilon_h) < 1,$$

for all $\nu \ge 2\gamma_h\epsilon_h$ such that $\gamma_h \Lambda_h(\nu) < 1$, there exists a unique solution U_h of the equation $H_h(U_h) = 0$, verifying:

$$\|U_h - \overline{U}_h\|_Y \le \nu.$$

Moreover, we have the estimate:

$$\|U_h - \overline{U}_h\|_Y \leq \frac{\gamma_h}{1-\gamma_h \Lambda_h(\nu)} \|H_h(\overline{U}_h)\|_Y.$$

In the next, we assume that the triplet (u, p, T) solution of the continuous problem has the following regularity:

$$u \in (H^\sigma(\Omega))^d, \quad p \in H^{\sigma-1}(\Omega) \quad \text{and } T \in H^\sigma(\Omega),$$

where $\dfrac{d}{2} < \sigma \leq 2$.

3.1. Existence, uniqueness and a priori error estimate. First of all, we have the following technical proposition whose the proof is similar to Bernardi et al ([2], lemma 3.12, page 917).

Proposition 3.1. *We have*

$$\lim_{h\longrightarrow 0} \sup_{q_h \in W_h} \frac{\|F'(T)q_h - I_h(F'(T))q_h\|_{0,\Omega}}{\|q_h\|_{1,\Omega}} = 0 \tag{3.1}$$

and

$$\lim_{h\longrightarrow 0} \sup_{q_h \in W_h} \frac{\|f'(T)q_h - i_h(f'(T))q_h\|_{0,\Omega}}{\|q_h\|_{1,\Omega}} = 0. \tag{3.2}$$

Moreover, for all $(r_h, q_h) \in W_h^2$. We have

$$\|I_h(F'(r_h) - F'(i_h T))q_h\|_{0,\Omega} \leq C\|r_h - i_h T\|_{1,\Omega}\|q_h\|_{1,\Omega}, \tag{3.3}$$

$$\|i_h(f'(r_h) - f'(i_h T))q_h\|_{0,\Omega} \leq C\|r_h - i_h T\|_{1,\Omega}\|q_h\|_{1,\Omega} \tag{3.4}$$

and

$$\|I_h(F(r_h) - F(s_h))\|_{0,\Omega} \leq C\|r_h - s_h\|_{0,\Omega}. \tag{3.5}$$

In the next, let $\overline{U}_h$ the element of $X_h \times W_h$ defined by

$$\overline{U}_h = (\mathcal{I}_h u, i_h T) \in X_h \times W_h.$$

Lemma 3.1. *There exist a constant h_0 such that, for all $h \leq h_0$, we have*

$$DH_h(\overline{U}_h) \in Isom(X_h \times W_h, X_h \times W_h).$$

Moreover, γ_h is bounded above by a constant γ independent of h.

Proof. An immediate consequence of the not-singularity of the continuous solution is that there is a positive constant C such that, for all $W_h = (w_h, s_h) \in X_h \times W_h$,

$$\|DH(U).W_h\|_Y \geq C\|W_h\|_Y.$$

the lemma will be proved, if we prove:

$$\lim_{h\longrightarrow 0} \sup_{W_h \in X_h \times W_h, \|W_h\| \leq 1} \|DH(U).W_h - DH_h(\overline{U}_h).W_h\|_Y = 0. \tag{3.6}$$

Remark that for all couples $V = (v, r)$ and $W = (w, q)$ of Y, we have

$$DH_h(V).W = (w - S_h[\, I_h[F'(r)q] - (v.\nabla)w - (w.\nabla)v], q - L_h[i_h[f'(r)q)] + v.\nabla q + w.\nabla r]).$$

For $\overline{U}_h = (\mathcal{I}_h u, i_h T)$ and $W_h = (w_h, q_h) \in X_h \times W_h$, we have

$$\begin{cases} DH_h(\overline{U}_h).W_h = DH_h(U).W_h + \\ ((S - S_h)[\ F'(T)q_h - (u.\nabla)w_h - (w_h.\nabla)u], (L_h - L)[f'(T)q_h + u.\nabla q_h + w_h.\nabla T)]) \\ +(S_h\Phi, L_h\psi), \end{cases}$$

where

$$\Phi = \ F'(T)q_h - I_h(\ F'(T)q_h) - ((u - \mathcal{I}_h u).\nabla).w_h - (w_h.\nabla)(u - \mathcal{I}_h u)$$

and

$$\psi = f'(T)q_h - i_h(f'(T)q_h) - (u - \mathcal{I}_h u).\nabla q_h - w_h.\nabla(T - i_h T).$$

The lemma will be proved, if we prove the convergence toward 0 of the two terms appearing in the last formula.

