INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 19, Number 5, Pages 685–708

A LEAST-SQUARES STABILIZATION VIRTUAL ELEMENT METHOD FOR THE STOKES PROBLEM ON POLYGONAL MESHES

YANG LI, CHAOLANG HU AND MINFU FENG*

Abstract. This paper studies the virtual element method for Stokes problem with a least-squares type stabilization. The method cannot only circumvent the Babuška-Brezzi condition, but also make use of general polygonal meshes, as opposed to more standard triangular grids. Moreover, it is suitable for arbitrary combinations of the velocity and pressure, including equal-order virtual element. We obtain the corresponding energy norm error estimates and L^2 norm error estimates for velocity. Finally, a series of numerical experiments are performed to verify the method has good behaviors.

Key words. Virtual element method, Stokes problem, Least-squares stabilization.

1. Introduction

Recently, there has been increasing interest in developing numerical methods that can make use of non-traditional meshes (e.g., polygons and polyhedra). Indeed, the use of polygonal meshes brings a number of advantages, including more efficient approximation of geometric data features, better domain meshing capabilities, more robustness to mesh deformation, and others. Several numerical methods for polygon meshes have been proposed, such as Hybrid High-Order method [1], Mimetic Finite Difference(MFD) method [2], Weak Galerkin method [3–5], Virtual Element method(VEM) [6].

Among them, the VEM follows the main ideas of MFD method [7–11]. By avoiding the explicit expression of basis function and the use of quadrature formula, it can handle the problem on general polygonal meshes. Indeed, virtual element space contains space of polynomials, with addition of suitable non-polynomial functions. The polynomial part is used to insure the consistency of the scheme, while the non-polynomial part is used to insure the stability, see [6, 12] for details. So far, the VEM has been used to solve various problems, such as elliptic [13] and parabolic equations [14], linear elastodynamics [15, 16], Cahn-Hilliard equation [17], Stokes [18–28] and Navier-Stokes(N-S) equation [29]. In addition, several different numerical methods have been analysed, such as H(div) and H(curl)-conforming VEM [30], discontinuous Galerkin VEM [31], nonconforming VEM [32, 33], mixed VEM [34].

All these mixed virtual element methods are constructed in such a way that only the velocity is approximated by a virtual element space while the pressure discretization is based on a traditional finite element space. The basis of their analysis is that the Babuška-Brezzi(B-B) condition must be satisfied, which is not easy in some cases. In order to deal with this problem, how to combine VEM

Received by the editors November 4, 2021 and, in revised form, June 12, 2022; accepted June 13, 2022.

2000 *Mathematics Subject Classification.* 65N12, 65N30.

*Corresponding author: fmf@scu.edu.cn.

with stabilization method [35, 36] is an interesting problem. Recently, a stabilized VEM for N-S problem has been proposed in [37], although numerical tests show this method is stable, there is no theoretical analysis. In [38, 39], the authors proposed a SUPG-stabilized conforming/nonconforming VEM for convection-diffusion-reaction problem. Moreover, a projection-based stabilized VEM for the Stokes problem has been considered in [40], which constitutes a low-cost solver. Nevertheless, it's only weakly consistent.

As we know least-squares technique has been frequently used to simulate solutions of partial differential equations. Its numerical stability is not sensitive to the choice of spaces or meshes. In addition, it's not subject to the B-B condition and leads to a symmetric and positive definite algebraic system. We believe the attractive characteristics of the VEM and least-squares method should remain if both of them are combined. This motivates us to consider the least-squares stabilization VEM for solving the Stokes problem. The approximation of mixed formulations is suitable for arbitrary finite element combinations of the two variables. That is, the method is applicative when velocity is approximated by virtual element, regardless of pressure is approximated by virtual element or traditional finite element. In particular, when both velocity and pressure are approximated by classical finite element pairs, the method becomes that proposed in [36] by Franca et al. We establish the stability and corresponding error estimates, including the energy norm and the L^2 norm for velocity. And several numerical tests confirm the theoretical convergence results.

The structure of the paper is as follows. Section 2 states the Stokes problem and its weak formulation. In Section 3, we outline the virtual element discretization of the problem. Section 4 considers the stability of the proposed method. The error estimates in the energy norm and the L^2 norm for velocity are investigated in Section 5. Section 6 gives some numerical tests which confirm the theoretical results. Finally, we present short conclusions.

Notation Throughout the paper C will denote a common constant, that is independent of the mesh size h. We use the standard notations for Sobolev space, norm, and semi-norm. More specifically, given bounded Lipschitz domain $D \subset \mathbb{R}^d$, we define by $W^{k,p}(D)$ the space of all L^p integrable functions in D whose weak derivatives less than or equal to order k are also L^p integrable. For $p = 2$, we denote $H^k(D) := W^{k,2}(D)$, and we utilize $\|\cdot\|_{k,D}$ and $|\cdot|_{k,D}$ to represent the corresponding norm and semi-norm, respectively. In the Sobolev space $H^k(D)$, while $(\cdot,\cdot)_{k,D}$ denote the corresponding inner product, and the standard $L^2(D)$ inner product is denoted by $(\cdot,\cdot)_D$ with corresponding norm $\|\cdot\|_{0,D}$.

2. The continuous problem

Let $\Omega \subset \mathbb{R}^2$ be a convex, bounded polygonal domain with homogeneous Dirichlet boundary condition. We consider the Stokes problem:

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\mathbf{u}$ and p denotes the velocity and pressure fields, respectively. And $\mathbf{f}$ denotes the external force, ν is a constant that represents the viscosity.

We define the velocity vector space $\mathbf{V}$ and the pressure scalar space Q by

$$\mathbf{V} := \mathbf{H}_0^1(\Omega), \quad Q := \left\{ q \in L^2(\Omega) \cap H^1(\Omega) : \int_\Omega q \, d\,\Omega = 0 \right\},$$

where we set $\mathbf{H}^s(\Omega) := (H^s(\Omega))^d$. We usually use bold fonts to indicate vector variables, operators, and spaces throughout this paper.

A weak formulation of (1) reads as follows: find $(\mathbf{u}, p) \in \mathbf{V} \times Q$, s.t.

$$B(\mathbf{u}, p; \mathbf{v}, q) = F(\mathbf{v}, q), \ \forall \ (\mathbf{v}, p) \in \mathbf{V} \times Q, \tag{2}$$

where we defined the operators

$$\begin{aligned} &B(\mathbf{u}, p; \mathbf{v}, q) := -\nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + (p, \nabla \cdot \mathbf{v}) + (q, \nabla \cdot \mathbf{u}), \\ &F(\mathbf{v}, q) := (-\mathbf{f}, \mathbf{v}). \end{aligned}$$

Obviously, the bilinear form $\nu(\nabla \mathbf{u}, \nabla \mathbf{v})$ is coercive and continuous on $\mathbf{V} \times \mathbf{V}$, i.e.,

$$\nu(\nabla \mathbf{v}, \nabla \mathbf{v}) \geq \nu \|\mathbf{v}\|_1^2 \text{ and } |\nu(\nabla \mathbf{u}, \nabla \mathbf{v})| \leq C \|\mathbf{u}\|_1 \|\mathbf{v}\|_1, \quad \forall \ \mathbf{u}, \mathbf{v} \in \mathbf{V},$$

the bilinear form $(q, \nabla \cdot \mathbf{v})$ is continuous on $\mathbf{V} \times Q$ and satisfies the inf-sup condition,

$$|(q, \nabla \cdot \mathbf{v})| \leq C \|\mathbf{v}\|_1 \|q\|_0 \text{ and } \sup_{\mathbf{v} \neq \mathbf{0}, \mathbf{v} \in \mathbf{V}} \frac{(q, \nabla \cdot \mathbf{v})}{\|\mathbf{v}\|_1} \geq \beta \|q\|_0 \quad \forall \ \mathbf{v} \in \mathbf{V}, \ q \in Q.$$

Therefore, the problem (2) has a unique solution $(\mathbf{u}, p) \in \mathbf{V} \times Q$; see [41] for the details.

We now introduce on the product space $\mathbf{V} \times Q$ the bilinear form given by

$$B_{CBB}(\mathbf{u}, p; \mathbf{v}, q) := a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) + b(q, \mathbf{u}) + c(\mathbf{u}, \mathbf{v}) + d(p, q) + e(\mathbf{u}, \mathbf{v}), \tag{3}$$

where the bilinear forms are defined as

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) := -\nu(\nabla \mathbf{u}, \nabla \mathbf{v}), \\ b(p, \mathbf{v}) := (p, \nabla \cdot \mathbf{v}) - \nu \tau h^2 (\nabla p, \Delta \mathbf{v}), \\ c(\mathbf{u}, \mathbf{v}) := -\delta(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}), \\ d(p, q) := \tau h^2 (\nabla p, \nabla q), \\ e(\mathbf{u}, \mathbf{v}) := \nu^2 \tau h^2 (\Delta \mathbf{u}, \Delta \mathbf{v}), \end{cases} \tag{4}$$

and

$$F_{CBB}(\mathbf{v}, q) := F(\mathbf{v}, q) + \tau h^2 (-\mathbf{f}, \nu \Delta \mathbf{v} - \nabla q). \tag{5}$$

The weak formulation (2) can be equivalent rewritten as: find $(\mathbf{u}, p) \in \mathbf{V} \times Q$, such that,

$$B_{CBB}(\mathbf{u}, p; \mathbf{v}, q) = F_{CBB}(\mathbf{v}, q) \quad \forall \ (\mathbf{v}, p) \in \mathbf{V} \times Q. \tag{6}$$

Theorem 2.1. *There exists a positive constant C_1 such that for all $(\mathbf{u}, p) \in \mathbf{V} \times Q$,*

$$B_{CBB}(\mathbf{u}, p; -\mathbf{u}, p) \geqslant C_1 \|\nabla \mathbf{u}\|_0^2 + \tau \sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla p\|_{0,K}^2.$$

One can refer to [36, 42] for details. The continuity of B_{CBB} and F_{CBB} are straightforward by using the Cauchy-Schwarz inequality. The problem (6) therefore has a unique solution.

3. The virtual element method

In this section, we discuss the stabilized virtual element discretization of (6). Let $\{\mathcal{T}_h\}_h$ be a sequence of decompositions of Ω into general polygonal elements K. By $\mathcal{E}_h$ and $\mathcal{V}_h$, we denote the set of all edges e and vertices V of the decomposition $\mathcal{T}_h$, respectively. For every element K and for every edge e, we set $h_K := \text{diameter}(K)$, $h_e := |e|$. To approximate the solution of (6), we give the following mesh regularity assumptions [12]:

Assumption 3.1. *We suppose that for all h, each element K satisfies the following assumptions:*
- *every element K is star-shaped with respect to all the points of a sphere of radius $\geq \varrho h_K$,*
- *the distance between any two vertexes of K is $\geq c h_K$,*

where ϱ and c are positive constants.

Since we will mostly work on a generic element K, we will denote by $a^K(\cdot,\cdot)$, $b^K(\cdot,\cdot)$, $c^K(\cdot,\cdot)$, $d^K(\cdot,\cdot)$, $e^K(\cdot,\cdot)$ the restriction to K of the corresponding bilinear forms defined in (4). First of all, we decompose into local contributions the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, $c(\cdot,\cdot)$, $d(\cdot,\cdot)$, $e(\cdot,\cdot)$ and the norms $\|\cdot\|_{\mathbf{V}}$, $\|\cdot\|_Q$ by defining, for all $\mathbf{u},\mathbf{v} \in \mathbf{V}, p, q \in Q$

(7)
$$a(\mathbf{u},\mathbf{v}) =: \sum_{K\in\mathcal{T}_h} a^K(\mathbf{u},\mathbf{v}), \quad b(p,\mathbf{v}) =: \sum_{K\in\mathcal{T}_h} b^K(p,\mathbf{v}), \quad c(\mathbf{u},\mathbf{v}) =: \sum_{K\in\mathcal{T}_h} c^K(\mathbf{u},\mathbf{v}),$$
$$d(p,q) =: \sum_{K\in\mathcal{T}_h} d^K(p,q), \quad e(\mathbf{u},\mathbf{v}) =: \sum_{K\in\mathcal{T}_h} e^K(\mathbf{u},\mathbf{v}),$$

and

$$\|\mathbf{v}\|_{\mathbf{V}} =: \left(\sum_{K\in\mathcal{T}_h} \|\mathbf{v}\|^2_{\mathbf{V},K}\right)^{1/2}, \quad \|q\|_Q =: \left(\sum_{K\in\mathcal{T}_h} \|q\|^2_{Q,K}\right)^{1/2}. \tag{8}$$

3.1. The virtual element discretization. Now, we introduce the local discrete virtual element spaces corresponding to $\mathbf{V}$ and Q, a set of local projectors mapping from these virtual element spaces into polynomial spaces, and finally, the related global counterparts. First of all, using standard VEM notations, for $k \in \mathbb{N}$, let us define the spaces
- $\mathbb{P}_k(K)$: the set of polynomials on K of degree $\leq k$,
- $\mathbb{B}_k(\partial K) := \{\omega \in C^0(\partial K) \text{ s.t. } \omega_{|e} \in \mathbb{P}_k(e), \quad \forall\, e \subset \partial K\}$,

and concerning geometric objects (and related items), we will use the following notations. For a geometric object D of dimension $d (d = 1, 2)$, we will denote by x_D its barycenter. Then we denote by $\mathcal{M}_k(D)$ the set of polynomials

$$\mathcal{M}_k(D) := \bigcup_{i=0}^{k} \mathcal{M}_i^*(D) \subseteq \mathbb{P}_k(D),$$

where $\mathcal{M}_i^*(D) := \left\{\left(\frac{x-x_D}{h_D}\right)^s, |s| = i\right\}$, and $s \in \mathbb{N}^d$ is a multi-index with $|s| = s_1 + ... + s_d$ and $x^s := x_1^{s_1} ... x_d^{s_d}$.

Then, we introduce on each element K the original local virtual element space [6]:

$$W_h(K) = \{\omega_h \in H^1(K) : \omega_{h|\partial K} \in \mathbb{B}_k(\partial K) \text{ and } \Delta\omega_h \in \mathbb{P}_{k-2}(K), \forall K \in \mathcal{T}_h\}. \tag{9}$$

The corresponding degrees of freedom are chosen, always at the element level, prescribing, for each $\omega_h \in W_h(K)$,
(**D1**) the values of ω_h at the vertices,
(**D2**) the values of ω_h at $k-1$ same interval points on e of K,
(**D3**) the moments $\frac{1}{|K|}(m, \omega_h)_{0,K}$, $\forall\, m \in \mathcal{M}_{k-2}(K)$.

