INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 19, Number 5, Pages 669–684

LOW ORDER MIXED FINITE ELEMENT APPROXIMATIONS OF THE MONGE-AMPÈRE EQUATION

JAMAL ADETOLA, BERNADIN AHOUNOU, GERARD AWANOU, AND HAILONG GUO

Abstract. In this paper, we are interested in the analysis of the convergence of a low order mixed finite element method for the Monge-Ampère equation. The unknowns in the formulation are the scalar variable and the discrete Hessian. The distinguished feature of the method is that the unknowns are discretized using only piecewise linear functions. A superconvergent gradient recovery technique is first applied to the scalar variable, then a piecewise gradient is taken, the projection of which gives the discrete Hessian matrix. For the analysis we make a discrete elliptic regularity assumption, supported by numerical experiments, for the discretization based on gradient recovery of an equation in non divergence form. A numerical example which confirms the theoretical results is presented.

Key words. Monge-Ampère, mixed finite element, gradient recovery, non divergence form.

1. Introduction

In this paper, we analyze a linear finite element discretization of the elliptic Monge-Ampère equation for smooth solutions on a convex polygonal domain. The method is a variant of the method introduced in [15] for which numerical experiments for both smooth and non-smooth solutions were reported in [20]. Let Ω be a convex polygonal domain in $\mathbb{R}^2$ endowed with a triangulation $\mathcal{T}_h$ which is conforming and quasi-uniform. For the purpose of our analysis, we further assume the triangulation to be uniform, i.e. two triangles sharing an edge form a parallelogram. Let V_h denote the space of piecewise linear continuous functions on Ω and let Σ_h denote the space of piecewise linear continuous 2×2 matrix fields on Ω. Our goal is to seek an element $u_h \in V_h$ which approximates the unique strictly convex $C^4(\overline{\Omega})$ solution u (when it exists) of the problem

$$\begin{aligned} \det(D^2u) &= f \text{ in } \Omega, \\ u &= g \text{ on } \partial\Omega. \end{aligned} \tag{1}$$

The right hand side function $f \in C^2(\overline{\Omega})$ is assumed to satisfy $f > 0$. The boundary function $g \in C(\partial\Omega)$ is also given and assumed to extend to a $C^4(\overline{\Omega})$ convex function. Here we use $\det(D^2u)$ to denote the determinant of the Hessian matrix $D^2u = \big(\partial^2 u/(\partial x_i \partial x_j)\big)_{i,j=1,2}$.

The discrete problem is to find $u_h \in V_h$ such that $u_h = g_h$ on $\partial\Omega$ and

$$\int_\Omega (f - \det H(u_h)) v \, dx = 0, \forall v \in V_h \cap H_0^1(\Omega), \tag{2}$$

where $H(u_h)$, the discrete Hessian of u_h, is an element of Σ_h defined by

$$\int_\Omega H(u_h) : \mu \, dx = \int_\Omega (DG_h u_h) : \mu \, dx, \forall \mu \in \Sigma_h. \tag{3}$$

The operator $G_h : V_h \to V_h \times V_h$ in (3) is taken as the weighted average gradient recovery operator and is somehow a substitution for the gradient operator. The

Received by the editors April 2, 2020 and, in revised form, May 30, 2022; accepted June 17, 2022.
2000 *Mathematics Subject Classification.* 65N30, 65N15, 35J96.

finite element function g_h is the standard finite element interpolation of the continuous function g in V_h. For two matrices A and B, $A : B$ denotes their Frobenius inner product. We denote by $\mathcal{E}_h^i$ the set of interior edges of $\mathcal{T}_h$and by $\mathcal{N}_h$ the set of vertices of $\mathcal{T}_h$. For a vector field v, Dv denotes its piecewise gradient vector, the matrix field with rows the gradients of the corresponding components of v,.

The Monge-Ampère operator appears in a number of problems where the solution is known to be smooth. For example, it appears in the study of von Kármán model for plate buckling [5]. It is argued in [17] that for meteorological applications for which legacy finite element codes are used for the discretization of other differential operators, it could be advantageous to use a finite element discretization as well for the Monge-Ampère operator. The readers are referred to [9, 21] and the references therein for a review of numerical methods for Monge-Ampère type equations.

Problem (2) with the discrete Hessian (3) is equivalent to the following mixed formulation: find $(u_h, \sigma_h) \in V_h \times \Sigma_h$ such that $u_h = g_h$ on $\partial\Omega$

$$
\begin{aligned}
&\int_\Omega (f - \det \sigma_h) v \, dx = 0, \quad \forall v \in V_h \cap H_0^1(\Omega) \\
&\int_\Omega \sigma_h : \mu \, dx = \int_\Omega (DG_h u_h) : \mu \, dx, \quad \forall \mu \in \Sigma_h.
\end{aligned} \tag{4}
$$

Analysis of discretizations similar to (2) and (4) for cubic and higher order elements were conducted in [20, 3]. Problem (2), with the gradient recovery operator replaced by the piecewise gradient in a weak formulation of (3), was proposed in [15, 20] for quadratic and higher order approximations, c.f. Remark 3.2 below. See also [20] for a version with linear approximations. Related ideas can be found in [14, 10, 16, 11]. Our error analysis is based on the above formulation (4). We use the same argument as in [20, 3].

In addition, we make a discrete elliptic regularity assumption for the discretization based on a gradient recovery operator of the non divergence form of a linear elliptic equation. We support this assumption with numerical experiments. The linear elliptic equation considered is the linearization of the Monge-Ampère equation and can be written in both divergence and non divergence forms. A discrete elliptic regularity approach for a linear equation in divergence form was first used in [19] for interior penalty methods for the Monge-Ampère equation on a smooth domain. It was recently used in [2] for a mixed method under an assumption of elliptic regularity for the linearization of the continuous problem.

We show that the piecewise gradient of the recovered gradient of the finite element solution converges at a rate $\mathcal{O}(h)$ to the piecewise gradient of the recovered gradient of the interpolant in the L^p norm with $|\ln h| \leq p \leq 2|\ln h|$, and the discrete Hessian converges at a rate $\mathcal{O}(h)$ in the L^∞ norm.

Our analysis is limited to uniform partitions of a convex polygonal domain so that we can take advantage of a superconvergent approximation property for the gradient recovery operator proved in [23], c.f. (11) below. We want to emphasize that although we only give the analysis on uniform meshes, numerical results indicate that the results may hold on general Delaunay triangulations. Elements of Σ_h can be required to be symmetric matrix fields to reduce the number of unknowns. The analysis of this paper also holds in that case.

The rest of the paper is organized as follows. In Section 2, we present some additional notation and preliminaries. In Section 3, we conduct the error estimate for the discrete Monge-Ampère equation. In section 4, we give numerical results for a smooth solution to support our theoretical results. Some conclusions are drawn in section 5. In an appendix, we collect some detailed calculations and give numerical results to support our discrete elliptic regularity assumption.

Remark 1.1. *Part of this paper is based on the Ph.D. thesis of Jamal Adetola* [1].

2. Preliminaries

For a subdomain $\mathcal{S}$ of Ω and a given real number $1 \leq p \leq \infty$, let $W^{k,p}(\mathcal{S})$ denote the Sobolev space with norm $\| \cdot \|_{W^{k,p}}$ and seminorm $| \cdot |_{W^{k,p}}$. The Sobolev space $W^{k,p}(\mathcal{S})$ reduces to the standard Lebesgue space $L^p(\mathcal{S})$ and its norm is denoted $\| \cdot \|_{L^p(\mathcal{S})}$ when $k = 0$. When $p = 2$, we denote simply $W^{k,2}(\mathcal{S})$ by $H^k(\mathcal{S})$ and the corresponding norm is denoted $\| \cdot \|_{H^k}$. In addition, we let $H_0^1(\mathcal{S})$ be the subset of $H^1(\mathcal{S})$ of elements with vanishing traces. Similarly, $W_0^{k,p}(\mathcal{S})$ denotes the subset of $W^{k,p}(\mathcal{S})$ of elements with vanishing traces.

