INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 19, Number 6, Pages 761–776

© 2022 Institute for Scientific
Computing and Information

ANALYSIS OF WEAK GALERKIN FINITE ELEMENT METHODS WITH SUPERCLOSENESS

AHMED AL-TAWEEL[1,2], SAQIB HUSSAIN, AND XIAOSHEN WANG

Abstract. In [15], the computational performance of various weak Galerkin finite element methods in terms of stability, convergence, and supercloseness is explored and numerical results are listed in 31 tables. Some of the phenomena can be explained by the existing theoretical results and the others are to be explained. The main purpose of this paper is to provide a unified theoretical foundation to a class of WG schemes, where $\left(P_k(T), P_{k+1}(e), [P_{k+1}(T)]^2\right)$ elements are used for solving the second order elliptic equations (1)-(2) on a triangle grid in 2D. With this unified treatment, all of the existing results become special cases. The theoretical conclusions are corroborated by a number of numerical examples.

Key words. Weak Galerkin, finite element methods, weak gradient, second-order elliptic problems, supercloseness, superconvergence.

1. Introduction

A weak Galerkin finite element method was presented by Wang and Ye in [12] to model the elliptic problems and then has been applied to solve various partial differential equations [1, 4, 5, 6, 7, 8, 9, 10, 11, 14, 18, 19].

The main idea of weak Galerkin finite element methods is the use of weak functions and their corresponding weak derivatives in algorithm design. Weak functions have the form of $v = \{v_0, v_b\}$, where v_0 and v_b can be approximated by polynomials in $P_\ell(T)$ and $P_s(e)$ respectively, where T stands for an element and e the edge or face of T, ℓ and s are non-negative integers. Weak gradients are defined for weak function in the sense of distributions and can be approximated in the polynomial space $[P_m(T)]^2$. Various combination of $(P_\ell(T), P_s(e), [P_m(T)]^2)$ leads to different weak Galerkin methods tailored for specific partial differential equations.

In [15], the computational performance of various weak Galerkin finite element methods in terms of stability, convergence, and supercloseness is explored and numerical results are listed in 31 tables. Some of the phenomena can be explained by the existing theoretical results and the others are to be explained. Table 1 (Table 6.3, [15]) shows the numerical results of a class of weak Galerkin schemes, where $\left(P_k(T), P_{k+1}(e), [P_{k+1}(T)]^2\right)$ elements are used for solving the second order elliptic equations (1)-(2) on a triangle grid in 2D. Some of the results in that table have theoretical explanations (such as elements 6.3.4, 6.3.8, and 6.3.12), while others are posed as open questions. The goal of this paper is to answer these open questions with a unified treatment. Furthermore, with this unified treatment, all of the existing results become spacial cases. As one of the main contributions of this paper, it is shown that by using $\left(P_k(T), P_{k+1}(e), [P_{k+1}(T)]^2\right)$ elements, the error between L^2-projection of the exact solution and the numerical solution will be dramatically reduced if the right parameter is used. More precisely, by choosing the appropriate

Received by the editors November 1, 2021 and, in revised form, March 12, 2022; accepted March 28, 2022.

2000 *Mathematics Subject Classification.* Primary: 65N15, 65N30; Secondary: 35J50.

TABLE 1 (TABLE 6.3, [15]).

Element $(P_k(T), P_{k+1}(e), [P_{k+1}(T)]^2)$ on triangular mesh, $|||\cdot||| = \mathcal{O}(h^{r_1})$ and $\|\cdot\| = \mathcal{O}(h^{r_2})$, t is defined in 7.

element	$P_k(T)$	$P_{k+1}(e)$	$[P_{k+1}(T)]^2$	t	r_1	r_2	Proved
6.3.1				-1	0	0	N|N
6.3.2	$P_0(T)$	$P_1(e)$	$[P_1(T)]^2$	0	1	1	N|N
6.3.3				1	2	2	N|N
6.3.4				∞	2	2	Y|Y
6.3.5				-1	1	2	Y|N
6.3.6	$P_1(T)$	$P_2(e)$	$[P_2(T)]^2$	0	2	3	N|N
6.3.7				1	3	4	N|N
6.3.8				∞	3	4	Y|Y
6.3.9				-1	2	3	Y|N
6.3.10	$P_2(T)$	$P_3(e)$	$[P_3(T)]^2$	0	3	4	N|N
6.3.11				1	4	5	N|N
6.3.12				∞	4	5	Y|Y

parameter, order one and two supercloseness for $k = 0$ and $k \geq 1$, respectively, can be obtained.

In this paper, we are concerned with the second order elliptic problem that seeks an unknown function u satisfying

$$-\nabla \cdot (a\nabla u) = f \quad \text{in } \Omega, \tag{1}$$

$$u = g \quad \text{on } \partial\Omega, \tag{2}$$

where Ω is a polytopal domain in $\mathbb{R}^2$, ∇u denotes the gradient of the function u, and a is a symmetric 2×2 matrix-valued function in Ω. For simplicity, we shall assume that there exist two positive numbers $\lambda_1, \lambda_2 > 0$ such that

$$\lambda_1 \xi^t \xi \leq \xi^t a \xi \leq \lambda_2 \xi^t \xi, \qquad \forall \xi \in \mathbb{R}^2. \tag{3}$$

Here ξ is understood as a column vector and ξ^t is the transpose of ξ.

The paper is organized as follows. In Section 2, we shall describe a WG scheme for solving the second order elliptic equations (1)-(2). Section 3 is devoted to the discussion of the well posedness of the WG scheme. The error analysis for the WG solutions in an energy norm and in the L^2 norm will be investigated in Section 4 and Section 5, respectively. In Section 6, we shall present some numerical examples that confirm the theoretical estimates.

2. Weak Galerkin Finite Element Schemes

Suppose $\mathcal{T}_h$ is a quasi uniform triangular partition of Ω. For every element $T \in \mathcal{T}_h$, denote by h_T its diameter and $h = \max_{T\in\mathcal{T}_h} h_T$. Let $\mathcal{E}_h$ be the set of all the edges in $\mathcal{T}_h$.

First, we adopt the following notations,

$$\begin{aligned}
(v, w)_{\mathcal{T}_h} &= \sum_{T\in\mathcal{T}_h} (v, w)_T = \sum_{T\in\mathcal{T}_h} \int_T vw d\mathbf{x}, \\
\langle v, w\rangle_{\partial\mathcal{T}_h} &= \sum_{T\in\mathcal{T}_h} \langle v, w\rangle_{\partial T} = \sum_{T\in\mathcal{T}_h} \int_{\partial T} vw ds.
\end{aligned}$$

For a given integer $k \geq 0$, a weak Galerkin finite element space associated with $\mathcal{T}_h$ is defined as follows:

$$V_h = \{\{v_0, v_b\} : \ v_0|_T \in P_k(T), \ v_b|_T \in P_{k+1}(e), \ T \in \mathcal{T}_h, e \subset \partial T\}, \tag{4}$$

and its subspace V_h^0 is defined as

$$V_h^0 = \{v : \ v \in V_h, \ v_b = 0 \text{ on } \partial\Omega\}. \tag{5}$$

We would like to emphasize that any function $v \in V_h$ has a single value v_b on each edge $e \in \mathcal{E}_h$.

For any $v = \{v_0, v_b\} \in V_h + H^1(\Omega)$, a weak gradient $\nabla_w v \in [P_{k+1}(T)]^2$ is defined on T as the unique polynomial satisfying

$$(\nabla_w v, \mathbf{q})_T = -(v_0, \nabla \cdot \mathbf{q})_T + \langle v_b, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T} \qquad \forall \mathbf{q} \in [P_{k+1}(T)]^2. \tag{6}$$

where $\mathbf{n}$ is the unit outward normal vector of ∂T. In the equation (6), we let $v_0 = v$ and $v_b = v$ if $v \in H^1(\Omega)$.