Similar to [12], for the L^2-projection can be exactly computable by the DoFs, we enlarge the previous space as follows:

$$\widetilde{W}_h^k(K) = \{\omega_h \in H^1(K) : \omega_{h|\partial K} \in \mathbb{B}_k(\partial K) \text{ and } \Delta\omega_h \in \mathbb{P}_k(K), \forall K \in \mathcal{T}_h\}, \tag{10}$$

Remark 3.1. *([12, 13]) For each element K and for all k the operators **D1-D3** satisfy the following property:*

$$\{q \in \mathbb{P}_k(K)\} \text{ and } \{\boldsymbol{Di}(q) = 0, i = 1, 2, 3\} \text{ imply } \{q = 0\},$$

*it implies that on each element K we can easily construct a projection operator from $\widetilde{W}_h^k(K)$ to $\mathbb{P}_k(K)$ that depends only on **D1-D3** and is explicitly computable starting from them, see [12, 13] for details.*

For any $n \in \mathbb{N}$ and $K \in \mathcal{T}_h$, we introduce some useful polynomial projections [12]:
• the H^1 semi-norm projector $\Pi_{k,K}^{\nabla} : H^1(K) \to \mathbb{P}_k(K)$,

$$\begin{cases} \left(\nabla(\Pi_k^{\nabla,K}\omega_h - \omega_h), \nabla m\right)_K = 0, \ \forall\, m \in \mathcal{M}_k(K) \\ P_0^K\left(\Pi_k^{\nabla,K}\omega_h - \omega_h\right) = 0, \end{cases} \tag{11}$$

where let N be the number of vertices V_i of the element K and the projection operator $P_0^K : \widetilde{W}_h^k(K) \to \mathbb{P}_0(K)$ satisfies

$$\begin{cases} P_0^K\omega_h := \frac{1}{N}\sum_{i=1}^N \omega_h(V_i) & \text{if } k = 1, \\ P_0^K\omega_h := \frac{1}{|K|}\int_K \omega_h \, \mathrm{d}\mathbf{x} & \text{if } k \geq 2. \end{cases}$$

• The L^2 -projection $\Pi_k^{0,K} : L^2(K) \to \mathbb{P}_k(K)$ defined by

$$(\Pi_k^{0,K}\omega_h, m)_K = (\omega_h, m)_K \quad \forall\, m \in \mathcal{M}_k(K). \tag{12}$$

According to references [6,12,13], we point out that the operators $\Pi_{k,K}^{\nabla}$, $\Pi_k^{0,K}$ and $\Pi_{k-1}^{0,K}$ can be computed using only the degrees of freedom.

For simplicity, we still use the same symbol for their extension for vector and tensor functions.

Now we define the virtual element space $V_h^k(K)$ as the restriction of $\widetilde{W}_h^k(K)$ given by [12]

$$V_h^k(K) := \left\{\omega_h \in \widetilde{W}_h^k(K) : (q_k, \omega_h - \Pi_k^{\nabla,K}\omega_h)_K = 0, \forall q_k \in \mathcal{M}_{k-1}^*(K) \cup \mathcal{M}_k^*(K)\right\}, \tag{13}$$

and see again [12,13] that the set of linear operators **D1-D3** are a set of degrees of freedom for the virtual space $V_h(K)$. Then by imposing boundary conditions, we obtain the global spaces

$$\begin{aligned} \mathbf{V}_h &:= \{\mathbf{v}_h \in \mathbf{V} : \mathbf{v}_{h|K} \in \mathbf{V}_h^k(K), \ \forall\, K \in \mathcal{T}_h\}, \\ Q_h &:= \{q_h \in Q \cap H^1(\Omega) : q_{h|K} \in V_h^l(K), \ \forall\, K \in \mathcal{T}_h, \ 1 \leq l \leq k\}. \end{aligned} \tag{14}$$

Now, we define the local discrete version for (3)-(6) as follows: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ s.t.

$$B_{CBB,h}(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h) = F_{CBB,h}(\mathbf{v}_h, q_h), \qquad \forall\, (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h, \tag{15}$$

where

(16)
$$B_{CBB,h} := a_h(\mathbf{u}_h,\mathbf{v}_h)+b_h(p_h,\mathbf{v}_h)+b_h(q_h,\mathbf{u}_h)+c_h(\mathbf{u}_h,\mathbf{v}_h)+d_h(p_h,q_h)+e_h(\mathbf{u}_h,\mathbf{v}_h),$$

(17)
$$F_{CBB,h} := -\sum_{K\in\mathcal{T}_h}(\mathbf{f},\Pi^0_{k,K}\mathbf{v}_h)_K - \sum_{K\in\mathcal{T}_h}\tau h_K^2(\mathbf{f},\,\nu\Delta\Pi^0_{k,K}\mathbf{v}_h-\Pi^0_{k-1,K}\nabla q_h)_K,$$

with

(18)
$$\begin{cases} a_h(\mathbf{u}_h,\mathbf{v}_h) := -\sum\limits_{K\in\mathcal{T}_h}\Big[\nu(\Pi^0_{k-1,K}\nabla\mathbf{u}_h,\Pi^0_{k-1,K}\nabla\mathbf{v}_h)_K \\ \qquad\qquad +S_a^K((I-\Pi^\nabla_{k,K})\mathbf{u}_h,(I-\Pi^\nabla_{k,K})\mathbf{v}_h)\Big], \\ b_h(p_h,\mathbf{v}_h) := \sum\limits_{K\in\mathcal{T}_h}\big[(\Pi^0_{k,K}p_h,\Pi^0_{k-1,K}\nabla\cdot\mathbf{v}_h)_K - \nu\tau h_K^2(\Pi^0_{k-1,K}\nabla p_h,\Delta\Pi^0_{k,K}\mathbf{v}_h)_K\big], \\ c_h(\mathbf{u}_h,\mathbf{v}_h) := -\delta\sum\limits_{K\in\mathcal{T}_h}(\Pi^0_{k-1,K}\nabla\cdot\mathbf{u}_h,\Pi^0_{k-1,K}\nabla\cdot\mathbf{v}_h)_K, \\ d_h(p_h,q_h) := \sum\limits_{K\in\mathcal{T}_h}\big[\tau h_K^2(\Pi^0_{k-1,K}\nabla p_h,\Pi^0_{k-1,K}\nabla q_h)_K \\ \qquad\qquad +S_d^K((I-\Pi^\nabla_{k,K})p_h,(I-\Pi^\nabla_{k,K})q_h)\big], \\ e_h(\mathbf{u}_h,\mathbf{v}_h) := \sum\limits_{K\in\mathcal{T}_h}\nu^2\tau h_K^2(\Delta\Pi^0_{k,K}\mathbf{u}_h,\Delta\Pi^0_{k,K}\mathbf{v}_h)_K, \end{cases}$$

where $S_a^K(\cdot,\cdot):\mathbf{V}_h(K)\times\mathbf{V}_h(K)\to\mathbb{R}$ and $S_d^K(\cdot,\cdot):Q_h(K)\times Q_h(K)\to\mathbb{R}$ are any symmetric, positive definite and computable bilinear form that guarantees [1]

$$\alpha_* a^K(\mathbf{v}_h,\mathbf{v}_h)\le S_a^K(\mathbf{v}_h,\mathbf{v}_h)\le\alpha^* a^K(\mathbf{v}_h,\mathbf{v}_h)\quad\forall\,\mathbf{v}_h\in\ker(\Pi_k^{\nabla,K}),\tag{19}$$

$$\beta_* d^K(q_h,q_h)\le S_d^K(q_h,q_h)\le\beta^* d^K(q_h,q_h)\quad\forall\,q_h\in\ker(\Pi_k^{\nabla,K}),\tag{20}$$

for constants $0<\alpha_*\le\alpha^*$ and $0<\beta_*\le\beta^*$. It is straightforward to check that definition (11) and properties (19,20) imply

•**Consistency:** for all $h>0$ and for all $K\in\mathcal{T}_h$, it holds that

$$a_h^K(\mathbf{m},\mathbf{v}_h)=a^K(\mathbf{m},\mathbf{v}_h)\quad\forall\,\mathbf{m}\in(\mathbb{P}_k(K))^d,\mathbf{v}_h\in\mathbf{V}_h(K),\tag{21}$$

$$d_h^K(m,q_h)=d^K(m,q_h)\quad\forall\,m\in\mathbb{P}_k(K),q_h\in Q_h(K).\tag{22}$$

•**Stability:** there exist some positive constants α_*, α^* and β_*, β^*, independent of h, such that

$$\alpha_* a^K(\mathbf{v}_h,\mathbf{v}_h)\le a_h^K(\mathbf{v}_h,\mathbf{v}_h)\le\alpha^* a^K(\mathbf{v}_h,\mathbf{v}_h)\quad\forall\,\mathbf{v}_h\in\mathbf{V}_h(K),\tag{23}$$

$$\beta_* d^K(q_h,q_h)\le d_h^K(q_h,q_h)\le\beta^* d^K(q_h,q_h)\quad\forall\,q_h\in Q_h(K),\tag{24}$$

in addition, due to the symmetry of $S_a^K(\cdot,\cdot)$, $S_d^K(\cdot,\cdot)$ and (11), it follows that

(25)
$$S_a^K(\mathbf{u}_h,\mathbf{v}_h)\le(S_a^K(\mathbf{u}_h,\mathbf{u}_h))^{\frac12}\left(S_a^K(\mathbf{v}_h,\mathbf{v}_h)\right)^{\frac12}\le\alpha^*(a^K(\mathbf{u}_h,\mathbf{u}_h))^{\frac12}(a^K(\mathbf{v}_h,\mathbf{v}_h))^{\frac12},$$

$$S_d^K(p_h,q_h)\le(S_d^K(p_h,p_h))^{\frac12}\left(S_d^K(q_h,q_h)\right)^{\frac12}\le\beta^*(d^K(p_h,p_h))^{\frac12}(d^K(q_h,q_h))^{\frac12}.\tag{26}$$

Remark 3.2. *Conditions (19-20) essentially state that the stabilizing term $S_\dagger^K(\cdot,\cdot)$, $\dagger=a,d$ scales as $\dagger^K(\cdot,\cdot)$ on the kernel of $\Pi_k^{\nabla,K}$. For instance, following the most standard VEM choice (cf. [6, 12]), the choice of the bilinear form $S_\dagger^K$ would depend on* *the problem and on the degrees of freedom. It will be sufficient to take the simple choice*

$$S_a^K(\cdot,\cdot) =: \nu h_K^{d-2} \sum_{i=1}^{N_{dof}} \chi_i(\cdot)\chi_i(\cdot), \quad S_d^K(\cdot,\cdot) =: \tau h_K^2 \cdot h_K^{d-2} \sum_{i=1}^{N_{dof}} \chi_i(\cdot)\chi_i(\cdot),$$

where $\chi_i(\cdot)$ is the operator that selects the i-th degrees of freedom.

3.2. Preliminary results. We now present some preliminary results useful in the sequel. We denote by $\widetilde{D}$ the patch elements intersecting D. Then, we start by the following approximation Lemma, that mainly comes from the Assumption 3.1 and standard approximation results on polygonal domains (see for instance [6, 40, 43]).

Lemma 3.1. *Suppose that Assumption 3.1 is satisfied. For all $\omega \in H^s(\Omega)$ and $K \in \mathcal{T}_h$, there exists the polynomial functions $\Pi_k^{0,K}\omega, \Pi_k^{\nabla,K}\omega, \omega_\pi \in \mathbb{P}_k(K)$ holds*

$$\|\omega - \Pi_k^{0,K}\omega\|_{m,K} \le Ch_K^{s-m}|\omega|_{s,K}, \quad m, s \in \mathbb{N}, m \le s \le k+1,$$

$$\|\omega - \Pi_k^{\nabla,K}\omega\|_{m,K} \le Ch_K^{s-m}|\omega|_{s,K}, \quad m, s \in \mathbb{N}, m \le s \le k+1, s \ge 1,$$

$$\|\omega - \omega_\pi\|_{0,K} + h_K|\omega - \omega_\pi|_{1,K} \le Ch_K^{s}|\omega|_{s,K}, \quad s \in \mathbb{N}, 1 \le s \le k+1.$$

Lemma 3.2. *[40, 43] Suppose that Assumption 3.1 is satisfied. For $\omega \in H_0^1(\Omega) \cap H^s(\Omega)$ with $1 \le s \le k+1$, there exists a $\omega_I \in H_0^1(\Omega) \cap V_h$ such that for all $K \in \mathcal{T}_h$, we have*

$$\|\omega - \omega_I\|_{m,K} \le C(\varrho, k)h_K^{s-m}|\omega|_{s,\widetilde{K}} \quad m = 0, 1.$$

Lemma 3.3. *[43, 44] Suppose that Assumption 3.1 is satisfied. Let $K \in \mathcal{T}_h$ and let $v_h \in V_h^K$ be such that $\Delta v_h \in P_k(K)$. There exists a constant C_{inv}, independent of v_h, h and K, such that (Lemma 10 in [43])*

$$C_{inv}\|\Delta v_h\|_{0,K} \le h_K^{-1}\|\nabla v_h\|_{0,K}, \tag{27}$$

and the following inverse inequality holds: (Theorem 3.6 in [44])

$$\|\nabla v_h\|_{0,K} \lesssim h_K^{-1}\|v_h\|_{0,K}. \tag{28}$$

4. Stability

Now, let us define the following norm [36]:

$$|||(\mathbf{u}, p)|||^2 := \nu\|\nabla \mathbf{u}\|_0^2 + \|p\|_0^2 + \tau h^2\|\nabla p\|_0^2 + \delta\|\nabla \cdot \mathbf{u}\|_0^2, \tag{29}$$

and to facilitate the presentation of the following analysis, we set

$$\begin{aligned} \hat{b}_h(\mathbf{v}_h, p_h) &:= \sum_{K\in\mathcal{T}_h} (\Pi_{k,K}^0 p_h, \Pi_{k-1,K}^0 \nabla \cdot \mathbf{v}_h)_K, \\ \tilde{b}_h(\mathbf{v}_h, p_h) &:= -\sum_{K\in\mathcal{T}_h} \nu\tau h_K^2(\Pi_{k-1,K}^0 \nabla p_h, \Delta\Pi_{k,K}^0 \mathbf{v}_h)_K, \end{aligned} \tag{30}$$

i.e.

$$b_h(\mathbf{v}_h, p_h) = \hat{b}_h(\mathbf{v}_h, p_h) + \tilde{b}_h(\mathbf{v}_h, p_h).$$