Given a normed space X with norm $||.||_X$, let X^2 denote the space of vector fields with components in X and let $X^{2\times 2}$ denote the space of matrix fields with each component in X. If X is finite dimensional of dimension N, then X^2 has dimension $2N$ and $X^{2\times 2}$ has dimension $4N$. The inner products in $L^2(\Omega), L^2(\Omega)^2$, and $L^2(\Omega)^{2\times 2}$ are denoted by $(\cdot, \cdot)$ and the inner products on $L^2(\partial\Omega)$ and $L^2(\partial\Omega)^2$ are denoted $\langle \cdot, \cdot \rangle$. For $1 \leq p < \infty$ and $v = (v_i)_{i=1}^2 \in W^{k,p}(\mathcal{S})^2$, its norm is given by $(||v||_{W^{k,p}})^p = (||v_1||_{W^{k,p}})^p + (||v_2||_{W^{k,p}})^p$. Similarly $\sigma = (\sigma_{ij})_{i,j=1}^2 \in W^{k,p}(\mathcal{S})^{2\times 2}$ has norm given by $(||\sigma||_{W^{k,p}})^p = \sum_{i,j=1}^2 (||\sigma_{ij}||_{W^{k,p}})^p$.
Put $||v||_{W^{k,\infty}} = \max_{i=1,2} ||v_i||_{W^{k,\infty}}$ and $||\sigma||_{W^{k,\infty}} = \max_{i,j=1,2} ||\sigma_{ij}||_{W^{k,\infty}}$.

Let n denote the unit outward normal vector to $\partial\Omega$. For a matrix A with entries A_{ij}, recall that the cofactor matrix of A, denoted $\operatorname{cof} A$, is the matrix with entries $(\operatorname{cof} A)_{ij} = (-1)^{i+j} \det(A)_i^j$ where $\det(A)_i^j$ is the determinant of the matrix obtained from A by deleting its ith row and its jth column. For two matrices $A = (A_{ij})_{i,j=1,2}$ and $B = (B_{ij})_{i,j=1,2}$, their Frobenius inner product is given by $A : B = \sum_{i,j=1}^{2} A_{ij} B_{ij}$.

We define the divergence of a matrix field as the vector obtained by taking the divergence of each row. We denote by h_K the diameter of the element K and put $h = \max_{K \in \mathcal{T}_h} h_K$. We assume that $h \leq 1$ and denote by h_e the length of the edge e. We assume that the triangulation is conforming and quasi-uniform, i.e. there is a constant $C > 0$ such that $h \leq C\rho_K$ for all $K \in \mathcal{T}_h$, where ρ_K denotes the radius of the largest ball inside K. Finally, we require that two triangles sharing an edge form a parallelogram. Triangulations with the latter property are called uniform. Constants are named and unless indicated, are independent of h and p.

For a scalar function v, Dv denotes its piecewise gradient vector when it is defined. We will often use the inverse estimate [6, Theorem 4.5.11]

$$\|z_h\|_{W^{t,p}(\mathcal{T}_h)} \leq C_1 h^{s-t+\min(0,\frac{2}{p}-\frac{2}{q})} \|z_h\|_{W^{s,q}(\mathcal{T}_h)}, \tag{5}$$

for $0 \leq s \leq t, 1 \leq p, q \leq \infty$ and $z_h \in V_h$. As explained in [2] the constant C_1 in (5) is independent of h and p. In particular,

$$||Dv||_{L^\infty} \leq C_1 h^{-1} ||v||_{L^\infty}, \forall v \in V_h. \tag{6}$$

Let $I_h(v)$ denote the Lagrange interpolant of $v \in C(\Omega)$. We have

$$||v - I_h v||_{W^{j,p}} \leq C_2 h^{2-j} \|v\|_{W^{2,p}}, \forall v \in W^{2,p}(\Omega), j = 0, 1 \text{ and } 2 \leq p \leq \infty. \tag{7}$$

We will use the same constant C_2 for constants arising from an interpolation estimate. We note that for $v \in C(\Omega) \cap W^{1,p}(\Omega)$ and $p > 2$ the interpolation and stability estimates

$$||v - I_h v||_{L^p} \leq C_2 h ||v||_{W^{1,p}}$$
$$||DI_h v||_{L^p} \leq C_2 ||v||_{W^{1,p}}, \tag{8}$$

hold [6, Corollary 4.4.24], where we use for simplicity the same constant C_2 as in the interpolation error estimate (7).

We make the abuse of notation of denoting by $I_h \sigma$ the matrix field with components the corresponding Lagrange interpolants of the components of σ. Again by an abuse of notation, let $I_h(Dv)$ denote the Lagrange interpolant of $Dv \in C(\Omega)^2$. Applying (7) to each component of $\sigma - I_h \sigma$ we have

$$||\sigma - I_h \sigma||_{L^p(K)} \leq 2C_2 h_K^2 \|\sigma\|_{W^{2,p}}, \forall \sigma \in W^{2,p}(K)^{2\times 2}. \tag{9}$$

Recall that $G_h : V_h \to V_h \times V_h$ denotes the weighted average gradient recovery operator. For any vertex $P \in \mathcal{N}_h$, let $\omega_P := \{ \tau \in \mathcal{T}_h, P \in \overline{\tau} \}$ be the union of local elements attached to P. For $v_h \in V_h$, the recovered gradient at the vertex P is defined by

$$(G_h v_h)(P) = \frac{1}{|\omega_P|} \int_{\omega_P} Dv_h,$$

where $|\omega_P|$ is the measure of ω_P. The recovered gradient function is then defined as

$$G_h v_h = \sum_{P \in \mathcal{N}_h} (G_h v_h)(P) \phi_P,$$

where ϕ_P is the linear nodal basis function corresponding to P. It is known that this definition is equivalent on uniform meshes to the polynomial preserving recovery operator analyzed in [23]. Thus, if we assume that the mesh is uniform, as in [18, Theorem 3.2], we have for all $v \in V_h$ and $p \geq 2$

$$||G_h v||_{L^p} \leq C_3 ||Dv||_{L^p}. \tag{10}$$

Moreover, analogous to [12, Lemma 4.5], G_h is superconvergent in the sense that for $2 \leq p \leq \infty$

$$||Du - G_h I_h u||_{L^p} \leq C_4 h^2 ||u||_{W^{3,p}}. \tag{11}$$

We will also need the simpler convergence estimate (using the same constant C_4 for convenience)

$$||Du - G_h I_h u||_{L^p} \leq C_4 h ||u||_{W^{2,p}}, \tag{12}$$

the proof of which is similar to the proof of (11), based on the Bramble-Hilbert lemma.

Arguing as for the proof of [13, Theorem 3.2], we have for $2 \leq p \leq \infty$

$$\begin{aligned}
||D^2 u - DG_h I_h u||_{L^p} &\leq ||D^2 u - D(I_h(Du))||_{L^p} + ||D(I_h(Du)) - DG_h I_h u||_{L^p} \\
&\leq C_2 h ||Du||_{W^{2,p}} + C_1 h^{-1} ||I_h(Du) - G_h I_h u||_{L^p} \\
&\leq C_2 h ||Du||_{W^{2,p}} + C_1 h^{-1} ||I_h(Du) - Du||_{L^p} \\
&\qquad\qquad + C_1 h^{-1} ||Du - G_h I_h u||_{L^p} \\
&\leq C_2 h ||Du||_{W^{2,p}} + C_2 C_1 h ||Du||_{W^{2,p}} + C_4 C_1 h ||u||_{W^{3,p}} \\
&\leq (C_2 + C_2 C_1 + C_4 C_1) h ||u||_{W^{3,p}},
\end{aligned}$$

where in the first step we use a triangular inequality, (7) and (6) in the second step, a triangular inequality in the third step, then (6) and (11) in the fourth. We put

$C_5 = C_2 + C_2C_1 + C_4C_1$, and we have

$$||D^2u - DG_hI_hu||_{L^p} \leq C_5h||u||_{W^{3,p}}. \tag{13}$$

We have for $v \in V_h$

$$||\operatorname{div} G_hv||_{L^p} \leq ||DG_hv||_{L^p}.$$

Analogous to [8, Lemma 3], c.f. the appendix,for $v_h \in V_h \cap H_0^1(\Omega)$ and $p \geq 2$

$$\|G_hv_h - Dv_h\|_{L^p} \leq C_6h\| \operatorname{div}(G_hv_h)\|_{L^p}. \tag{14}$$

By Poincaré's inequality and [8, Lemma 4], for all $v \in V_h \cap H_0^1(\Omega)$

$$||v||_{L^2} \leq C_7||Dv||_{L^2} \leq C_8||DG_hv||_{L^2}. \tag{15}$$

Since for $p \geq 2$, $||DG_hv||_{L^2} \leq C_9||DG_hv||_{L^p}$, it follows from (15) that if $||DG_hv||_{L^p} = 0$ for $v_h \in V_h \cap H_0^1(\Omega)$, we have $v_h = 0$. We shall consider the following norm on $V_h \cap H_0^1(\Omega)$ for $p \geq 2$

$$\|v\|^p_{\widetilde{W}^{1,p}(\Omega)} := ||DG_hv||^p_{L^p}. \tag{16}$$

Let n_K denote the outward normal to an element K of $\mathcal{T}_h$ and let v_K denote the restriction of the field v on K. For any edge $e \subset \partial K$ such that $e = \partial K \cap \partial L$ for $L \in \mathcal{T}_h$, we define for a vector v the jump of v by $[\![v]\!]_e = v_K \cdot n_K + v_L \cdot n_L$. If e is a boundary edge, i.e. $e = \partial K \cap \partial\Omega$, we let $[\![v]\!]_e = v_K \cdot n_K$.

