INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 19, Number 5, Pages 656–668

A SECOND-ORDER EMBEDDED LOW-REGULARITY INTEGRATOR FOR THE QUADRATIC NONLINEAR SCHRÖDINGER EQUATION ON TORUS

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Abstract. A new embedded low-regularity integrator is proposed for the quadratic nonlinear Schrödinger equation on the one-dimensional torus. Second-order convergence in H^γ is proved for solutions in $C([0,T];H^\gamma)$ with $\gamma > \frac{3}{2}$, i.e., no additional regularity in the solution is required. The proposed method is fully explicit and can be computed by the fast Fourier transform with $\mathcal{O}(N\log N)$ operations at every time level, where N denotes the degrees of freedom in the spatial discretization. The method extends the first-order convergent low-regularity integrator in [14] to second-order time discretization in the case $\gamma > \frac{3}{2}$ without requiring additional regularity of the solution. Numerical experiments are presented to support the theoretical analysis by illustrating the convergence of the proposed method.

Key words. Quadratic nonlinear Schrödinger equation, low-regularity integrator, second-order convergence, fast Fourier transform.

1. Introduction

This paper is concerned with the development of low-regularity integrators for the quadratic nonlinear Schrödinger (NLS) equation on the one-dimensional torus, i.e.,

$$\begin{cases} i\partial_t u(t,x) + \partial_{xx}u(t,x) = \mu u^2(t,x), & t>0 \text{ and } x\in\mathbb{T}=[0,2\pi], \\ u(0,x) = u^0(x). \end{cases} \tag{1}$$

where $u:\mathbb{R}^+\times\mathbb{T}\to\mathbb{C}$ is a complex-valued unknown function with initial value $u^0\in H^\gamma(\mathbb{T})$, $\gamma\geq 0$, and $\mu\in\mathbb{R}$ is a given constant. The well-posedness of the equation has been proved in [1].

Time discretization of the nonlinear Schrödinger equation has been considered in many papers with different methods. In general, classical time discretizations require the solution to be in $C([0,T];H^{\gamma+2})$ and $C([0,T];H^{\gamma+4})$ in order to have first- and second-order convergence in H^γ, respectively, i.e., two additional derivatives in the solution are required for every order of convergence. The convergence of time discretizations under these (or stronger) regularity conditions has been proved for the finite difference methods [17], operator splitting [2, 5, 10], and exponential integrators [4].

In practical computations, the initial data may be polluted by nonsmooth noises from the measurements. Accordingly, the development of low-regularity integrators which can reduce the regularity requirement of the solution and has attracted much attention from numerical analysts. Ostermann and Schratz [14] proposed a new exponential-type integrator for the cubic NLS equation in the d-dimensional space, and proved its first-order convergence in H^γ for solutions in $C([0,T];H^{\gamma+1})$, with $\gamma > \frac{d}{2}$. In one dimension, Wu and Yao [18] proposed a new time discretization which has first-order convergence in H^γ for solutions

Received by the editors June 12, 2022 and in revised form, June 29, 2022.
2020 *Mathematics Subject Classification.* Primary 65M12, 65M15, 35Q55.

in $C([0,T];H^\gamma)$ with $\gamma > \frac{3}{2}$, without requiring any additional regularity in the solution. These articles are all concerned with first-order convergent low-regularity integrators.

Knöller, Ostermann and Schratz [7] proposed a second-order low-regularity integrator which requires two and three additional derivatives in the solution in one- and higher-dimensional spaces, respectively. In two- and higher-dimensional spaces, the regularity requirement was relaxed to two additional derivatives by Bruned and Schratz [3] and Ostermann, Wu and Yao [15] with different methods.

For convergence in L^2, Ostermann, Rousset and Schratz [12, 13] proved certain fractional-order convergence of some filtered methods for solutions in $C([0,T];H^\gamma)$ with $\gamma \in (0,1]$. Li and Wu [8] constructed a fully discrete low-regularity integrator with first-order convergence in both time and space for solutions in $C([0,T];H^1)$. Ostermann and Yao [16] proposed a different fully discrete method with an error estimate of $\mathcal{O}(\tau^{\frac{3}{2}\gamma-\frac{1}{2}-\varepsilon} + N^{-\gamma})$ for solutions in $C([0,T];H^\gamma)$ with $\gamma \in (\frac{1}{2},1]$.

More recently, Wu and Zhao [19, 20] introduced an embedded low-regularity integrator for the Korteweg-de Vries (KdV) equation with first- and second-order convergence in $H^\gamma(\mathbb{T})$ for solutions in $C([0,T];H^{\gamma+1})$ and $C([0,T];H^{\gamma+3})$, respectively. By using new harmonic analysis techniques, Li, Wu and Yao [9] proposed a method for the KdV equation with $\frac{1}{2}$-order convergence in H^γ for solutions in $C([0,T];H^\gamma)$ with $\gamma > \frac{3}{2}$, without requiring any additional derivatives in the solution. For the modified KdV equation, Ning, Wu and Zhao [11] proposed a new embedded low-regularity integrator and proved first-order convergence by requiring the boundedness of one additional spatial derivative of the solution.

For the quadratic nonlinear Schrödinger equation on the one-dimensional torus, Ostermann and Schratz [14] proposed a low-regularity integrator with first-order convergence in H^γ for solutions in $C([0,T];H^\gamma)$, $\gamma > \frac{1}{2}$. In the present paper, we propose a new embedded low-regularity integrator with second-order convergence in H^γ for solutions in $C([0,T];H^\gamma)$, $\gamma > \frac{3}{2}$. The construction of the method extends the low-regularity integrators in [19] and [15], which were originally proposed for the KdV equation and cubic nonlinear Schrödinger equation, respectively. The proof of convergence for the proposed method is based on harmonic analysis techniques.

The rest of this paper is organized as follows. The notations and the main result are presented in Section 2. The construction of the low-regularity integrator and the technical lemmas to be used in the convergence analysis are presented in Section 3. The proof of the main theorem is presented in Section 4. Numerical experiments are reported in Section 5. Some concluding remarks are presented in Section 6.

