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A NUMERICAL ANALYSIS OF THE COUPLED CAHN-HILLIARD/ALLEN-CAHN SYSTEM WITH DYNAMIC BOUNDARY CONDITIONS

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Abstract. The numerical analysis of the coupled Cahn-Hilliard/Allen-Cahn system endowed with dynamic boundary conditions is studied in this article. We consider a semi-discretisation in space using a finite element method and we derive error estimates between the exact and the approximate solution. Then, using the backward Euler scheme for the time variable, a fully discrete scheme is obtained and its stability is proved. Some numerical simulations illustrate the behavior of the solution under the influence of dynamical boundary conditions.

Key words. Cahn-Hilliard/Allen-Cahn equations, dynamic boundary conditions, finite element method, error estimates, backward Euler scheme, Lojasiewicz inequality.

1. Introduction

We consider the Cahn-Hilliard/Allen-Cahn system with dynamic boundary conditions

$$
(1)\qquad \begin{cases}
u_t = \Delta\mu, & x \in \Omega,\\
\mu = -\Delta u + f(u+v) + f(u-v), & x \in \Omega,\\
v_t = \Delta v - f(u+v) + f(u-v) - \alpha v, & x \in \Omega,\\
u_t = \delta\Delta_\Gamma \mu - \partial_n \mu, & x \in \Gamma,\\
\mu = -\sigma\Delta_\Gamma u + g(u+v) + g(u-v) + \partial_n u, & x \in \Gamma,\\
v_t + \partial_n v - \kappa\Delta_\Gamma v + g(u+v) - g(u-v) = 0, & x \in \Gamma,
\end{cases}
$$

where Ω is a $2d$ or $3d$ slab, i.e.

$$
\Omega = \prod_{i=1}^{d-1} (\mathbb{R}/(L_i\mathbb{Z})) \times (0, L_d), \qquad L_i > 0, i = 1, \cdots, d, \quad d = 2 \text{ or } 3,
$$

with smooth boundary

$$
\Gamma = \partial\Omega = \prod_{i=1}^{d-1} (\mathbb{R}/(L_i\mathbb{Z})) \times \{0, L_d\};
$$

in other words, when $d = 2$, Ω is the rectangle $(0, L_1) \times (0, L_2)$, u, μ and v are periodic in x_1-direction and the boundary conditions are valid for $x_2 = 0$ and $x_2 = L_2$; when $d = 3$, Ω is the parallelepiped $(0, L_1) \times (0, L_2) \times (0, L_3)$, u, μ and v are periodic in the x_1 and x_2-directions and the boundary conditions are valid for $x_3 = 0$ and $x_3 = L_3$. The function f is the derivative of some double-well potential (typically, $f(s) = s^3 - s$) and g is the derivative of a surface potential, (typically, $g(s) = a_\Gamma s - b_\Gamma,\ a_\Gamma > 0,\ b_\Gamma \in \mathbb{R}$).

In (1), u and v represent a conserved (typically an average concentration) and a non-conserved order parameter, respectively. See [5] for further relevant references. Furthermore, the parameter α reflects the location of the system within the

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phase diagram and may be either positive or negative. In what follows we consider, without any restriction of generality, α positive (the case α negative can be treated similarly, adapting certain *a priori* estimates). Moreover, δ, σ are nonnegative parameters related to the boundary diffusion and $\kappa > 0$ is a physical coefficient. Also, Δ_Γ is the Laplace-Beltrami operator on Γ and ∂_n is the outward normal derivative. The evolution boundary value problem (1) is completed by initial conditions $u(0) = u_0$ and $v(0) = v_0$. We remark that in the particular case that we consider here, when the domain is a slab, the Laplace-Beltrami operator on Γ reduces to $\partial^2_{x_1x_1}$ for the case $d = 2$ and to $\partial^2_{x_1x_1} + \partial^2_{x_2x_2}$ for the case $d = 3$.