3. Error analysis for smooth solutions

We need the following weak formulation of (1): find $(u, \sigma) \in W^{4,\infty}(\Omega) \times W^{2,\infty}(\Omega)^{2\times 2}$ such that for all $K \in \mathcal{T}_h$

$$\begin{aligned}(\sigma, \mu)_K + (\operatorname{div} \mu, Du)_K - \langle Du, \mu n\rangle_{\partial K} &= 0, \quad \forall \mu \in H^1(\Omega)^{2\times 2}\\ (\det \sigma, v) &= (f, v), \quad \forall v \in H_0^1(\Omega)\\ u &= g \quad \text{ on } \partial\Omega.\end{aligned} \tag{17}$$

It was proved in [3] that (17) is well defined. Also, if u is a smooth solution of (1), then (u, D^2u) solves (17). We first make an observation which will allow us to view (4) as a variant of a method proposed in [15, 20].

Lemma 3.1. *For $\mu \in \Sigma_h$ and $v \in V_h$*

$$(DG_hv, \mu) = -(\operatorname{div} \mu, G_hv) + \langle G_hv, \mu n\rangle_{\partial\Omega}.$$

Proof. Using an integration by parts

$$\begin{aligned}(\operatorname{div} \mu, G_hv) &= \sum_{K\in\mathcal{T}_h} (\operatorname{div} \mu, G_hv)_K = -\sum_{K\in\mathcal{T}_h} (\mu, DG_hv)_K + \sum_{K\in\mathcal{T}_h} \langle G_hv, \mu n_K\rangle_{\partial K}\\ &= -\sum_{K\in\mathcal{T}_h} (\mu, DG_hv)_K + \sum_{e\in\mathcal{E}_h^i} \int_e [\![\mu G_hv]\!]\, ds + \langle G_hv, \mu n\rangle_{\partial\Omega}.\end{aligned}$$

Since μ and G_hv are continuous, the result follows. □

Remark 3.2. *It follows from Lemma 3.1 that for $\mu \in \Sigma_h$*

$$(H(v), \mu) = -(\operatorname{div} \mu, G_hv) + \langle G_hv, \mu n\rangle_{\partial\Omega},$$

which is the definition of the discrete Hessian given in [15] *with G_hv replaced by Dv.*

In this paper, we make a discrete elliptic regularity assumption for the non divergence form of the linearization of (1), c.f. Assumption 3.5 below. To partially motivate such an assumption, we prove a discrete elliptic regularity assumption for the divergence form of the linearization of (1). First we make a regularity assumption about the continuous problem.

Assumption 3.3. *Let ϕ be the solution of*

$$-\operatorname{div}((\operatorname{cof} D^2u)D\phi) = r \text{ in } \Omega,\ \phi = 0 \text{ on } \partial\Omega. \tag{18}$$

For $r \in L^p(\Omega)$, $p \geq 2$, the weak solution ϕ of (18) is in $W^{2,p}(\Omega)$ and

$$\|\phi\|_{W^{2,p}} \leq C_{10}(D^2u)p\|r\|_{L^p}, \tag{19}$$

for a constant C_{10} which depends on D^2u. Moreover, if $r \in H^1(\Omega) \cap C(\Omega)$, then $D\phi \in C(\Omega)^2$.

It is known that (19) holds when Ω is smooth [7] and when Ω is a plane rectangular domain [1]. As for the C^1 continuity of ϕ, it is known that when Ω is an acute triangular domain [4, Section 4.1], for $r \in H^1(\Omega)$, $\phi \in H^3(\Omega)$, hence $D\phi \in (H^2(\Omega))^2$. Thus $D\phi \in C(\Omega)^2$.

Lemma 3.4 (A discrete elliptic regularity result)**.** *Let $r \in L^p(\Omega) \cap H^1(\Omega) \cap C(\Omega), p > 2$ and let $v \in V_h \cap W_0^{1,p}(\Omega)$ solve*

$$\big((\operatorname{cof} D^2u)Dv, Dw\big) = (r, w), \forall w \in V_h \cap W_0^{1,p}(\Omega). \tag{20}$$

We have

$$\|v\|_{\widetilde{W}^{1,p}(\mathcal{T}_h)} \leq C_{11}(D^2u)p\|r\|_{L^p},$$

for a constant C_{11} which depends on D^2u.

Proof. By Assumption 3.3, $\phi \in W^{2,p}(\Omega)$ and

$$||\phi||_{W^{2,p}} \leq C_{10}p||r||_{L^p}.$$

Let $P_h : W_0^{1,p}(\Omega) \to V_h \cap W_0^{1,p}(\Omega)$ be the projection defined by

$$\big((\operatorname{cof} D^2u)DP_hz, Dw\big) = \big((\operatorname{cof} D^2u)Dz, Dw\big), \quad \forall w \in V_h \cap W_0^{1,q}(\Omega), \frac{1}{p} + \frac{1}{q} = 1.$$

For $z \in W^{2,p}(\Omega)$ we have $\operatorname{div}\big((\operatorname{cof} D^2u)Dz\big) \in L^p(\Omega)$ and for $w \in W_0^{1,q}(\Omega)$

$$\big((\operatorname{cof} D^2u)Dz, Dw\big) = \bigg(\operatorname{div}\big((\operatorname{cof} D^2u)Dz\big), w\bigg).$$

We have, c.f. for example [6, (8.5.4)],

$$\|P_hw - w\|_{W^{1,p}} \leq C_{12}(D^2u)h\|w\|_{W^{2,p}} \text{ for } w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \tag{21}$$

for a constant C_{12} which depends on D^2u.

Analogous to the proof of (13), we have using (12), (8) and the continuity of $D\phi$

(22)

$$\begin{aligned}
||D^2\phi - DG_hI_h\phi||_{L^p} &\leq ||D^2\phi - D(I_h(D\phi))||_{L^p} + ||D(I_h(D\phi)) - DG_hI_h\phi||_{L^p} \\
&\leq ||\phi||_{W^{2,p}} + C_2||D\phi||_{W^{1,p}} + C_1h^{-1}||I_h(D\phi) - G_hI_h\phi||_{L^p} \\
&\leq ||\phi||_{W^{2,p}} + C_2||D\phi||_{W^{1,p}} + C_1h^{-1}||I_h(D\phi) - D\phi||_{L^p} \\
&\qquad + C_1h^{-1}||D\phi - G_hI_h\phi||_{L^p} \\
&\leq (1 + C_2)||\phi||_{W^{2,p}} + C_1C_2||\phi||_{W^{2,p}} + C_1C_4||\phi||_{W^{2,p}} \\
&\leq C_{13}||\phi||_{W^{2,p}},
\end{aligned}$$

with $C_{13} = 1 + C_2 + C_1C_2 + C_1C_4$. We have by (22) and (21)

$$\begin{aligned} ||DG_hP_h\phi||_{L^p} &\leq ||DG_hP_h\phi - DG_hI_h\phi||_{L^p} + ||DG_hI_h\phi - D^2\phi||_{L^p} + ||D^2\phi||_{L^p} \\ &\leq C_1h^{-1}||G_h(P_h\phi - I_h\phi)||_{L^p} + ||DG_hI_h\phi - D^2\phi||_{L^p} + ||D^2\phi||_{L^p} \\ &\leq C_1C_3h^{-1}||D(P_h\phi - I_h\phi)||_{L^p} + (1 + C_{13})||\phi||_{W^{2,p}} \\ &\leq C_1C_3h^{-1}||DP_h\phi - D\phi||_{L^p} + C_1C_3h^{-1}||D\phi - DI_h\phi||_{L^p} \\ &\qquad + (1 + C_{13})||\phi||_{W^{2,p}} \\ &\leq C_1C_3C_{12}\|\phi\|_{W^{2,p}} + C_1C_3C_2\|\phi\|_{W^{2,p}} + (1 + C_{13})||\phi||_{W^{2,p}}. \end{aligned}$$