2. Notations and main results

2.1. Some notations. We denote by $\langle \cdot,\cdot \rangle$ the inner product of $L^2 = L^2(\mathbb{T})$, i.e.,

$$\langle f,g \rangle = \int_{\mathbb{T}} f(x)\overline{g(x)}\,dx, \qquad f,\, g \in L^2.$$

The Fourier transform $(\hat{f}_k)_{k\in\mathbb{Z}}$ of a function $f\colon\, \mathbb{T} \to \mathbb{C}$ is defined by

$$\hat{f}_k = \frac{1}{2\pi}\int_{\mathbb{T}} e^{-ikx} f(x)\,dx.$$

The inverse Fourier transform formula is given by

$$f(x) = \sum_{k\in\mathbb{Z}} \hat{f}_k e^{ikx}.$$

The following standard properties of the Fourier transform are well known:

$$\|f\|_{L^2(\mathbb{T})}^2 = 2\pi \sum_{k\in\mathbb{Z}} \left|\hat{f}_k\right|^2, \qquad f \in L^2;$$

$$\widehat{(fg)}_k = \sum_{k=k_1+k_2} \hat{f}_{k_1}\hat{g}_{k_2}, \qquad f,\, g \in L^2.$$

We denote by H^s, $s \geq 0$, the space of functions in L^2 such that their s-order derivatives are also in L^2, equipped with the following norm:

$$\left\|f\right\|_{H^s}^2 = \left\|J^s f\right\|_{L^2}^2 = 2\pi \sum_{k\in\mathbb{Z}} (1+k^2)^s |\hat{f}_k|^2, \qquad J^s = (1-\partial_{xx})^{\frac{s}{2}}.$$

Further, we denote by ∂_x^{-1} the operator defined in Fourier space as

$$(\widehat{\partial_x^{-1} f})_k = \begin{cases} (ik)^{-1}\hat{f}_k & \text{if } k \neq 0, \\ 0 & \text{if } k = 0. \end{cases} \tag{2}$$

For the convenience of notations, we introduce the zero-mode operator $P_0 : L^2 \to \mathbb{C}$, defined by

$$P_0 f = \hat{f}_0 = \frac{1}{2\pi}\int_{\mathbb{T}} f(x)\, dx,$$

and we denote by $\mathbb{P}$ the orthogonal projection onto the space of mean-zero functions

$$\mathbb{P}f = f - P_0 f.$$

The following phase functions will be used in the convergence analysis:

$$\phi_1 = k^2 - k_1^2 - k_2^2, \tag{3}$$

$$\phi_2 = k^2 - k_1^2 - k_2^2 - k_3^2, \tag{4}$$

$$\phi_3 = k^2 - k_1^2 - (k_2+k_3)^2. \tag{5}$$

Furthermore, we define the following functions:

$$\varphi(z) = \begin{cases} \dfrac{e^z - 1}{z}, & z \neq 0, \\ 1, & z = 0, \end{cases} \qquad \psi(z) = \begin{cases} \dfrac{e^z - 1 - ze^z}{z^2}, & z \neq 0, \\ -\dfrac{1}{2}, & z = 0. \end{cases} \tag{6}$$

2.2. The numerical method and the main result. Let $t_n = n\tau$, $n = 0, 1, \ldots, L = T/\tau$, be a partition of the time interval $[0, T]$ with a uniform stepsize $\tau > 0$. The low-regularity integrator proposed in this paper for equation (1) is given by

$$u^{n+1} = \Phi(u^n), \tag{7}$$

with

$$\begin{aligned}
\Phi(u^n) := {} & \mathrm{e}^{i\tau\partial_x^2} u^n \big(1 + 2\mu^2\tau^2 (P_0 u^n)^2\big) - \big(i\mu + 4\mu^2\tau P_0 u^n\big) G_n(u^n, u^n) \\
& + i\mu^2 G_n(\partial_x^{-1} u^n, \partial_x^{-1} u^n \cdot u^n) - i\mu^2 \tau P_0\Big(\big(\partial_x^{-1} u^n\big)^2 \cdot \big(\varphi(2i\tau\partial_x^2) + \psi(2i\tau\partial_x^2)\big) u^n\Big) \\
& + i\mu^2\tau P_0\Big(\mathrm{e}^{-i\tau\partial_x^2}\big(\mathrm{e}^{i\tau\partial_x^2}\partial_x^{-1} u^n\big)^2 \cdot \psi(2i\tau\partial_x^2) u^n\Big) \\
& - i\mu^2 P_0\big(G_n(\partial_x^{-1} u^n, \partial_x^{-1} u^n)\big) \cdot \mathrm{e}^{i\tau\partial_x^2}\mathbb{P}u^n + \frac{i\mu^2}{\tau}\partial_x^{-1} G_n\big(\partial_x^{-1} G_n(\mathbb{P}u^n, \mathbb{P}u^n), \mathbb{P}u^n\big) \\
& + \frac{\mu^2}{2}\partial_x^{-1}\Big(\partial_x^{-1}\big(\mathrm{e}^{i\tau\partial_x^2}\partial_x^{-1} u^n\big)^2 \cdot \mathrm{e}^{i\tau\partial_x^2}\mathbb{P}u^n\Big) - \frac{\mu^2}{2}\mathrm{e}^{i\tau\partial_x^2}\partial_x^{-1}\Big(\partial_x^{-1}\big(\partial_x^{-1} u^n\big)^2 \cdot \mathbb{P}u^n\Big) \\
& + \frac{\mu^2}{3}\partial_x^{-1}\big(\mathrm{e}^{i\tau\partial_x^2}\partial_x^{-1} u^n\big)^3 - \frac{\mu^2}{3}\mathrm{e}^{i\tau\partial_x^2}\partial_x^{-1}(\partial_x^{-1} u^n)^3 + 2i\mu^2 P_0 u^n \cdot \partial_x^{-1} G_n(\partial_x^{-1} u^n, u^n),
\end{aligned}$$

where the bilinear functional G_n is defined by

$$G_n(u,v) = \frac{i}{2}\big(\mathrm{e}^{i\tau\partial_x^2}\partial_x^{-1}u\cdot \mathrm{e}^{i\tau\partial_x^2}\partial_x^{-1}v\big) - \frac{i}{2}\mathrm{e}^{i\tau\partial_x^2}\big(\partial_x^{-1}u\cdot\partial_x^{-1}v\big) \\ + \tau P_0 u\cdot \mathrm{e}^{i\tau\partial_x^2}v + \tau P_0 v\cdot \mathrm{e}^{i\tau\partial_x^2}u - \tau P_0 u\cdot P_0 v.$$

Although we focus on the analysis of the time discretization without considering spatial discretizations, we point out that the proposed time discretization in (7) can be implemented with FFT in the practical computation.

The main theoretical result of this paper is the following theorem.