1) Let us recall that if Co_1 and Co_2 are respectively compacts of $(H^{-1}(\Omega))^d$ and of $H^{-1}(\Omega)$, we have

$$\lim_{h \longrightarrow 0} \sup_{g \in Co_1} \|(S - S_h)g\|_{1,\Omega} = 0$$

and

$$\lim_{h \longrightarrow 0} \sup_{g \in Co_2} \|(L - L_h)g\|_{1,\Omega} = 0.$$

However, the unit ball image of $Y = (H_0^1(\Omega))^d \times H_0^1(\Omega)$ by the mapping

$$(w, s) \longrightarrow \ F'(T)s - (u.\nabla)w - (w.\nabla)u$$

is a compact of $(H^{-1}(\Omega))^d$, and its image by the mapping

$$(w, s) \longrightarrow f'(T)s + u.\nabla s + w.\nabla T$$

is a compact of $H^{-1}(\Omega)$. We deduce that

$$\lim_{h \longrightarrow 0} \sup_{W_h = (w_h, s_h), \|W_h\|_Y \le 1} \|(S - S_h)(\ F'(T)s_h - (u.\nabla)w_h - (w_h.\nabla)u)\|_{1,\Omega} = 0$$

and

$$\lim_{h \longrightarrow 0} \sup_{W_h = (w_h, s_h), \|W_h\|_Y \le 1} \|(L - L_h)(f'(T)s_h + u.\nabla s_h + w_h.\nabla T)\|_{1,\Omega} = 0.$$

2) By using (3.1)-(3.2), and

$$\lim_{h \longrightarrow 0} \|T - i_h T\|_{1,\Omega} = \lim_{h \longrightarrow 0} \|u - \mathcal{I}_h u\|_{1,\Omega} = 0,$$

we obtain

$$\lim_{h \longrightarrow 0} \|S_h \Phi\|_{1,\Omega} = \lim_{h \longrightarrow 0} \|L_h \psi\|_{1,\Omega} = 0.$$

□

Finally, by using the two results, we obtain the lemma.

Lemma 3.2. *There exist $h_1 > 0$ and a constant C such that*

$$\forall h \le h_1,\ \forall \nu > 0; \quad \Lambda_h(\nu) \le C\nu.$$

Proof. Remark that, for all $V_h = (v_h, r_h)$ and $W_h = (w_h, q_h)$ in $X_h \times W_h$, we have

$$DH_h(\overline{U}_h).W_h - DH_h(V_h).W_h = (S_h \eta, L_h \zeta),$$

where

$$\eta = \ I_h[(F'(r_h) - F'(i_h))q_h] - ((v_h - (\mathcal{I}_h u).\nabla).w_h - (w_h.\nabla)((v_h - (\mathcal{I}_h u)$$

and

$$\zeta = -i_h[(f'(r_h) - f'(i_h))q_h] - (v_h - \mathcal{I}_h u).\nabla q_h - w_h.\nabla(r_h - i_h T).$$

If we assume that

$$\|\mathcal{I}_h u - u\|_{1,\Omega} + \|T - i_h T\|_{1,\Omega} \leq 2\nu,$$

by using the stability of the operators S_h and L_h and (3.3)-(3.4), we deduce easily, that there exist $h_1 > 0$ and a constant C, such that

$$\forall h \leq h_1,\ \forall \nu > 0, \quad \Lambda_h(\nu) \leq C\nu.$$

□

Lemma 3.3. *There exist a constant depending only on (u, p, T), such that*

$$\epsilon_h := \|H(\overline{U}_h)\|_{((H_0^1(\Omega))^d \times H_0^1(\Omega)} \leq Ch^{\sigma-1}.$$