Therefore for a constant $C_{14} := C_1C_3(C_{12}(D^2u) + C_2) + 1 + C_{13}$ which depends on D^2u, we have

$$||DG_hP_h\phi||_{L^p} \leq C_{14}||\phi||_{W^{2,p}}. \tag{23}$$

We obtain $\|v\|_{\widetilde{W}^{1,p}(\mathcal{T}_h)} \leq C_{14}||\phi||_{W^{2,p}}$. We have $\|\phi\|_{W^{2,p}} \leq C_{10}p||r||_{L^p}$ by (19), from which the result follows. □

We will not use the above discrete elliptic regularity result in this paper. What is needed is the discrete elliptic regularity assumption below. Numerical results supporting such an assumption are given in the appendix. The solvability and error estimates for (25) is the main difficult part. With estimates such as (21) for a suitable projection for the non divergence form of the equation, the proof of the discrete elliptic regularity assumption would be similar to the one proved in Theorem 3.4.

Since $\operatorname{div}\operatorname{cof} D^2u = 0$, $\operatorname{div}((\operatorname{cof} D^2u)D\phi) = r \ in \ \Omega$ can be written

$$A : D^2\phi = r \ in \ \Omega, \tag{24}$$

where

$$A = \operatorname{cof} D^2u.$$

We next consider a discretization of the above form (24), known as non divergence form, of the linearization of (1).

Assumption 3.5 (Discrete elliptic regularity for non divergence form)**.** *For $r \in L^p(\Omega), p \geq 2$, there exists a unique $v \in V_h \cap W_0^{1,p}(\Omega)$ which solves*

$$(A : H(v), w) = (r, w), \forall w \in V_h \cap W_0^{1,p}(\Omega). \tag{25}$$

Moreover

$$\|v\|_{\widetilde{W}^{1,p}(\Omega)} \leq C_{15}(D^2u)p\|r\|_{L^p}, \tag{26}$$

for a constant C_{15} which depends on D^2u.

The proof of the following lemma is the same as the proof of [2, Lemma 2.3].

Lemma 3.6. *For $p \geq 2$ and q such that $1/p + 1/q = 1$, we have*

$$\|r\|_{L^p} \leq C_{16} \sup_{\substack{\chi\neq 0 \\ \chi\in V_h}} \frac{|(r,\chi)|}{\|\chi\|_{L^q}}, \qquad r \in V_h$$

$$\|r\|_{L^p} \leq C_{16} \sup_{\substack{\chi\neq 0 \\ \chi\in V_h\cap H_0^1(\Omega)}} \frac{|(r,\chi)|}{\|\chi\|_{L^q}}, \qquad r \in V_h \cap H_0^1(\Omega),$$

where we use the same constant C_{17} in both estimates for convenience.

Lemma 3.6 also holds for matrix valued fields. One starts with a converse Hölder inequality for matrix fields

$$\|\eta\|_{L^p} = \sup_{\substack{\mu \neq 0 \\ \mu \in (L^q(\Omega))^{2\times 2}}} \frac{|(\eta, \mu)|}{\|\mu\|_{L^q}}, \qquad \eta \in (L^p(\Omega))^{2\times 2}, \frac{1}{p} + \frac{1}{q} = 1, 1 < p < \infty. \tag{27}$$

We give the proof of (27) in the appendix. One then uses projections as in the proof of [2, Lemma 2.3] to obtain

$$\|\eta\|_{L^p} \leq C_{16} \sup_{\substack{\mu \neq 0 \\ \mu \in \Sigma_h}} \frac{|(\eta, \mu)|}{\|\mu\|_{L^q}}, \qquad \eta \in \Sigma_h, \frac{1}{p} + \frac{1}{q} = 1. \tag{28}$$

Our strategy is to use the discrete elliptic regularity approach taken in [19]. In the remainder of the paper, we assume p satisfies

$$|\ln h| \leq p \leq 2|\ln h|, p > 2. \tag{29}$$

For results which do not necessarily use (29), we will state when the constants do not depend on p. We can now analyze the discretization (4). We are interested in finding the solution $(w_h, \eta_h) \in V_h \times \Sigma_h$ satisfying

$$(\eta_h, \mu) = (DG_h w_h, \mu), \forall \mu \in \Sigma_h. \tag{30}$$

It follows from Hölder inequality and the Lax-Milgram lemma that, given $w_h \in V_h$, the discrete Hessian of w_h

$$H(w_h) := \eta_h,$$

is well defined by (30).

Lemma 3.7. *There exists a positive constant $C_{17} \geq 1$ such that for w_h and $z_h \in V_h$*

$$||H(w_h) - H(z_h)||_{L^\infty} \leq C_{17}||w_h - z_h||_{\widetilde{W}^{1,p}(\Omega)}.$$

Proof. Let w_h and $z_h \in V_h$. By (30) we have for $p \geq 2$ and $1/q = 1 - 1/p$

$$\begin{aligned} |(H(w_h) - H(z_h), \mu)| &= \big|(DG_h(w_h - z_h), \mu)\big| \\ &\leq ||DG_h(w_h - z_h)||_{L^p}||\mu||_{L^q}. \end{aligned}$$

Since by definition, $||DG_h(w_h - z_h)||_{L^p} \leq ||w_h - z_h||_{\widetilde{W}^{1,p}(\Omega)}$, by (28) we have

$$||H(w_h) - H(z_h)||_{L^p} \leq C_{16}||w_h - z_h||_{\widetilde{W}^{1,p}(\Omega)}.$$

By an inverse estimate and since p satisfies (29) we have

$$||H(w_h) - H(z_h)||_{L^\infty} \leq C_1 h^{-\frac{2}{p}}||H(w_h) - H(z_h)||_{L^p} \leq C_{17}||w_h - z_h||_{\widetilde{W}^{1,p}(\Omega)},$$

where $C_{17} = \max\{\, C_1 C_{16} \exp(2), 1 \,\}$ where we note that as $h \leq 1$, $|\ln h| = -\ln h$ and since $|\ln h| \leq p \leq 2|\ln h|$, $h^{-\frac{2}{p}} = \exp(-2/p \ln h) = \exp(2|\ln h|/p) \leq \exp(2)$. □

For $\rho > 0$ we define

$$\bar{B}_h(\rho) = \{(w_h, \eta_h) \in V_h \times \Sigma_h,\ \|w_h - I_h u\|_{\widetilde{W}^{1,p}(\mathcal{T}_h)} \leq \rho,\ \|\eta_h - I_h \sigma\|_{L^\infty} \leq \rho\}.$$

We also define

$$Z_h = \{\, (w_h, \eta_h) \in V_h \times \Sigma_h, w_h = g_h \text{ on } \partial\Omega, (w_h, \eta_h) \text{ solves (30)} \,\} \text{ and}$$

$$B_h(\rho) = \bar{B}_h(\rho) \cap Z_h.$$

Lemma 3.8. *We have $B_h(\rho) \neq \emptyset$, for h sufficiently small and $\rho = C_{18} h \|u\|_{W^{4,\infty}}$, for a positive constant $C_{18} > 0$ independent of h. More precisely, we have $\|H(I_h u) - I_h \sigma\|_{L^\infty} \leq C_{18} h \|u\|_{W^{4,\infty}}$ with $\sigma = D^2 u$.*

Proof. Let $\eta_h \in \Sigma_h$ denote the discrete Hessian of $I_h u$ given by (30). We show that $(I_h u, \eta_h) \in B_h(\rho)$ for h sufficiently small and $\rho = C_{18} h \|u\|_{W^{4,\infty}}$ for a constant C_{18}. By (30), $(\eta_h, \mu) = (DG_h I_h u, \mu)\ \forall \mu \in \Sigma_h$. Therefore

$$(\eta_h - I_h\sigma, \mu) = (\eta_h - \sigma, \mu) + (\sigma - I_h\sigma, \mu) = (DG_h I_h u - D^2 u, \mu) + (\sigma - I_h \sigma, \mu).$$