The Cahn-Hilliard/Allen-Cahn system endowed with Neumann boundary conditions was introduced in [4, 5], in order to describe simultaneous order-disorder and phase separation in binary alloys on a BCC lattice in the neighborhood of the triple point. For further references on the physical pertinence of the model, we refer the interested reader to [2]. The authors of [5] explored two phenomenological approaches leading to systems of coupled Allen-Cahn/Cahn-Hilliard (AC/CH) equations. Another important application of the coupled (AC/CH) equations is that under appropriate compositional conditions, ordering can be induced in a previously homogeneous material. If the composition differs slightly from these conditions, the excess composition can emerge as droplets along the boundaries between the ordered regions. This phenomena can be modeled by a coupled (AC/CH) system with degenerate mobilities. In similar applications, surface diffusion coupled with motion by mean curvature appears quite naturally. There are additional effects which are often neglected and which arguably should be included. However, the coupled motion, by itself, is not finally understood and it was thus reasonable to isolate it and study it, even given its limitations (see [9]).

In [4], the authors prove the well-posedness and the existence of maximal attractors and inertial sets (i.e., exponential attractors) for the usual cubic nonlinear term $f(s) = s^3 - \beta s$ in three space dimensions when Neumann boundary conditions are considered. The numerical study using a finite element approximation was treated in [3] for the case of a degenerate Allen-Cahn/Cahn-Hilliard system under Neumann boundary conditions.

A similar system, with a non-constant mobility, was treated in [10] where the existence of weak solutions for a degenerate parabolic system consisting of a fourth-order and a second-order equation with singular lower-order terms in one space dimension with Neumann boundary conditions was proved. In addition, asymptotics for a similar system with a non-constant mobility, proposed as a diffuse interface model for simultaneous order-disorder and phase separation, was studied in [19]. There, A. Novick-Cohen focused on the motion in the plane. This framework yields both sharp interface and diffuse interface models of sintering of small grains and thermal grains boundary grooving in polycrystalline films. This work was extended in [20], where the authors studied the partial wetting case, and their analysis accounts for motion in three space dimensions.

The Cahn-Hilliard/Allen-Cahn system (1) is derived from the following Ginzburg-Landau free energy

$$\begin{aligned} J(u,v) =& \frac{1}{2}(\|\nabla u\|^2_\Omega + \|\nabla v\|^2_\Omega) + \frac{\alpha}{2}\|v\|^2_\Omega \\ &+ \int_\Omega \{F(u+v) + F(u-v)\}\, dx + \frac{\sigma}{2}\|\nabla_\Gamma u\|^2_\Gamma \\ &+ \frac{\kappa}{2}\|\nabla_\Gamma v\|^2_\Gamma + \int_\Gamma \{G(u+v) + G(u-v)\}\, d\Gamma, \end{aligned} \tag{2}$$

where $\|\cdot\|_\Omega$ (resp. $\|\cdot\|_\Gamma$) designates the norm on $L^2(\Omega)$ (resp. on $L^2(\Gamma)$) and ∇_Γ is the tangential gradient operator in Γ, F is the double-well potential and G the surface potential.

The first line in (2) represent the Ginzburg-Landau (bulk) free energy and the remaining terms represent the surface energy. If (u,μ,v) is a regular solution of (1), then u, v dissipate J since

$$\frac{d}{dt}J(u(t),v(t)) = -\|\nabla\mu\|_\Omega^2 - \delta\|\nabla_\Gamma\mu\|_\Gamma^2 - \left\|\frac{\partial v}{\partial t}\right\|_\Omega^2 - \left\|\frac{\partial v}{\partial t}\right\|_\Gamma^2 \leqslant 0, \quad t \geqslant 0. \tag{3}$$

Moreover, the total mass of u in the bulk and on the boundary is conserved:

$$\frac{d}{dt}\left(\int_\Omega u\,dx + \int_\Gamma u\,d\Gamma\right) = 0, \quad t \geqslant 0,$$

which is immediately obtained by integrating $(1)_1$ and $(1)_4$ respectively over Ω and Γ and adding the resulting equations.

We mention that numerical methods to solve coupled (AC/CH) systems were studied in, e.g., [5, 17, 21, 24, 25, 26]. Furthemore, a NKS-method for the implicit solution of a coupled (AC/CH) system was studied in [27].

The main result of our paper, Theorem 7.1, states optimal error estimates for the differences $u^h - u$ and $v^h - v$ in energy norms and weaker norms as the mesh step h tends to 0, where (u^h, v^h) is the solution of a finite element space semidiscrete scheme and (u,v) is the solution of the continuous problem, which is supposed regular enough.