For each $T \in \mathcal{T}_h$, let Q_0 be the element-wise defined L^2 projections onto $P_k(T)$ and Q_b be the element-wise defined L^2 projections onto $P_{k+1}(e)$ with $e \subset \partial T$. Define $Q_h u = \{Q_0 u, Q_b u\} \in V_h$. Let $\mathbb{Q}_h$ be the element-wise defined L^2 projection onto $[P_{k+1}(T)]^2$ on each element $T \in \mathcal{T}_h$.

The stabilizer form is introduced for $v, w \in V_h$, as follows:

$$s_t(v, w) = \sum_{T \in \mathcal{T}_h} h_T^t \langle Q_b v_0 - v_b, Q_b w_0 - w_b \rangle_{\partial T}, \tag{7}$$

where $t \geq -1$ is a parameter. When $t = \infty$, we set $s_t(v, w) = 0$.

The WG finite element scheme for the elliptic equations (1)-(2) is as follows:

Algorithm 1 Weak Galerkin Algorithm

The weak Galerkin finite element: A numerical approximation for (1)-(2) can be obtained by finding $u_h = \{u_0, u_b\} \in V_h$, such that $u_b = Q_b g$ on $\partial\Omega$ and the following equation holds

$$(a\nabla_w u_h, \nabla_w v)_{\mathcal{T}_h} + s_t(u_h, v) = (f, \ v_0), \tag{8}$$

for all $v = \{v_0, v_b\} \in V_h^0$. Here $s_t(u_h, v)$ is defined in (7).

We define the following energy norm $|\!|\!| \cdot |\!|\!|$ on V_h:

$$|\!|\!| v |\!|\!|^2 = \sum_{T \in \mathcal{T}_h} \|\nabla_w v\|_T^2 + \sum_{T \in \mathcal{T}_h} h_T^t \|Q_b v_0 - v_b\|_{\partial T}^2. \tag{9}$$

The following lemmas will be needed in the error analysis.

Lemma 1. *Let $\phi \in H^1(\Omega) \cap V_h$. Then for each element $T \in \mathcal{T}_h$, we have*

$$(\mathbb{Q}_h \nabla\phi, \mathbf{q})_T = (\nabla_w Q_h \phi, \mathbf{q})_T + \langle \phi - Q_b\phi, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T} \quad \forall \mathbf{q} \in [P_{k+1}(T)]^2, \tag{10}$$

$$(\mathbb{Q}_h \nabla\phi, \mathbf{q})_T = (\nabla_w \phi, \mathbf{q})_T. \tag{11}$$

Proof. By definition (6) and integration by parts, for each $\mathbf{q} \in [P_{k+1}(T)]^2$ we have

$$\begin{aligned}
(\nabla_w Q_h \phi, \mathbf{q})_T &= -(Q_0\phi, \nabla \cdot \mathbf{q})_T + \langle Q_b\phi, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T} \\
&= -(\phi, \nabla \cdot \mathbf{q})_T + \langle Q_b\phi, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T} \\
&= (\nabla\phi, \mathbf{q})_T - \langle \phi - Q_b\phi, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T} \\
&= (\mathbb{Q}_h(\nabla\phi), \mathbf{q})_T - \langle \phi - Q_b\phi, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T},
\end{aligned}$$

which implies (10). Similarly,

$$\begin{aligned}(\nabla_w \phi, \mathbf{q})_T &= (\nabla \phi, \mathbf{q})_T + \langle Q_b(\phi - \phi), \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T} \\ &= (\mathbb{Q}_h \nabla \phi, \mathbf{q})_T,\end{aligned}$$

which implies (11). □

Lemma 2. *(trace inequality) For any function $\varphi \in H^1(T)$, the following inequality holds true (see [13] for details):*

$$\|\varphi\|_e^2 \le C\left(h_T^{-1}\|\varphi\|_T^2 + h_T\|\nabla\varphi\|_T^2\right). \tag{12}$$

3. Well Posedness

We introduce a discrete H^1 semi-norm on V_h as follows:

$$\|v\|_{1,h}^2 = \sum_{T\in\mathcal{T}_h}\left(\|\nabla v_0\|_T^2 + h_T^{-1}\|v_0 - v_b\|_{\partial T}^2\right). \tag{13}$$

It is easy to show that $\|v\|_{1,h}$ defines a norm in V_h^0.

We need the following lemma.

Lemma 3. *For any $v \in V_h$, we have*

$$\|v\|_{1,h}^2 \le C(\nabla_w v, \nabla_w v), \tag{14}$$

(*for further details, see* [2, 3, 16]).

Lemma 4. *The weak Galerkin finite element scheme 8 has one and only one solution when h is small enough.*

Proof. If $u_h^{(1)}$ and $u_h^{(2)}$ are two solutions of (8), then $\varrho_h = u_h^{(1)} - u_h^{(2)} \in V_h^0$ would satisfy the following equation

$$(a\nabla_w \varrho_h, \nabla_w v)_{T\in\mathcal{T}_h} + s_t(\varrho_h, v) = 0 \qquad \forall v \in V_h^0. \tag{15}$$

Note that $\varrho_h \in V_h^0$. Then by letting $v = \varrho_h$ in the equation (15) and (9), we have

$$C{|\mskip-2mu|\mskip-2mu|}\varrho_h{|\mskip-2mu|\mskip-2mu|}^2 \le (a\nabla_w \varrho_h, \nabla_w \varrho_h)_{T\in\mathcal{T}_h} + s_t(\varrho_h, \varrho_h) = 0.$$

It follows from Lemma 3 that $\varrho_h = 0$, and thus completes the proof. □

4. Error Estimates in Energy Norm

The goal of this section is to derive some error estimates for the WG finite element solution u_h arising from (8). For the sake of simplicity, we will confine our attention to the case where the coefficient tensor a in (1) is a piecewise constant matrix with respect to the finite element partition $\mathcal{T}_h$.

Next, we derive an error equation. First, we define bilinear forms $\ell_a(u,v)$ and $\ell_b(u,v)$ by

$$\begin{aligned}\ell_a(u,v) &= \sum_{T\in\mathcal{T}_h}\langle a(\nabla u - \mathbb{Q}_h\nabla u)\cdot\mathbf{n},\ v_0 - v_b\rangle_{\partial T}, \\ \ell_b(u,v) &= \sum_{T\in\mathcal{T}_h}\langle u - Q_b u, a\nabla_w v\cdot\mathbf{n}\rangle_{\partial T}.\end{aligned}$$

Lemma 5. *Let $e_h = Q_h u - u_h \in V_h$. For any $v \in V_h^0$, the error e_h satisfies the following equation*

$$(a\nabla_w e_h, \nabla_w v)_{\mathcal{T}_h} + s_t(e_h, v) = s_t(Q_h u, v) + \ell_a(u,v) - \ell_b(u,v). \tag{16}$$

Proof. For $v = \{v_0, v_b\} \in V_h^0$, testing (1) by v_0 and using integration by parts gives

$$(a\nabla u, \nabla v_0)_{\mathcal{T}_h} - \langle a\nabla u \cdot \mathbf{n}, v_0 - v_b\rangle_{\partial\mathcal{T}_h} = (f, v_0)_{\mathcal{T}_h}, \tag{17}$$

where we have used the fact that $\langle a\nabla u \cdot \mathbf{n}, v_b\rangle_{\partial\mathcal{T}_h} = 0$.