It follows from (28) that for $p \geq 2$

$$\|\eta_h - I_h\sigma\|_{L^p} \leq C_{16}(||DG_h I_h u - D^2 u||_{L^p} + ||I_h\sigma - \sigma||_{L^p}).$$

Therefore by (9), (13) and since $\sigma = D^2 u$

$$\begin{aligned}\|\eta_h - I_h\sigma\|_{L^p} &\leq C_{16} C_5 h \|u\|_{W^{3,p}} + 2C_{16} C_2 h^2 \|u\|_{W^{4,p}} \\ &\leq (C_{16} C_5 \|u\|_{W^{3,p}} + 2C_{16} C_2 \|u\|_{W^{4,p}}) h.\end{aligned}$$

By an inverse estimate and since p satisfies (29) we have

$$\|\eta_h - I_h\sigma\|_{L^\infty} \leq C_1 h^{-\frac{2}{p}} \|\eta_h - I_h\sigma\|_{L^p} \leq C_1 C_{16} (C_5 + 2C_2) \exp(2) h \|u\|_{W^{4,p}},$$

from which the result follows with $C_{18} = C_1 C_{16}(C_5 + 2C_2)\exp(2)\max(|\Omega|, 1)$, and we recall that $\eta_h = H(I_h u)$. ☐

As in [3], we consider the linearized problem : find $(w_h, \eta_h) \in V_h \cap H^1_0 \times \Sigma_h$

$$\begin{aligned}(\eta_h, \mu) &= (DG_h w_h, \mu) \forall \mu \in \Sigma_h \\ (A : \eta_h, v) &= (f, v),\ \forall v \in V_h \cap H^1_0(\Omega) \\ w_h &= g_h \text{ on } \partial\Omega.\end{aligned}$$

By the strict convexity of u, w_h is well defined and $\eta_h = H(w_h)$. We can therefore define the mapping $T : V_h \times \Sigma_h \to V_h \times \Sigma_h$ by

$$T(w_h, \eta_h) = (T_1(w_h, \eta_h), T_2(w_h, \eta_h)),$$

where $T_1(w_h, \eta_h)$ and $T_2(w_h, \eta_h)$ satisfy

$$w_h - T_1(w_h, \eta_h) = 0 \quad \text{on} \quad \partial\Omega \tag{31}$$

$$(T_2(w_h, \eta_h), \mu) = (DG_h T_1(w_h, \eta_h), \mu), \forall \mu \in \Sigma_h \tag{32}$$

$$(A : H(w_h - T_1(w_h, \eta_h)), v) = -(f, v) + (\det \eta_h, v), \forall\ v \in V_h \cap H^1_0(\Omega). \tag{33}$$

A fixed point of T with $w_h = g_h$ on $\partial\Omega$ is a solution of the nonlinear problem (4). Since $T_2(w_h, \eta_h) = H(T_1(w_h, \eta_h))$, we have the following corollary of Lemma 3.7.

Lemma 3.9. *For $\rho > 0$ and (w_1, η_1) and (w_2, η_2) in $B_h(\rho)$, we have*

$$||T_2(w_1, \eta_1) - T_2(w_2, \eta_2)||_{L^\infty} \leq C_{17} ||T_1(w_1, \eta_1) - T_1(w_2, \eta_2)||_{\widetilde{W}^{1,p}(\mathcal{T}_h)}, \tag{34}$$

where C_{17} is the constant defined in Lemma 3.7.

Lemma 3.10. *We have for h sufficiently small*

$$\|I_h u - T_1(I_h u, I_h\sigma)\|_{\widetilde{W}^{1,p}(\Omega)} \leq C_{19}(\sigma) h \tag{35}$$

$$\|I_h\sigma - T_2(I_h u, I_h\sigma)\|_{L^\infty} \leq C_{20}(\sigma) h, \tag{36}$$

for $\sigma = D^2 u$ and positive constants C_{19} and C_{20} which depends on σ.

Proof. Since $T_1(I_h u, I_h\sigma) = I_h u$ on $\partial\Omega$, we have $v = I_h u - T_1(I_h u, I_h\sigma) \in V_h \cap H^1_0(\Omega)$. For $w_h = I_h u$ and $\eta_h = I_h\sigma$, using (33) , $\det D^2 u = \det \sigma = f$, and discrete elliptic regularity, we have

$$\|I_h u - T_1(I_h u, I_h\sigma)\|_{\widetilde{W}^{1,p}(\Omega)} \leq C_{15}(D^2 u) p \|\det \sigma - \det I_h\sigma\|_{L^p}. \tag{37}$$

Since on each element K, $\det I_h\sigma - \det\sigma = \frac{1}{2}(\text{cof}(I_h\sigma) + \text{cof}(\sigma)) : (I_h\sigma - \sigma)$, we have

$$\begin{aligned}\|\det I_h\sigma - \det\sigma\|_{L^p(K)} &\leq C_{21}\|\det I_h\sigma - \det\sigma\|_{L^\infty(K)}\\ &\leq 2C_{21}\|I_h\sigma + \sigma\|_{L^\infty(K)}\|I_h\sigma - \sigma\|_{L^\infty(K)},\end{aligned}$$

where $C_{21} = \max(|\Omega|, 1/4)$. Since I_h is a linear finite element interpolation, we have $\|I_h\sigma\|_{L^\infty} \leq \|\sigma\|_{L^\infty}$. Therefore, using (9)

(38)
$$\|\det I_h\sigma - \det\sigma\|_{L^p(K)} \leq 4C_{21}\|\sigma\|_{L^\infty}\|I_h\sigma - \sigma\|_{L^\infty(K)} \leq 8C_{21}C_2\|\sigma\|_{L^\infty}h^2\|\sigma\|_{W^{2,\infty}}.$$

This implies by (37) and (29) that

$$\|I_hu - T_1(I_hu, I_h\sigma)\|_{\widetilde{W}^{1,2}(\Omega)} \leq 16C_{21}C_2C_{15}\|\sigma\|_{L^\infty}\|\sigma\|_{W^{2,\infty}}h^2|\ln h|.$$

We conclude that there exists a constant C_{19} which depends on σ such that

$$\|I_hu - T_1(I_hu, I_h\sigma)\|_{\widetilde{W}^{1,p}(\Omega)} \leq C_{19}(\sigma)h,$$

for h sufficiently small. By Lemma 3.7 and (30) we have

$$||H(I_hu) - T_2(I_hu, I_h\sigma)||_{L^\infty} \leq C_{17}\|I_hu - T_1(I_hu, I_h\sigma)\|_{\widetilde{W}^{1,p}(\Omega)} \leq C_{17}C_{19}(\sigma)h.$$

By triangular inequality, we have :

$$||I_h\sigma - T_2(I_hu, I_h\sigma)||_{L^\infty} \leq ||I_h\sigma - H(I_hu)||_{L^\infty} + ||H(I_hu) - T_2(I_hu, I_h\sigma)||_{L^\infty}.$$

Since we proved in Lemma 3.8 that $||H(I_hu) - I_h\sigma||_{L^\infty} \leq C_{18}h\|u\|_{W^{4,\infty}}$, we obtain

$$\|I_h\sigma - T_2(I_hu, I_h\sigma)\|_{L^\infty} \leq C_{20}h,$$

where $C_{20} = C_{18}\|u\|_{W^{4,\infty}} + C_{17}C_{19}(\sigma)$. □

Lemma 3.11. *For h sufficiently small and for (w_1, η_1) and (w_2, η_2) in $B_h(\rho)$, $\rho \geq C_{18}h\|u\|_{W^{4,\infty}}$, we have*

$$\|T_1(w_1, \eta_1) - T_1(w_2, \eta_2)\|_{\widetilde{W}^{1,p}(\Omega)} \leq C_{22}\rho|\ln h|\,\|\eta_1 - \eta_2\|_{L^\infty},$$

for a positive constant C_{22} which depends on $D^2(u)$.