When (u,v) are less regular, we prove in Theorem 5.1 that u^h and v^h tends to u and v respectively in some weak sense, if the nonlinearity f has a subcritical or critical growth. We also propose a fully discrete scheme obtained from the previous one by using the backward Euler scheme for the time discretization: the solution (u_h^n, v_h^n) of the fully discrete scheme is shown to be unconditionally stable (i.e., without mesh-dependent restriction on the time step) and to converge to equilibrium as $n \to +\infty$.

2. Notation and assumptions

As mentioned previously, in what follows, we propose a numerical study for the AC/CH model when dynamic boundary conditions are considered.

We introduce the space $H = L^2(\Omega) \times L^2(\Gamma)$ and we endow the space H with the scalar product $(\rho,\omega)_H = (\rho_1,\omega_1)_\Omega + (\rho_2,\omega_2)_\Gamma$, $\forall\rho = (\rho_1,\rho_2) \in H$ and $\omega = (\omega_1,\omega_2) \in H$ and the corresponding norm is $\|\cdot\|_H = (\cdot,\cdot)^{\frac{1}{2}}$.

We also introduce the space $W = \{\rho \in H_p^1(\Omega),\ \rho_{|\Gamma} \in H_{per}^1(\Gamma)\}$, which is a Hilbert space for the norm $\|\rho\|_W = \left(\|\rho\|_{H^1(\Omega)}^2 + \|\nabla\rho\|_{H^1(\Gamma)}^2\right)^{1/2}$ and the space $V_2 = \{\rho \in H_p^2(\Omega),\ \rho_{|\Gamma} \in H_{per}^2(\Gamma)\}$.

We mention that by $H_p^m(\Omega)$, with $m \geqslant 1$, we understand the functions that belong to $H^m(\Omega)$ and which are periodic in the $x_1, \cdots, x_{d-1}$-directions. More precisely, when $d = 2$, a function $\rho \in W$ is a function belonging to $H^1(\Omega)$, which is periodic in the x_1-direction and for which we have that *trace* $v|_{x=0} \in H_{per}^1(0,L_1)$ and trace $v|_{x=L_2} \in H_{per}^1(0,L_1)$, where

$$H_{per}^1(0,L_1) = \{\rho \in H^1(0,L_1), \quad \rho \text{ is } L_1\text{-periodic}\}.$$

A similar definition holds for $d = 3$ and for V_2.

We will also use the notation $m(\rho) = \frac{1}{|\Omega|+|\Gamma|}(\rho, \mathbf{1})_H$, where $\mathbf{1} = (1,1)$ and $\dot{H} = \{\rho \in H; m(\rho) = 0\}$. More generally, for an arbitrary set X, we will note $\dot{X} = X \cap \dot{H}$. We also denote by W' the dual space of W.

The functions f and g belong to $\mathcal{C}^2(\mathbb{R}, \mathbb{R})$ and satisfy the following standard dissipativity assumptions:

$$\liminf_{|s|\to\infty} f'(s) > 0, \qquad \liminf_{|s|\to\infty} g'(s) > 0, \tag{4}$$

which imply the existence of two constants $c_1 > 0,\ c_2 \geqslant 0$ such that

$$F(s) \geqslant c_1 s^2 - c_2 \quad \text{and} \quad G(s) \geqslant c_1 s^2 - c_2 \tag{5}$$

where F and G are the antiderivatives of the functions f and g. Moreover, in order to prove Theorem 2.2 (and only at that point) we will assume that f and g have a subcritical growth. More precisely, we assume that there exists a positive constant c_3 such that

$$|f(s)| \leqslant c_3(1 + |s|^{p-1}), \quad \forall s \in \mathbb{R}, \tag{6}$$

with $p \in [2, 4]$ when $d = 3$ and $p \geqslant 2$ arbitrary when $d = 2$. We also assume that there exists a positive constant c_4 such that

$$|g(s)| \leqslant c_4(1 + |s|^{q-1}), \quad \forall s \in \mathbb{R}, \tag{7}$$

with $q \geqslant 2$ arbitrary. We note that the cubic nonlinearity for f satisfies this assumption with $p = 4$ and $q = 2$.