Proof. From the equation $H(U) = 0$, we deduce that

$$H_h(\overline{U}_h) = \overline{U}_h - U - ((S_h - S)(\ F(T) - (u.\nabla)u), (L - L_h)(f(T) + u.\nabla T))$$

$$-(S_h v, L_h s),$$

where

$$v = \ F(T) - I_h(F(T)) - (u.\nabla)u + (\mathcal{I}_h u.\nabla).\mathcal{I}_h u$$

and

$$s = f(T) - f_h(T) + u.\nabla T - \mathcal{I}_h u.\nabla i_h T,$$

then

$$\epsilon_h \leq C\{\|T - i_h T\|_{1,\Omega} + \|u - \mathcal{I}_h u\|_{1,\Omega} + \|(S - S_h)(\ F(T) - (u.\nabla)u)\|_{(H_0^1(\Omega))^d}$$

$$+\|(L - L_h)(f(T) + u.\nabla T)\|_{H_0^1(\Omega)} + \|S_h\|_{\mathcal{L}((H^{-1}(\Omega))^d,(H_0^1(\Omega))^d}\|v\|_{(H^{-1}(\Omega))^d}$$

$$+\|L_h\|_{\mathcal{L}(H^{-1}(\Omega),H_0^1(\Omega))}\|s\|_{H^{-1}(\Omega)}.$$

By using the regularity of (u, T), we have

$$\|T - i_T\|_{1,\Omega} \leq Ch^{\sigma-1}\|T\|_{\sigma,\Omega} \quad \text{and } \|u - \mathcal{I}_h u\|_{1,\Omega} \leq Ch^{\sigma-1}\|u\|_{\sigma,\Omega}.$$

Since

$$T = -L(f(T) + u.\nabla T) \quad \text{and} \quad u = S(\ F(T) - (u.\nabla)u),$$

we have

$$\|(L - L_h)(f(T) + u.\nabla T)\|_{H_0^1(\Omega)} \leq Ch^{\sigma-1}\|L(f(T) + u.\nabla T)\|_{\sigma,\Omega} := Ch^{\sigma-1}\|T\|_{\sigma,\Omega}$$

and

$$\|(S - S_h)(F(T) - (u.\nabla)u)\|_{(H_0^1(\Omega))^d} \leq Ch^{\sigma-1}(\|u\|_{\sigma,\Omega} + \|p\|_{\sigma-1,\Omega}).$$

Finally, by using the stability of the operators S_h and L_h, the regularity of the (u, T), the following equalities:

$$(u.\nabla)u - (\mathcal{I}_h u.\nabla).(\mathcal{I}_h u = ((u - \mathcal{I}_h u).\nabla).u + (u.\nabla).(u - \mathcal{I}_h u) - ((u - \mathcal{I}_h u).\nabla).(u - \mathcal{I}_h u)$$

and

$$u.\nabla T - \mathcal{I}_h u.\nabla i_h T = (u - \mathcal{I}_h u).\nabla T + u.\nabla(T - i_h T) - (u - \mathcal{I}_h u).\nabla(T - i_h T),$$

we obtain

$$\|L_h\|_{\mathcal{L}(H^{-1}(\Omega),H_0^1(\Omega))}\|s\|_{H^{-1}(\Omega)} \leq Ch^{\sigma-1}(\|T\|_{\sigma,\Omega} + \|u\|_{\sigma,\Omega})$$

and

$$\|S_h\|_{\mathcal{L}((H^{-1}(\Omega))^d,(H_0^1(\Omega))^d}\|v\|_{(H^{-1}(\Omega))^d} \leq Ch^{\sigma-1}(\|T\|_{\sigma,\Omega} + \|u\|_{\sigma,\Omega}).$$

By using the estimates above, we prove the lemma. □

Theorem 3.2. *Let (u, p, T) the solution of the problem (P) verifying the assumptions of regularity. there exist a real H such that for all $h \leq H$, the discrete problem admits a solution $U_h = (u_h, T_h) \in X_h \times W_h$. However, this solution verifies the following estimate:*

$$\|u - u_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} + \|T - T_h\|_{1,\Omega} \leq Ch^{\sigma-1}.$$