It is immediate to see that for all $(\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$, there holds

$$\begin{aligned} &|B_{CBB,h}(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h)| \le C|||(\mathbf{u}_h, p_h)|||\,|||(\mathbf{v}_h, q_h)|||, \\ &|F_{CBB,h}(\mathbf{v}_h; q_h)| \le C|||(\mathbf{v}_h, q_h)|||. \end{aligned} \tag{31}$$

4.1. The weak inf-sup condition.

Theorem 4.1. *(weak inf-sup condition) Assume that Assumption 3.1 is satisfied. Then, there exist positive constants β_1 and β_2 such that*

$$\sup_{\mathbf{v}_h\in\mathbf{V}_h\ \mathbf{v}_h\neq\mathbf{0}}\frac{\hat{b}_h(p_h,\mathbf{v}_h)}{\|\mathbf{v}_h\|_1}\geq\beta_1\|p_h\|_0-\beta_2\left(\sum_{K\in\mathcal{T}_h}h_K^2\|\nabla p_h\|_{0,K}^2\right)^{1/2}\qquad\forall p_h\in Q_h.$$

Proof. Firstly, by the surjectivity of the divergence operator, there exists a $\mathbf{w}\in\mathbf{H}_0^1(\Omega)$ such that

$$(\nabla\cdot\mathbf{w},p_h)\geq\beta_1\|\mathbf{w}\|_1\,\|p_h\|_0\qquad\forall p_h\in Q_h.$$

Let $\mathbf{w}_I$ be the interpolant of $\mathbf{w}$ in $\mathbf{V}_h$, and we can easily get $\|\mathbf{w}_I\|_1\leq C\|\mathbf{w}\|_1$. Using the definition of the projection 12 many times, we have

$$\begin{aligned}\sup_{\mathbf{v}_h\neq\mathbf{0}\in\mathbf{V}_h}\frac{\hat{b}_h(\mathbf{v}_h,p_h)}{\|\mathbf{v}_h\|_1}&=\sup_{\mathbf{v}_h\neq\mathbf{0}\in\mathbf{V}_h}\frac{\sum\limits_{K\in\mathcal{T}_h}(\Pi_{k-1}^{0,K}\nabla\cdot\mathbf{v}_h,\Pi_k^{0,K}p_h)_K}{\|\mathbf{v}_h\|_1}\\&\geq\frac{\sum\limits_{K\in\mathcal{T}_h}(\Pi_{k-1}^{0}\nabla\cdot\mathbf{w}_I,\Pi_k^{0}p_h)_K}{\|\mathbf{w}_I\|_1}=\frac{\sum\limits_{K\in\mathcal{T}_h}(\nabla\cdot\mathbf{w}_I,\Pi_{k-1}^{0}p_h)_K}{\|\mathbf{w}_I\|_1}\\&\geq\frac{\sum\limits_{K\in\mathcal{T}_h}(\nabla\cdot\mathbf{w}_I,\Pi_{k-1}^{0}p_h)_K}{C\|\mathbf{w}\|_1},\end{aligned}$$

then, using triangle inequality, integration by parts and Lemma 3.2, we obtain

$$\begin{aligned}&\sup_{\mathbf{v}_h\neq\mathbf{0}\in\mathbf{V}_h}\frac{\hat{b}_h\left(\mathbf{v}_h,p_h\right)}{\|\mathbf{v}_h\|_1}\\&\geq-\frac{\left|\sum\limits_{K\in\mathcal{T}_h}\left(\nabla\cdot\mathbf{w}_I,p_h-\Pi_{k-1}^0p_h\right)_K\right|}{C\|\mathbf{w}\|_1}-\frac{|(\nabla\cdot(\mathbf{w}-\mathbf{w}_I),p_h)|}{C\|\mathbf{w}\|_1}+\frac{(\nabla\cdot\mathbf{w},p_h)}{C\|\mathbf{w}\|_1}\\&\geq-C\sum_{K\in\mathcal{T}_h}\left\|p_h-\Pi_{k-1}^0p_h\right\|_{0,K}-Ch\left\|\nabla p_h\right\|_0+\beta_1\left\|p_h\right\|_0\\&\geq\beta_1\left\|p_h\right\|_0-\beta_2\left(\sum_{K\in\mathcal{T}_h}h_K^2\|\nabla p_h\|_{0,K}^2\right)^{1/2}.\end{aligned}$$

□

4.2. Well-posedness of discrete formulation.

Theorem 4.2. *Assume that Assumption 3.1 is satisfied. For each $(\mathbf{u}_h,p_h)\in\mathbf{V}_h\times Q_h$, there is a positive constant $\hat{\beta}$, such that*

$$\inf_{(\mathbf{u}_h,p_h)\in\mathbf{V}_h\times Q_h}\ \sup_{(\mathbf{w}_h,r_h)\in\mathbf{V}_h\times Q_h}\frac{B_{CBB,h}(\mathbf{u}_h,p_h;\mathbf{w}_h,r_h)}{|||(\mathbf{u}_h,p_h)|||\cdot|||(\mathbf{w}_h,r_h)|||}\geq\hat{\beta}>0.\tag{32}$$

Proof. Let us consider an arbitrary $(\mathbf{u}_h,p_h)\in\mathbf{V}_h\times Q_h$ and let $X^2:=\nu\|\nabla\mathbf{u}_h\|_0^2+\tau h^2\|\nabla p_h\|_0^2+\delta\|\nabla\cdot\mathbf{u}_h\|_0^2$.

Firstly, we choose $(\mathbf{w}_h, r_h) = (-\mathbf{u}_h, p_h)$. Using (19-20) and Lemma 3.3, we have

$$
\begin{aligned}
&B_{CBB,h}(\mathbf{u}_h, p_h; -\mathbf{u}_h, p_h) \\
&\geq \sum_{K\in\mathcal{T}_h} \Big[\nu \|\Pi^0_{k-1,K}\nabla\mathbf{u}_h\|^2_{0,K} + S^K_a\left((I-\Pi^\nabla_{k,K})\mathbf{u}_h, (I-\Pi^\nabla_{k,K})\mathbf{u}_h\right) \\
&\qquad + \tau h_K^2 \|\Pi^0_{k-1,K}\nabla p_h\|^2_{0,K} + S^K_d\left((I-\Pi^\nabla_{k,K})p_h, (I-\Pi^\nabla_{k,K})p_h\right) \\
&\qquad - \nu^2\tau h_K^2 \|\Delta\Pi^0_{k,K}\mathbf{u}_h\|^2_{0,K}\Big] \\
&\geq \sum_{K\in\mathcal{T}_h} \Big[\min\{1,\alpha_*\}\nu\left(\|\Pi^0_{k-1,K}\nabla\mathbf{u}_h\|^2_{0,K} + \|\nabla\mathbf{u}_h - \Pi^0_{k-1,K}\nabla\mathbf{u}_h\|^2_{0,K}\right) \\
&\qquad + \min\{1,\beta_*\}\tau h_K^2\left(\|\Pi^0_{k-1,K}\nabla p_h\|^2_{0,K} + \|\nabla p_h - \Pi^0_{k-1,K}\nabla p_h\|^2_{0,K}\right) \\
&\qquad - C\nu^2\tau\|\nabla\mathbf{u}_h\|^2_{0,K}\Big] \\
&\geq \sum_{K\in\mathcal{T}_h} \left[\min\{1,\alpha_*\}\frac{\nu}{2}\|\nabla\mathbf{u}_h\|^2_{0,K} + \min\{1,\beta_*\}\frac{\tau h_K^2}{2}\|\nabla p_h\|^2_{0,K} - C\nu^2\tau\|\nabla\mathbf{u}_h\|^2_{0,K}\right],
\end{aligned}
$$

and taking $0 < \tau \leq \frac{\min\{1,\alpha_*\}}{4C\nu}$, we have

$$
\begin{aligned}
&B_{CBB,h}(\mathbf{u}_h, p_h; -\mathbf{u}_h, p_h) \\
&\geq \sum_{K\in\mathcal{T}_h} \left[\min\{1,\alpha_*\}\frac{\nu}{4}\|\nabla\mathbf{u}_h\|^2_{0,K} + \min\{1,\beta_*\}\frac{\tau h_K^2}{2}\|\nabla p_h\|^2_{0,K}\right],
\end{aligned}
$$

then using $\|\nabla\cdot\mathbf{u}_h\|_{0,K} \leq \tilde{C}\|\nabla\mathbf{u}_h\|_{0,K}$ and taking $0 < \delta \leq \frac{\nu}{8\tilde{C}}$, we get

$$
\begin{aligned}
&B_{CBB,h}(\mathbf{u}_h, p_h; -\mathbf{u}_h, p_h) \\
&\geq \sum_{K\in\mathcal{T}_h} \min\left\{\frac{1}{8}, \frac{\alpha_*}{8}, \frac{\beta_*}{2}\right\}\left(\nu\|\nabla\mathbf{u}_h\|^2_{0,K} + \delta\|\nabla\cdot\mathbf{u}_h\|^2_{0,K} + \tau h_K^2\|\nabla p_h\|^2_{0,K}\right) \\
&\geq C_0 X^2.
\end{aligned}
\tag{33}
$$

Secondly, setting $(\mathbf{w}_h, r_h) = (-\mathbf{v}_h, 0)$, we get

$$
\begin{aligned}
&B_{CBB,h}\left(\mathbf{u}_h, p_h; -\mathbf{v}_h, 0\right) \\
&= -a_h(\mathbf{u}_h,\mathbf{v}_h) - \hat{b}_h(p_h,\mathbf{v}_h) - \tilde{b}_h(p_h,\mathbf{v}_h) - c_h(\mathbf{u}_h,\mathbf{v}_h) - e_h(\mathbf{u}_h,\mathbf{v}_h),
\end{aligned}
$$

and based on Theorem 4.1, for a given $p_h \in Q_h$ there exists $\widetilde{\mathbf{v}}_h \in \mathbf{V}_h$ such that

$$
\hat{b}_h\left(\widetilde{\mathbf{v}}_h, p_h\right) \geq \|\widetilde{\mathbf{v}}_h\|_1 \left[\beta_1\|p_h\|_0 - \beta_2\left(\sum_{K\in\mathcal{T}_h} h_K^2\|\nabla p_h\|^2_{0,K}\right)^{1/2}\right],
$$

taking $\mathbf{v}_h = \frac{\|p_h\|_0}{\|\widetilde{\mathbf{v}}_h\|_1}\widetilde{\mathbf{v}}_h$, that is $\|\mathbf{v}_h\|_1 = \|p_h\|_0$, then using Young inequality yields

$$
\begin{aligned}
\hat{b}_h(\mathbf{v}_h, p_h) &\geq \beta_1\|p_h\|_0^2 - \beta_2\left(\sum_{K\in\mathcal{T}_h} h_K^2\|\nabla p_h\|^2_{0,K}\right)^{\frac{1}{2}}\|p_h\|_0 \\
&\geq \left(\beta_1 - \frac{\beta_2^2}{l_0}\right)\|p_h\|_0^2 - \frac{l_0}{4}\sum_{K\in\mathcal{T}_h} h_K^2\|\nabla p_h\|^2_{0,K},
\end{aligned}
$$

i.e.,

$$-\hat{b}_h(\mathbf{v}_h, p_h) \le \left(\frac{\beta_2^2}{l_0} - \beta_1\right) \|p_h\|_0^2 + \frac{l_0}{4}\left(\sum_{K\in\mathcal{T}_h} h_K^2 \|\nabla p_h\|_{0,K}^2\right),$$

where $l_0 > 0$ will be determined by the needs of later analysis. Thus, we can easily obtain

(34)
$$\begin{aligned} B_{CBB,h}(\mathbf{u}_h, p_h; -\mathbf{v}_h, 0) \le &\left|a_h(\mathbf{u}_h, \mathbf{v}_h) + \tilde{b}_h(p_h, \mathbf{v}_h) + c_h(\mathbf{u}_h, \mathbf{v}_h) + e_h(\mathbf{u}_h, \mathbf{v}_h)\right| \\ &+ \left(\frac{\beta_2^2}{l_0} - \beta_1\right) \|p_h\|_0^2 + \frac{l_0}{4}\left(\sum_{K\in\mathcal{T}_h} h_K^2 \|\nabla p_h\|_{0,K}^2\right). \end{aligned}$$

Next, let's analyze each terms in (34).

$$\begin{aligned} &\left|a_h(\mathbf{u}_h, \mathbf{v}_h) + \tilde{b}_h(p_h, \mathbf{v}_h) + c_h(\mathbf{u}_h, \mathbf{v}_h) + e_h(\mathbf{u}_h, \mathbf{v}_h)\right| \\ &\le \left(\min\{1, \alpha^*\}\nu\|\nabla \mathbf{u}_h\|_0 + \nu\tau h\|\nabla p_h\|_0 + \delta\|\nabla\cdot\mathbf{u}_h\|_0 + \nu^2\tau\|\nabla\mathbf{u}_h\|_0\right)\cdot\|\mathbf{v}_h\|_1 \\ &\le \frac{\beta_1}{8}\|p_h\|_0^2 + CX^2. \end{aligned}$$

Then choosing $l_0 = \frac{8\beta_2^2}{3\beta_1}$, we get

$$\begin{aligned} B_{CBB,h}(\mathbf{u}_h, p_h; -\mathbf{v}_h, 0) &\le \left(\frac{\beta_2^2}{l_0} - \frac{7\beta_1}{8}\right)\|p_h\|_0^2 + CX^2 + \frac{l_0}{4}h^2\|\nabla p_h\|_0^2 \\ &= -\frac{\beta_1}{2}\|p_h\|_0^2 + CX^2 + \frac{2\beta_2^2}{3\beta_1}h^2\|\nabla p_h\|_0^2, \end{aligned}$$

thus

$$B_{CBB,h}(\mathbf{u}_h, p_h; \mathbf{v}_h, 0) \ge \frac{\beta_1}{2}\|p_h\|_0^2 - CX^2 - C\tau h^2\|\nabla p_h\|_0^2, \tag{35}$$

and we multiply (35) by $\frac{2}{\beta_1}$,

$$B_{CBB,h}(\mathbf{u}_h, p_h; \frac{2}{\beta_1}\mathbf{v}_h, 0) \ge \|p_h\|_0^2 - C_1X^2, \tag{36}$$

with a suitable positive constant C_1. We define for an arbitrary $(\mathbf{w}_h, r_h) \in \mathbf{V}_h \times Q_h$

$$(\mathbf{w}_h, r_h) := (-\mathbf{u}_h, p_h) + \frac{2C_0}{\beta_1(C_0 + C_1)}(\mathbf{v}_h, 0) \in \mathbf{V}_h \times Q_h. \tag{37}$$

Then, we have

$$\begin{aligned} B_{CBB,h}(\mathbf{u}_h, p_h; \mathbf{w}_h, r_h) &\ge \frac{C_0}{C_0 + C_1}\|p_h\|_0^2 + \frac{C_0^2}{C_0 + C_1}X^2 \\ &\ge \min\{1, C_0\}\frac{C_0}{C_0 + C_1}|||(\mathbf{u}_h, p_h)|||^2, \end{aligned} \tag{38}$$

and

$$\begin{aligned} |||(\mathbf{w}_h, r_h)||| &\le |||(\mathbf{u}_h, p_h)||| + \frac{2C_0}{\beta_1(C_0 + C_1)}|||(\mathbf{v}_h, 0)||| \\ &\le |||(\mathbf{u}_h, p_h)||| + C\|\mathbf{v}_h\|_1 = |||(\mathbf{u}_h, p_h)||| + C\|p_h\|_0 \\ &\le C_2|||(\mathbf{u}_h, p_h)|||, \end{aligned} \tag{39}$$

from (38) and (39), we conclude (32) with $\hat{\beta} = \min\{1, C_0\}\frac{C_0}{C_2(C_0+C_1)} > 0$.