Proof. The proof is analogous to the one of [3, Lemma 3.10]. Using (33), we have with $A = \text{cof}\, D^2u$

$$(A : H(T_1(w_1, \eta_1) - T_1(w_2, \eta_2)), v) = (A : H(w_1 - w_2), v) - (\det\eta_1 - \det\eta_2, v).$$

By [3, Lemma 2.4], on each element K we have

$$\det\eta_1 - \det\eta_2 = \text{cof}\left(\frac{1}{2}\eta_1 + \frac{1}{2}\eta_2\right) : (\eta_1 - \eta_2).$$

Therefore, on each element K and using $\sigma = D^2u$, $H(w_1) = \eta_1$, we have

$$\begin{aligned}(\text{cof}\, D^2u) : (\eta_1 - \eta_2) - (\det\eta_1 - \det\eta_2) &=\\ &\left((\text{cof}\, D^2u) - \text{cof}\left(\frac{1}{2}\eta_1 + \frac{1}{2}\eta_2\right)\right) : (\eta_1 - \eta_2)\\ &= \text{cof}\left(\sigma - \frac{1}{2}\eta_1 - \frac{1}{2}\eta_2\right) : (\eta_1 - \eta_2)\\ &= \text{cof}\left(\sigma - I_h\sigma + \frac{1}{2}I_h\sigma - \frac{1}{2}\eta_1 + \frac{1}{2}I_h\sigma - \frac{1}{2}\eta_2\right) : (\eta_1 - \eta_2).\end{aligned}$$

We have

$$\left\|\operatorname{cof}\left(\sigma - I_h\sigma + \frac{1}{2}I_h\sigma - \frac{1}{2}\eta_1 + \frac{1}{2}I_h\sigma - \frac{1}{2}\eta_2\right)\right\|_{L^\infty(K)} \leq$$
$$\|\sigma - I_h\sigma\|_{L^\infty(K)} + \frac{1}{2}\|I_h\sigma - \eta_1\|_{L^\infty(K)} + \frac{1}{2}\|I_h\sigma - \eta_2\|_{L^\infty(K)}$$
$$\leq 2C_2\|u\|_{W^{4,\infty}}h^2 + \rho.$$

Therefore,

$$||(\operatorname{cof} D^2u) : (\eta_1 - \eta_2) + \det\eta_1 - \det\eta_2||_{L^p} \leq (2C_2\|u\|_{W^{4,\infty}}h^2 + \rho)||\eta_1 - \eta_2||_{L^p}.$$

Therefore, by discrete elliptic regularity and (29)

$$\|T_1(w_1,\eta_1) - T_1(w_2,\eta_2)\|_{\widetilde{W}^{1,p}(\Omega)}$$
$$\leq C_{15}(D^2u)p||(\operatorname{cof} D^2u) : (\eta_1 - \eta_2) + \det\eta_1 - \det\eta_2||_{L^p}$$
$$\leq 2C_{15}(D^2u)|\ln h|(2C_2\|u\|_{W^{4,\infty}}h^2 + \rho)\max(|\Omega|,1)||\eta_1 - \eta_2||_{L^\infty}.$$

The result follows for h sufficiently small. We assumed that $\rho \geq C_{18}h\|u\|_{W^{4,\infty}}$, since by Lemma 3.8 $B_h(\rho) \neq \emptyset$ for $\rho = C_{18}h\|u\|_{W^{4,\infty}}$. □

Lemma 3.12. *Let* $\rho = 4C_{23}(D^2u)h$ *where* $C_{23}(D^2u) = \max(C_{18}\|u\|_{W^{4,\infty}}, C_{19}, C_{20})$. *For* h *sufficiently small, the mapping* T *leaves invariant the ball* $B_h(\rho)$. *That is, for* (w_h,η_h) *in* $B_h(\rho)$, *we have*

$$\|T_1(w_h,\eta_h) - I_hu\|_{\widetilde{W}^{1,p}(\Omega)} \leq \rho \tag{39}$$
$$\|T_2(w_h,\eta_h) - I_h\sigma\|_{L^\infty} \leq \rho. \tag{40}$$

Proof. Let $(w_h,\eta_h) \in B_h(\rho)$. Recall that $\|w_h - I_hu\|_{\widetilde{W}^{1,p}(\Omega)} \leq \rho$ and $\|\eta_h - I_h\sigma\|_{L^\infty} \leq \rho$. We have by triangle inequality, Lemmas 3.11 and 3.10

$$\|T_1(w_h,\eta_h) - I_hu\|_{\widetilde{W}^{1,p}(\Omega)} \leq \|T_1(w_h,\eta_h) - T_1(I_hu, I_h\sigma)\|_{\widetilde{W}^{1,p}(\Omega)}$$
$$+ \|T_1(I_hu, I_h\sigma) - I_hu\|_{\widetilde{W}^{1,p}(\Omega)}$$
$$\leq 4C_{23}h|\ln h|\|\eta_h - I_h\sigma\|_{L^\infty} + C_{19}h.$$

For h sufficiently small, $4C_{23}h|\ln h| \leq 1/4$ and by construction $C_{19}h \leq \rho/4$. Therefore

$$\|T_1(w_h,\eta_h) - I_hu\|_{\widetilde{W}^{1,p}(\Omega)} \leq \frac{1}{4}\rho + \frac{1}{4}\rho \leq \frac{\rho}{2} \leq \rho.$$

This proves (39). By triangle inequality

$$\|T_2(w_h,\eta_h) - I_h\sigma\|_{L^\infty} \leq \|T_2(w_h,\eta_h) - T_2(I_hu, I_h\sigma)\|_{L^\infty} + \|T_2(I_hu, I_h\sigma) - I_h\sigma\|_{L^\infty}.$$

Thus, by Lemma 3.9

$$\|T_2(w_h,\eta_h) - I_h\sigma\|_{L^\infty} \leq C_{17}\|T_1(w_h,\eta_h) - T_1(I_hu, I_h\sigma)\|_{\widetilde{W}^{1,p}(\Omega)} + C_{20}h$$
$$\leq 4C_{17}C_{23}h|\ln h|\|\eta_h - I_h\sigma\|_{L^\infty} + C_{20}h.$$

Furthermore, for h sufficiently small and since $\|\eta_h - I_h\sigma\|_{L^\infty} \leq \rho$

$$\|T_2(w_h,\eta_h) - I_h\sigma\|_{L^\infty} \leq \frac{1}{4}\rho + \frac{1}{4}\rho \leq \rho.$$

This proves (40). □

Lemma 3.13. *The mapping* T *is continuous on* $B_h(\rho)$ *for* ρ *as defined in Lemma 3.12.*

Proof. Let (w_1, η_1) and (w_2, η_2) in $B_h(\rho)$. We have by Lemmas 3.9 and 3.11, for h sufficiently small

$$\begin{aligned}\|T_1(w_1,\eta_1) - T_1(w_2,\eta_2)\|_{\widetilde{W}^{1,p}(\Omega)} + \|T_2(w_1,\eta_1) - T_2(w_2,\eta_2)\|_{L^\infty} \leq \\ (C_{17}+1)\|T_1(w_1,\eta_1) - T_1(w_2,\eta_2)\|_{\widetilde{W}^{1,p}(\Omega)} \leq 4(C_{17}+1)C_{22}C_{23}h|\ln h|\|\eta_1 - \eta_2\|_{L^\infty} \\ \leq 2\|\eta_1 - \eta_2\|_{L^\infty},\end{aligned}$$

which proves the result. □

Now, we are ready to show the well-posedness of the discrete problem (4).