It follows from integration by parts, (6) and (10) that

$$\begin{aligned}
(a\nabla u, \nabla v_0)_{\mathcal{T}_h} &= (a\mathbb{Q}_h\nabla u, \nabla v_0)_{\mathcal{T}_h} \\
&= -(v_0, \nabla\cdot(a\mathbb{Q}_h\nabla u))_{\mathcal{T}_h} + \langle v_0, a\mathbb{Q}_h\nabla u\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} \\
&= (a\mathbb{Q}_h\nabla u, \nabla_w v)_{\mathcal{T}_h} + \langle v_0 - v_b, a\mathbb{Q}_h\nabla u\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} \\
&= (a\nabla_w Q_h u, \nabla_w v)_{\mathcal{T}_h} + \langle v_0 - v_b, a\mathbb{Q}_h\nabla u\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} \\
&\quad + \langle u - Q_b u, a\nabla_w v\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}.
\end{aligned} \tag{18}$$

Combining (17) and (18) gives

$$(a\nabla_w Q_h u, \nabla_w v)_{\mathcal{T}_h} = (f, v_0) + \ell_a(u, v) - \ell_b(u, v). \tag{19}$$

Adding the stabilizer term $s_t(Q_h u, v)$ to both sides of the above equation, we have

$$\begin{aligned}
(a\nabla_w Q_h u, \nabla_w v)_{\mathcal{T}_h} + s_t(Q_h u, v) &= s_t(Q_h u, v) \\
&\quad + (f, v_0) + \ell_a(u, v) - \ell_b(u, v).
\end{aligned} \tag{20}$$

The error equation follows from subtracting (8) from (20):

$$(a\nabla_w e_h, \nabla_w v)_{\mathcal{T}_h} + s_t(e_h, v) = s_t(Q_h u, v) + \ell_a(u, v) - \ell_b(u, v), \quad \forall v \in V_h^0.$$

This completes the proof of the lemma. □

For the sake of simplicity, we will use

$$\hat{t} = \begin{cases} t & \text{if } -1 \le t \le 1, \\ 1 & \text{if } t > 1, \end{cases}$$

and

$$\hat{k} = \begin{cases} -1 & \text{if } k = 0, \\ k & \text{if } k > 0, \end{cases}$$

in the remaining part of this paper.

Theorem 1. *Let $u_h \in V_h$ be the WG finite element solution of (8). In addition, assuming the regularity of exact solution $u \in H^{k+3}(\Omega)$, then there exists a constant C such that when $0 < h \le 1$*

$$\|\!|Q_h u - u_h|\!\| \le Ch^{k+\hat{t}+1}\|u\|_{k+3}. \tag{21}$$

Proof. By letting $v = e_h$ in (16) and using (3), we have

$$\begin{aligned}
\|\!|e_h|\!\|^2 \le (a\nabla_w e_h, \nabla_w e_h)_{\mathcal{T}_h} + s_t(e_h, e_h) &\le |s_t(Q_h u, e_h)| \\
&\quad + |\ell_a(u, e_h)| + |\ell_b(u, e_h)|.
\end{aligned} \tag{22}$$

By using the definition Q_b, (12), and (14), we have

$$\begin{aligned}
|s_t(Q_h u, e_h)| &= \left| \sum_{T\in\mathcal{T}_h} h_T^t \langle Q_0 u - Q_b u, e_0 - e_b\rangle_{\partial T} \right| \\
&= \left| \sum_{T\in\mathcal{T}_h} h_T^{t+1-1} \langle Q_0 u - u, e_0 - e_b\rangle_{\partial T} \right| \\
&\leq \left| C h^{t+1} \sum_{T\in\mathcal{T}_h} h_T^{-1} \langle Q_0 u - u, e_0 - e_b\rangle_{\partial T} \right| \\
&\leq C h^{t+1} \left(\sum_{T\in\mathcal{T}_h} h_T^{-2} \|Q_0 u - u\|_T^2 + \|\nabla (Q_0 u - u)\|_T^2 \right)^{\frac{1}{2}} \cdot \\
&\quad \left(\sum_{T\in\mathcal{T}_h} h_T^{-1} \|e_0 - e_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
&\leq C h^{k+t+1} \|u\|_{k+1} |\!|\!| e_h |\!|\!|. \qquad (23)
\end{aligned}$$

Using the Cauchy-Schwarz inequality, the trace inequality (12), (3), and Lemma 3, we have

$$\begin{aligned}
|\ell_a(u, e_h)| &= \left| \sum_{T\in\mathcal{T}_h} \langle a(\nabla u - \mathbb{Q}_h \nabla u)\cdot \mathbf{n}, e_0 - e_b\rangle_{\partial T} \right| \\
&\leq C \sum_{T\in\mathcal{T}_h} \|\nabla u - \mathbb{Q}_h \nabla u\|_{\partial T} \|e_0 - e_b\|_{\partial T} \\
&\leq C \left(\sum_{T\in\mathcal{T}_h} h_T \|(\nabla u - \mathbb{Q}_h \nabla u)\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T\in\mathcal{T}_h} h_T^{-1} \|e_0 - e_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
&\leq C h^{k+2} \|u\|_{k+3} |\!|\!| e_h |\!|\!|. \qquad (24)
\end{aligned}$$

Similarly, Using the Cauchy-Schwarz inequality, the trace inequality (12), (3), and Lemma 3, we have

$$\begin{aligned}
|\ell_b(u, e_h)| &= \left| \sum_{T\in\mathcal{T}_h} \langle u - Q_b u, a\nabla_w e_h \cdot \mathbf{n}\rangle_{\partial T} \right| \\
&\leq C \sum_{T\in\mathcal{T}_h} \|u - Q_b u\|_{\partial T} \|\nabla_w e_h\|_{\partial T} \\
&\leq C \left(\sum_{T\in\mathcal{T}_h} h_T^{-1} \|u - Q_b u\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T\in\mathcal{T}_h} h_T \|\nabla_w e_h\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
&\leq C h^{k+2} \|u\|_{k+2} |\!|\!| e_h |\!|\!|. \qquad (25)
\end{aligned}$$

It follows from (22), (23), (24), and (25) that

$$|\!|\!| e_h |\!|\!|^2 \leq C h^{k+\hat{t}+1} \|u\|_{k+3} |\!|\!| e_h |\!|\!|.$$

This completes the proof. □

Remark 1. *It is easy to see that Theorem 1 covers all of the cases for r_1 in Table 1 (Table 6.3, [15]).*

5. Error Estimates in L^2 Norm

The duality argument is used to get L^2 error estimate. Recall $e_h = \{e_0, e_b\} = Q_h u - u_h = \{Q_0 u - u_0, Q_b u - u_b\}$. Considered the dual problem seeks $\Phi \in H_0^1(\Omega)$ satisfying

$$-\nabla \cdot a\nabla\Phi = e_0 \quad \text{in } \Omega. \tag{26}$$

Suppose that the following H^2-regularity conditions holds

$$\|\Phi\|_2 \le C\|e_0\|. \tag{27}$$

The following lemma is from [17], which will be needed to get the optimal L^2 convergence.