□

5. Error estimates for the discrete formulation

We will give some error estimates for our scheme and get the optimal error estimate in this section.

5.1. Error estimate in energy norm.

Theorem 5.1. *Assume that Assumption 3.1 is satisfied. Let $(\mathbf{u}, p) \in \mathbf{V} \times Q$ be the solution of (6) and $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ be the solution of (15). Then, there is a positive constant C independent of h such that*

$$\begin{aligned}|||(\mathbf{u}-\mathbf{u}_h, p-p_h)||| \le C\Big(&\|\mathbf{u}-\mathbf{u}_\pi\|_1 + \|\mathbf{u}-\mathbf{u}_I\|_1 + h^{-1}\|\mathbf{u}-\mathbf{u}_I\|_0 \\ &+ \|p-p_I\|_0 + h\|p-p_I\|_1 + h\|p-p_\pi\|_1 + N\Big),\end{aligned} \tag{40}$$

where

$$\begin{aligned}N := \sum_{K\in\mathcal{T}_h}\Big(&\|p-\Pi^0_{k-1,K}p\|_{0,K} + h_K\|\mathbf{u}-\Pi^0_{k,K}\mathbf{u}\|_{2,K} + \|\mathbf{u}-\Pi^0_{k,K}\mathbf{u}\|_{1,K} \\ &+ \|\nabla\cdot\mathbf{u}-\Pi^0_{k-1,K}\nabla\cdot\mathbf{u}\|_{0,K} + h_K\|\mathbf{f}-\Pi^0_{k-1,K}\mathbf{f}\|_{0,K}\Big).\end{aligned}$$

More specifically, assuming that $\mathbf{f} \in \mathbf{H}^{k-1}(\Omega)$, $\mathbf{u} \in \mathbf{V}\cap\mathbf{H}^{k+1}(\Omega)$ and $p \in Q\cap H^k(\Omega)$. Then, there exists a constant C independent of h such that

$$|||(\mathbf{u}-\mathbf{u}_h, p-p_h)||| \le Ch^k(\|\mathbf{u}\|_{k+1} + \|p\|_k + \|\mathbf{f}\|_{k-1}). \tag{41}$$

Proof. Let $(\mathbf{u}_I, p_I) \in \mathbf{V}_h \times Q_h$ be the approximation to $(\mathbf{u}, p)$ satisfying Lemma 3.2 and $(\mathbf{u}_\pi, p_\pi) \in (\mathbb{P}_k)^d \times \mathbb{P}_k$ be the approximation to $(\mathbf{u}, p)$ satisfying Lemma 3.1.

Firstly, we set

$$\mathbf{u}-\mathbf{u}_h = (\mathbf{u}-\mathbf{u}_I) + (\mathbf{u}_I-\mathbf{u}_h) =: \theta_\mathbf{u} + \xi_\mathbf{u},$$
$$p-p_h = (p-p_I) + (p_I-p_h) =: \theta_p + \xi_p,$$

where $\theta_\mathbf{u}, \theta_p$ can be estimated by Lemma 3.2. Thus, we are just going to estimate the rest.

According to the result of Theorem 4.2, it is obvious that

$$\begin{aligned}&C|||(\xi_\mathbf{u},\xi_p)|||\cdot|||(\mathbf{w}_h,r_h)||| \\ &\le B_{CBB,h}(\xi_\mathbf{u},\xi_p;\mathbf{w}_h,r_h) = B_{CBB,h}(\mathbf{u}_h,p_h;\mathbf{w}_h,r_h) - B_{CBB,h}(\mathbf{u}_I,p_I;\mathbf{w}_h,r_h) \\ &= F_{CBB,h}(\mathbf{w}_h,r_h) - F_{CBB}(\mathbf{w}_h,r_h) + B_{CBB}(\mathbf{u},p;\mathbf{w}_h,r_h) - B_{CBB,h}(\mathbf{u}_I,p_I;\mathbf{w}_h,r_h) \\ &= [a(\mathbf{u},\mathbf{w}_h) - a_h(\mathbf{u}_I,\mathbf{w}_h)] + [b(r_h,\mathbf{u}) - b_h(r_h,\mathbf{u}_I)] + [d(p,r_h) - d_h(p_I,r_h)] \\ &\quad + [c(\mathbf{u},\mathbf{w}_h) - c_h(\mathbf{u}_I,\mathbf{w}_h)] + I,\end{aligned} \tag{42}$$

where,

$$\begin{aligned}I &:= [b(p,\mathbf{w}_h) - b_h(p_I,\mathbf{w}_h)] + [e(\mathbf{u},\mathbf{w}_h) - e_h(\mathbf{u}_I,\mathbf{w}_h)] + [F_{CBB,h}(\mathbf{w}_h,r_h) - F_{CBB}(\mathbf{w}_h,r_h)] \\ &=: I_1 + I_2 + I_3,\end{aligned}$$

let's observe the relationship between I_1, I_2 and I_3:

$$\begin{aligned}I_1 :&= b(p,\mathbf{w}_h) - b_h(p_I,\mathbf{w}_h) = b(p,\mathbf{w}_h) - b_h(p,\mathbf{w}_h) + b_h(p-p_I,\mathbf{w}_h) \\ &= \sum_{K\in\mathcal{T}_h}\Big[(p-\Pi^0_{k-1,K}p, \nabla\cdot\mathbf{w}_h - \Pi^0_{k-1,K}\nabla\cdot\mathbf{w}_h)_K + b_h^K(p-p_I,\mathbf{w}_h) \\ &\qquad - \nu\tau h_K^2(\nabla p, \Delta\mathbf{w}_h - \Delta\Pi^0_{k,K}\mathbf{w}_h)_K\Big],\end{aligned}$$

$$I_2 := e(\mathbf{u},\mathbf{w}_h) - e_h(\mathbf{u}_I,\mathbf{w}_h) = e(\mathbf{u},\mathbf{w}_h) - e_h(\mathbf{u},\mathbf{w}_h) + e_h(\mathbf{u}-\mathbf{u}_I,\mathbf{w}_h)$$
$$= \sum_{K\in\mathcal{T}_h}\Big[\nu^2\tau h_K^2(\Delta\mathbf{u} - \Delta\Pi^0_{k,K}\mathbf{u}, \Delta\Pi^0_{k,K}\mathbf{w}_h)_K + e_h^K(\mathbf{u}-\mathbf{u}_I,\mathbf{w}_h)$$
$$+ \nu\tau h_K^2(\nu\Delta\mathbf{u}, \Delta\mathbf{w}_h - \Delta\Pi^0_{k,K}\mathbf{w}_h)_K\Big],$$

and

$$I_3 := F_{CBB,h}(\mathbf{w}_h, r_h) - F_{CBB}(\mathbf{w}_h, r_h)$$
$$= \sum_{K\in\mathcal{T}_h}\Big[(\mathbf{f}, \mathbf{w}_h - \Pi^0_{k,K}\mathbf{w}_h)_K - \nu\tau h_K^2(\mathbf{f}, \Delta\Pi^0_{k,K}\mathbf{w}_h)_K + \nu\tau h_K^2(\mathbf{f}, \Delta\mathbf{w}_h)_K$$
$$+ \tau h_K^2(\mathbf{f}, \Pi^0_{k-1,K}\nabla r_h - \nabla r_h)_K\Big]$$
$$= \sum_{K\in\mathcal{T}_h}\Big[(\mathbf{f} - \Pi^0_{k-1,K}\mathbf{f}, \mathbf{w}_h - \Pi^0_{k,K}\mathbf{w}_h)_K + \tau h_K^2(\mathbf{f} - \Pi^0_{k-1,K}\mathbf{f}, (\Pi^0_{k-1,K} - I)\nabla r_h)_K$$
$$+ \nu\tau h_K^2(\mathbf{f}, \Delta\mathbf{w}_h - \Delta\Pi^0_{k,K}\mathbf{w}_h)_K\Big],$$

by adding I_1-I_3 and according to (1), we obtain

$$I = \sum_{K\in\mathcal{T}_h}\Big[\nu^2\tau h_K^2(\Delta\mathbf{u} - \Delta\Pi^0_{k,K}\mathbf{u}, \Delta\Pi^0_{k,K}\mathbf{w}_h)_K + e_h^K(\mathbf{u}-\mathbf{u}_I,\mathbf{w}_h)$$
$$+ (\mathbf{f} - \Pi^0_{k-1,K}\mathbf{f}, \mathbf{w}_h - \Pi^0_{k,K}\mathbf{w}_h)_K + \tau h_K^2(\mathbf{f} - \Pi^0_{k-1,K}\mathbf{f}, \Pi^0_{k-1,K}\nabla r_h - \nabla r_h)_K$$
$$+ (p - \Pi^0_{k-1,K}p, (I - \Pi^0_{k-1,K})\nabla\cdot\mathbf{w}_h)_K + b_h^K(p - p_I, \mathbf{w}_h)\Big],$$

then using Cauchy-Schwarz inequality and Lemma 3.3, we have the following estimate

(43)
$$|I| \le C\sum_{K\in\mathcal{T}_h}\big(h_K\|\mathbf{u} - \Pi^0_{k,K}\mathbf{u}\|_{2,K} + \|p - \Pi^0_{k-1,K}p\|_{0,K} + \|\mathbf{u} - \mathbf{u}_I\|_{1,K}$$
$$+ \|p - p_I\|_{0,K} + h_K\|p - p_I\|_{1,K} + h_K\|\mathbf{f} - \Pi^0_{k-1,K}\mathbf{f}\|_{0,K}\big)\cdot |||(\mathbf{w}_h, r_h)|||.$$

Then, according to the consistency assumption (20) and continuity of a, a_h, we have

$$|a(\mathbf{u},\mathbf{w}_h) - a_h(\mathbf{u}_I,\mathbf{w}_h)| = \left|\sum_{K\in\mathcal{T}_h}(a^K(\mathbf{u}-\mathbf{u}_\pi,\mathbf{w}_h) + a_h^K(\mathbf{u}_\pi - \mathbf{u}_I,\mathbf{w}_h))\right|$$
$$\le C(\|\mathbf{u}-\mathbf{u}_\pi\|_1 + \|\mathbf{u}-\mathbf{u}_I\|_1)\cdot|||(\mathbf{w}_h, r_h)|||, \tag{44}$$

similarly, we can get

$$|d(p,r_h) - d_h(p_I,r_h)| = \left|\sum_{K\in\mathcal{T}_h}(d^K(p-p_\pi,r_h) + d_h^K(p_\pi - p_I,r_h))\right|$$
$$\le Ch(\|p-p_I\|_1 + \|p-p_\pi\|_1)\cdot|||(\mathbf{w}_h, r_h)|||, \tag{45}$$

for the other terms of (42), we have

$$
\begin{aligned}
&|b(r_h,\mathbf{u})-b_h(r_h,\mathbf{u}_I)| = |b(r_h,\mathbf{u}-\mathbf{u}_I)+b(r_h,\mathbf{u}_I)-b_h(r_h,\mathbf{u}_I)|\\
&=\Bigg|\sum_{K\in\mathcal{T}_h}\Big[(r_h,\nabla\cdot\mathbf{u}_I)_K-\nu\tau h_K^2(\nabla r_h,\Delta\mathbf{u}_I)_K-(\Pi^0_{k,K}r_h,\Pi^0_{k-1,K}\nabla\cdot\mathbf{u}_I)_K\\
&\qquad+\nu\tau h_K^2(\Pi^0_{k-1,K}\nabla r_h,\Delta\Pi^0_{k,K}\mathbf{u}_I)_K\Big]+b(r_h,\mathbf{u}-\mathbf{u}_I)\Bigg|\\
&=\Bigg|\sum_{K\in\mathcal{T}_h}\Big[(r_h-\Pi^0_{k-1,K}r_h,\nabla\cdot\mathbf{u}_I-\Pi^0_{k-1,K}\nabla\cdot\mathbf{u}_I)_K\\
&\qquad-\nu\tau h_K^2(\nabla r_h,\Delta\mathbf{u}_I-\Delta\Pi^0_{k,K}\mathbf{u}_I)_K\Big]+b(r_h,\mathbf{u}-\mathbf{u}_I)\Bigg|,
\end{aligned}
$$

then using the fact $\Pi^0_{k,K}q=q,\forall q\in\mathbb{P}_k(K)$, we have

$$
\begin{aligned}
&\left|(r_h-\Pi^0_{k-1,K}r_h,\nabla\cdot\mathbf{u}_I-\Pi^0_{k-1,K}\nabla\cdot\mathbf{u}_I)_K\right|\\
&\le Ch_K\|r_h\|_{1,K}\cdot\|(I-\Pi^0_{k-1,K})(\nabla\cdot(\mathbf{u}_I-\mathbf{u}_\pi))\|_{0,K}\\
&\le Ch_K\|r_h\|_{1,K}\cdot(\|\mathbf{u}-\mathbf{u}_I\|_{1,K}+\|\mathbf{u}-\mathbf{u}_\pi\|_{1,K}),
\end{aligned}
$$