Proof. We will apply the discrete implicit function Theorem. First of all, for h rather small, we have

$$DH_h(\overline{U}_h) \in Isom(X_h \times W_h, X_h \times W_h).$$

By using the lemmas (2.2) and (2.3), we deduce that there exist a H such that

$$\forall h \leq H; \qquad 2\gamma_h \Lambda_h(2\gamma_h \epsilon_h) < 1,$$

in a manner that the discrete problem admits a solution (u_h, T_h). Finally, by using the compatibility assumption of spaces (X_h, M_h), we deduce that there exist a unique element p_h of M_h such that (u_h, p_h, T_h) will be solution of the problem (P_h). However

$$\|u - u_h\|_{1,\Omega} + \|T - T_h\|_{1,\Omega} \leq C\epsilon_h.$$

Then by using the lemma (2.3), we obtain

$$\|u - u_h\|_{1,\Omega} + \|T - T_h\|_{1,\Omega} \leq Ch^{\sigma-1}.$$

The estimates over the pressure is obtained by using the compatibility of spaces (X_h, M_h) and (3.5) ([2], Prop 3.8, page 907). □

4. A posteriori error estimate .

Let F an operator defined as following:

$$F : Y := (H_0^1(\Omega))^d \times L_0^2(\Omega) \times H_0^1(\Omega) \longrightarrow Y^* := (H^{-1}(\Omega)^d \times L_0^2(\Omega) \times H^{-1}(\Omega)),$$

such that

$$\forall (v, q, s) \in (H_0^1(\Omega))^d \times L_0^2(\Omega) \times H_0^1(\Omega),$$

$$\begin{aligned} < F(u, p, T), (v, q, s) > &= \mu \int_\Omega \nabla u.\nabla v dx + \int_\Omega (u.\nabla)u.v dx - \int_\Omega F(T).v dx \\ &- \int_\Omega p div\ v dx + \int_\Omega q div\ u dx \\ &+ \int_\Omega \nabla T.\nabla s dx + \int_\Omega (u.\nabla T) s dx + \int_\Omega f(T) s dx. \end{aligned}$$

It is obvious that the triplet $(u, p, T) \in (H_0^1(\Omega))^d \times L_0^2(\Omega) \times H_0^1(\Omega)$ is solution of the problem P if and only if $F(u, p, T) = 0$.
Assume that the triplet $(u, p, T) \in (H_0^1(\Omega))^d \times L_0^2(\Omega) \times H_0^1(\Omega)$ is a regular solution in the sense that

$$DF(u, p, T) \in Isom((H_0^1(\Omega))^d \times L_0^2(\Omega) \times H_0^1(\Omega), (H^{-1}(\Omega)^d \times L_0^2(\Omega) \times H^{-1}(\Omega))$$

and

$$DF \text{ is Lipschitz-continuous in } (u, p, T).$$

Let $(u_h, p_h, T_h) \in X_h \times M_h \times W_h$ a triplet, not necessary solution of the discrete problem. Assume that

$$\lim_{h \longrightarrow 0} \|(u, p, T) - (u_h, p_h, T_h)\|_Y = 0.$$

By using the proposition 2.1 of Verfurth [13]. We have, for h rather small:

$$\|(u,p,T)-(u_h,p_h,T_h)\|_Y \leq 2\|(DF(u,p,T))^{-1}\|_{\mathcal{L}(Y,Y^*)}\|F(u_h,p_h,T_h)\|_{Y^*}. \tag{4.1}$$

For all $K \in \mathcal{T}_h$, we note by ∂K a set of internal edges (faces), by n_e the exterior normal vector on e, by $[v]_e$ the jump function v on the edge (face) e and by $\Delta(K)$ the triangles union (tetrahedrons) having a common vertex with K. Let $X_h(K)$ a set of $v_h \in X_h$ having support in $\Delta(K)$. We define by the same manner the sets $M_h(K)$ and $W_h(K)$. We assume that

(1) There exist an operator P_h of $L^2_0(\Omega)$ onto M_h such that

$$\forall q \in L^2_0(\Omega); \quad \|P_h q\|_{0,\Omega} \leq C\|q\|_{0,\Omega}.$$