Lemma 6. *Let $\psi \in H^{k+1}(\Omega)$, then*

$$|\!|\!|\psi - Q_h\psi|\!|\!| \le Ch^k|\psi|_{k+1}. \tag{28}$$

Theorem 2. *Let $u_h = \{u_0, u_b\} \in V_h$ be the WG finite element solution of (8). Assume that the exact solution $u \in H^{k+3}(\Omega)$ and (27) holds true. Then, there exists a constant C such that when $0 < h \le 1$,*

$$\|Q_0 u - u_0\| \le Ch^{\hat{k}+\hat{t}+2}\|u\|_{k+3}. \tag{29}$$

Proof. By testing (26) with e_0 and integrating by parts, we obtain

$$\begin{aligned}
\|e_0\|^2 &= -(\nabla\cdot(a\nabla\Phi), e_0)\\
&= (a\nabla\Phi,\ \nabla e_0)_{\mathcal{T}_h} - \langle a\nabla\Phi\cdot\mathbf{n},\ e_0 - e_b\rangle_{\partial\mathcal{T}_h},
\end{aligned} \tag{30}$$

where we have used the fact that $e_b = 0$ on $\partial\Omega$. Setting $u = \Phi$ and $v = e_h$ in (18) yields

$$\begin{aligned}
(a\nabla\Phi,\ \nabla e_0)_{\mathcal{T}_h} &= (a\nabla_w Q_h\Phi,\ \nabla_w e_h)_{\mathcal{T}_h} + \langle (a\mathbb{Q}_h\nabla\Phi)\cdot\mathbf{n},\ e_0 - e_b\rangle_{\partial\mathcal{T}_h}\\
&+ \langle \Phi - Q_b\Phi, a\nabla_w e_h\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}.
\end{aligned} \tag{31}$$

Substituting (31) into (30) gives

$$\begin{aligned}
\|e_0\|^2 &= (a\nabla_w e_h,\ \nabla_w Q_h\Phi)_{\mathcal{T}_h} + \langle a(\mathbb{Q}_h\nabla\Phi - \nabla\Phi)\cdot\mathbf{n},\ e_0 - e_b\rangle_{\partial\mathcal{T}_h}\\
&+ \langle \Phi - Q_b\Phi, a\nabla_w e_h\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}\\
&= (a\nabla_w e_h,\ \nabla_w Q_h\Phi)_{\mathcal{T}_h} - \ell_a(\Phi, e_h) + \ell_b(\Phi, e_h)\\
&= (a\nabla_w e_h,\ \nabla_w Q_h\Phi)_{\mathcal{T}_h} + s_t(e_h, Q_h\Phi) - s_t(e_h, Q_h\Phi) - \ell_a(\Phi, e_h) + \ell_b(\Phi, e_h)\\
&= \ell_a(u, Q_h\Phi) - \ell_b(u, Q_h\Phi) - \ell_a(\Phi, e_h) + \ell_b(\Phi, e_h)\\
&- s_t(e_h, Q_h\Phi) + s_t(Q_h u, Q_h\Phi).
\end{aligned} \tag{32}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
|\ell_a(u, Q_h\Phi)| &= \left|\sum_{T\in\mathcal{T}_h}\langle a(\nabla u - \mathbb{Q}_h\nabla u)\cdot\mathbf{n},\ Q_0\Phi - Q_b\Phi\rangle_{\partial T}\right|\\
&\le C\sum_{T\in\mathcal{T}_h}\|\nabla u - \mathbb{Q}_h\nabla u\|_{\partial T}\|Q_0\Phi - Q_b\Phi\|_{\partial T}\\
&\le C\left(\sum_{T\in\mathcal{T}_h}\|\nabla u - \mathbb{Q}_h\nabla u\|_{\partial T}^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h}\|Q_0\Phi - Q_b\Phi\|_{\partial T}^2\right)^{1/2}.
\end{aligned} \tag{33}$$

From the trace inequality (12) and the definition of Q_b, we have

$$\begin{aligned}\left(\sum_{T\in\mathcal{T}_h}\|Q_0\Phi-Q_b\Phi\|_{\partial T}^2\right)^{1/2} &\le \left(\sum_{T\in\mathcal{T}_h}\|Q_0\Phi-\Phi\|_{\partial T}^2+\|\Phi-Q_b\Phi\|_{\partial T}^2\right)^{1/2}\\ &\le C\left(\sum_{T\in\mathcal{T}_h}\|Q_0\Phi-\Phi\|_{\partial T}^2\right)^{1/2}\le Ch^{\frac{3}{2}-i}\|\Phi\|_2,\end{aligned}$$

where $i=1$, when $k=0$ and $i=0$ when $k\ge 1$.

It is easy to see that

$$\left(\sum_{T\in\mathcal{T}_h}\|a(\nabla u-\mathbb{Q}_h\nabla u)\|_{\partial T}^2\right)^{1/2}\le Ch^{k+\frac{3}{2}}\|u\|_{k+3}.$$

Combining the above two estimates with (33) gives

$$|\ell_a(u,Q_h\Phi)| \le Ch^{\hat{k}+3}|u|_{k+3}\|\Phi\|_2 \tag{34}$$

Using the Cauchy-Schwarz inequality, the trace inequality (12), (3), (14), and (21), we have

$$\begin{aligned}|\ell_a(\Phi,e_h)| &= \left|\sum_{T\in\mathcal{T}_h}\langle a(\nabla\Phi-\mathbb{Q}_h\nabla\Phi)\cdot\mathbf{n},e_0-e_b\rangle_{\partial T}\right|\\ &\le C\left(\sum_{T\in\mathcal{T}_h}h_T\|(\nabla\Phi-\mathbb{Q}_h\nabla\Phi)\|_{\partial T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|e_0-e_b\|_{\partial T}^2\right)^{\frac{1}{2}}\\ &\le Ch\|\Phi\|_2|\!|\!|e_h|\!|\!|\\ &\le Ch^{k+\hat{t}+2}|u|_{k+3}\|\Phi\|_2.\end{aligned}\tag{35}$$

Using the Cauchy-Schwarz inequality, the trace inequality (12), (3), (14), and (21), we have

$$\begin{aligned}|\ell_b(u,Q_h\Phi)| &= \left|\sum_{T\in\mathcal{T}_h}\langle u-Q_bu,a\nabla_wQ_h\Phi\cdot\mathbf{n}\rangle_{\partial T}\right|\\ &\le \left|\sum_{T\in\mathcal{T}_h}\langle u-Q_bu,a\nabla_w(Q_h\Phi-\Phi)\cdot\mathbf{n}\rangle_{\partial T}\right|\\ &+ \left|\sum_{T\in\mathcal{T}_h}\langle u-Q_bu,a\nabla\Phi\cdot\mathbf{n}\rangle_{\partial T}\right|.\end{aligned}\tag{36}$$

To estimate the terms on the right hand side of (36), let $\mathbb{Q}_0$ and Q_1 be the element-wise defined L^2 projection onto $[P_0(T)]^2$ and $P_{k+1}(T)$, respectively, on each element $T\in\mathcal{T}_h$. It follows from the definition of Q_b and (28) that

$$\begin{aligned}\left|\sum_{T\in\mathcal{T}_h}\langle u-Q_bu,a\nabla_w(Q_h\Phi-\Phi)\cdot\mathbf{n}\rangle_{\partial T}\right| &\le C\sum_{T\in\mathcal{T}_h}\|u-Q_1u\|_{\partial T}\|\nabla_w(Q_h\Phi-\Phi)\|_{\partial T}\\ &\le Ch^{k+3}|u|_{k+2}\|\Phi\|_2.\end{aligned}\tag{37}$$

Next,

$$
\begin{aligned}
\left|\sum_{T\in\mathcal{T}_h}\langle u-Q_bu, a\nabla\Phi\cdot\mathbf{n}\rangle_{\partial T}\right| &= \left|\sum_{T\in\mathcal{T}_h}\langle u-Q_bu, a(\nabla\Phi-\mathbb{Q}_0\nabla\Phi)\cdot\mathbf{n}\rangle_{\partial T}\right| \\
&\leq C\sum_{T\in\mathcal{T}_h}\|u-Q_1u\|_{\partial T}\|\nabla\Phi-Q_0\nabla\Phi\|_{\partial T} \\
&\leq Ch^{k+3}|u|_{k+2}\|\Phi\|_2, \qquad (38)
\end{aligned}
$$

Combining (37) and (38), we have

$$
\begin{aligned}
|\ell_b(u,Q_h\Phi)| &\leq Ch^{k+3}|u|_{k+2}\|\Phi\|_2 \\
&\leq Ch^{k+\hat{t}+2}|u|_{k+2}\|\Phi\|_2, \qquad (39)
\end{aligned}
$$

when $0<h\leq 1$.