We will often use the Poincaré inequality in the form (see [22])

$$\|\rho - m(\rho)\|_H \ \leqslant\ c_p\ |\rho|_{1,W}, \quad \forall \rho \in W, \tag{8}$$

with $|\rho|^2_{1,W} = \|\nabla\rho\|^2_\Omega + \|\nabla_\Gamma \rho\|^2_\Gamma$,

as well as the following inequality:

$$\|\rho\|^2_\Omega \ \leqslant\ C_p\ (\|\nabla\rho\|^2_\Omega + \|\rho\|^2_\Gamma), \quad \forall \rho \in H^1_p(\Omega). \tag{9}$$

Inequality (9) ensures the fact that $(\|\nabla\rho\|^2_\Omega + \|\rho\|^2_\Gamma)^{1/2}$ is a norm on $H^1_p(\Omega)$, equivalent to the usual one.

Using these notations, we can introduce the variational formulation of (1) which reads:

For f satisfying (4) *and $u_0, v_0 \in W$, find $u, v \in L^\infty(0,T,W)$ and $\mu \in L^2(0,T,W)$ such that $u(0) = u_0$, $v(0) = v_0$ and:*

$$\left\{\begin{array}{l} (u_t, \Psi)_H + (\nabla\mu, \nabla\Psi)_\Omega + \delta(\nabla_\Gamma\mu, \nabla_\Gamma\Psi)_\Gamma = 0, \\ (\mu, \Phi)_H = (\nabla u, \nabla\Phi)_\Omega + (f(u+v) + f(u-v), \Phi)_\Omega \\ \qquad\qquad +\sigma(\nabla_\Gamma u, \nabla_\Gamma\Phi)_\Gamma + (g(u+v) + g(u-v), \Phi)_\Gamma, \\ (v_t, \varphi)_H + (\nabla v, \nabla\varphi) + \kappa(\nabla_\Gamma v, \nabla_\Gamma\varphi)_\Gamma + \alpha(v, \varphi)_\Omega \\ \qquad\qquad = -(f(u+v) - f(u-v), \varphi)_\Omega - (g(u+v) - g(u-v), \varphi)_\Gamma, \end{array}\right. \tag{10}$$

for all Ψ, Φ and $\varphi \in W$.

3. The space semidiscrete scheme

For the space discretization of these equations, we first let $\{\Omega^h\}$ be a quasiuniform family of decompositions of $\bar{\Omega}$. Every decomposition Ω^h is composed of d-simplices uniquely (i.e. triangles if $d = 2$ and tetrahedrons if $d = 3$); the decomposition takes into account the periodic boundary conditions on Ω, so that $\{\Omega^h\}$ is in fact a triangulation of $\bar{\Omega}$. The triangulation Ω^h of $\bar{\Omega}$ induces a triangulation

Γ^h of Γ into $d-1$ simplices in a natural way. We associate to $\Omega^h = \bigcup_{T\in\Omega^h} T$ the conforming P^1 finite element space

$$W^h = \left\{\rho^h \in \mathcal{C}^0(\bar{\Omega}),\ \rho^h|_T \text{ is affine } \ \forall T \in \Omega^h\right\}.$$

Note that for every $\rho^h \in W^h$, the restriction $\varphi^h = \rho^h_{|\Gamma}$ on the boundary is a P^1 finite element on the $(d-1)$-dimensional domain Γ. In fact, the space of such functions φ^h is the usual P^1 conforming finite element discretization of the space $H^1_{per}(\Gamma)$ built on the triangulation Γ^h. It is thus well-known that W^h has finite dimension, that $W^h \subset W$ and that W^h satisfies

$$\text{for all } h>0,\ W^h \text{ contains the constants;} \tag{11}$$

$$\bigcup_{h>0} W^h \quad \text{is dense in } W. \tag{12}$$

Note that the same space W^h is used for both the discretization of $H^1_p(\Omega)$ and that of W.

For $\rho \in \mathcal{C}^0(\bar{\Omega})$, let $I^h\rho$ denotes the P^1 interpolate of ρ on Ω^h, i.e. $I^h\rho$ is the unique function in W^h such that $I^h\rho(x_i) = \rho(x_i)$ for every node x_i of the triangulation Ω^h. Note that $(I^h\rho)_{|\Gamma}$ is the P^1 interpolate of $\rho_{|\Gamma}$ on Γ^h.

Throughout the paper, the letters C and c will denote constants independent of h which may vary from line to line.