Theorem 2.1. *We assume that the solution of* (1) *satisfies* $u\in C([0,T];H^\gamma)$ *for some* $\gamma>\frac{3}{2}$, *and consider the numerical solution* u^n, $n=0,1,\ldots,L$, *given by* (7). *Then there exist positive constants* τ_0 *and* C_0 *such that for any stepsize* $\tau\in(0,\tau_0]$ *the following error bound holds:*

$$\|u(t_n,\cdot)-u^n\|_{H^\gamma}\le C_0\tau^2, \tag{8}$$

where the constants τ_0 *and* C_0 *depend only on* T *and* $\|u\|_{C([0,T];H^\gamma)}$.

For the simplicity of notation, we denote by $A\lesssim B$ or $B\gtrsim A$ the statement "$A\le CB$ for some positive constant C", where C may be different at each occurrence but is always independent of τ and n.

3. The construction of the scheme and some technical lemmas

In this section, we present the construction of the low-regularity integrator for the quadratic Schrödinger equation, and then state some technical lemmas to be used in the error analysis.

3.1. The construction of the scheme. By using Duhamel's formula, we can write the solution of the quadratic Schrödinger equation as

$$u(t_{n+1}) = \mathrm{e}^{i\tau\partial_x^2}u(t_n) - i\mu\int_0^\tau \mathrm{e}^{i(t_{n+1}-(t_n+s))\partial_x^2}u^2(t_n+s)\,ds.$$

Then, by introducing the twisted function $v(t)=\mathrm{e}^{-it\partial_x^2}u(t)$, the formula above can be expressed as

$$v(t_{n+1}) = v(t_n) - i\mu\int_0^\tau \mathrm{e}^{-i(t_n+s)\partial_x^2}\big(\mathrm{e}^{i(t_n+s)\partial_x^2}v(t_n+s)\big)^2\,ds. \tag{9}$$

We are interested in the construction of a second-order convergent method for the quadratic Schrödinger equation. To this end, we consider the following approximation to the term $v(t_n+s)$ in (9):

$$v(t_n+s)\approx v(t_n) - i\mu F_n(v(t_n),v(t_n),s),$$

where F_n is defined as

$$F_n(f,g,s) = \int_0^s \mathrm{e}^{-i(t_n+t)\partial_x^2}\big(\mathrm{e}^{i(t_n+t)\partial_x^2}f\cdot \mathrm{e}^{i(t_n+t)\partial_x^2}g\big)\,dt. \tag{10}$$

This type of approximations was proposed in [19] for the KdV equation. By inserting this approximation into (9), we get

$$v(t_{n+1}) = v(t_n) - i\mu F_n(v(t_n),v(t_n),\tau) + I(v(t_n)) + \mathcal{R}_1(v(t_n)), \tag{11}$$

where $\mathcal{R}_1(v(t_n))$ is the remainder term and $I(v(t_n))$ denotes

$$I(v(t_n)) := -2\mu^2\int_0^\tau \mathrm{e}^{-i(t_n+s)\partial_x^2}\Big(\mathrm{e}^{i(t_n+s)\partial_x^2}v(t_n)\cdot \mathrm{e}^{i(t_n+s)\partial_x^2}F_n(v(t_n),v(t_n),s)\Big)\,ds.$$

First, we consider the term $F_n(v(t_n), v(t_n), \tau)$ in (11). To simplify the notation, we use v instead of $v(t_n)$ in the rest of this paper. Then, by applying the Fourier transform to (10), we have

$$\hat{F}_n(v, v, \tau)_k = \int_0^\tau \sum_{k=k_1+k_2} \mathrm{e}^{i(t_n+s)\phi_1}\, \hat{v}_{k_1}\hat{v}_{k_2}\, ds, \tag{12}$$

where $\hat{v}_k$ denotes the kth Fourier coefficient of $v(t_n)$ and ϕ_1 is the phase function defined in (3). Under condition $k = k_1 + k_2$, the following relation holds:

$$\phi_1 = k^2 - k_1^2 - k_2^2 = 2k_1k_2. \tag{13}$$

Hence, $F_n(v, v, \tau)$ can be explicitly integrated. In particular, integrating it with respect to s yields the following expression:

$$\hat{F}_n(v, v, \tau)_k = \sum_{k=k_1+k_2} \Big[\frac{1}{2ik_1k_2}\Big(\mathrm{e}^{it_{n+1}\phi_1} - \mathrm{e}^{it_n\phi_1}\Big)\, \hat{v}_{k_1}\hat{v}_{k_2}\Big] + 2\tau\hat{v}_0\hat{v}_k \quad \text{for } k \neq 0.$$

$$\hat{F}_n(v, v, \tau)_0 = \sum_{0=k_1+k_2} \Big[\frac{1}{2ik_1k_2}\Big(\mathrm{e}^{it_{n+1}\phi_1} - \mathrm{e}^{it_n\phi_1}\Big)\, \hat{v}_{k_1}\hat{v}_{k_2}\Big] + \tau(\hat{v}_0)^2.$$

The inverse Fourier transform of this expression gives that

$$F_n(v, v, \tau) = \frac{i}{2}\mathrm{e}^{-it_{n+1}\partial_x^2}\big(\mathrm{e}^{it_{n+1}\partial_x^2}\partial_x^{-1}v\big)^2 - \frac{i}{2}\mathrm{e}^{-it_n\partial_x^2}\big(\mathrm{e}^{it_n\partial_x^2}\partial_x^{-1}v\big)^2 + 2\tau\hat{v}_0 v - \tau(\hat{v}_0)^2. \tag{14}$$

Next, we consider the term $I(v)$ in (11). Applying the Fourier transform, we have

$$\hat{I}_k(v) = -2\mu^2 \int_0^\tau \sum_{k=k_1+k_2} \mathrm{e}^{i(t_n+s)\phi_1}\, \hat{v}_{k_1}\hat{F}_n(v, v, s)_{k_2}\, ds.$$