by Lemma 3.3, we get

$$
\begin{aligned}
&\left|\nu\tau h_K^2(\nabla r_h,\Delta\mathbf{u}_I-\Delta\Pi^0_{k,K}\mathbf{u}_I)_K\right|\le C\|\nabla r_h\|_{0,K}\cdot\|(\mathbf{u}_I-\mathbf{u}_\pi)-\Pi^0_{k,K}(\mathbf{u}_I-\mathbf{u}_\pi)\|_{0,K}\\
&\le C(\|\mathbf{u}-\mathbf{u}_I\|_{1,K}+\|\mathbf{u}-\mathbf{u}_\pi\|_{1,K})\cdot h_K\|\nabla r_h\|_{0,K},
\end{aligned}
$$

due to $\mathbf{u}_I\in\mathbf{V}_h\subset\mathbf{V}=\mathbf{H}^1_0(\Omega)$, we have $\mathbf{u}-\mathbf{u}_I\in\mathbf{H}^1_0(\Omega)$ and $\Delta\mathbf{u}_I|_K\in\mathbb{P}_k(K)$, then similar to the analysis of [38,39] and using Lemma 3.3,

$$
\begin{aligned}
|b(r_h,\mathbf{u}-\mathbf{u}_I)| &= \left|-(\nabla r_h,\mathbf{u}-\mathbf{u}_I)-\nu\tau h^2(\nabla r_h,\Delta(\mathbf{u}-\mathbf{u}_I))\right|\\
&\le C\Big[h^{-1}\|\mathbf{u}-\mathbf{u}_I\|_0+\sum_{K\in\mathcal{T}_h}\big(h\|\Delta\mathbf{u}-\Delta\Pi^0_{k,K}\mathbf{u}\|_{0,K}\\
&\qquad+h\|\Delta\Pi^0_{k,K}\mathbf{u}-\Delta\mathbf{u}_I\|_{0,K}\big)\Big]\cdot h\|\nabla r_h\|_0\\
&\le C\Big[h^{-1}\|\mathbf{u}-\mathbf{u}_I\|_0+\sum_{K\in\mathcal{T}_h}\big(h\|\mathbf{u}-\Pi^0_{k,K}\mathbf{u}\|_{2,K}+\|\mathbf{u}-\Pi^0_{k,K}\mathbf{u}\|_{1,K}\\
&\qquad+\|\mathbf{u}-\mathbf{u}_I\|_{1,K}\big)\Big]\cdot h\|\nabla r_h\|_0,
\end{aligned}
$$

therefore, we have

(46)
$$
\begin{aligned}
|b(r_h,\mathbf{u})-b_h(r_h,\mathbf{u}_I)|\le C\sum_{K\in\mathcal{T}_h}\Big[&\|\mathbf{u}-\mathbf{u}_I\|_{1,K}+h_K^{-1}\|\mathbf{u}-\mathbf{u}_I\|_{0,K}+\|\mathbf{u}-\mathbf{u}_\pi\|_{1,K}\\
&+h\|\mathbf{u}-\Pi^0_{k,K}\mathbf{u}\|_{2,K}+\|\mathbf{u}-\Pi^0_{k,K}\mathbf{u}\|_{1,K}\Big]\cdot|||(\xi_{\mathbf{u}},r_h)|||.
\end{aligned}
$$

And similarly

(47)

$$
\begin{aligned}
&|c(\mathbf{u}, \mathbf{w}_h) - c_h(\mathbf{u}_I, \mathbf{w}_h)| = |c(\mathbf{u}, \mathbf{w}_h) - c_h(\mathbf{u}, \mathbf{w}_h) + c_h(\mathbf{u} - \mathbf{u}_I, \mathbf{w}_h)| \\
&= \Big| \sum_{K \in \mathcal{T}_h} \left[-\delta(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{w}_h)_K + \delta(\Pi^0_{k-1,K} \nabla \cdot \mathbf{u}, \Pi^0_{k-1,K} \nabla \cdot \mathbf{w}_h)_K \right] + c_h(\mathbf{u} - \mathbf{u}_I, \mathbf{w}_h) \Big| \\
&= \Big| \sum_{K \in \mathcal{T}_h} \delta(\nabla \cdot \mathbf{u} - \Pi^0_{k-1,K} \nabla \cdot \mathbf{u}, \Pi^0_{k-1,K} \nabla \cdot \mathbf{w}_h - \nabla \cdot \mathbf{w}_h)_K + c_h(\mathbf{u} - \mathbf{u}_I, \mathbf{w}_h) \Big|, \\
&\le C \Big(\sum_{K \in \mathcal{T}_h} \|\nabla \cdot \mathbf{u} - \Pi^0_{k-1,K} \nabla \cdot \mathbf{u}\|_{0,K} + \|\mathbf{u} - \mathbf{u}_I\|_1 \Big) \cdot |||(\mathbf{w}_h, r_h)|||.
\end{aligned}
$$

Therefore, combining (42) with (43)-(47), we obtain

(48)

$$
\begin{aligned}
&|||(\xi_{\mathbf{u}}, \xi_p)||| \\
&\le C \Big[\|\mathbf{u} - \mathbf{u}_\pi\|_1 + \|\mathbf{u} - \mathbf{u}_I\|_1 + h^{-1}\|\mathbf{u} - \mathbf{u}_I\|_0 + \|p - p_I\|_0 + h\|p - p_I\|_1 \\
&\quad + h\|p - p_\pi\|_1 + \sum_{K \in \mathcal{T}_h} \big(\|p - \Pi^0_{k-1,K} p\|_{0,K} + h_K \|\mathbf{u} - \Pi^0_{k,K}\mathbf{u}\|_{2,K} \\
&\quad + \|\mathbf{u} - \Pi^0_{k,K}\mathbf{u}\|_{1,K} + \|\nabla \cdot \mathbf{u} - \Pi^0_{k-1,K} \nabla \cdot \mathbf{u}\|_{0,K} + h_K \|\mathbf{f} - \Pi^0_{k-1,K}\mathbf{f}\|_{0,K} \big) \Big].
\end{aligned}
$$

Hence, we can obtain the results (40)-(41) by using the above estimates and Lemma 3.1-3.2.

□

5.2. Error estimate for velocity in L^2 norm. To obtain the L^2 error estimate of $\mathbf{u}$, the following dual problem is discussed: [45]

$$
\left\{
\begin{array}{ll}
-\nu \Delta \mathbf{\Phi} + \nabla \psi = \mathbf{u} - \mathbf{u}_h, & \text{in } \Omega, \\
\nabla \cdot \mathbf{\Phi} = 0 & \text{in } \Omega, \\
\mathbf{\Phi} = \mathbf{0} & \text{on } \partial\Omega,
\end{array}
\right. \tag{49}
$$

where $(\mathbf{\Phi}, \psi) \in [\mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega)] \times [H^1(\Omega) \cap L^2_0(\Omega)]$. Let Ω be a convex polygon in $\mathbb{R}^2$, similar to the statement in [40, 45, 46], it satisfies

$$
\|\mathbf{\Phi}\|_2 + \|\psi\|_1 \le C\|\mathbf{u} - \mathbf{u}_h\|_0. \tag{50}
$$

Theorem 5.2. *Assume that the same assumptions of Theorem 5.1, then there holds*

$$
\|\mathbf{u} - \mathbf{u}_h\|_0 \le C h^{k+1} \left(\|\mathbf{u}\|_{k+1} + \|p\|_k + \|\mathbf{f}\|_{k-1} \right). \tag{51}
$$

Proof. Let $(\mathbf{\Phi}_I, \psi_I) \in \mathbf{V}_h \times Q_h$ and $(\mathbf{\Phi}_\pi, \psi_\pi) \in (\mathbb{P}_k)^d \times P_k$ be the approximations to $(\mathbf{\Phi}, \psi)$ satisfying Lemma 3.1 and Lemma 3.2, respectively. Then, combining the fact $\nabla \cdot \mathbf{\Phi} = 0$, it is clear from (49) that

(52)

$$
\begin{aligned}
&\|\mathbf{u} - \mathbf{u}_h\|_0^2 \\
&:= a(\mathbf{\Phi}, \mathbf{u}_h - \mathbf{u}) + b(\psi, \mathbf{u}_h - \mathbf{u}) + b(p_h - p, \mathbf{\Phi}) + c(\mathbf{\Phi}, \mathbf{u}_h - \mathbf{u}) + d(\psi, p_h - p) \\
&\quad + e(\mathbf{\Phi}, \mathbf{u}_h - \mathbf{u}) + \nu\tau h^2(\mathbf{u} - \mathbf{u}_h, \Delta(\mathbf{u}_h - \mathbf{u})) + \tau h^2(\nabla(p_h - p), \nu\Delta\mathbf{\Phi} - \nabla\psi).
\end{aligned}
$$

Next, by taking $(\mathbf{v}, q) = (\mathbf{\Phi}_I, \psi_I)$ in (6), $(\mathbf{v}_h, q_h) = (\mathbf{\Phi}_I, \psi_I)$ in (15) and combining (52) with the above equations and using the symmetry of a, c, d, e, we can easily get

(53)

$$
\begin{aligned}
\|\mathbf{u}-\mathbf{u}_h\|_0^2 := {} & a(\mathbf{\Phi}-\mathbf{\Phi}_I,\mathbf{u}_h-\mathbf{u})+b(\psi-\psi_I,\mathbf{u}_h-\mathbf{u})+b(p_h-p,\mathbf{\Phi}-\mathbf{\Phi}_I)\\
&+c(\mathbf{\Phi}-\mathbf{\Phi}_I,\mathbf{u}_h-\mathbf{u})+d(\psi-\psi_I,p_h-p)+e(\mathbf{\Phi}-\mathbf{\Phi}_I,\mathbf{u}_h-\mathbf{u})\\
&+[a(\mathbf{\Phi}_I,\mathbf{u}_h)-a_h(\mathbf{\Phi}_I,\mathbf{u}_h)]+[b(\psi_I,\mathbf{u}_h)-b_h(\psi_I,\mathbf{u}_h)]\\
&+[c(\mathbf{\Phi}_I,\mathbf{u}_h)-c_h(\mathbf{\Phi}_I,\mathbf{u}_h)]+[d(\psi_I,p_h)-d_h(\psi_I,p_h)]\\
&+\nu\tau h^2(\mathbf{u}-\mathbf{u}_h,\Delta(\mathbf{u}_h-\mathbf{u}))+\tau h^2(\nabla(p_h-p),\nu\Delta\mathbf{\Phi}-\nabla\psi)+H,
\end{aligned}
$$

where we set

$$
\begin{aligned}
H := {} & [b(p_h,\mathbf{\Phi}_I)-b_h(p_h,\mathbf{\Phi}_I)]+[e(\mathbf{\Phi}_I,\mathbf{u}_h)-e_h(\mathbf{\Phi}_I,\mathbf{u}_h)]\\
&+[F_{CBB,h}(\mathbf{\Phi}_I,\psi_I)-F_{CBB}(\mathbf{\Phi}_I,\psi_I)]\,,
\end{aligned}
$$

and using the definition of projection and (1), we can easily rewrite as

$$
\begin{aligned}
H := {} & b(p_h-p,\mathbf{\Phi}_I)-b_h(p_h-p,\mathbf{\Phi}_I)+e(\mathbf{u}_h-\mathbf{u},\mathbf{\Phi}_I)-e_h(\mathbf{u}_h-\mathbf{u},\mathbf{\Phi}_I)\\
&+\sum_{K\in\mathcal{T}_h}\Big[(p,\nabla\cdot\mathbf{\Phi}_I-\Pi^0_{k-1,K}\nabla\cdot\mathbf{\Phi}_I)_K+\nu^2\tau h_K^2(\Delta\mathbf{u}-\Delta\Pi^0_{k,K}\mathbf{u},\Delta\Pi^0_{k,K}\mathbf{\Phi}_I)_K\\
&\qquad+(\mathbf{f},\mathbf{\Phi}_I-\Pi^0_{k,K}\mathbf{\Phi}_I)_K-\tau h_K^2(\mathbf{f},\nabla\psi_I-\Pi^0_{k-1,K}\nabla\psi_I)_K\Big].
\end{aligned}
$$

First, using the triangle inequality, the inverse estimates in Lemma 3.3, the properties of Lemma 3.1-3.2 and the continuity of projection $\Pi^0_{k,K}$, we have the following estimate

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h}\|\Delta\Pi^0_{k,K}\mathbf{\Phi}_I\|_{0,K}\\
&\le\sum_{K\in\mathcal{T}_h}\big(\|\Delta\Pi^0_{k,K}(\mathbf{\Phi}_I-\mathbf{\Phi})\|_{0,K}+\|\Delta(\Pi^0_{k,K}\mathbf{\Phi}-\mathbf{\Phi})\|_{0,K}\big)+\|\Delta\mathbf{\Phi}\|_0 \qquad (54)\\
&\le C\sum_{K\in\mathcal{T}_h}h_K^{-1}\|\Pi^0_{k,K}(\mathbf{\Phi}_I-\mathbf{\Phi})\|_{1,K}+C\|\mathbf{\Phi}\|_2\le C\|\mathbf{\Phi}\|_2\le C\|\mathbf{u}-\mathbf{u}_h\|_0,
\end{aligned}
$$

then using the continuity of projection and $\mathbf{\Phi}-\mathbf{\Phi}_I\in\mathbf{V}$, we have

$$
\begin{aligned}
&|b(p_h-p,\mathbf{\Phi}_I)-b_h(p_h-p,\mathbf{\Phi}_I)+b(p_h-p,\mathbf{\Phi}-\mathbf{\Phi}_I)|\\
&=\Big|\sum_{K\in\mathcal{T}_h}\Big[((p_h-p)-\Pi^0_{k-1,K}(p_h-p),\nabla\cdot\mathbf{\Phi}_I-\Pi^0_{k-1,K}\nabla\cdot\mathbf{\Phi}_I)_K\\
&\qquad-\nu\tau h_K^2(\nabla(p_h-p),\Delta\mathbf{\Phi}_I)_K+\nu\tau h_K^2(\nabla(p_h-p),\Delta\Pi^0_{k,K}\mathbf{\Phi}_I)_K\\
&\qquad-(\nabla(p_h-p),\mathbf{\Phi}-\mathbf{\Phi}_I)-\nu\tau h^2(\nabla(p_h-p),\Delta(\mathbf{\Phi}-\mathbf{\Phi}_I))\Big]\Big|\\
&=\Big|\sum_{K\in\mathcal{T}_h}\Big[((I-\Pi^0_{k-1,K})(p_h-p),(I-\Pi^0_{k-1,K})(\nabla\cdot\mathbf{\Phi}_I-\nabla\cdot\mathbf{\Phi}))_K\\
&\qquad-(\nabla(p_h-p),\mathbf{\Phi}-\mathbf{\Phi}_I)_K+\nu\tau h_K^2(\nabla(p_h-p),\Delta\Pi^0_{k,K}\mathbf{\Phi}_I)_K\Big]\\
&\qquad-\nu\tau h^2(\nabla(p_h-p),\Delta\mathbf{\Phi})\Big|\\
&\le Ch^2\|p-p_h\|_1\cdot\|\mathbf{u}-\mathbf{u}_h\|_0,
\end{aligned}
$$