Theorem 3.14. *The discrete problem* (4) *has a unique solution* (u_h, σ_h) *in* $B_h(\rho)$ *for* h *sufficiently small and* ρ *as defined in Lemma 3.12.*

Proof. By Lemma 3.13, T is continuous on $B_h(\rho)$ and by Lemma 3.12, T maps $B_h(\rho)$ into itself. Therefore by the Brouwer fixed point theorem it has a fixed point (w_h, η_h) in $B_h(\rho)$. Assume that there exist two fixed points (w_h^1, η_h^1) and (w_h^2, η_h^2) of T. We then have $T_1(w_h^1, \eta_h^1) = w_h^1$ and $T_1(w_h^2, \eta_h^2) = w_h^2$. Also, $T_2(w_h^1, \eta_h^1) = \eta_h^1$ and $T_2(w_h^2, \eta_h^2) = \eta_h^2$. For h sufficiently small, by Lemmas 3.7 and 3.11

$$\begin{aligned}\|T_2(w_h^1,\eta_h^1) - T_2(w_h^2,\eta_h^2)\|_{L^\infty} &\leq C_{17}\|T_1(w_h^1,\eta_h^1) - T_1(w_h^2,\eta_h^2)\|_{\widetilde{W}^{1,p}(\Omega)} \\ &\leq \frac{1}{4}\|\eta_h^1 - \eta_h^2\|_{L^\infty}.\end{aligned}$$

Therefore

$$\|\eta_h^1 - \eta_h^2\|_{L^\infty} \leq C_{17}\|w_h^1 - w_h^2\|_{\widetilde{W}^{1,p}(\Omega)} \leq \frac{1}{4}\|\eta_h^1 - \eta_h^2\|_{L^\infty}.$$

We conclude that $\eta_h^1 = \eta_h^2$ and thus $w_h^1 = w_h^2$. □

With the previous preparation, we are now in a perfect position to present our main error estimation.

Theorem 3.15. *Let* $(u, \sigma) \in W^{4,\infty}(\Omega) \times W^{2,\infty}(\Omega)^{2\times 2}$ *be the unique strictly convex solution of* (1) *and let* (u_h, σ_h) *the unique solution in* $B_h(\rho)$ *of* (4) *for* h *sufficiently small and* $\rho = 4C_{19}h$. *We have*

$$\begin{aligned}\|I_h u - u_h\|_{\widetilde{W}^{1,p}(\Omega)} &\leq C_{24}(u)h \\ \|\sigma - \sigma_h\|_{L^\infty} &\leq C_{25}(u)h.\end{aligned}$$

Moreover, $||u - u_h||_{W^{1,2}} = \mathcal{O}(h)$.

Proof. By Theorem 3.14, which states that the solution is in $B_h(\rho)$, the definition of $B_h(\rho)$, the choice of ρ and with $C_{24} = 4C_{23}$ and depends on u, we have $\|I_h u - u_h\|_{\widetilde{W}^{1,p}(\Omega)} \leq C_{24}(u)h$.

Similarly, $\|I_h\sigma - \sigma_h\|_{L^\infty} \leq C_{24}(u)h$. By a triangular inequality and using (9), we obtain $\|\sigma - \sigma_h\|_{L^\infty} \leq C_{25}(u)h$, for a constant $C_{25}(u)$.

By (15), (5), $p > 2$ we have

$$\begin{aligned}||I_h u - u_h||_{L^2} \leq C_8||DG_h(I_h u - u_h)||_{L^2} &\leq C_8C_1||DG_h(I_h u - u_h)||_{L^p} \\ &= C_8C_1\|I_h u - u_h\|_{\widetilde{W}^{1,p}(\Omega)} \leq C_8C_1C_{24}(u)h.\end{aligned}$$

With a similar argument, $||DI_h u - Du_h||_{L^2} = \mathcal{O}(h)$. Thus by a triangular inequality and (7) we obtain $||I_h u - u||_{L^2} = \mathcal{O}(h)$ and $||DI_h u - Du||_{L^2} = \mathcal{O}(h)$ which gives $||u - u_h||_{W^{1,2}} = \mathcal{O}(h)$. □

Remark 3.16. *The analysis may extend with a few technicalities to three dimensions. We mention that the definition of uniform meshes is different in dimension 3 and dealing with* $\operatorname{cof}\sigma - \operatorname{cof}\eta$ *requires the mean value theorem. For the extension to general domains, one may use the penalty approach to the boundary conditions proposed in* [19].

4. Numerical results

In this section, we present a numerical example to verify and validate the theoretical results. To solve the nonlinear problem, we solve the discrete problem (2) using Newton's method. Although we only established the theoretical results on uniform meshes, we want to show with an example that the method works for general unstructured meshes. We consider two different types meshes: regular type uniform meshes and Delaunay meshes.

In the numerical example, we choose the test function as $u(x,y) = e^{(x^2+y^2)/2}$. Thus $f(x,y) = (1+x^2+y^2)e^{(x^2+y^2)}$ and $g(x,y) = e^{(x^2+y^2)/2}$. The initial guess is obtained by solving the mixed finite element approximation of the problem

$$\Delta u_0 = 2\sqrt{f}, \text{ in } U \quad u_0 = g \text{ on } \partial\Omega,$$

that is (2) with the determinant operator replaced by the trace operator and f replaced by $2\sqrt{f}$.

To summarize the numerical results, we consider the following four different (discrete) norms:

$$D_0 e = \|u - u_h\|_{L^\infty}, \qquad D_1 e = \|Du - Du_h\|_{L^\infty};$$
$$D_1^r e = \|Du - G_h u_h\|_{L^\infty}, \quad D_2 e = \|D^2 u - \sigma_h\|_{L^\infty}.$$

TABLE 1. Numerical result on regular type uniform meshes.

Dof	D_0e	order	D_1e	Order	D_1^re	Order	D_2e	Order
289	1.54e-03	0.00	2.32e-01	0.00	1.77e-02	0.00	5.97e-01	0.00
1089	3.68e-04	2.16	1.21e-01	0.98	4.86e-03	1.95	3.26e-01	0.91
4225	9.02e-05	2.07	6.21e-02	0.99	1.28e-03	1.97	1.71e-01	0.95
16641	2.23e-05	2.04	3.15e-02	0.99	3.30e-04	1.98	8.79e-02	0.97
66049	5.56e-06	2.02	1.58e-02	1.00	8.36e-05	1.99	4.46e-02	0.99

TABLE 2. Numerical result on Delaunay meshes.

Dof	D_0e	order	D_1e	Order	D_1^re	Order	D_2e	Order
139	2.76e-03	0.00	2.44e-01	0.00	2.37e-02	0.00	6.51e-01	0.00
513	6.98e-04	2.11	1.26e-01	1.02	6.97e-03	1.87	3.22e-01	1.08
1969	1.78e-04	2.03	6.45e-02	0.99	1.80e-03	2.02	1.65e-01	1.00
7713	4.40e-05	2.05	3.27e-02	1.00	4.44e-04	2.05	7.34e-02	1.18
30529	1.08e-05	2.03	1.64e-02	1.00	1.04e-04	2.11	2.66e-02	1.48

We display the numerical convergence history in Tables 1 and 2 for regular type uniform meshes and Delaunay meshes respectively. From those two tables, we can see that the L^∞ errors for σ converge at the rate $\mathcal{O}(h)$ indicated in Theorem 3.15. We also observe that L^∞ errors for Du converge at a rate $\mathcal{O}(h)$. This confirms the rate $\mathcal{O}(h)$ for the L^2 error for Du. The recovered gradient converges to the exact

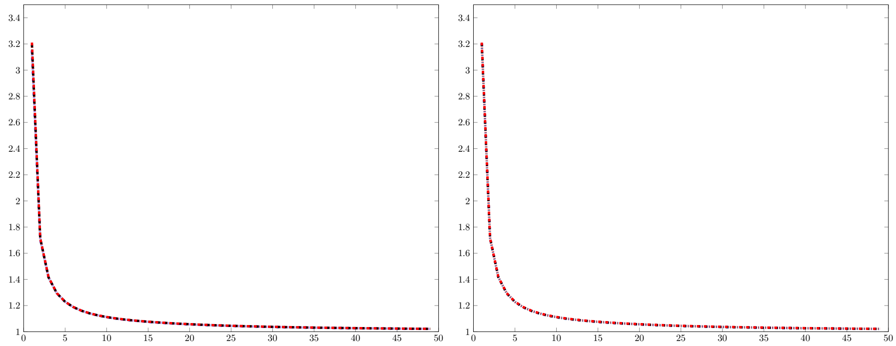

FIGURE 1. $(p+1)Z(p)/(pZ(p))$ as a function of p on a uniform mesh (left) and on an unstructured mesh.

gradient at a superconvergent rate $\mathcal{O}(h^2)$. The experimental rate for the L^∞ error in u is $\mathcal{O}(h^2)$. This suggests that our L^2 error estimate is suboptimal.

5. Conclusion

In this paper, we proposed a linear finite element method for solving the Monge-Ampère equation with a smooth solution and using a mixed finite element formulation. The Hessian matrix is calculated using the gradient recovery technique. The theoretical results are verified with a numerical example.