From (14) we see that the above formula can be written as

$$\begin{aligned}
\hat{I}_k(v) = {} & i\mu^2 \int_0^\tau \sum_{k=k_1+k_2+k_3} \frac{1}{k_1k_2}\mathrm{e}^{i(t_n+s)\phi_2}\, \hat{v}_{k_1}\hat{v}_{k_2}\hat{v}_{k_3}\, ds \\
& - i\mu^2 \int_0^\tau \sum_{k=k_1+k_2+k_3} \frac{1}{k_1k_2}\mathrm{e}^{it_n\phi_2}\mathrm{e}^{is\phi_3}\, \hat{v}_{k_1}\hat{v}_{k_2}\hat{v}_{k_3}\, ds \\
& - 4\mu^2\tau\hat{v}_0 \int_0^\tau \sum_{k=k_1+k_2} \mathrm{e}^{i(t_n+s)\phi_1}\, \hat{v}_{k_1}\hat{v}_{k_2}\, ds + 2\mu^2\tau^2(\hat{v}_0)^2\hat{v}_k \\
=: {} & \hat{I}_{1,k}(v) + \hat{I}_{2,k}(v) + \hat{I}_{3,k}(v) + 2\mu^2\tau^2(\hat{v}_0)^2\hat{v}_k,
\end{aligned}$$

where ϕ_2 and ϕ_3 are phase functions defined in (4) and (5), i.e.,

$$\phi_2 = k^2 - k_1^2 - k_2^2 - k_3^2, \qquad \phi_3 = k^2 - k_1^2 - (k_2 + k_3)^2.$$

Note that

$$\begin{aligned}
I_2(v) = {} & i\mu^2 F_n\big(\partial_x^{-1}v, \mathrm{e}^{-it_n\partial_x^2}[(\mathrm{e}^{it_n\partial_x^2}\partial_x^{-1}v)(\mathrm{e}^{it_n\partial_x^2}v)], \tau\big), \\
I_3(v) = {} & -4\mu^2\tau\hat{v}_0 F_n(v, v, \tau).
\end{aligned}$$

We consider $\hat{I}_{1,k}(v)$ in the following two cases.

Case 1: $k = 0$. From the definition of $\hat{I}_{1,k}(v)$ we see that

$$\hat{I}_{1,0}(v) = i\mu^2 \int_0^\tau \sum_{0=k_1+k_2+k_3} \frac{1}{k_1 k_2} \mathrm{e}^{i(t_n+s)(-k_1^2-k_2^2-k_3^2)}\, \hat{v}_{k_1}\hat{v}_{k_2}\hat{v}_{k_3}\, ds.$$

Under the assumption $0 = k_1 + k_2 + k_3$, we can express the phase function as

$$-k_1^2 - k_2^2 - k_3^2 = -2k_3^2 + 2k_1k_2.$$

We recall the following formula introduced in [15]:

$$\int_0^\tau \mathrm{e}^{is(\alpha+\beta)}\, ds = \tau\varphi(i\tau\alpha) - \tau\big(\mathrm{e}^{i\tau\beta} - 1\big)\psi(i\tau\alpha) + \mathcal{R}(\alpha,\beta,\tau), \tag{15}$$

where $|\mathcal{R}(\alpha,\beta,\tau)| \lesssim \tau^3|\beta|^2$, φ and ψ are defined in (6).

Utilizing this formula, choosing $\alpha = -2k_3^2$ and $\beta = 2k_1k_2$, we note that $\hat{I}_{1,0}(v)$ is approximately integrable, with the following expression:

$$\begin{aligned}\hat{I}_{1,0}(v) = &-i\mu^2\tau P_0\Big(\big(\mathrm{e}^{it_n\partial_x^2}\partial_x^{-1}v\big)^2 \cdot \big(\varphi(2i\tau\partial_x^2) + \psi(2i\tau\partial_x^2)\big)\mathrm{e}^{it_n\partial_x^2}v\Big)\\ &+ i\mu^2\tau P_0\Big(\mathrm{e}^{-i\tau\partial_x^2}\big(\mathrm{e}^{it_{n+1}\partial_x^2}\partial_x^{-1}v\big)^2 \cdot \psi(2i\tau\partial_x^2)\mathrm{e}^{it_n\partial_x^2}v\Big) + \mathcal{R}_2(v),\end{aligned}$$

where

$$|\mathcal{R}_2(v)| \lesssim \tau^3 \sum_{0=k_1+k_2+k_3} |k_1||k_2|\, |\hat{v}_{k_1}||\hat{v}_{k_2}||\hat{v}_{k_3}|. \tag{16}$$

Case 2: $k \neq 0$. By using the identity $1 = \dfrac{k_1+k_2+k_3}{k}$ and symmetry, we can decompose $\hat{I}_{1,k}$ into the following two parts:

$$\begin{aligned}\hat{I}_{1,k} &= i\mu^2 \int_0^\tau \sum_{k=k_1+k_2+k_3} \frac{k_3}{kk_1k_2}\mathrm{e}^{i(t_n+s)\phi_2}\, \hat{v}_{k_1}\hat{v}_{k_2}\hat{v}_{k_3}\, ds\\ &\quad + 2i\mu^2 \int_0^\tau \sum_{\substack{k=k_1+k_2+k_3\\ k_2\neq 0}} \frac{1}{kk_1}\mathrm{e}^{i(t_n+s)\phi_2}\, \hat{v}_{k_1}\hat{v}_{k_2}\hat{v}_{k_3}\, ds\\ &=: \hat{I}_{11,k} + \hat{I}_{12,k}.\end{aligned}$$

In the estimation of $\hat{I}_{11,k}$, we consider the following decomposition of ϕ_2:

$$\phi_2 = 2k_1k_2 + 2k_1k_3 + 2k_2k_3 = 2k_3(k_1+k_2) + 2k_1k_2.$$

By considering the two cases $k_1 + k_2 = 0$ and $k_1 + k_2 \neq 0$, separately, we can further divide $\hat{I}_{11,k}$ into two parts, i.e.,

$$\hat{I}_{11,k} = \hat{I}_{11a,k} + \hat{I}_{11b,k},$$

where

$$\begin{aligned}\hat{I}_{11a,k} &= i\mu^2 \int_0^\tau \sum_{0=k_1+k_2} \frac{1}{k_1k_2}\mathrm{e}^{i(t_n+s)(-k_1^2-k_2^2)}\, \hat{v}_{k_1}\hat{v}_{k_2}\, ds \cdot \hat{v}_k,\\ \hat{I}_{11b,k} &= i\mu^2 \int_0^\tau \sum_{\substack{k=k_1+k_2+k_3\\ k_1+k_2\neq 0}} \frac{k_3}{kk_1k_2}\mathrm{e}^{i(t_n+s)\phi_2}\, \hat{v}_{k_1}\hat{v}_{k_2}\hat{v}_{k_3}\, ds.\end{aligned}$$