and

$$
\begin{aligned}
&|e(\mathbf{u}_h-\mathbf{u},\mathbf{\Phi}_I)-e_h(\mathbf{u}_h-\mathbf{u},\mathbf{\Phi}_I)+e(\mathbf{\Phi}-\mathbf{\Phi}_I,\mathbf{u}_h-\mathbf{u})| \\
&=|\sum_{K\in\mathcal{T}_h}\big[\nu^2\tau h_K^2(\Delta(\mathbf{u}_h-\mathbf{u}),\Delta\mathbf{\Phi}_I)_K-\nu^2\tau h_K^2(\Delta\Pi^0_{k,K}(\mathbf{u}_h-\mathbf{u}),\Delta\Pi^0_{k,K}\mathbf{\Phi}_I)_K \\
&\qquad+\tau h^2(\nu\Delta(\mathbf{\Phi}-\mathbf{\Phi}_I),\nu\Delta(\mathbf{u}_h-\mathbf{u}))\big]| \\
&=\left|\nu^2\tau h^2(\Delta(\mathbf{u}_h-\mathbf{u}),\Delta\mathbf{\Phi})-\sum_{K\in T_h}\nu^2\tau h_K^2(\Delta\Pi^0_{k,K}(\mathbf{u}_h-\mathbf{u}),\Delta\Pi^0_{k,K}\mathbf{\Phi}_I)_K\right| \\
&\le\sum_{K\in\mathcal{T}_h}\nu^2\tau h_K^2\left(\|\nabla\cdot\nabla(\mathbf{u}_h-\Pi^0_{k,K}\mathbf{u})\|_{0,K}+\|\Delta\Pi^0_{k,K}\mathbf{u}-\Delta\mathbf{u}\|_{0,K}\right)\cdot\|\mathbf{u}-\mathbf{u}_h\|_0 \\
&\le C\left(h\|\mathbf{u}-\mathbf{u}_h\|_1+h^{k+1}\|\mathbf{u}\|_{k+1}\right)\cdot\|\mathbf{u}-\mathbf{u}_h\|_0,
\end{aligned}
$$

and for other terms of H, we have

$$
\begin{aligned}
&\left|\sum_{K\in\mathcal{T}_h}(p,\nabla\cdot\mathbf{\Phi}_I-\Pi^0_{k-1,K}\nabla\cdot\mathbf{\Phi}_I)_K\right| \\
&=\left|\sum_{K\in\mathcal{T}_h}(p-\Pi^0_{k-1,K}p,\nabla\cdot\mathbf{\Phi}_I-\Pi^0_{k-1,K}\nabla\cdot\mathbf{\Phi}_I)_K\right|\le Ch^{k+1}\|p\|_k\cdot\|\mathbf{u}-\mathbf{u}_h\|_0,
\end{aligned}
$$

$$
\left|\sum_{K\in\mathcal{T}_h}\nu^2\tau h_K^2(\Delta\mathbf{u}-\Delta\Pi^0_{k,K}\mathbf{u},\Delta\Pi^0_{k,K}\mathbf{\Phi}_I)_K\right|\le Ch^{k+1}\|\mathbf{u}\|_{k+1}\cdot\|\mathbf{u}-\mathbf{u}_h\|_0,
$$

$$
\begin{aligned}
\left|\sum_{K\in\mathcal{T}_h}(\mathbf{f},\mathbf{\Phi}_I-\Pi^0_{k,K}\mathbf{\Phi}_I)_K\right|&=\left|\sum_{K\in\mathcal{T}_h}(\mathbf{f}-\Pi^0_{k,K}\mathbf{f},\mathbf{\Phi}_I-\Pi^0_{k,K}\mathbf{\Phi}_I)_K\right| \\
&\le Ch^{k+1}\|\mathbf{f}\|_{k-1}\|\mathbf{u}-\mathbf{u}_h\|_0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\sum_{K\in\mathcal{T}_h}\tau h_K^2(\mathbf{f},\nabla\psi_I-\Pi^0_{k-1,K}\nabla\psi_I)_K\right| \\
&=\left|\sum_{K\in\mathcal{T}_h}\tau h_K^2(\mathbf{f}-\Pi^0_{k-1,K}\mathbf{f},\nabla\psi_I-\Pi^0_{k-1,K}\nabla\psi_I)_K\right| \\
&\le C\sum_{K\in\mathcal{T}_h}h_K^{k+1}\|\mathbf{f}\|_{k-1,K}\cdot(\|\psi\|_{1,K}+\|\psi_I-\psi\|_{1,K})\le Ch^{k+1}\|\mathbf{f}\|_{k-1}\cdot\|\mathbf{u}-\mathbf{u}_h\|_0,
\end{aligned}
$$

hence, we have

$$
|H|\le Ch^{k+1}\left(\|\mathbf{u}\|_{k+1}+\|p\|_k+\|\mathbf{f}\|_{k-1}\right)\cdot\|\mathbf{u}-\mathbf{u}_h\|_0. \tag{55}
$$

Next, by applying the continuity together with (21) twice, we get

$$
\begin{aligned}
&|a(\mathbf{u}_h,\mathbf{\Phi}_I)-a_h(\mathbf{u}_h,\mathbf{\Phi}_I)| \\
&=\left|\sum_{K\in\mathcal{T}_h}\left(a^K(\mathbf{u}_h-\mathbf{u}_\pi,\mathbf{\Phi}_I-\mathbf{\Phi}_\pi)-a_h^K(\mathbf{u}_h-\mathbf{u}_\pi,\mathbf{\Phi}_I-\mathbf{\Phi}_\pi)\right)\right| \\
&\le C\left(|\mathbf{u}-\mathbf{u}_h|_1+|\mathbf{u}-\mathbf{u}_\pi|_1\right)\cdot\left(|\mathbf{\Phi}-\mathbf{\Phi}_I|_1+|\mathbf{\Phi}-\mathbf{\Phi}_\pi|_1\right) \\
&\le Ch^{k+1}\left(\|\mathbf{u}\|_{k+1}+\|p\|_k+\|\mathbf{f}\|_{k-1}\right)\cdot\|\mathbf{u}-\mathbf{u}_h\|_0,
\end{aligned} \tag{56}
$$

similarly, using the same technique, we have

$$
\begin{aligned}
&|d(p_h,\psi_I) - d_h(p_h,\psi_I)| \\
&= \left| \sum_{K\in\mathcal{T}_h} \left(d^K(p_h - p_\pi, \psi_I - \psi_\pi) - d_h^K(p_h - p_\pi, \psi_I - \psi_\pi) \right) \right| \\
&\le Ch^2(|p-p_h|_1 + |p-p_\pi|_1)\cdot(|\psi-\psi_I|_1 + |\psi - \psi_\pi|_1) \\
&\le Ch^{k+1}(\|\mathbf{u}\|_{k+1} + \|p\|_k + \|\mathbf{f}\|_{k-1})\cdot \|\mathbf{u}-\mathbf{u}_h\|_0.
\end{aligned} \tag{57}
$$

For the other terms of (53), we have

$$
\begin{aligned}
&|b(\psi_I,\mathbf{u}_h) - b_h(\psi_I,\mathbf{u}_h)| \\
&= \left| \sum_{K\in\mathcal{T}_h} \left[b^K(\psi_I,\mathbf{u}) - b_h^K(\psi_I,\mathbf{u}) + b_h^K(\psi_I,\mathbf{u}-\mathbf{u}_h) - b^K(\psi_I,\mathbf{u}-\mathbf{u}_h) \right] \right|,
\end{aligned}
$$

where

$$
\begin{aligned}
&\left| b^K(\psi_I,\mathbf{u}) - b_h^K(\psi_I,\mathbf{u}) \right| \\
&= \left| (\psi_I, \nabla\cdot\mathbf{u} - \Pi^0_{k-1,K}\nabla\cdot\mathbf{u})_K - \nu\tau h_K^2(\nabla\psi_I, \Delta\mathbf{u} - \Delta\Pi^0_{k,K}\mathbf{u})_K \right| \\
&\le \|\psi_I - \Pi^0_{k-1,K}\psi_I\|_{0,K} h_K^k \|\mathbf{u}\|_{k+1,K} + \nu\tau h_K^{k+1}\|\psi_I\|_{1,K}\|\mathbf{u}\|_{k+1,K} \\
&\le Ch_K^{k+1}\|\mathbf{u}\|_{k+1,K}(\|\psi\|_{1,K} + \|\psi_I - \psi\|_{1,K}) \\
&\le Ch_K^{k+1}\|\mathbf{u}\|_{k+1,K}\cdot\|\mathbf{u}-\mathbf{u}_h\|_{0,K},
\end{aligned}
$$

and Moreover, we have

$$
\begin{aligned}
&\left| b_h^K(\psi_I,\mathbf{u}-\mathbf{u}_h) - b^K(\psi_I,\mathbf{u}-\mathbf{u}_h) \right| \\
&= \Big| (\psi_I, \nabla\cdot(\mathbf{u}-\mathbf{u}_h) - \Pi^0_{k-1,K}\nabla\cdot(\mathbf{u}-\mathbf{u}_h))_K - \nu\tau h_K^2(\nabla\psi_I, \Delta(\mathbf{u}-\mathbf{u}_h) \\
&\quad - \Delta\Pi^0_{k,K}(\mathbf{u}-\mathbf{u}_h))_K \Big| \\
&\le \|\psi_I - \Pi^0_{k-1,K}\psi_I\|_{0,K}\|\mathbf{u}-\mathbf{u}_h\|_{1,K} + \nu\tau h_K^2\|\psi_I\|_{1,K}\big(\|\mathbf{u}-\Pi^0_{k,K}\mathbf{u}\|_{2,K} \\
&\quad + \|\Pi^0_{k,K}\mathbf{u} - \mathbf{u}_h\|_{2,K}\big) \\
&\le C(h_K\|\mathbf{u}-\mathbf{u}_h\|_{1,K} + h_K^{k+1}\|\mathbf{u}\|_{k+1,K})\cdot\|\mathbf{u}-\mathbf{u}_h\|_{0,K},
\end{aligned}
$$

therefore

$$
|b(\psi_I,\mathbf{u}_h) - b_h(\psi_I,\mathbf{u}_h)| \le Ch^{k+1}(\|\mathbf{u}\|_{k+1} + \|p\|_k + \|\mathbf{f}\|_{k-1})\cdot\|\mathbf{u}-\mathbf{u}_h\|_0. \tag{58}
$$

Moreover, we have

$$
\begin{aligned}
&|c(\mathbf{u}_h,\mathbf{\Phi}_I) - c_h(\mathbf{u}_h,\mathbf{\Phi}_I)| \\
&= \left| \sum_{K\in\mathcal{T}_h} (c^K(\mathbf{u},\mathbf{\Phi}_I) - c_h^K(\mathbf{u},\mathbf{\Phi}_I) + c_h^K(\mathbf{u}-\mathbf{u}_h,\mathbf{\Phi}_I) - c^K(\mathbf{u}-\mathbf{u}_h,\mathbf{\Phi}_I)) \right| \\
&= \Bigg| \sum_{K\in\mathcal{T}_h} \Big[-\delta(\nabla\cdot\mathbf{u}, \nabla\cdot\mathbf{\Phi}_I - \Pi^0_{k-1,K}\nabla\cdot\mathbf{\Phi}_I)_K \\
&\qquad - \delta(\nabla\cdot(\mathbf{u}-\mathbf{u}_h), \nabla\cdot\mathbf{\Phi}_I - \Pi^0_{k-1,K}\nabla\cdot\mathbf{\Phi}_I)_K \Big] \Bigg| \\
&\le C(h^{k+1}\|\mathbf{u}\|_{k+1} + h\|\mathbf{u}-\mathbf{u}_h\|_1)\cdot\|\mathbf{u}-\mathbf{u}_h\|_0,
\end{aligned} \tag{59}
$$

then, estimating the rest terms of (53)

$$|a(\mathbf{\Phi}-\mathbf{\Phi}_I,\mathbf{u}_h-\mathbf{u})|=|\nu(\nabla(\mathbf{\Phi}-\mathbf{\Phi}_I),\nabla(\mathbf{u}_h-\mathbf{u}))|\le Ch\|\mathbf{u}-\mathbf{u}_h\|_1\cdot\|\mathbf{u}-\mathbf{u}_h\|_0, \tag{60}$$

$$\begin{aligned}&|b(\psi-\psi_I,\mathbf{u}_h-\mathbf{u})|\\&=\left|(\psi-\psi_I,\nabla\cdot(\mathbf{u}_h-\mathbf{u}))-\nu\tau h^2(\nabla(\psi-\psi_I),\Delta(\mathbf{u}_h-\mathbf{u}))\right|\\&\le C\|\psi\|_1\cdot\left(h\|\mathbf{u}_h-\mathbf{u}\|_1+h^2\|\mathbf{u}_h-\Pi^0_{k,K}\mathbf{u}\|_2+h^2\|\Pi^0_{k,K}\mathbf{u}-\mathbf{u}\|_2\right)\\&\le C\left(h\|\mathbf{u}_h-\mathbf{u}\|_1+h^{k+1}\|\mathbf{u}\|_{k+1}\right)\cdot\|\mathbf{u}-\mathbf{u}_h\|_0,\end{aligned} \tag{61}$$

$$|c(\mathbf{\Phi}-\mathbf{\Phi}_I,\mathbf{u}_h-\mathbf{u})|=|\delta(\nabla\cdot(\mathbf{\Phi}-\mathbf{\Phi}_I),\nabla\cdot(\mathbf{u}_h-\mathbf{u}))|\le Ch\|\mathbf{u}_h-\mathbf{u}\|_1\cdot\|\mathbf{u}-\mathbf{u}_h\|_0, \tag{62}$$

$$|d(\psi-\psi_I,p_h-p)|=|\tau h^2(\nabla(\psi-\psi_I),\nabla(p_h-p))|\le Ch^2\|p-p_h\|_1\cdot\|\mathbf{u}-\mathbf{u}_h\|_0, \tag{63}$$

then similar to the analytical technique above, we have

$$\left|\nu\tau h^2(\mathbf{u}-\mathbf{u}_h,\Delta(\mathbf{u}_h-\mathbf{u}))\right|\le C(h\|\mathbf{u}-\mathbf{u}_h\|_1+h^{k+1}\|\mathbf{u}\|_{k+1})\cdot\|\mathbf{u}-\mathbf{u}_h\|_0, \tag{64}$$

$$\left|\tau h^2(\nabla(p_h-p),\nu\Delta\mathbf{\Phi}-\nabla\psi)\right|\le Ch^2\|p-p_h\|_1\cdot\|\mathbf{u}-\mathbf{u}_h\|_0. \tag{65}$$

Hence, by using the above (55)-(65), Lemma 3.1-3.2 and Theorem 5.1, the proof is concluded.