$$
\begin{aligned}
|\ell_b(\Phi,e_h)| &= \left|\sum_{T\in\mathcal{T}_h}\langle \Phi-Q_b\Phi, a\nabla_w e_h\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}\right| \\
&\leq C\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|\Phi-Q_1\Phi\|_{\partial T}^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h}h_T\|\nabla_w e_h\|_{\partial T}^2\right)^{1/2} \\
&\leq Ch\|\Phi\|_2{|\kern-0.25ex|\kern-0.25ex|}e_h{|\kern-0.25ex|\kern-0.25ex|} \\
&\leq Ch^{k+\hat{t}+2}|u|_{k+1}\|\Phi\|_2. \qquad (40)
\end{aligned}
$$

Using the definition of Q_b and (12), we obtain

$$
\begin{aligned}
|s_t(Q_hu,Q_h\Phi)| &= \left|\sum_{T\in\mathcal{T}_h}h_T^t\langle Q_0u-Q_bu, Q_0\Phi-Q_b\Phi\rangle_{\partial T}\right| \\
&= \left|\sum_{T\in\mathcal{T}_h}h_T^{t+1-1}\langle Q_0u-Q_bu, Q_0\Phi-Q_b\Phi\rangle_{\partial T}\right| \\
&\leq \left|Ch^{t+1}\sum_{T\in\mathcal{T}_h}h_T^{-1}\langle Q_0u-Q_bu, Q_0\Phi-Q_b\Phi\rangle_{\partial T}\right| \\
&\leq Ch^{t+1}\left(\sum_{T\in\mathcal{T}_h}h_T^{-2}\|Q_0u-u\|_T^2+\|\nabla(Q_0u-u)\|_T^2\right)^{1/2}\cdot \\
&\quad \left(\sum_{T\in\mathcal{T}_h}h_T^{-2}\|Q_0\Phi-\Phi\|_T^2+\|\nabla(Q_0\Phi-\Phi)\|_T^2\right)^{1/2} \\
&\leq Ch^{k+\hat{t}+2}|u|_{k+1}\|\Phi\|_2. \qquad (41)
\end{aligned}
$$

It follows from (3) and (21) that

$$\begin{aligned}
|s_t(Q_h\Phi, e_h)| &= \left|\sum_{T\in\mathcal{T}_h} h_T^t \,\langle Q_0\Phi - Q_b\Phi, e_0 - e_b\rangle_{\partial T}\right| \\
&\leq \left|\sum_{T\in\mathcal{T}_h} h_T^{t+1-1} \,\langle Q_0\Phi - \Phi, e_0 - e_b\rangle_{\partial T}\right| \\
&\leq \left|Ch^{t+1}\sum_{T\in\mathcal{T}_h} h_T^{-1} \,\langle Q_0\Phi - \Phi, e_0 - e_b\rangle_{\partial T}\right| \\
&\leq Ch^{t+2}\|\Phi\|_2 |\!|\!| e_h |\!|\!| \\
&\leq Ch^{k+\hat{t}+2}|u|_{k+3}\|\Phi\|_2. \qquad (42)
\end{aligned}$$

Substituting (34), (35), (40), (39), (41) and (42) into (32) yields

$$\|e_0\|^2 \leq Ch^{\hat{k}+\hat{t}+2}\|u\|_{k+3}\|\Phi\|_2,$$

when $0 < h \leq 1$. Using the regularity assumption (27) gives the error estimate (29). □

Remark 2. *It is easy to see that Theorem 2 covers all of the cases for r_2 in Table 1 (Table 6.3, [15]).*

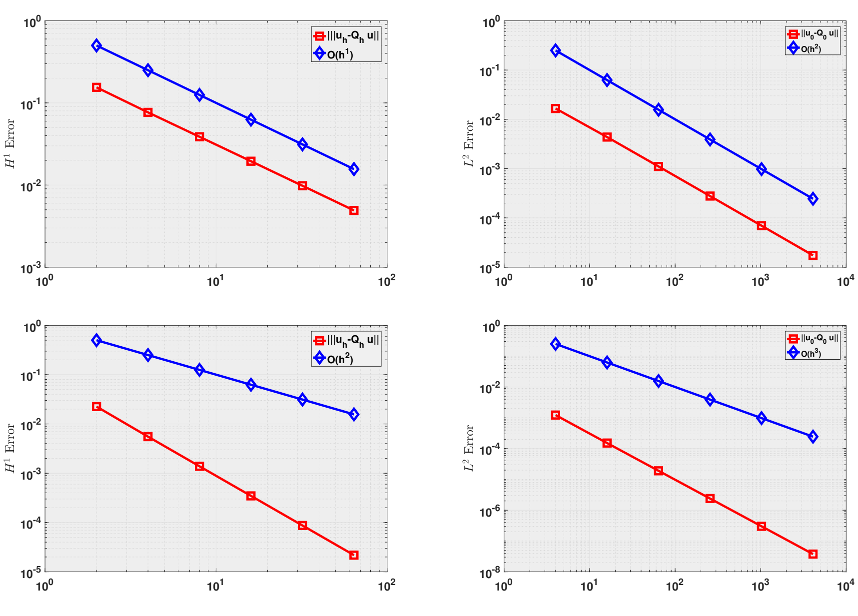

FIGURE 1. Example 1: Plot of the errors and convergence rate for errors measured by $\|Q_0u - u_0\|$ and $|\!|\!| Q_hu - u_h |\!|\!|$ with $t = -1$ and $h = 1/64$: $\big(P_1(T), P_2(e), [P_2(T)]^2\big)$ element (top); $\big(P_2(T), P_3(e), [P_3(T)]^2\big)$ element (bottom).

TABLE 2. Example 1: Error profiles and convergence rates.

h	$\|\|\|u_h - Q_h u\|\|\|$	rate	$\|\|u_h - Q_h u\|\|_0$	rate
	by the $P_0(T)$-$P_1(e)$-$[P_1(T)]^2$ element and $t = 0$			
1/2	4.0876E-01	-	1.6398E-02	-
1/4	2.3901E-01	0.77	1.6577E-02	-0.02
1/8	1.2238E-01	0.97	1.0022E-02	0.73
1/16	6.1226E-02	1.00	5.2927E-03	0.92
1/32	3.0530E-02	1.00	2.6918E-03	0.98
1/64	1.5232E-02	1.00	1.3538E-03	0.99
	by the $P_0(T)$-$P_1(e)$-$[P_1(T)]^2$ element and $t = \infty$ stabilizer free			
1/2	7.5114E-01	-	7.1931E-02	-
1/4	1.2680E-00	-0.76	9.4366E-02	-0.39
1/8	3.6930E-01	1.78	3.3120E-02	1.51
1/16	9.6181E-02	1.94	9.0440E-03	1.87
1/32	2.4305E-02	1.98	2.3122E-03	1.97
1/64	6.0937E-03	2.00	5.8131E-04	1.99
	by the $P_1(T)$-$P_2(e)$-$[P_2(T)]^2$ element and $t = 0$			
1/2	1.1768E-01	-	1.1279E-02	-
1/4	3.0280E-02	1.96	1.5810E-03	2.83
1/8	7.6031E-03	1.99	2.0383E-04	2.96
1/16	1.8992E-03	2.00	2.5564E-05	3.00
1/32	4.7420E-04	2.00	3.1925E-06	3.00
1/64	1.1844E-04	2.00	3.9857E-07	3.00
	by the $P_1(T)$-$P_2(e)$-$[P_2(T)]^2$ element and $t = \infty$ stabilizer free			
1/2	1.4619E-00	-	1.1836E-01	-
1/4	2.0798E-01	2.81	1.1841E-02	3.32
1/8	2.8113E-02	2.89	9.0564E-04	3.96
1/16	3.5995E-03	2.97	5.9666E-05	3.71
1/32	4.5325E-04	2.99	3.7710E-06	3.98
1/64	5.6791E-05	3.00	2.3707E-07	3.99
	by the $P_2(T)$-$P_3(e)$-$[P_3(T)]^2$ element and $t = 0$			
1/2	1.6022E-02	-	7.9960E-04	-
1/4	2.0059E-03	3.00	5.1460E-05	3.96
1/8	2.4776E-04	3.02	3.2377E-06	3.99
1/16	3.0718E-05	3.01	2.0289E-07	4.00
1/32	3.8225E-06	3.00	1.2695E-08	4.00
1/64	4.7671E-07	3.00	7.9382E-10	4.00
	by the $P_2(T)$-$P_3(e)$-$[P_3(T)]^2$ element and $t = \infty$ stabilizer free			
1/2	1.4173E-01	-	8.8638E-03	-
1/4	3.4657E-02	2.03	8.0415E-04	3.46
1/8	2.2842E-03	3.92	2.3562E-05	5.09
1/16	1.4413E-04	3.99	7.1949E-07	5.03
1/32	9.0288E-06	4.00	2.2357E-08	5.01
1/64	5.6473E-07	4.00	6.9772E-10	5.00