Using (12) and the formula of inverse Fourier transform, we obtain

$$I_{11a} = -i\mu^2 P_0(F_n(\partial_x^{-1}v, \partial_x^{-1}v, \tau)) \cdot \hat{v}_k.$$

If $\alpha \neq 0$ then the expressions in (6) imply that

$$\tau\varphi(i\tau\alpha) - \tau\big(\mathrm{e}^{i\tau\beta}-1\big)\psi(i\tau\alpha) = \tau\frac{e^{i\tau\alpha}-1}{i\tau\alpha} - \tau\big(\mathrm{e}^{i\tau\beta}-1\big)\frac{e^{i\tau\alpha}-1-i\tau\alpha e^{i\tau\alpha}}{(i\tau\alpha)^2}$$
$$= \tau\frac{e^{i\tau(\alpha+\beta)}-1}{i\tau\alpha} - \tau\frac{\big(e^{i\tau\alpha}-1\big)\big(e^{i\tau\beta}-1\big)}{(i\tau\alpha)^2}.$$

Hence, we can choose $\alpha = 2k_3(k_1+k_2)$, $\beta = 2k_1k_2$, and use (15) to get the following expression:

$$\hat{I}_{11b,k} = \frac{\mu^2}{2}\sum_{\substack{k=k_1+k_2+k_3\\ k_3\neq 0}} \frac{\mathrm{e}^{i\tau\phi_2}-1}{kk_1k_2(k_1+k_2)}\mathrm{e}^{it_n\phi_2}\,\hat{v}_{k_1}\hat{v}_{k_2}\hat{v}_{k_3}$$
$$+\frac{i\mu^2}{4\tau}\sum_{k=k_1+k_2+k_3}\frac{\big(\mathrm{e}^{2i\tau k_1k_2}-1\big)\big(\mathrm{e}^{2i\tau k_3(k_1+k_2)}-1\big)}{kk_1k_2k_3(k_1+k_2)^2}\cdot\mathrm{e}^{it_n\phi_2}\,\hat{v}_{k_1}\hat{v}_{k_2}\hat{v}_{k_3}+\hat{\mathcal{R}}_{3,k}(v),$$

where

$$|\hat{\mathcal{R}}_{3,k}(v)| \lesssim \tau^3\sum_{k=k_1+k_2+k_3}\frac{|k_1k_2k_3|}{|k|}\,|\hat{v}_{k_1}||\hat{v}_{k_2}||\hat{v}_{k_3}|. \tag{17}$$

The first term on right-hand side can be integrated explicitly. From (14) we derive that

$$\hat{F}_n(\mathbb{P}v,\mathbb{P}v,\tau)_k = \sum_{k=k_1+k_2}\frac{1}{2ik_1k_2}\Big(\mathrm{e}^{2it_{n+1}k_1k_2}-\mathrm{e}^{2it_nk_1k_2}\Big)\,\hat{v}_{k_1}\hat{v}_{k_2}.$$

Therefore, the second term on right-hand side can be written as

$$-\frac{\mu^2}{2\tau}\sum_{k=\tilde{k}+k_3}\frac{\mathrm{e}^{2i\tau k_3\tilde{k}}-1}{kk_3\tilde{k}^2}\cdot\mathrm{e}^{2it_nk_3\tilde{k}}\,\hat{F}_n(\mathbb{P}v,\mathbb{P}v,\tau)_{\tilde{k}}\hat{v}_{k_3},$$

which is the kth Fourier coefficient of the function

$$\frac{i\mu^2}{\tau}\partial_x^{-1}F_n\big(\partial_x^{-1}F_n(\mathbb{P}v,\mathbb{P}v,\tau),\mathbb{P}v\big).$$

Hence, we obtain

$$I_{11b} = \frac{\mu^2}{2}\mathrm{e}^{-i(t_n+s)\partial_x^2}\partial_x^{-1}\Big(\partial_x^{-1}\big(\mathrm{e}^{i(t_n+s)\partial_x^2}\partial_x^{-1}v\big)^2\cdot\mathrm{e}^{i(t_n+s)\partial_x^2}\mathbb{P}v\Big)\Big|_{s=0}^{s=\tau}$$
$$+\frac{i\mu^2}{\tau}\partial_x^{-1}F_n\big(\partial_x^{-1}F_n(\mathbb{P}v,\mathbb{P}v,\tau),\mathbb{P}v\big).$$

We rewrite $\hat{I}_{12,k}$ as

$$\hat{I}_{12,k} = J_a + J_b, \tag{18}$$

where

$$J_a = 2i\mu^2\int_0^\tau\sum_{k=k_1+k_2+k_3}\frac{1}{kk_1}\mathrm{e}^{i(t_n+s)\phi_2}\,\hat{v}_{k_1}\hat{v}_{k_2}\hat{v}_{k_3}\,ds,$$
$$J_b = -2i\mu^2\int_0^\tau\sum_{k=k_1+k_2}\frac{1}{kk_1}\mathrm{e}^{i(t_n+s)\phi_1}\,\hat{v}_{k_1}\hat{v}_{k_2}\,ds\cdot\hat{v}_0.$$

By using the symmetry among k_1, k_2 and k_3, we can rewrite J_a as

$$J_a = \frac{2i\mu^2}{3}\int_0^\tau\sum_{k=k_1+k_2+k_3}\frac{1}{k}\Big(\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}\Big)\mathrm{e}^{i(t_n+s)\phi_2}\,\hat{v}_{k_1}\hat{v}_{k_2}\hat{v}_{k_3}\,ds.$$

In particular, J_a can be integrated explicitly by using the identity

$$\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}=\frac{\phi_2}{2k_1k_2k_3},$$

with

$$J_a=\frac{\mu^2}{3}\sum_{k=k_1+k_2+k_3}\frac{1}{kk_1k_2k_3}\Big(\mathrm{e}^{it_{n+1}\phi_2}-\mathrm{e}^{it_n\phi_2}\Big)\,\hat v_{k_1}\hat v_{k_2}\hat v_{k_3}.$$

J_b can be expressed as follows by using the definition of F_n in (14):

$$J_b=2i\mu^2\hat v_0\mathscr{F}\partial_x^{-1}F_n(\partial_x^{-1}v,v,\tau)(k).$$

Combining with the above formulas and taking inverse Fourier transform, we obtain

$$I_{12}=\frac{\mu^2}{3}\mathrm{e}^{-i(t_n+s)\partial_x^2}\partial_x^{-1}\big(\mathrm{e}^{i(t_n+s)\partial_x^2}\partial_x^{-1}v\big)^3\Big|_{s=0}^{s=\tau}+2i\mu^2\hat v_0\partial_x^{-1}F_n(\partial_x^{-1}v,v,\tau).$$