□

6. Numerical experiments

In this section, we consider some numerical experiments to test the actual performance of the method. In order to compute the VEM errors, we consider the computable error quantities:

- H^1-norm for velocity: $e_{\mathbf{u},1}=\sqrt{\sum_{K\in\mathcal{T}_h}\|\nabla\mathbf{u}-\Pi^0_{k-1,K}\nabla\mathbf{u}_h\|^2_{0,K}}$,
- L^2-norm for velocity: $e_{\mathbf{u},0}=\sqrt{\sum_{K\in\mathcal{T}_h}\|\mathbf{u}-\Pi^0_{k,K}\mathbf{u}_h\|^2_{0,K}}$,
- L^2-norm for pressure: $e_{p,0}=\sqrt{\sum_{K\in\mathcal{T}_h}\|p-\Pi^0_{k,K}p_h\|^2_{0,K}}$.

In the experiments, we consider the computational domains $\Omega=[0,1]^2$. The square domain Ω is partitioned by the following sequences of polygonal meshes:

- $\mathcal{T}_h^1$: Hexagons mesh with $h=1/8,1/16,1/32,1/64,1/128$;
- $\mathcal{T}_h^2$: Distorted hexagons mesh with $h=1/8,1/16,1/32,1/64,1/128$;
- $\mathcal{T}_h^3$: Triangle mesh with $h=1/8,1/16,1/32,1/64,1/128$,

as shown in Figure 1.

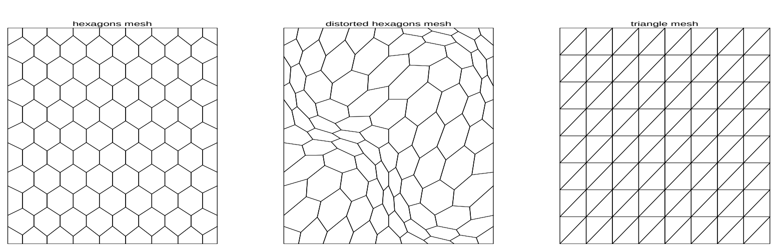


FIGURE 1. Sample meshes: $\mathcal{T}_h^1$ (left), $\mathcal{T}_h^2$ (center) and $\mathcal{T}_h^3$ (right).

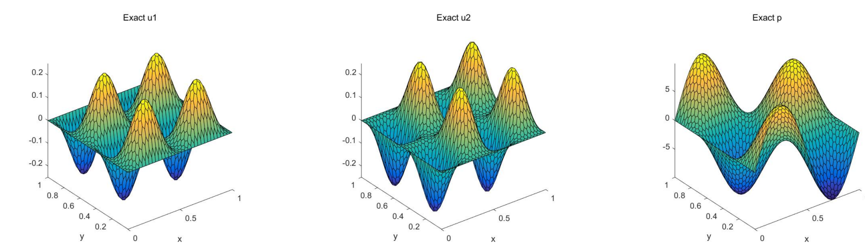
We consider problem (1) with $\nu = 1$. The stabilization parameters τ, δ are selected. The exact solutions are

$$\mathbf{u} = \left(\frac{\cos(2\pi y)\sin^2(2\pi x)\sin(2\pi y)}{2}, -\frac{\cos(2\pi x)\sin^2(2\pi y)\sin(2\pi x)}{2} \right),$$

$$p = \pi^2 \cos(2\pi y)\sin(2\pi x),$$

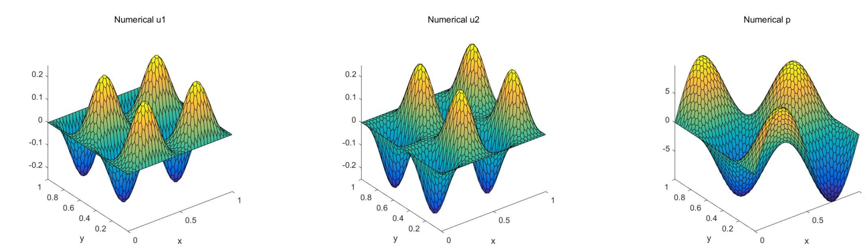
then we can obtain Dirichlet boundary data and source term $\mathbf{f}$ by exact solutions.

We show its exact and numerical solutions on $\mathcal{T}_h^1$. Moreover, we test the case without the least-square stabilization terms, i.e. $\tau = \delta = 0$, as shown in Figures 2-4 below.



(a) exact solution u_1 (b) exact solution u_2 (c) exact solution p

FIGURE 2. Exact solutions $\mathbf{u}$ and p on $\mathcal{T}_h^1$ for $h = 1/64$.



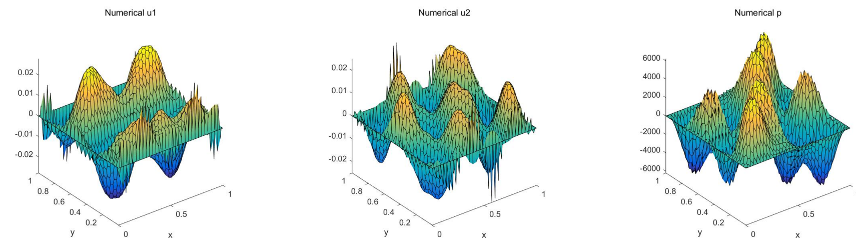
(a) exact solution u_1 (b) exact solution u_2 (c) exact solution p

FIGURE 3. Numerical solutions $\mathbf{u}$ and p on $\mathcal{T}_h^1$ for $\tau = \delta = 0.1,\ h = 1/64$.

Tables 1-3 and Table 4 below show that error estimations and convergence rates for stabilization VEM on hexagonal mesh $\mathcal{T}_h^1$ and distorted hexagons mesh $\mathcal{T}_h^2$, respectively.

TABLE 1. Numerical results of the CBB scheme on hexagonal mesh $\mathcal{T}_h^1$ for order=1.

$1/h$	$\tau = 1,\ \delta = 0$						$\tau = 0.1,\ \delta = 0.1$					
	$e_{\mathbf{u},0}$	rate	$e_{\mathbf{u},1}$	rate	$e_{p,0}$	rate	$e_{\mathbf{u},0}$	rate	$e_{\mathbf{u},1}$	rate	$e_{p,0}$	rate
8	0.1656		2.2786		1.5163		0.1355		2.1351		1.2987	
16	0.0363	1.77	1.0418	0.91	0.3812	1.61	0.0256	1.95	1.0056	0.88	0.2667	1.85
32	0.0124	1.82	0.5574	1.06	0.2751	0.55	0.0069	2.22	0.5287	1.09	0.1074	1.54
64	0.0054	1.02	0.2840	0.83	0.1498	0.75	0.0017	1.71	0.2631	0.86	0.0335	1.43
128	0.0020	1.72	0.1398	1.21	0.0612	1.53	0.0004	2.31	0.1318	1.18	0.0115	1.83



(a) numerical solution u_1 (b) numerical solution u_2 (c) numerical solution p

FIGURE 4. Numerical solutions $\mathbf{u}$ and p on $\mathcal{T}_h^1$ for $\tau = \delta = 0,\ h = 1/64$.

TABLE 2. Numerical results of the CBB scheme on hexagonal mesh $\mathcal{T}_h^1$ for order=2.

$1/h$	$\tau=1,\ \delta=0$						$\tau=0.1,\ \delta=0.1$					
	$e_{\mathbf{u},0}$	rate	$e_{\mathbf{u},1}$	rate	$e_{p,0}$	rate	$e_{\mathbf{u},0}$	rate	$e_{\mathbf{u},1}$	rate	$e_{p,0}$	rate
8	0.0861		1.4073		1.4404		0.0501		1.0405		0.7920	
16	0.0126	2.25	0.3623	1.58	0.3371	1.70	0.0047	2.77	0.2569	1.63	0.1395	2.03
32	0.0012	3.91	0.0726	2.72	0.0475	3.32	0.0005	3.65	0.0662	2.30	0.0300	2.60
64	0.0001	3.04	0.0164	1.82	0.0079	2.20	6e-5	2.66	0.0161	1.73	0.0069	1.81
128	1e-5	4.18	0.0041	2.39	0.0017	2.60	8e-6	3.57	0.0041	2.36	0.0017	2.40

TABLE 3. Numerical results of the CBB scheme on hexagonal mesh $\mathcal{T}_h^1$ for order=3.

$1/h$	$\tau=1,\ \delta=0$						$\tau=0.1,\ \delta=0.1$					
	$e_{\mathbf{u},0}$	rate	$e_{\mathbf{u},1}$	rate	$e_{p,0}$	rate	$e_{\mathbf{u},0}$	rate	$e_{\mathbf{u},1}$	rate	$e_{p,0}$	rate
8	0.0266		0.5876		0.5192		0.0216		0.5057		0.2600	
16	0.0017	3.21	0.0782	2.36	0.0643	2.44	0.0014	3.17	0.0661	2.38	0.0342	2.37
32	9e-5	4.91	0.0092	3.63	0.0069	3.77	9e-5	4.65	0.0086	3.44	0.0043	3.52
64	5e-6	3.60	0.0010	2.66	0.0008	2.68	5e-6	3.45	0.0010	2.62	0.0004	2.82
128	3e-7	4.86	0.0001	3.66	0.00008	3.85	3e-7	4.75	0.0001	3.57	0.00005	3.65

TABLE 4. Numerical results of the CBB scheme with $\tau = 0.1,\ \delta = 0.1$ on distorted hexagons mesh $\mathcal{T}_h^2$ for order=1, 2, 3.

$1/h$	order	$\tau=0.1,\ \delta=0.1$					
		$e_{\mathbf{u},0}$	rate	$e_{\mathbf{u},1}$	rate	$e_{p,0}$	rate
8	1	0.1429		2.115		1.5112	
16		0.046571	2.02	1.1814	1.05	0.51355	1.94
32		0.014708	1.77	0.63691	0.95	0.16361	1.75
64		0.0039511	1.92	0.31852	1.01	0.046279	1.84
128		0.0010295	1.95	0.15978	1.00	0.012444	1.90
8	2	0.06686		1.2246		0.793	
16		0.012342	3.05	0.40492	1.99	0.25149	2.07
32		0.0016408	3.10	0.11042	1.99	0.055013	2.33
64		0.00020654	3.02	0.027247	2.04	0.011852	2.24
128		3.5226e-05	2.56	0.0069229	1.98	0.0029794	2.00
8	3	0.039838		0.73001		0.4911	
16		0.0036908	4.29	0.13101	3.10	0.092532	3.01
32		0.00028852	3.90	0.019896	2.89	0.015577	2.73
64		1.8365e-05	4.02	0.0024378	3.06	0.0021388	2.90
128		1.3474e-06	3.78	0.00033696	2.87	0.00039827	2.44

We also take corresponding tests for traditional triangle mesh $\mathcal{T}_h^3$. Tables 5-7 below show the numerical results for order=1,2,3. Table 8 below shows the error estimates and convergence rates without the least-square stabilization terms, i.e., $\tau = \delta = 0$.

TABLE 5. Numerical results of the CBB scheme on triangle mesh $\mathcal{T}_h^3$ for order=1.

$1/h$	$\tau = 1,\ \delta = 0$						$\tau = 0.1,\ \delta = 0.1$					
	$e_{\mathbf{u},0}$	rate	$e_{\mathbf{u},1}$	rate	$e_{p,0}$	rate	$e_{\mathbf{u},0}$	rate	$e_{\mathbf{u},1}$	rate	$e_{p,0}$	rate
8	0.0660		1.3352		0.6784		0.0547		1.2693		0.4207	
16	0.0200	1.72	0.7148	0.90	0.3005	1.17	0.0167	1.71	0.6895	0.88	0.0994	2.08
32	0.0079	1.34	0.3774	0.92	0.1876	0.68	0.0045	1.89	0.3534	0.96	0.0303	1.72
64	0.0030	1.39	0.1890	1.00	0.0847	1.15	0.0015	1.97	0.1778	0.99	0.0097	1.63
128	0.0010	1.70	0.0922	1.04	0.0299	1.50	0.0003	1.99	0.0890	1.00	0.0032	1.59

TABLE 6. Numerical results of the CBB scheme on triangle mesh $\mathcal{T}_h^3$ for order=2.

$1/h$	$\tau = 1,\ \delta = 0$						$\tau = 0.1,\ \delta = 0.1$					
	$e_{\mathbf{u},0}$	rate	$e_{\mathbf{u},1}$	rate	$e_{p,0}$	rate	$e_{\mathbf{u},0}$	rate	$e_{\mathbf{u},1}$	rate	$e_{p,0}$	rate
8	0.0089		0.434		0.268		0.0090		0.439		0.254	
16	0.0011	3.04	0.115	1.92	0.067	2.00	0.0011	2.97	0.119	1.878	0.066	1.95
32	0.0001	3.03	0.029	1.99	0.017	2.02	0.0001	2.999	0.031	1.968	0.017	1.99
64	2e-05	3.01	0.007	2.00	0.004	2.01	2e-05	3.00	0.008	1.991	0.004	2.00
128	2e-06	3.00	0.002	2.00	0.001	2.00	2e-06	3.00	0.002	1.998	0.001	2.00

TABLE 7. Numerical results of the CBB scheme on triangle mesh $\mathcal{T}_h^3$ for order=3.

$1/h$	$\tau = 1,\ \delta = 0$						$\tau = 0.1,\ \delta = 0.1$					
	$e_{\mathbf{u},0}$	rate	$e_{\mathbf{u},1}$	rate	$e_{p,0}$	rate	$e_{\mathbf{u},0}$	rate	$e_{\mathbf{u},1}$	rate	$e_{p,0}$	rate
8	0.0117		0.4137		0.2099		0.0140		0.4788		0.2598	
16	0.0013	3.16	0.0850	2.28	0.0502	2.06	0.0013	3.40	0.0856	2.48	0.0487	2.42
32	0.0001	3.61	0.0128	2.73	0.0089	2.50	0.0001	3.72	0.0122	2.80	0.0081	2.59
64	8e-6	3.80	0.0017	2.88	0.0013	2.81	7e-6	3.86	0.0016	2.92	0.0011	2.85
128	5e-7	3.93	0.0002	2.96	0.0002	2.94	4e-7	3.95	0.0002	2.97	0.0001	2.95

TABLE 8. Numerical results on triangle mesh $\mathcal{T}_h^3$ for $\tau = \delta = 0$ and order=2.