(2) There exist an operator i_h of $H^1_0(\Omega)$ onto W_h such that

$$\forall s \in H^1_0(\Omega),\ \forall K \in \mathcal{T}_h; \quad |i_h s|_{1,\Omega} \leq C|s|_{1,\Omega} \quad \text{and } \|s - i_h s\|_{0,K} \leq Ch_K\|s\|_{1,\Delta(K)}.$$

(3) There exist an operator I_h de $(H^1_0(\Omega))^d$ onto $(W_h)^d$ such that

$$\forall v \in (H^1_0(\Omega))^d,\ \forall K \in \mathcal{T}_h; \quad |I_h v|_{1,\Omega} \leq C|v|_{1,\Omega} \quad \text{and } \|v - I_h v\|_{0,K} \leq Ch_K\|v\|_{1,\Delta(K)}.$$

The finite element spaces used before verify the assumptions (1)-(3) ([4],[12]).
We set

$$\begin{aligned} e_{1,K} :\ &= h_K^2\| -\mu\Delta u_h + (u_h.\nabla)u_h.u_h + \nabla p_h - F(T_h)\|^2_{0,K} + \|div\ u_h\|^2_{0,K} \\ &+ \sum_{e\subset\partial K} h_e\|[-\mu\frac{\partial u_h}{\partial n_e}]_e + [p_h]_e n_e\|^2_{0,e}, \end{aligned}$$

$$e_{2,K} := h_K^2\| -\Delta T_h + u_h.\nabla T_h + f(T_h)\|^2_{0,K} + \sum_{e\subset\partial K} h_e\|[\frac{\partial T_h}{\partial n_e}]_e\|^2_{0,e},$$

$$\epsilon_{1,K} := \sup_{v_h\in X_h(K)} \frac{\mu\int_\Omega \nabla u_h.\nabla v_h + \int_\Omega (u_h.\nabla)u_h.v_h - \int_\Omega p_h div\ v_h - \int_\Omega F(T_h)v_h}{|v_h|_{1,\Delta(K)}},$$

$$\epsilon_{2,K} := \sup_{q_h\in M_h(K)} \frac{\int_\Omega q_h div\ u_h}{|q_h|_{0,\Delta(K)}}$$

and

$$\epsilon_{3,k} := \sup_{s_h\in W_h(K)} \frac{\int_\Omega \nabla T_h.\nabla s_h dx + \int_\Omega (u_h.\nabla T_h)s_h dx + \int_\Omega f(T_h)s_h dx}{|s_h|_{1,\Delta(K)}}.$$

Theorem 4.1. *There exist h_0, such that, for all $h \leq h_0$, we have*

$$\|(u_h,p_h,T_h)-(u,p,T)\|_Y \leq C((\sum_{i=1}^{2}\sum_{K\in\mathcal{T}_h} e^2_{i,K})^{\frac{1}{2}} + (\sum_{i=1}^{3}\sum_{K\in\mathcal{T}_h} \epsilon^2_{i,K})^{\frac{1}{2}}).$$

Moreover, we have

$$e_{1,K} \leq C(|u-u_h|_{1,\Delta(K)} + |p-p_h|_{0,\Delta(K)} + |T-T_h|_{1,\Delta(K)} +$$

$$(\textstyle\sum_{K\in\Delta(K)} h_K^2\|F(T_h)-F_h(T_h)\|^2_{0,K})^{\frac{1}{2}},$$

$$e_{2,K} \leq C(|u-u_h|_{1,\Delta(K)} + |p-p_h|_{0,\Delta(K)} + |T-T_h|_{1,\Delta(K)} +$$

$$(\textstyle\sum_{K\in\Delta(K)} h_K^2\|f(T_h)-f_h(T_h)\|^2_{0,K})^{\frac{1}{2}}$$

and

$$\forall i = 1,..,3, \qquad \epsilon_{i,K} \le C(|u-u_h|_{1,\Delta(K)} + |p-p_h|_{0,\Delta(K)} + |T-T_h|_{1,\Delta(K)}.$$