To summarize, we obtain the following formula:

$$v(t_{n+1})=\Phi^n(v(t_n))+\mathcal{R}_1(v(t_n))+\mathcal{R}_2(v(t_n))+\mathcal{R}_3(v(t_n)),\tag{19}$$

where $\Phi^n(v)$ is defined by

$$\begin{aligned}\Phi^n(v)&=v-(i\mu+4\mu^2\tau\hat v_0)F_n(v,v,\tau)+i\mu^2F_n\big(\partial_x^{-1}v,\mathrm{e}^{-it_n\partial_x^2}[(\mathrm{e}^{it_n\partial_x^2}\partial_x^{-1}v)(\mathrm{e}^{it_n\partial_x^2}v)],\tau\big)\\&\quad+2\mu^2\tau^2(\hat v_0)^2v-i\mu^2\tau P_0\Big(\big(\mathrm{e}^{it_n\partial_x^2}\partial_x^{-1}v\big)^2\cdot\big(\varphi(2i\tau\partial_x^2)+\psi(2i\tau\partial_x^2)\big)\mathrm{e}^{it_n\partial_x^2}v\Big)\\&\quad+i\mu^2\tau P_0\Big(\mathrm{e}^{-i\tau\partial_x^2}\big(\mathrm{e}^{it_{n+1}\partial_x^2}\partial_x^{-1}v\big)^2\cdot\psi(2i\tau\partial_x^2)\mathrm{e}^{it_n\partial_x^2}v\Big)\\&\quad-i\mu^2P_0(F_n(\partial_x^{-1}v,\partial_x^{-1}v,\tau))\cdot\mathbb{P}v+\frac{i\mu^2}{\tau}\partial_x^{-1}F_n\big(\partial_x^{-1}F_n(\mathbb{P}v,\mathbb{P}v,\tau),\mathbb{P}v\big)\\&\quad+\frac{\mu^2}{2}\mathrm{e}^{-i(t_n+s)\partial_x^2}\partial_x^{-1}\Big(\partial_x^{-1}\big(\mathrm{e}^{i(t_n+s)\partial_x^2}\partial_x^{-1}v\big)^2\cdot\mathrm{e}^{i(t_n+s)\partial_x^2}\mathbb{P}v\Big)\Big|_{s=0}^{s=\tau}\\&\quad+\frac{\mu^2}{3}\mathrm{e}^{-i(t_n+s)\partial_x^2}\partial_x^{-1}\big(\mathrm{e}^{i(t_n+s)\partial_x^2}\partial_x^{-1}v\big)^3\Big|_{s=0}^{s=\tau}+2i\mu^2\hat v_0\partial_x^{-1}F_n(\partial_x^{-1}v,v,\tau).\end{aligned}\tag{20}$$

By dropping the remainders $\mathcal{R}_1(v(t_n))$, $\mathcal{R}_2(v(t_n))$ and $\mathcal{R}_3(v(t_n))$, we obtain the following numerical scheme: For any given v^n, compute

$$v^{n+1}=\Phi^n\big(v^n\big),\quad n=0,1,\ldots,L;\quad v^0=u_0.\tag{21}$$

Substituting $u^n:=\mathrm{e}^{it_n\partial_x^2}v^n$ into (21) yields the numerical scheme in (7).

3.2. Some technical estimates. In this subsection we present two technical estimates which will be used in the error estimation.

Lemma 3.1 (Kato-Ponce inequality, [6])**.** *Let $f,g\in H^\gamma$ for some $\gamma>\frac{1}{2}$. Then the following inequality holds:*

$$\|fg\|_{H^\gamma}\lesssim\|f\|_{H^\gamma}\|g\|_{H^\gamma}.$$

Lemma 3.2 (Some trilinear estimates)**.**

(i) Assume that f, g, h are functions in H^1, and

$$|T(f,g,h)|\lesssim\sum_{0=k_1+k_2+k_3}|k_1||k_2|\,|\hat f_{k_1}||\hat g_{k_2}||\hat h_{k_3}|.$$

Then the following estimate holds:

$$|T(f,g,h)|\lesssim\|f\|_{H^1}\|g\|_{H^1}\|h\|_{H^1}.$$

(ii) Assume that $f, g, h \in H^\gamma$ with $\gamma > \frac{3}{2}$, and

$$|\hat{T}_k(f,g,h)| \lesssim \sum_{k=k_1+k_2+k_3} \frac{|k_1k_2k_3|}{|k|} |\hat{f}_{k_1}||\hat{g}_{k_2}||\hat{h}_{k_3}|.$$

Then the following estimate holds:

$$\|T(f,g,h)\|_{H^\gamma} \lesssim \|f\|_{H^\gamma}\|g\|_{H^\gamma}\|h\|_{H^\gamma}.$$

Proof. We denote

$$\tilde{f}(x) = \sum_{k\in\mathbb{Z}} e^{ikx}|\hat{f}_k|, \qquad \tilde{g}(x) = \sum_{k\in\mathbb{Z}} e^{ikx}|\hat{g}_k|, \qquad \tilde{h}(x) = \sum_{k\in\mathbb{Z}} e^{ikx}|\hat{h}_k|.$$

Then $\widehat{\tilde{f}}_k(t) = |\hat{f}_k(t)|$ and the following identities hold (for all $s \geq 0$):

$$\begin{aligned} \|\tilde{f}\|_{H^s}^2 &= 2\pi\sum_{k\in\mathbb{Z}}(1+k^2)^s|\widehat{\tilde{f}}_k|^2 = \|f\|_{H^s}^2, \\ \|\tilde{g}\|_{H^s}^2 &= \|g\|_{H^s}^2, \\ \|\tilde{h}\|_{H^s}^2 &= \|h\|_{H^s}^2. \end{aligned} \tag{22}$$

(i) The condition guarantees that

$$\begin{aligned} |T(f,g,h)| &\lesssim \sum_{0=k_1+k_2+k_3} |k_1||k_2|\,|\hat{f}_{k_1}||\hat{g}_{k_2}||\hat{h}_{k_3}| \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} |\nabla|\tilde{f}\cdot|\nabla|\tilde{g}\cdot\tilde{h}\,dx, \end{aligned}$$

where the last equality is obtained by using the definition of the Fourier transform. Then, by using Hölder's inequality and (22), we obtain

$$|T(f,g,h)| \lesssim \|\tilde{f}\|_{H^1}\|\tilde{g}\|_{H^1}\|\tilde{h}\|_{H^1} = \|f\|_{H^1}\|g\|_{H^1}\|h\|_{H^1}.$$