6. Numerical Experiments

In this section, various numerical examples in 2D uniform triangular meshes are presented to support our theoretical findings. We perform the Weak Galerkin Algorithm 1 using $(P_k(T), P_{k+1}(e), [P_{k+1}(T)]^2), k = 0, 1, 2$ elements and choose various stabilizing parameters for comparison in the computation.

6.1. Example 1. In this example, we consider problem (1) with $\Omega = (0,1)^2$ and $a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The source term f and the boundary value condition g are chosen so that the exact solution is

$$u(x, y) = \cos(x)\cos(\pi y).$$

It can be observed from Figure 1 that the error between numerical solutions obtained by WG Algorithm 1 and the L^2 projection of u, e_h, with $t = -1$ and $k \geq 1$ converge to zero at the rate of $\mathcal{O}(h^k)$ in H^1-norm and $\mathcal{O}(h^{k+1})$ in L^2-norm, respectively. On the other hand, with $t = \infty$ and $k \geq 0$ the corresponding rates are $\mathcal{O}(h^{\hat{k}+2})$ and $\mathcal{O}(h^{\hat{k}+3})$ in H^1-norm and L^2 norm, respectively, as can be seen from Table 2. Table 2 shows that the WG scheme 8 with $t = 0$ and $k \geq 1$ has one orders

TABLE 3. Example 1: Error profiles and convergence rates.

h	$\|u_h - Q_h u\|$	rate	$\|u_h - Q_h u\|_0$	rate
	by the $P_0(T)$-$P_1(e)$-$[P_1(T)]^2$ element and $t = 1$			
1/2	3.1524E-01	-	1.2024E-02	-
1/4	1.0716E-01	1.56	7.7129E-03	0.64
1/8	2.9220E-02	1.87	2.4617E-03	1.65
1/16	7.5031E-03	1.96	6.5479E-04	1.91
1/32	1.8927E-03	1.99	1.6627E-04	1.98
1/64	4.7476E-04	2.00	4.1732E-05	1.99
	by the $P_0(T)$-$P_1(e)$-$[P_1(T)]^2$ element and $t = 2$			
1/2	8.2330E-01	-	7.8732E-02	-
1/4	1.3259E-00	-0.69	9.7907E-02	-0.31
1/8	3.7929E-01	1.81	3.4019E-02	1.53
1/16	9.7545E-02	1.96	9.1782E-03	1.89
1/32	2.4480E-02	1.99	2.3297E-03	1.98
1/64	6.1156E-03	2.00	5.8353E-04	2.00
	by the $P_1(T)$-$P_2(e)$-$[P_2(T)]^2$ element and $t = 1$			
1/2	8.4214E-02	-	7.9833E-03	-
1/4	1.2077E-02	2.80	6.2769E-04	3.67
1/8	1.5898E-03	2.93	4.1840E-05	3.91
1/16	2.0284E-04	2.97	2.6628E-06	3.97
1/32	2.5577E-05	2.99	1.6728E-07	3.99
1/64	3.2097E-06	2.99	1.0471E-08	4.00
	by the $P_1(T)$-$P_2(e)$-$[P_2(T)]^2$ element and $t = 2$			
1/2	1.6358E-00	-	1.3024E-01	-
1/4	2.2364E-01	2.87	1.2710E-03	3.35
1/8	2.9227E-02	2.94	9.4591E-04	3.76
1/16	3.6717E-03	2.99	6.1022E-05	3.95
1/32	4.5781E-04	3.00	3.8232E-06	4.00
1/64	5.7077E-05	3.00	2.3843E-07	4.00
	by the $P_2(T)$-$P_3(e)$-$[P_3(T)]^2$ element and $t = 1$			
1/2	1.1100E-02	-	5.4262E-04	-
1/4	7.5839E-04	3.87	1.8902E-05	4.84
1/8	4.8677E-05	3.96	6.1210E-07	4.95
1/16	3.0718E-06	3.99	1.9407E-08	4.98
1/32	1.9277E-07	3.99	6.1035E-10	4.99
1/64	1.2071E-08	4.00	1.9130E-11	5.00
	by the $P_2(T)$-$P_3(e)$-$[P_3(T)]^2$ element and $t = 2$			
1/2	1.6637E-01	-	1.0384E-02	-
1/4	3.8211E-02	2.12	8.9594E-04	3.53
1/8	2.4048E-03	3.99	2.4977E-05	5.16
1/16	1.4795E-04	4.02	7.4123E-07	5.07
1/32	9.1487E-06	4.02	2.2695E-08	5.03
1/64	5.6848E-07	4.01	7.0299E-10	5.01

of supercloseness in both H^1 norm and L^2 norm. As we can see from Table 3 that the WG scheme 8 with $t \geq 1$ and $k \geq 1$ has two orders of supercloseness in both energy norm and L^2 norm.

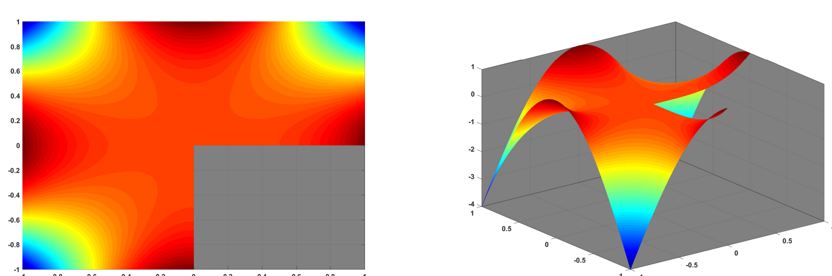

FIGURE 2. Example 2: Plot of numerical solutions for $\left(P_1(T), P_2(e), [P_2(T)]^2\right)$ element using WG method (8) with $t = 1$ and $h = 1/64$: (left) 2D plot; (right) 3D plot.