6. Appendix

6.1. Proof of the duality relation (27). The proof is analogous to the scalar case c.f. [22, p. 130]. Let $\eta \in (L^p(\Omega))^{2\times 2}, 1 < p < \infty$ and choose q such that $\frac{1}{p} + \frac{1}{q} = 1$. And let $\mu \in (L^q(\Omega))^{2\times 2}$. We have

$$|(\eta, \mu)| \leq ||\eta||_{L^p} ||\mu||_{L^q}. \tag{41}$$

Recall that for $x \in \mathbb{R}$, sgn$(x) = 1$ if $x > 0$, sgn$(x) = 0$ if $x = 0$ and sgn$(x) = -1$ if $x < 0$. For i, j in $\{1, 2\}$ we define

$$\eta_{ij} = |\mu_{ij}|^{\frac{q}{p}} sgn(\mu_{ij}).$$

Then $|\eta_{ij}|^p = |\mu_{ij}|^q = \eta_{ij}\mu_{ij}$. Hence $\eta_{ij} \in L^p(\Omega)$ and $||\eta_{ij}||_{L^p} = ||\mu_{ij}||^{\frac{q}{p}}_{L^q}$. Therefore $||\eta||^p_{L^p} = \sum_{i,j=1}^2 ||\eta_{ij}||^p_{L^p} = \sum_{i,j=1}^2 ||\mu_{ij}||^q_{L^q} = ||\mu||^q_{L^q}$. That is, $||\eta||_{L^p} = ||\mu||^{\frac{q}{p}}_{L^q}$. Next,

$$(\eta, \mu) = \sum_{i,j=1}^{2} \int_\Omega \eta_{ij}\mu_{ij} = \sum_{i,j=1}^{2} ||\mu_{ij}||^q_{L^q} = ||\mu||^q_{L^q} = ||\mu||_{L^q} ||\mu||^{q-1}_{L^q} = ||\mu||_{L^q} ||\mu||^{\frac{q}{p}}_{L^q}$$
$$= ||\mu||_{L^q} ||\eta||_{L^p}.$$

This completes the proof.

6.2. Numerical evidence of discrete elliptic regularity for non divergence form Assumption 3.5. We take

$$u(x) = \exp((x_1^2 + x_2^2)/2), \tag{42}$$

from which we compute $A = D^2 u$. The solution v is taken as $\sin(\pi x_1)\sin(\pi x_2)$. The right hand side function r was computed from A and v. Numerical evidence indicate that (25) is solvable.

Let $Z(p) = \|r\|_{L^p}/\|v\|_{\widetilde{W}^{1,p}(\Omega)}$. Should the discrete elliptic regularity hold as predicted, the ratios $(p+1)Z(p)/(pZ(p))$ should be equal to 1. In Figure 1 we plot this ratio as a function of p on a uniform mesh and on an unstructured mesh. The results confirm our predictions.

Acknowledgments

Gerard Awanou was partially supported by NSF grants DMS-1319640 and DMS-1720276. The authors thank the reviewers for remarks that improved significantly the paper.

References

[1] J. Adetola, Analyses d'erreur pour des méthodes d'éléments finis de l'équation elliptique de Monge-Ampère et du problème couplé stationnaire Navier-Stokes/Darcy., Ph.D. Dissertation, Université d'Abomey-Calavi, Benin, 2017.

[2] G. Awanou, Erratum to: Quadratic mixed finite element approximations of the Monge-Ampère equation in 2D, Calcolo, (2016), pp. 1–17.

[3] G. Awanou and H. Li, Error analysis of a mixed finite element method for the Monge-Ampère equation, Int. J. Num. Analysis and Modeling, 11 (2014), pp. 745–761.

[4] C. Bacuta, J. H. Bramble, and J. Xu, Regularity estimates for elliptic boundary value problems with smooth data on polygonal domains, J. Numer. Math., 11 (2003), pp. 75–94.

[5] S. C. Brenner, M. Neilan, A. Reiser, and L.-Y. Sung, A C^0 interior penalty method for a von Kármán plate, Numer. Math., 135 (2017), pp. 803–832.

[6] S. C. Brenner and L. R. Scott, The mathematical theory of finite element methods, vol. 15 of Texts in Applied Mathematics, Springer, New York, third ed., 2008.

[7] A. W. Cameron, Estimates for solutions of elliptic partial differential equations with explicit constants and aspects of the finite element method for second-order equations, Ph.D Dissertation, Cornell University, 2010.

[8] H. Chen, X. Feng, and Z. Zhang, A recovery-based linear C^0 finite element method for a fourth-order singularly perturbed Monge-Ampère equation, Adv. Comput. Math., 47 (2021), pp. Paper No. 21, 37.

[9] X. Feng, R. Glowinski, and M. Neilan, Recent developments in numerical methods for fully nonlinear second order partial differential equations, SIAM Rev., 55 (2013), pp. 205–267.

[10] R. Glowinski, S. Leung, H. Liu, and J. Qian, On the numerical solution of nonlinear eigenvalue problems for the Monge-Ampère operator, ESAIM Control Optim. Calc. Var., 26 (2020), pp. Paper No. 118, 25.

[11] R. Glowinski, H. Liu, S. Leung, and J. Qian, A finite element/operator-splitting method for the numerical solution of the two dimensional elliptic Monge-Ampère equation, J. Sci. Comput., 79 (2019), pp. 1–47.

[12] H. Guo and X. Yang, Polynomial preserving recovery for high frequency wave propagation, Journal of Scientific Computing, 71 (2017), pp. 594–614.

[13] H. Guo, Z. Zhang, and Q. Zou, A C^0 linear finite element method for biharmonic problems, J. Sci. Comput., 74 (2018), pp. 1397–1422.

[14] E. Kawecki, O. Lakkis, and T. Pryer, A finite element method for the Monge–Ampère equation with transport boundary conditions. 2018.

[15] O. Lakkis and T. Pryer, A finite element method for nonlinear elliptic problems, SIAM J. Sci. Comput., 35 (2013), pp. A2025–A2045.

[16] H. Liu, R. Glowinski, S. Leung, and J. Qian, A finite element/operator-splitting method for the numerical solution of the three dimensional Monge-Ampère equation, J. Sci. Comput., 81 (2019), pp. 2271–2302.

[17] A. T. T. McRae, C. J. Cotter, and C. J. Budd, Optimal-transport-based mesh adaptivity on the plane and sphere using finite elements, SIAM J. Sci. Comput., 40 (2018), pp. A1121–A1148.

[18] A. Naga and Z. Zhang, A posteriori error estimates based on the polynomial preserving recovery, SIAM J. Numer. Anal., 42 (2004), pp. 1780–1800.

[19] M. Neilan, Quadratic finite element approximations of the Monge-Ampère equation, J. Sci. Comput., 54 (2013), pp. 200–226.

[20] M. Neilan, Finite element methods for fully nonlinear second order PDEs based on a discrete Hessian with applications to the Monge–Ampère equation, J. Comput. Appl. Math., 263 (2014), pp. 351–369.
[21] M. Neilan, A. J. Salgado, and W. Zhang, The Monge-Ampére equation, in Geometric partial differential equations. Part I, vol. 21 of Handb. Numer. Anal., Elsevier/North-Holland, Amsterdam, [2020] ©2020, pp. 105–219.
[22] H. L. Royden, Real analysis, Macmillan Publishing Company, New York, third ed., 1988.
[23] Z. Zhang and A. Naga, A new finite element gradient recovery method: superconvergence property, SIAM J. Sci. Comput., 26 (2005), pp. 1192–1213.

École Nationale de Génie Mathématique et Modélisation, Université Nationale des Sciences, Technologies, Ingénieries et Mathématique, Republic of Benin
E-mail: adetolajamal58@yahoo.com

Department de Mathématiques, Université d'Abomey-Calavi (UAC), Abomey-Calavi, Republic of Benin
E-mail: bahounou@yahoo.fr

Department of Mathematics, Statistics, and Computer Science, M/C 249. University of Illinois at Chicago, Chicago, IL 60607-7045, USA
E-mail: awanou@uic.edu
URL: http://homepages.math.uic.edu/ awanou/

School of Mathematics and Statistics, The University of Melbourne, Parkville, VIC 3010, Australia
E-mail: hailong.guo@unimelb.edu.au
URL: https://hessianguo.github.io