We can thus introduced the following space semidiscrete variational formulation:

Find $(u^h, \mu^h, v^h):\ [0,T] \to W^h \times W^h \times W^h$ *such that*

(13)
$$\begin{cases} (u^h_t, \Psi)_H + (\nabla\mu^h, \nabla\Psi)_\Omega + \delta(\nabla_\Gamma\mu^h, \nabla_\Gamma\Psi)_\Gamma = 0, \\ (\mu^h, \Phi)_H = (\nabla u^h, \nabla\Phi)_\Omega + (f(u^h+v^h) + f(u^h-v^h), \Phi)_\Omega \\ \qquad\qquad +\sigma(\nabla_\Gamma u^h, \nabla_\Gamma\Phi)_\Gamma + (g(u^h+v^h) + g(u^h-v^h), \Phi)_\Gamma, \\ (v^h_t, \varphi)_H + (\nabla v^h, \nabla\varphi)_\Omega + \kappa(\nabla_\Gamma v^h, \nabla_\Gamma\varphi)_\Gamma + \alpha(v^h, \varphi)_\Omega \\ \qquad\qquad = -(f(u^h+v^h) - f(u^h-v^h), \varphi)_\Omega - (g(u^h+v^h) - g(u^h-v^h), \varphi)_\Gamma, \end{cases}$$

for all Ψ, Φ *and* $\varphi \in W^h$.

Taking $\Psi = 1$ in $(13)_1$, we find

$$m(u^h_t(t)) = 0 \quad \text{which implies} \quad m(u^h(t)) = \text{cst}, \quad \forall t \geqslant 0, \tag{14}$$

thus, the finite element approximation u^h conserves the mass conservation property of the exact solution u.

4. Discrete energy estimate and convergence to steady state

Let us first prove the existence and uniqueness of an approximate solution (u^h, μ^h, v^h) given by the space semi-discrete problem (13). More exactly, we have the following result:

Theorem 4.1. *For every* $u^h_0, v^h_0 \in W^h$*, problem* (13) *has a unique solution*

$$(u^h, \mu^h, v^h) \in \mathcal{C}^1([0,+\infty), W^h \times W^h \times W^h)$$

such that $u^h(0) = u^h_0$ *and* $v^h(0) = v^h_0$*. Moreover, the following energy estimate holds*

$$J(u^h(t), v^h(t)) + \int_0^t \left\{\|\nabla\mu^h\|^2_\Omega + \delta\|\nabla_\Gamma\mu^h\|^2_\Gamma + \|v^h_t\|^2_H\right\} ds \leqslant J(u^h_0, v^h_0), \ \ \forall t \geqslant 0, \tag{15}$$

where J *is defined in* (2).

Proof. Let $(\varphi_1,\dots,\varphi_M)$ be an orthonormal basis of W^h for the H-scalar product and such that $\varphi_1 \equiv \text{cst}$. We seek for

$$u^h(t)=\sum_{i=1}^M u_i(t)\varphi_i,\quad \mu^h(t)=\sum_{i=1}^M \mu_i(t)\varphi_i \quad\text{and}\quad v^h(t)=\sum_{i=1}^M v_i(t)\varphi_i.$$

We define the matrices

$$(A)_{ij}=(\nabla\varphi_i,\nabla\varphi_j)_\Omega \text{ and } \quad (A_\Gamma)_{ij}=(\nabla_\Gamma\varphi_i,\nabla_\Gamma\varphi_j)_\Gamma,\quad 1\leqslant i,j\leqslant M,$$

the vectors

$$U=\begin{pmatrix}u_1\\ \vdots\\ u_M\end{pmatrix},\quad Z=\begin{pmatrix}\mu_1\\ \vdots\\ \mu_M\end{pmatrix},\quad V=\begin{pmatrix}v_1\\ \vdots\\ v_M\end{pmatrix}$$

and the functions

$$F_1(U,V)=\begin{pmatrix}(f(u^h+v^h),\varphi_1)_\Omega\\ \vdots\\ (f(u^h+v^h),\varphi_M)_\Omega\end{pmatrix},\quad F_2(U,V)=\begin{pmatrix}(f(u^h-v^h),\varphi_1)_\Omega\\ \vdots\\ (f(u^h-v^h),\varphi_M)_\Omega\end{pmatrix}$$

and

$$G_1(U,V)=\begin{pmatrix}(g(u^h+v^h),\varphi_1)_\Gamma\\ \vdots\\ (g(u^h+v^h),\varphi_M)_\Gamma\end{pmatrix},\quad G_2(U,V)=\begin{pmatrix}(g(u^h-v^h),\varphi_1)_\Gamma\\ \vdots\\ (g(u^h-v^h),\varphi_M)_\Gamma\end{pmatrix}.$$