(ii) The following result can be obtained by using Plancherel's identity:

$$\begin{aligned} \big\|T(f,g,h)\big\|_{H^\gamma} &\lesssim \Big\| \sum_{k=k_1+k_2+k_3\neq 0} |k|^{\gamma-1}|k_1||k_2||k_3|\,|\hat{f}_{k_1}(t)||\hat{g}_{k_2}(t)||\hat{h}_{k_3}(t)|\Big\|_{L^\infty((0,T);l^2)} \\ &\lesssim \big\||\nabla|\tilde{f}\cdot|\nabla|\tilde{g}\cdot|\nabla|\tilde{h}\big\|_{L^\infty((0,T);H^{\gamma-1})}. \end{aligned}$$

Since $\gamma - 1 > \frac{1}{2}$, the following result holds according to Lemma 3.1:

$$\big\|T(f,g,h)\big\|_{H^\gamma} \lesssim \|\tilde{f}\|_{H^\gamma}\|\tilde{g}\|_{H^\gamma}\|\tilde{h}\|_{H^\gamma} = \|f\|_{H^\gamma}\|g\|_{H^\gamma}\|h\|_{H^\gamma}.$$

□

4. Proof of Theorem 2.1

By considering the difference between (21) and (19), we obtain the following error equation:

$$v(t_{n+1}) - v^{n+1} \triangleq \mathcal{L}^n + \Phi^n\big(v(t_n)\big) - \Phi^n\big(v^n\big), \quad \text{for all } n = 0,1,\ldots,L, \tag{23}$$

where

$$\mathcal{L}^n = \mathcal{R}_1(v(t_n)) + \mathcal{R}_2(v(t_n)) + \mathcal{R}_3(v(t_n))$$

is the consistency error.

For $\gamma > \frac{3}{2}$, the following estimate is a consequence of (16), (17) and Lemma 3.2:

$$\|\mathcal{R}_2(v(t_n))\|_{H^\gamma} + \|\mathcal{R}_3(v(t_n))\|_{H^\gamma} \leq C\tau^3, \tag{24}$$

where the constant C depends only on $\|u\|_{L^\infty((0,T);H^\gamma)}$.

The specific expression of $\mathcal{R}_1(v(t_n))$ can be found by comparing (9) with (11), which imply that

$$\mathcal{R}_1(v(t_n)) = \mathcal{R}_{11}(v(t_n)) + \mathcal{R}_{12}(v(t_n)),$$

where

$$\begin{aligned}\mathcal{R}_{11}(v(t_n)) =& - i\mu \int_0^\tau \mathrm{e}^{-i(t_n+s)\partial_x^2} \Big(\mathrm{e}^{i(t_n+s)\partial_x^2} v(t_n+s)\Big)^2 \, ds \\ &+ i\mu \int_0^\tau \mathrm{e}^{-i(t_n+s)\partial_x^2} \Big(\mathrm{e}^{i(t_n+s)\partial_x^2} \Big(v(t_n) - i\mu F_n\big(v(t_n), v(t_n), s\big)\Big)\Big)^2 \, ds, \\ \mathcal{R}_{12}(v(t_n)) =& i\mu^3 \int_0^\tau \mathrm{e}^{-i(t_n+s)\partial_x^2} \Big(\mathrm{e}^{i(t_n+s)\partial_x^2} F_n\big(v(t_n), v(t_n), s\big)\Big)^2 \, ds.\end{aligned}$$

The term $\mathcal{R}_{11}(v(t_n))$ can be rewritten as

$$\begin{aligned}\mathcal{R}_{11}(v(t_n)) =& - i\mu \int_0^\tau \mathrm{e}^{-i(t_n+s)\partial_x^2} \Big(\mathrm{e}^{i(t_n+s)\partial_x^2} \big(v(t_n+s) - v(t_n) + i\mu F_n\big(v(t_n), v(t_n), s\big)\big) \\ &\cdot \mathrm{e}^{i(t_n+s)\partial_x^2} \big(v(t_n+s) + v(t_n) - i\mu F_n\big(v(t_n), v(t_n), s\big)\big)\Big) \, ds.\end{aligned}$$

From (9) and (10) we can obtain the following estimate:

$$\begin{aligned}&\big\|v(t_n+s) - v(t_n) + i\mu F_n\big(v(t_n), v(t_n), s\big)\big\|_{H^\gamma} \\ &= \Big\| - i\mu \int_0^s \mathrm{e}^{-i(t_n+t)\partial_x^2} \Big(\mathrm{e}^{i(t_n+t)\partial_x^2} \big(v(t_n+t) - v(t_n)\big) \cdot \mathrm{e}^{i(t_n+t)\partial_x^2} \big(v(t_n+t) + v(t_n)\big)\Big) \Big\|_{H^\gamma} \\ &\lesssim \tau \|v(t_n+t) - v(t_n)\|_{H^\gamma} \|v(t_n+t) + v(t_n)\|_{H^\gamma}.\end{aligned}$$

Furthermore, by using (9), (10) and Lemma 3.1, we derive that

$$\|v(t_n+t) - v(t_n)\|_{L^\infty H^\gamma} \lesssim \|F_n(v(t_n+s), v(t_n+s), t)\|_{L^\infty H^\gamma} \lesssim \tau \|v\|^2_{L^\infty H^\gamma}.$$

The following result can be obtained by combining the estimates above:

$$\|\mathcal{R}_{11}(v(t_n))\|_{L^\infty H^\gamma} \le C\tau^3,$$

where the constant C depends only on $\|u\|_{L^\infty((0,T);H^\gamma)}$. The term $\mathcal{R}_{12}(v(t_n))$ can be estimated by using Lemma 3.1, i.e.,

$$\|\mathcal{R}_{12}(v(t_n))\|_{L^\infty H^\gamma} \lesssim \tau \|F_n(v(t_n), v(t_n), s)\|_{L^\infty H^\gamma} \lesssim \tau^3 \|v\|^4_{L^\infty H^\gamma}.$$

The two estimates above imply the following result:

$$\|\mathcal{L}^n\|_{L^\infty H^\gamma} \le C\tau^3, \tag{25}$$

where the constant C depends only on $\|u\|_{L^\infty((0,T);H^\gamma)}$.