TABLE 4. Example 2: Error profiles and convergence rates.

h	$\lVert u_h - Q_h u \rVert$	rate	$\lVert u_h - Q_h u\rVert_0$	rate
	by the $P_0(T)$-$P_1(e)$-$[P_1(T)]^2$ element with $t = 0$			
1/2	9.2889E-01	-	9.5123E-02	-
1/4	5.5178E-01	0.75	6.2774E-02	0.60
1/8	3.1283E-01	0.82	3.6873E-02	0.77
1/16	1.6754E-01	0.90	1.9841E-02	0.89
1/32	8.6771E-02	0.95	1.0265E-02	0.95
1/64	4.4163E-02	0.97	5.2168E-03	0.98
	by the $P_1(T)$-$P_2(e)$-$[P_2(T)]^2$ element with $t = -1$			
1/2	4.9788E-01	-	4.6137E-02	-
1/4	2.6553E-01	0.91	1.2682E-02	1.86
1/8	1.3652E-01	0.96	3.2617E-03	1.96
1/16	6.9147E-02	0.98	8.2268E-04	1.99
1/32	3.4787E-02	0.99	2.0628E-04	2.00
1/64	1.7446E-02	1.00	5.1624E-05	2.00
	by the $P_1(T)$-$P_2(e)$-$[P_2(T)]^2$ element with $t = 0$			
1/2	3.0484E-01	-	2.8089E-02	-
1/4	8.3716E-02	1.86	4.1194E-03	2.77
1/8	2.1912E-02	1.935	5.4997E-04	2.91
1/16	5.6075E-03	1.97	7.0790E-05	2.96
1/32	1.4186E-03	1.98	8.9699E-06	2.99
1/64	3.5678E-04	1.99	1.1285E-06	2.99
	by the $P_2(T)$-$P_3(e)$-$[P_3(T)]^2$ element with $t = -1$			
1/2	5.7839E-02	-	2.7134E-03	-
1/4	1.4873E-02	1.96	3.1672E-04	3.10
1/8	3.7604E-03	1.98	3.8384E-05	3.04
1/16	9.4492E-04	1.99	4.7369E-06	3.02
1/32	2.3681E-04	2.00	5.8883E-07	3.01
1/64	5.9275E-05	2.00	7.3418E-08	3.00
	by the $P_2(T)$-$P_3(e)$-$[P_3(T)]^2$ element with $t = 0$			
1/2	3.4661E-02	-	1.6253E-03	-
1/4	4.5488E-03	2.93	9.9213E-05	4.03
1/8	5.8124E-04	2.97	6.1650E-06	4.01
1/16	7.3436E-05	2.98	3.8556E-07	4.00
1/32	9.2285E-06	2.99	2.4132E-08	4.00
1/64	1.1566E-06	3.00	1.5098E-09	4.00

TABLE 5. Example 2: Error profiles and convergence rates.

h	$\lVert u_h - Q_h u \rVert$	rate	$\lVert u_h - Q_h u\rVert_0$	rate
	by the $P_0(T)$-$P_1(e)$-$[P_1(T)]^2$ element with $t = 1$			
1/2	6.6535E-01	-	6.5264E-02	-
1/4	1.9301E-01	1.79	2.1231E-02	1.62
1/8	5.1610E-02	1.90	5.9462E-03	1.84
1/16	1.3285E-02	1.96	1.5350E-03	1.95
1/32	3.3654E-03	1.99	3.8697E-04	1.99
1/64	8.4659E-04	1.99	9.6948E-05	2.00
	by the $P_1(T)$-$P_2(e)$-$[P_2(T)]^2$ element with $t = 1$			
1/2	1.9182E-01	-	1.7154E-02	-
1/4	2.6325E-02	2.87	1.2624E-03	3.76
1/8	3.4064E-03	2.95	8.2959E-05	3.93
1/16	4.3192E-04	2.98	5.2684E-06	3.98
1/32	5.4333E-05	2.99	3.3105E-07	3.99
1/64	6.8117E-06	3.00	2.0731E-08	4.00
	by the $P_2(T)$-$P_3(e)$-$[P_3(T)]^2$ element with $t = 1$			
1/2	2.1362E-02	-	9.8370E-04	-
1/4	1.3995E-03	3.93	3.0125E-05	5.03
1/8	8.8833E-05	3.98	9.2809E-07	5.02
1/16	5.5848E-06	3.99	2.8815E-08	5.00
1/32	3.4993E-07	4.00	8.9786E-10	5.00
1/64	2.1896E-08	4.00	2.8019E-11	5.00

6.2. Example 2. We solve problem (1) on an L-shaped domain $\Omega = [-1,1]^2 \setminus (0,1)\times(-1,0)$. The source term f and the boundary value g are chosen so that the exact solution is

$$u(x,y) = x^4 - 6x^2y^2 + y^4.$$

TABLE 6. Example 3: Error profiles and convergence rates.

h	$\|\|\|u_h - Q_h u\|\|\|$	rate	$\|u_h - Q_h u\|_0$	rate
	by the $P_0(T)$-$P_1(e)$-$[P_1(T)]^2$ element and $t = 0$			
1/2	4.6019E-00	-	2.6110E-01	-
1/4	2.4447E-00	0.91	1.5017E-01	0.80
1/8	1.1569E-00	1.08	7.8271E-02	0.94
1/16	5.6114E-01	1.04	3.9296E-02	0.99
1/32	2.7741E-01	1.02	1.9580E-02	1.00
1/64	1.3816E-01	1.01	9.7582E-03	1.00
	by the $P_1(T)$-$P_2(e)$-$[P_2(T)]^2$ element and $t = 0$			
1/2	1.2361E-00	-	6.4221E-02	-
1/4	3.1496E-01	1.97	9.2275E-03	2.80
1/8	7.8314E-02	2.01	1.2108E-03	2.93
1/16	1.9461E-02	2.01	1.5240E-04	2.99
1/32	4.8470E-03	2.01	1.9002E-05	3.00
1/64	1.2092E-03	2.00	2.3684E-06	3.00
	by the $P_2(T)$-$P_3(e)$-$[P_3(T)]^2$ element and $t = 0$			
1/2	2.0374E-01	-	6.9202E-03	-
1/4	2.5261E-02	3.01	4.2044E-04	4.04
1/8	3.0918E-03	3.03	2.4654E-05	4.09
1/16	3.8234E-04	3.02	1.4831E-06	4.06
1/32	4.7580E-05	3.01	9.1002E-08	4.03
1/64	5.9363E-06	3.00	5.6382E-09	4.01

TABLE 7. Example 3: Error profiles and convergence rates.

h	$\|\|\|u_h - Q_h u\|\|\|$	rate	$\|u_h - Q_h u\|_0$	rate
	by the $P_0(T)$-$P_1(e)$-$[P_1(T)]^2$ element and $t = 1$			
1/2	3.6905E-00	-	1.8124E-01	-
1/4	1.2522E-00	1.56	3.8705E-02	2.28
1/8	3.4963E-01	1.84	7.9855E-03	2.28
1/16	9.1765E-02	1.93	1.8254E-03	2.13
1/32	2.3474E-02	1.97	4.4350E-04	2.04
1/64	5.9345E-03	1.98	1.1001E-04	2.01
	by the $P_0(T)$-$P_1(e)$-$[P_1(T)]^2$ element and $t = 3$			
1/2	4.0971E-00	-	2.1528E-01	-
1/4	1.5111E-00	1.44	6.8234E-02	1.66
1/8	4.2724E-01	1.82	1.8089E-02	1.92
1/16	1.1206E-01	1.93	4.5826E-03	1.98
1/32	2.8610E-02	1.97	1.1492E-03	2.00
1/64	7.2229E-03	1.99	2.8750E-04	2.00
	by the $P_1(T)$-$P_2(e)$-$[P_2(T)]^2$ element and $t = 1$			
1/2	1.0060E-00	-	4.6731E-02	-
1/4	1.5657E-01	2.68	3.8281E-03	3.61
1/8	2.1198E-02	2.88	2.6792E-04	3.84
1/16	2.7308E-03	2.96	1.7407E-05	3.94
1/32	3.4551E-04	2.98	1.1018E-06	3.98
1/64	4.3414E-05	2.99	6.9158E-08	3.99
	by the $P_1(T)$-$P_2(e)$-$[P_2(T)]^2$ element and $t = 3$			
1/2	8.2701E-01	-	3.2212E-02	-
1/4	1.1317E-01	2.87	2.0076E-03	4.00
1/8	1.4802E-02	2.93	1.2823E-04	3.97
1/16	1.8915E-03	2.97	8.1255E-06	3.98
1/32	2.3899E-04	2.98	5.1102E-07	3.98
1/64	3.0033E-05	2.99	3.2026E-08	4.00
	by the $P_2(T)$-$P_3(e)$-$[P_3(T)]^2$ element and $t = 1$			
1/2	1.5821E-01	-	4.9670E-03	-
1/4	1.1597E-02	3.77	1.7351E-04	4.84
1/8	7.6057E-04	3.93	5.4703E-06	4.99
1/16	4.8301E-05	3.98	1.7043E-07	5.00
1/32	3.0372E-06	3.99	5.3162E-09	5.00
1/64	1.9030E-07	4.00	1.6600E-10	5.00
	by the $P_2(T)$-$P_3(e)$-$[P_3(T)]^2$ element and $t = 3$			
1/2	1.2357E-01	-	3.3879E-03	-
1/4	7.9071E-03	3.97	9.6460E-05	5.13
1/8	5.0474E-04	3.97	2.9395E-06	5.04
1/16	3.1882E-05	3.98	9.1166E-08	5.01
1/32	2.0027E-06	3.99	2.8413E-09	5.00
1/64	1.2546E-07	4.00	8.8686E-11	5.00