Proof. Remark that, for all $(v,q,s) \in Y$, we have

$$< F(u_h,p_h,T_h),(v,q,s) >=< F(u_h,p_h,T_h),(v-I_hv,q-P_hq,s-i_hs) >$$

$$+ < F(u_h,p_h,T_h),(I_hv,P_hq,i_hs) > .$$

By using the stability of the operators I_h, P_h and i_h, we have

$$\sup_{(v,q,s)\in Y} \frac{< F(u_h,p_h,T_h),(I_hv,P_hq,i_hs) >}{\|(v,q,s)\|_Y} \le (\sum_{i=1}^{3}\sum_{K\in\mathcal{T}_h} \epsilon_{i,K}^2)^{\frac{1}{2}}.$$

By using Green formula, we have

$$< F(u_h,p_h,T_h),(v-I_hv,q-P_hq,s-i_hs) >$$

$$= \sum_{K\in\mathcal{T}_h}\{\int_K(-\nu\Delta u_h + (u_h.\nabla)u_h.u_h + \nabla p_h - \ F(T_h))(v-I_hv)dx$$

$$+\int_K q_h div\ u_h dx + \sum_{e\subset\partial K}\int_e([-\mu\frac{\partial u_h}{\partial n_e}]_e + [p_h]_e n_e)(v-I_hv)dx\}$$

$$+\sum_{K\in\mathcal{T}_h}\{\int_K(-\Delta T_h + u_h.\nabla T_h + \nabla p_h + f(T_h))(s-i_hs)dx$$

$$+\sum_{e\subset\partial K}\int_e([\frac{\partial T_h}{\partial n_e}]_e)(s-i_hs)dx\}.$$

From the following inequalities:

$$\|v-I_hv\|_{0,K} \le h_k\|v\|_{1,\Delta(K)}, \qquad \|v-I_hv\|_{0,e} \le h_e^{\frac{1}{2}}\|v\|_{1,\Delta(K)}$$

and

$$\|s-i_hs\|_{0,K} \le h_k\|s\|_{1,\Delta(K)}, \qquad \|s-i_hs\|_{0,e} \le h_e^{\frac{1}{2}}\|s\|_{1,\Delta(K)}.$$

We deduce

$$\sup_{(v,q,s)\in Y} \frac{< F(u_h,p_h,T_h),(v-I_hv,q-P_hq,s-i_hs) >}{\|(v,q,s)\|_Y} \le C(\sum_{i=1}^{2}\sum_{K\in\mathcal{T}_h} e_{i,K}^2)^{\frac{1}{2}}.$$

Finally, by using the relation (4.1). We have

$$\|(u_h,p_h,T_h)-(u,p,T)\|_Y \le C((\sum_{i=1}^{2}\sum_{K\in\mathcal{T}_h} e_{i,K}^2)^{\frac{1}{2}} + (\sum_{i=1}^{3}\sum_{K\in\mathcal{T}_h} \epsilon_{i,K}^2)^{\frac{1}{2}}).$$

Now we will prove the opposite inequalities. First of all, by using the continuity and the equality $F(u,p,T)=0$, we have

$$\forall i = 1,..,3, \qquad \epsilon_{i,K} \le C(|u-u_h|_{1,\Delta(K)} + |p-p_h|_{0,\Delta(K)} + |T-T_h|_{1,\Delta(K)}).$$

Let b_K the bubble function on K such that $max_K b_K = 1$ [13]. We set

$$v_K = -\Delta T_h + u_h.\nabla T_h + f_h(T_h)$$

and

$$w_K = b_K(-\Delta T_h + u_h.\nabla T_h + f_h(T_h)) \in H_0^1(\Omega).$$

By using the norms equivalence $\|.\|_{0,K}$ and $\|b_K.\|_{0,K}$ over functions space with finite dimension, we have

$$e_{2,K}^2 \leq C \int_K v_K w_K dx + \|f(T_h) - f_h(T_h)\|_{0,K}.$$

However

$$\int_K v_K w_K dx = \int_K \nabla T_h.\nabla w_K dx + \int_K (u_h.\nabla T_h + f_h(T_h)) w_K dx$$

and

$$\int_K \nabla T.\nabla w_K dx + \int_K (u.\nabla T + f(T)) w_K dx = 0.$$

Then

$$\int_K v_K w_K dx = \int_K \nabla(T_h - T).\nabla w_K dx + \int_K ((u_h - u).\nabla(T_h - T) + u.\nabla(T_h - T)$$

$$+(u_h - u).\nabla T) w_k dx + \int_K (f_h(T_h) - f(T_h)) dx + \int_K (f(T_h) - f(T)) w_K dx.$$