Then (13) can be written as

(16)
$$\begin{cases} U' &= -(A+\delta A_\Gamma)Z,\\ Z &= \sigma A_\Gamma U + AU + F_1(U,V)+F_2(U,V)+G_1(U,V)+G_2(U,V),\\ V' &= -AV-\kappa A_\Gamma V-\alpha V-F_1(U,V)+F_2(U,V)-G_1(U,V)+G_2(U,V)\end{cases}$$

and thus

(17)
$$\begin{cases} U'=-(A+\delta A_\Gamma)\Big(\sigma A_\Gamma U+AU+F_1(U,V)+F_2(U,V)+G_1(U,V)+G_2(U,V)\Big),\\ V'=-AV-\kappa A_\Gamma V-\alpha V-F_1(U,V)+F_2(U,V)-G_1(U,V)+G_2(U,V).\end{cases}$$

Therefore, by the Cauchy-Lipschitz theorem, problem (13) has a unique maximal solution $(u^h,\mu^h,v^h)\in\mathcal{C}^1([0,T^+);W^h\times W^h\times W^h)$ such that $u^h(0)=u_0^h$ and $v^h(0)=v_0^h$.

Taking $\Psi=\mu^h$ (resp. $\Phi=u_t^h$) in the first (resp. the second) equation of (13) and summing the two resultant equations, we get

(18)
$$\frac{1}{2}\frac{d}{dt}\{\|\nabla u^h\|_\Omega^2+\sigma\|\nabla_\Gamma u^h\|_\Gamma^2\}+\|\nabla\mu^h\|_\Omega^2+\delta\|\nabla_\Gamma\mu^h\|_\Gamma^2 + (f(u^h+v^h)+f(u^h-v^h),u_t^h)_\Omega+(g(u^h+v^h)+g(u^h-v^h),u_t^h)_\Gamma=0.$$

We take now $\varphi=v_t^h$ in third equation of (13) and obtain

(19)
$$\frac{1}{2}\frac{d}{dt}\left\{\|\nabla v^h\|_\Omega^2+\kappa\|\nabla_\Gamma v^h\|_\Gamma^2+\alpha\|v^h\|_\Omega^2\right\}+\|v_t^h\|_H^2 + (f(u^h+v^h)-f(u^h-v^h),v_t^h)_\Omega+(g(u^h+v^h)-g(u^h-v^h),v_t^h)_\Gamma=0.$$

Summing (18) and (19), we obtain

$$\frac{d}{dt}\{J(u^h, v^h)\} + \|\nabla\mu^h\|_\Omega^2 + \delta\|\nabla_\Gamma\mu^h\|_\Gamma^2 + \|v_t^h\|_H^2 = 0, \tag{20}$$

so (u^h, μ^h, v^h) satisfies the energy estimate (15) of Theorem 4.1. Therefore, using (5), we find $u^h, v^h \in L^\infty(0, T^+; W)$ and we conclude that $T^+ = +\infty$ (i.e. the solution is global).

□

In order to obtain the convergence to steady state we will first introduce the following definition:

Definition 4.1. *A steady state for* (13) *with initial condition* (u_0^h, v_0^h) *is a triplet* $(\bar{u}^h, \bar{\mu}^h, \bar{v}^h) \in W^h \times W^h \times W^h$ *such that*

$$\begin{cases} (\bar{u}^h, \boldsymbol{1})_H \ = (u_0^h, \boldsymbol{1})_H, \\ (\bar{\mu}^h, \Phi)_H = (\nabla\bar{u}^h, \nabla\Phi)_\Omega + \sigma(\nabla_\Gamma\bar{u}^h, \nabla_\Gamma\Phi)_\Gamma \\ \qquad +(f(\bar{u}^h + \bar{v}^h) + f(\bar{u}^h - \bar{v}^h), \Phi)_\Omega + (g(\bar{u}^h + \bar{v}^h) + g(\bar{u}^h - \bar{v}^h), \Phi)_\Gamma, \\ 0 = (\nabla\bar{v}^h, \nabla\varphi)_\Omega + \kappa(\nabla_\Gamma\bar{v}^h, \nabla_\Gamma\varphi)_\Gamma + \alpha(\bar{v}^h, \varphi)_\Omega + (f(\bar{u}^h + \bar{v}^h) \\ \qquad -f(\bar{u}^h - \bar{v}^h), \varphi)_\Omega + (g(\bar{u}^h + \bar{v}^h) - g(\bar{u}^h - \bar{v}^h), \varphi)_\Gamma. \end{cases} \tag{21}$$

$\forall\Phi,\ \varphi \in W^h$.