$1/h$	$e_{\mathbf{u},0}$	rate	$e_{\mathbf{u},1}$	rate	$e_{p,0}$	rate
8	0.07		1		1.0e+16*21	
16	27.40	-8.68	1334	-10.08	1.0e+16*8645	-8.72
32	379666.25	-13.76	47365122	-15.12	1.0e+23*3.1	-11.83
64	1.0e+13*1.41	-25.15	1.0e+15*3.6	-26.19	1.0e+31*4.2	-26.995
128	1.0e+13*0.0002	12.76	1.0e+15*0.001	11.73	1.0e+27*1.5	14.76

As can be seen from the above Figures 2-4 and Tables 1-8, the least-square stabilization terms we add are indispensable and our method is effective. Then, the log-log curves of the error quantities for both velocity and pressure versus the mesh sizes h are shown in Figures 5-7. From these figures, the desired results can be observed.

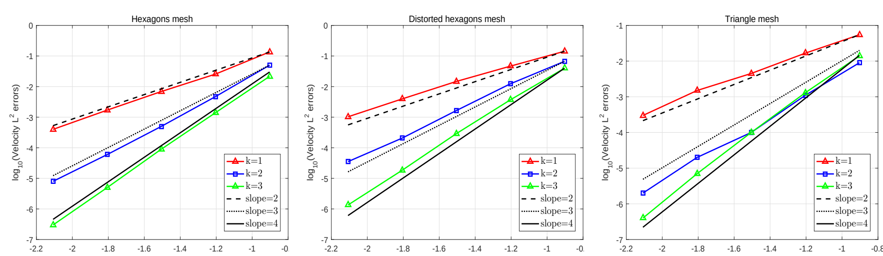


FIGURE 5. Behaviour of $e_{\mathbf{u},0}$ on $\mathcal{T}_h^1$, $\mathcal{T}_h^2$ and $\mathcal{T}_h^3$ for $\tau = 0.1,\ \delta = 0.1$ and order=1, 2, 3.

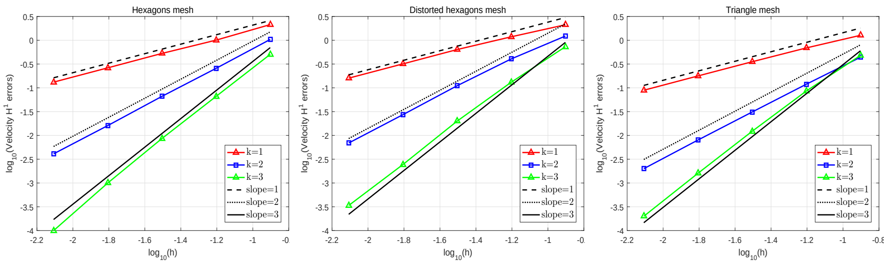


FIGURE 6. Behaviour of $e_{\mathbf{u},1}$ on $\mathcal{T}_h^1$, $\mathcal{T}_h^2$ and $\mathcal{T}_h^3$ for $\tau = 0.1,\ \delta = 0.1$ and order=1, 2, 3.

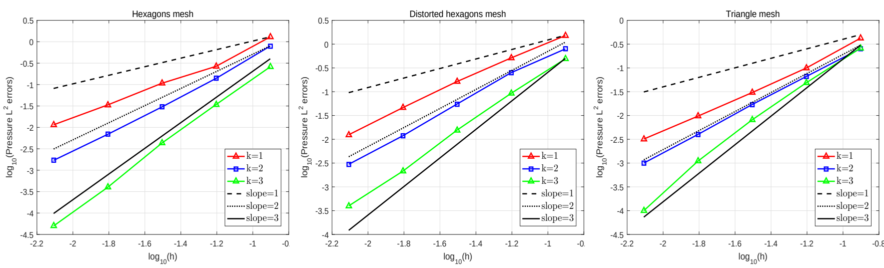


FIGURE 7. Behaviour of $e_{p,0}$ on $\mathcal{T}_h^1$, $\mathcal{T}_h^2$ and $\mathcal{T}_h^3$ for $\tau = 0.1,\ \delta = 0.1$ and order=1, 2, 3.

7. Conclusions

In this paper, we considered a least-square stabilization virtual element method for the Stokes problem on general polygonal meshes. The VEM approximation, which includes both the standard Galerkin terms and the stabilization terms, has been described in detail. This method can not only circumvent the Babuška-Brezzi condition, but also make use of general polygonal meshes, as opposed to more standard triangular grids. Based on appropriate assumptions of coefficients and meshes, the stability and optimal error estimates in energy norm and L^2 norm for velocity were obtained. The numerical experiments agree with the theoretical analysis. The ideas can also be used to solve other problems, which will be considered in our future work.

Acknowledgments

This research was supported by the Natural Science Foundation of China (No.11971337). In particular, our deepest gratitude goes to the anonymous reviewers for their careful work and thoughtful suggestions that have helped improve this paper substantially.

References

[1] D. A. Di Pietro and A. Ern. A hybrid high-order locking-free method for linear elasticity on general meshes. Comput. Methods Appl. Mech. Engrg., 283:1–21, 2015.

[2] F. Brezzi, K. Lipnikov, and M. Shashkov. Convergence of the mimetic finite difference method for diffusion problems on polyhedral meshes. SIAM J. Numer. Anal., 43(5):1872–1896, 2005.

[3] J. Wang and X. Ye. A weak galerkin mixed finite element method for second order elliptic problems. Math. Comp., 83(289):2101–2126, 2014.

[4] G. Chen, M. Feng, and X. Xie. A robust wg finite element method for convection-diffusion-reaction equations. J. Comput. Appl. Math., 315:107–125, 2017.
[5] L. Zhang, M. Feng, and J. Zhang. A globally divergence-free weak galerkin method for brinkman equations. Appl. Numer. Math., 137:213–229, 2019.
[6] L. Beirão da Veiga, F. Brezzi, A. Cangiani, G. Manzini, L. D. Marini, and A. Russo. Basic principles of virtual element methods. Math. Models Methods Appl. Sci., 23(1):199–214, 2013.
[7] F. Brezzi, K. Lipnikov, and M. Shashkov. Convergence of mimetic finite difference method for diffusion problems on polyhedral meshes with curved faces. Math. Models Methods Appl. Sci., 16(2):275–297, 2006.
[8] F. Brezzi, K. Lipnikov, and V. Simoncini. A family of mimetic finite difference methods on polygonal and polyhedral meshes. Math. Models Methods Appl. Sci., 15(10):1533–1551, 2005.
[9] P. Bochev and J. M. Hyman. Principles of mimetic discretizations of differential operators. In Compatible spatial discretizations, volume 142 of IMA Vol. Math. Appl., pages 89–119. Springer, New York, 2006.
[10] F. Brezzi, K. Lipnikov, M. Shashkov, and V. Simoncini. A new discretization methodology for diffusion problems on generalized polyhedral meshes. Comput. Methods Appl. Mech. Engrg., 196(37-40):3682–3692, 2007.
[11] A. Cangiani, G. Manzini, and A. Russo. Convergence analysis of the mimetic finite difference method for elliptic problems. SIAM J. Numer. Anal., 47(4):2612–2637, 2009.
[12] B. Ahmad, A. Alsaedi, F. Brezzi, L. D. Marini, and A. Russo. Equivalent projectors for virtual element methods. Comput. Math. Appl., 66(3):376–391, 2013.
[13] L. Beirão da Veiga, F. Brezzi, L. D. Marini, and A. Russo. Virtual element method for general second-order elliptic problems on polygonal meshes. Math. Models Methods Appl. Sci., 26(4):729–750, 2016.
[14] G. Vacca and L. Beirão da Veiga. Virtual element methods for parabolic problems on polygonal meshes. Numer. Methods Partial Differential Equations, 31(6):2110–2134, 2015.
[15] B. Zhang, Y. Yang, and M. Feng. Mixed virtual element methods for elastodynamics with weak symmetry. J. Comput. Appl. Math., 353:49–71, 2019.
[16] B. Zhang and M. Feng. Virtual element method for two-dimensional linear elasticity problem in mixed weakly symmetric formulation. Appl. Math. Comput., 328:1–25, 2018.
[17] P. F. Antonietti, L. Beirão da Veiga, S. Scacchi, and M. Verani. A C^1 virtual element method for the cahn-hilliard equation with polygonal meshes. SIAM J. Numer. Anal., 54(1):34–56, 2016.
[18] L. Beirão da Veiga, C. Lovadina, and G. Vacca. Divergence free virtual elements for the stokes problem on polygonal meshes. ESAIM Math. Model. Numer. Anal., 51(2):509–535, 2017.
[19] P. F. Antonietti, L. Beirão da Veiga, D. Mora, and M. Verani. A stream virtual element formulation of the Stokes problem on polygonal meshes. SIAM J. Numer. Anal., 52(1):386–404, 2014.
[20] L. Beirão da Veiga, F. Dassi, and G. Vacca. The Stokes complex for virtual elements in three dimensions. Math. Models Methods Appl. Sci., 30(3):477–512, 2020.
[21] G. Wang, F. Wang, L. Chen, and Y. He. A divergence free weak virtual element method for the Stokes-Darcy problem on general meshes. Comput. Methods Appl. Mech. Engrg., 344:998–1020, 2019.
[22] E. Cáceres and G. N. Gatica. A mixed virtual element method for the pseudostress-velocity formulation of the Stokes problem. IMA J. Numer. Anal., 37(1):296–331, 2017.
[23] A. Chernov, C. Marcati, and L. Mascotto. *p*- and *hp*-virtual elements for the Stokes problem. Adv. Comput. Math., 47(2):Paper No. 24, 31, 2021.
[24] G. Manzini and A. Mazzia. A virtual element generalization on polygonal meshes of the scott-vogelius finite element method for the 2-d stokes problem. United States: N. p., 2021.
[25] X. Liu, J. Li, and Z. Chen. A nonconforming virtual element method for the Stokes problem on general meshes. Comput. Methods Appl. Mech. Engrg., 320:694–711, 2017.
[26] G. N. Gatica, M. Munar, and F. A. Sequeira. A mixed virtual element method for the Navier-Stokes equations. Math. Models Methods Appl. Sci., 28(14):2719–2762, 2018.
[27] L. Beirão da Veiga, D. Mora, and G. Vacca. The Stokes complex for virtual elements with application to Navier-Stokes flows. J. Sci. Comput., 81(2):990–1018, 2019.
[28] X. Liu and Z. Chen. The nonconforming virtual element method for the Navier-Stokes equations. Adv. Comput. Math., 45(1):51–74, 2019.

[29] L. Beirão da Veiga, C. Lovadina, and G. Vacca. Virtual elements for the navier-stokes problem on polygonal meshes. SIAM J. Numer. Anal., 56(3):1210–1242, 2018.
[30] L. Beirão da Veiga, F. Brezzi, L. D. Marini, and A. Russo. $H(\mathrm{div})$ and $H(\mathbf{curl})$-conforming virtual element methods. Numer. Math., 133(2):303–332, 2016.
[31] F. Brezzi and L. D. Marini. Virtual element and discontinuous Galerkin methods. In Recent developments in discontinuous Galerkin finite element methods for partial differential equations, volume 157 of IMA Vol. Math. Appl., pages 209–221. Springer, Cham, 2014.
[32] B. Ayuso de Dios, K. Lipnikov, and G. Manzini. The nonconforming virtual element method. ESAIM Math. Model. Numer. Anal., 50(3):879–904, 2016.
[33] X. Liu, J. Li, and Z. Chen. A nonconforming virtual element method for the Stokes problem on general meshes. Comput. Methods Appl. Mech. Engrg., 320:694–711, 2017.
[34] F. Brezzi, R. S. Falk, and L. D. Marini. Basic principles of mixed virtual element methods. ESAIM Math. Model. Numer. Anal., 48(4):1227–1240, 2014.
[35] J. Wen, M. Feng, and Y. He. Convergence analysis of a new multiscale finite element method with the P_1/P_0 element for the incompressible flow. Comput. Methods Appl. Mech. Engrg., 258:13–25, 2013.
[36] L. P. Franca and T. J. R. Hughes. Two classes of mixed finite element methods. Comput. Methods Appl. Mech. Engrg., 69(1):89–129, 1988.
[37] D. Irisarri and G. Hauke. Stabilized virtual element methods for the unsteady incompressible Navier-Stokes equations. Calcolo, 56(4):Paper No. 38, 21, 2019.
[38] S. Berrone, A. Borio, and G. Manzini. SUPG stabilization for the nonconforming virtual element method for advection-diffusion-reaction equations. Comput. Methods Appl. Mech. Engrg., 340:500–529, 2018.
[39] M. F. Benedetto, S. Berrone, A. Borio, S. Pieraccini, and S. Scialò. Order preserving SUPG stabilization for the virtual element formulation of advection-diffusion problems. Comput. Methods Appl. Mech. Engrg., 311:18–40, 2016.
[40] J. Guo and M. Feng. A new projection-based stabilized virtual element method for the Stokes problem. J. Sci. Comput., 85(1):Paper No. 16, 28, 2020.
[41] V. Girault and P.A. Raviart. Finite element methods for Navier-Stokes equations, volume 5 of Springer Series in Computational Mathematics. Springer-Verlag, Berlin, 1986. Theory and algorithms.
[42] L. P. Franca, S. Frey, and T. J. R. Hughes. Stabilized finite element methods. I. Application to the advective-diffusive model. Comput. Methods Appl. Mech. Engrg., 95(2):253–276, 1992.
[43] A. Cangiani, E. H. Georgoulis, T. Pryer, and O. J. Sutton. A posteriori error estimates for the virtual element method. math.NA., 2016.
[44] L. Chen and J. Huang. Some error analysis on virtual element methods. Calcolo, 55(1):Paper No. 5, 23, 2018.
[45] J. Douglas, Jr. and J. Wang. An absolutely stabilized finite element method for the Stokes problem. Math. Comp., 52(186):495–508, 1989.
[46] R. B. Kellogg and J. E. Osborn. A regularity result for the Stokes problem in a convex polygon. J. Functional Analysis, 21(4):397–431, 1976.

College of Mathematics, Sichuan University, 610065, Chengdu, China
E-mail: yang1020zz@163.com, huchaolang@scu.edu.cn and fmf@scu.edu.cn