Tables 4 and 5 list errors and convergence rates in $|||\cdot|||$-norm and L^2-norm. It can be observed from Table 4 that the error between numerical solutions obtained by the WG Algorithm 1 and the L^2 projection of u, e_h, with $t = -1$ and $k \geq 1$ converge to zero at the rates of k and $k+1$ in H^1-norm and L^2-norm, respectively. On the other hand we do have one order of supercloseness in both $|||\cdot|||$-norm and L^2 norm with $t = 0$ and $k \geq 1$. If the WG method (8) is used with $t \geq 1$, e_h converges to zero at the rates of $\hat{k}+2$ in H^1-norm and $\hat{k}+3$ in L^2-norm, respectively, as can be seen from Table 5. We observe from Table 5 that the numerical performance is the same as those in Tables 3, two orders of supercloseness in both L^2-norm and $|||\cdot|||$-norm. The numerical solutions for the WG are plotted in Figure 2.

6.3. Example 3. Consider problem (1) with $\Omega = (0,1)^2$ and $a = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. The source term f and the boundary value g are chosen so that the exact solution is

$$u(x,y) = e^{\pi x}\cos(\pi y).$$

As we can see in Table 7 that the error between the numerical solution obtained by using the WG method (8) and the L^2-projection of u, e_h, with $t \geq 1$ and $k \geq 0$, is $\|Q_0 u - u_0\| = \mathcal{O}(h^{\hat{k}+3})$. If the WG method with $t = 0$ and $k \geq 1$ is used, $\|Q_0 u - u_0\| = \mathcal{O}(h^{k+2})$, as can be seen from Table 6. It can be observed from Table 7 that the error between numerical solutions obtained by the WG algorithm 1 and the L^2 projection of u, e_h, with $t \geq 1$ converge to zero at the rate of $\mathcal{O}(h^{\hat{k}+2})$ in H^1-norm. On the other hand, the corresponding rates is $\mathcal{O}(h^{k+1})$ in H^1-norm with $t = 0$ and $k \geq 0$. We can capture one and two order of supercloseness in both L^2-norm and H^1-norm by using WG algorithm 1 with $t = 0, 1, 3$ and $k \geq 1$, as can be see from Table 6 and Table 7, respectively.

References

[1] A. Al-Taweel, S. Hussain, X. Wang, and B. Jones, A P0–P0 weak Galerkin finite element method for solving singularly perturbed reaction–diffusion problems, Numerical Methods for Partial Differential Equations, 36 (2020), 213-227.

[2] A. Al-Taweel, X. Wang, The lowest-order stabilizer free Weak Galerkin Finite Element Methods, Applied Numerical Mathematics,157 (2020) 434-445.

[3] A. Al-Taweel, X. Wang, X. Ye and S. Zhang, A stabilizer free weak Galerkin finite element method with supercloseness of order two, Numerical Methods for Partial Differential Equations, (2020), 1-18.

[4] X. Hu, L. Mu and X. Ye, A weak Galerkin finite element method for the Navier-Stokes equations on polytopal meshes, J. of Computational and Applied Mathematics, 362 (2019), 614-625.

[5] R. Lin, X. Ye, S. Zhang and P. Zhu, A weak Galerkin finite element method for singularly perturbed convection-diffusion-reaction problems, SINUM, 56 (2018) 1482-1497.

[6] L. Mu, J. Wang and X. Ye, weak Galerkin finite element method for the Helmholtz equation with large wave number on polytopal meshes, IMA J. Numer. Anal. 35 (2015), 1228-1255.

[7] L. Mu, J. Wang, and X. Ye, A stable numerical algorithm for the Brinkman equations by weak Galerkin finite element methods, J. of Computational Physics, 273 (2014), 327-342.

[8] L. Mu, J. Wang and X. Ye, A weak Galerkin finite element method for biharmonic equations on polytopal meshes, Numer. Meth. Partial Diff. Eq. 30 (2014), 1003-1029.

[9] L. Mu, J. Wang and X. Ye, Weak Galerkin finite element method for second-order elliptic problems on polytopal meshes, Int. J. Numer. Anal. Model. 12 (2015) 31-53.

[10] L. Mu, J. Wang, X. Ye and S. Zhang, A weak Galerkin finite element method for the Maxwell equations, J. Sci. Comput. 65 (2015), 363-386.

[11] L. Mu, J. Wang, X. Ye and S. Zhao, A new weak Galerkin finite element method for elliptic interface problems, J. Comput. Phy. 325 (2016), 157-173.

[12] J. Wang and X. Ye, A weak Galerkin finite element method for second-order elliptic problems. J. Comput. Appl. Math. 241 (2013), 103-115.
[13] J. Wang and X. Ye, A Weak Galerkin mixed finite element method for second-order elliptic problems, Math. Comp., 83 (2014), 2101-2126.
[14] J. Wang and X. Ye, A weak Galerkin finite element method for the Stokes equations, Adv. in Comput. Math. 42 (2016), 155-174.
[15] J. Wang, X. Ye and S. Zhang, Numerical investigation on weak Galerkin finite elements, International Journal of Numerical and Analysis and Modeling. 17 (4) (2020), 517-531.
[16] X. Ye and S. Zhang, A stabilizer-free weak Galerkin finite element method on polytopal meshes, J. Comput. Appl. Math., 372 (2020), 112699, arXiv:1906.06634.
[17] X. Ye and S. Zhang, A new weak gradient for the stabilizer free weak Galerkin method with polynomial reduction, (2020), arXiv:2004.13019 .
[18] X. Ye and S. Zhang and Y.Zhu, Stabilizer-free weak Galerkin methods for monotone quasi-linear elliptic PDEs, Results in Applied Mathematics, 8 (2020), 100097.
[19] P.Zhu, and X. Wang, A Least Square Based Weak Galerkin Finite Element Method for Second Order Elliptic Equations in Non-Divergence Form, Acta Mathematica Scientia, 40 (2020) 1553-1562.

[1]Department of Mathematics and Statistics, University of Arkansas at Little Rock, Little Rock, AR, 72204, USA, [2]Department of Mathematics, University of Al-Qadisiyah, Al Diwaniyah, Iraq
E-mail: `asaltaweel@ualr.edu`

Department of Mathematics and Physics, Texas A&M International University, Laredo, TX 78041, USA
E-mail: `saqib.hussain@tamiu.edu`

Department of Mathematics and Statistics, University of Arkansas at Little Rock, Little Rock, AR, 72204, USA
E-mail: `xxwang@ualr.edu`