We obtain

$$\int_K v_K w_K dx \leq C\{|u - u_h|_K + |T - T_h|_{1,K}\}\|w_K\|_{1,K} + \|f(T_h) - f_h(T_h)\|_{0,K}.$$

Finally:

$$|w_K|_{1,K} \leq C h_K^{-1} \|w_K\|_{0,K} \leq C h_K^{-1} \|v_K\|_{0,K}.$$

So

$$h_K \| - \Delta T_h + u_h.\nabla T_h + f(T_h)\|_{0,K}$$
$$\leq C\{|u - u_h|_K + |T - T_h|_{1,K}\} + h_K \|f(T_h) - f_h(T_h)\|_{0,K}.$$

Let e an internal face (edge) of K and b_e the bubble function of K zero over $\partial K/e$. By using the extension operator $P_e : \mathcal{C}^0(e) \longrightarrow \mathcal{C}^0(K_+ \cup K_-)$ [13], we have

$$\|[\frac{\partial T_h}{\partial n_e}]_e\|_{0,e}^2 \leq \int_e \frac{\partial T_h}{\partial n_e} P_e(s) d\sigma = \int_e \frac{\partial T_h}{\partial n_e} P_e(s) d\sigma$$

$$= \int_{K_+ \cup K_-} (\nabla T_h.\nabla P_e(s) + \Delta u_h P_e(s)) dx,$$

where $s = [\frac{\partial T_h}{\partial n_e}]_e$.

By using the same arguments as before, we obtain

$$\int_{K_+ \cup K_-} (\nabla T_h.\nabla P_e(s) + \Delta u_h P_e(s)) dx$$

$$= \int_{K_+ \cup K_-} (\nabla T_h.\nabla P_e(s) + (u_h.\nabla T_h + f(T_h)) P_e(s) dx$$

$$- \int_{K_+ \cup K_-} (-\Delta T_h + u_h.\nabla T_h + f(T_h)) P_e(s) dx$$

$$\leq C\{\|u - u_h\|_{1,\Delta(K)} + \|T - T_h\|_{1,\Omega}\}\|P_e(s)\|_{1,\Omega}$$

$$+\| - \Delta T_h + u_h.\nabla T_h + f(T_h)\|_{0,\Omega}\|P_e(s)\|_{0,\Omega}.$$

However

$$\|P_e(s)\|_{1,K_+\cup K_-} \le h_e^{-1}\|P_e(s)\|_{0,K_+\cup K_-} \le Ch_e^{-\frac{1}{2}}\|s\|_{0,e}.$$

So

$$h_e^{\frac{1}{2}}\|[\frac{\partial T_h}{\partial n_e}]_e\|_{0,e} \le C(|T-T_h|_{1,\Delta(K)} + |u-u_h|_{1,\Delta(K)}),$$

therefore

$$\begin{aligned} e_{2,K} \quad &\le \quad C(|u-u_h|_{1,\Delta(K)} + |p-p_h|_{0,\Delta(K)} + |T-T_h|_{1,\Delta(K)}) \\ &\quad +(\sum_{K\in\Delta(K)} h_K^2\|f(T_h)-f_h(T_h)\|_{0,K}^2)^{\frac{1}{2}}. \end{aligned}$$

Finally, by the same arguments, we have the following inequality:

$$\begin{aligned} e_{1,K} \quad &\le \quad C(|u-u_h|_{1,\Delta(K)} + |p-p_h|_{0,\Delta(K)} + |T-T_h|_{1,\Delta(K)}) \\ &\quad +(\sum_{K\in\Delta(K)} h_K^2\|F(T_h)-F_h(T_h)\|_{0,K}^2)^{\frac{1}{2}}. \end{aligned}$$

□

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