In view of (11) and (19), the functional Φ^n defined in (20) can be rewritten into the following integral form:

$$\Phi^n(v) = v - i\mu F_n(v, v, \tau) + I(v) - \mathcal{R}_2(v) - \mathcal{R}_3(v). \tag{26}$$

Using the definition of F_n in (10), we have

$$\begin{aligned}&F_n(v(t_n), v(t_n), \tau) - F_n(v^n, v^n, \tau) \\ &= - \int_0^\tau \mathrm{e}^{-i(t_n+s)\partial_x^2} \Big(\mathrm{e}^{i(t_n+s)\partial_x^2} \big(v(t_n) - v^n\big)\Big)^2 \, dt \\ &\quad + 2 \int_0^\tau \mathrm{e}^{-i(t_n+s)\partial_x^2} \Big(\mathrm{e}^{i(t_n+s)\partial_x^2} \big(v(t_n) - v^n\big) \cdot \mathrm{e}^{i(t_n+s)\partial_x^2} v(t_n)\Big).\end{aligned}$$

The following inequality can be obtained by applying the Kato–Ponce inequality in Lemma 3.1:

$$\|F_n(v(t_n), v(t_n), \tau) - F_n(v^n, v^n, \tau)\|_{L^\infty H^\gamma} \lesssim \tau\|v(t_n) - v^n\|^2_{L^\infty H^\gamma} + \tau\|v(t_n) - v^n\|_{L^\infty H^\gamma}\|v(t_n)\|_{L^\infty H^\gamma}.$$

The other term can be estimated in a similar way. Specifically, the following result holds for any $\gamma > \frac{3}{2}$:

$$\|I(v(t_n)) - I(v^n)\|_{L^\infty H^\gamma} \leq C\tau\big(\|v(t_n) - v^n\|_{L^\infty H^\gamma} + \|v(t_n) - v^n\|^3_{L^\infty H^\gamma}\big),$$

where the constant C depends only on $\|u\|_{L^\infty((0,T);H^\gamma)}$. The following estimate can be obtained by combining (24) with the estimates above:

$$\|\Phi^n(v(t_n)) - \Phi^n(v^n)\|_{L^\infty H^\gamma} \leq (1 + C\tau)\|v(t_n) - v^n\|_{L^\infty H^\gamma} + C\tau\|v(t_n) - v^n\|^3_{L^\infty H^\gamma}, \tag{27}$$

where the constant C depends only on $\|u\|_{L^\infty((0,T);H^\gamma)}$.

Substituting (25) and (27) into the error equation in (23), and applying discrete Gronwall's inequality, we obtain the following error estimate:

$$\left\|v(t_{n+1}) - v^{n+1}\right\|_{H^\gamma} \leq C\tau^2,$$

where the constant C depends only on $\|u\|_{L^\infty((0,T);H^\gamma)}$. This proves Theorem 2.1. □

5. Numerical experiments

In this section, we present numerical experiments to illustrate the convergence of the proposed method under different regularity conditions. We consider the quadratic nonlinear Schrödinger equation (1) with the following initial value:

$$u_0(x) := \frac{|\partial_{x,N}|^{-\gamma}\mathcal{U}^N}{\||\partial_{x,N}|^{-\gamma}\mathcal{U}^N\|_{L^\infty}}, \qquad \text{with} \quad \mathcal{U}^N = \mathrm{rand}(N,1) + i\mathrm{rand}(N,1), \tag{28}$$

where γ is chosen to determine the regularity of the initial data, and the pseudo-differential operator $|\partial_{x,N}|^{-\gamma}$ for $\gamma \geq 0$ is defined as: for Fourier modes $l = -N/2, \ldots, N/2-1$,

$$\big(|\partial_{x,N}|^{-\gamma}\big)_l = \begin{cases} |l|^{-\gamma} & \text{if } l \neq 0, \\ 0 & \text{if } l = 0. \end{cases}$$

$\mathcal{U}^N$ are uniformly distributed random variables in $[0,1] + i[0,1]$. This choice of the initial data guarantees $u^0 \in H^\gamma$ and therefore $u \in C([0,T]; H^\gamma)$.

The errors of the numerical solutions with $\gamma = \frac{3}{2}$ and $\gamma = 2$ are presented in Figure 1, where the spatial discretization is performed with a Fourier spectral method using FFT based on the grid points $x_j = j\frac{2\pi}{N}$ for $j = 0, 1, \ldots, N = 2^6$. The numerical results in Figure 1 indicate that the proposed method has the second-order convergence in the H^γ norm for initial data in H^γ. This is consistent with the theoretical result proved in Theorem 2.1.

6. Conclusion

We have constructed a low-regularity integrator for the quadratic NLS equation on the one-dimensional torus based on harmonic analysis techniques. The proposed low-regularity integrator can be implemented with FFT using the Fourier spectral method with $\mathcal{O}(N \log N)$ operations at every time level. Theoretically, we have proved that the proposed low-regularity integrator has second-order convergence in H^γ for solutions in $C([0,T]; H^\gamma)$, for any $\gamma > \frac{3}{2}$, i.e. no additional regularity in the

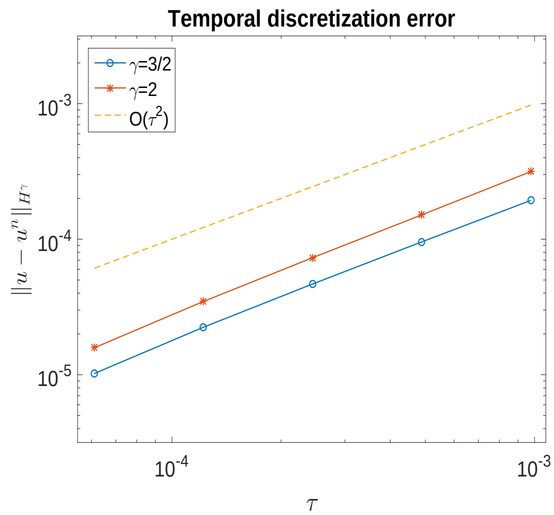

FIGURE 1. H^γ-norm errors of the numerical solutions at $T = 2$ for various different τ and γ, with $N = 2^6$ degrees of freedom in a spatial discretization using the Fourier spectral method.

solution is required. The numerical experiments are consistent with the theoretical analysis.

Acknowledgements

This research is supported by the NSFC key project under the grant number 11831003, and by NSFC under the grant number 11971356.

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