Corollary 4.1. *The flow* $S^h(t)(u_0^h, v_0^h) = (u^h(t), v^h(t))$ *defined in Theorem 4.1 is a gradient flow for* J *in the affine space* $\tilde{W}^h = \{u^h \in W^h : m(u^h) = m(u_0^h)\}$*, i.e.*

$$\langle(u_t^h, v_t^h), (\Psi, \Phi)\rangle^h = -\frac{dJ}{d\beta}\left((u^h(t), v^h(t)) + \beta(\Psi^h, \Phi^h)\right)|_{\beta=0},\ \ \forall(\Psi^h, \Phi^h) \in \dot{W}^h \times W^h,$$

where $\langle\cdot,\cdot\rangle^h$ *is a scalar product on* $\dot{W}^h \times W^h$ *given by* (25)*. In particular, if* f *and* g *are real analytic functions, there exists a steady state* $(\bar{u}^h, \bar{\mu}^h, \bar{v}^h) \in W^h \times W^h \times W^h$ *such that* $(u^h(t), \mu^h(t), v^h(t)) \to (\bar{u}^h, \bar{\mu}^h, \bar{v}^h)$ *as* $t \to +\infty$.

Proof. Let us start by noticing that, since $\varphi_1 = \text{cst}$, the first line and the first column of matrix $A + \delta A_\Gamma$ are filled only with null elements and thus the matrix is not invertible. In order to handle this difficulty, we need to rewrite system (16) into a more convenient form. For a vector $X = (x_i)_{1\leqslant i\leqslant M} \in \mathbb{R}^M$, we write $\dot{X} = (x_i)_{2\leqslant i\leqslant M} \in \mathbb{R}^{M-1}$ (recall that the first component x_1 is associated to the constant φ_1). For a square matrix $C = (c_{ij})_{1\leqslant i,j\leqslant M}$ of size M, we write $\dot{C} = (c_{ij})_{2\leqslant i,j\leqslant M}$. Equation (16) without the two lines corresponding to φ_1 becomes

$$\begin{cases} \dot{U}' \ \ = -(\dot{A} + \delta\dot{A}_\Gamma)\dot{Z}, \\ \dot{Z} \ \ = \sigma\dot{A}_\Gamma\dot{U} + \dot{A}\dot{U} + \dot{F}_1(u_1, \dot{U}, V) + \dot{F}_2(u_1, \dot{U}, V) \\ \qquad +\dot{G}_1(u_1, \dot{U}, V) + \dot{G}_2(u_1, \dot{U}, V), \\ V' \ \ = -AV - \kappa A_\Gamma V - \alpha V - F_1(u_1, \dot{U}, V) + F_2(u_1, \dot{U}, V) \\ \qquad -G_1(u_1, \dot{U}, V) + G_2(u_1, \dot{U}, V), \end{cases} \tag{22}$$

with u_1 obtained from the conservation of mass as $u_1 = (u^h(0), \varphi_1)_H$.

Now, the matrix $\dot{A} + \delta\dot{A}_\Gamma$ is invertible, thus we obtain

$$\begin{cases} (\dot{A} + \delta\dot{A}_\Gamma)^{-1}\dot{U}' = -(\sigma\dot{A}_\Gamma\dot{U} + \dot{A}\dot{U} + \dot{F}_1(u_1, \dot{U}, V) + \dot{F}_2(u_1, \dot{U}, V) \\ \qquad +\dot{G}_1(u_1, \dot{U}, V) + \dot{G}_2(u_1, \dot{U}, V)), \\ V' = -AV - \kappa A_\Gamma V - \alpha V - \dot{F}_1(u_1, \dot{U}, V) + \dot{F}_2(u_1, \dot{U}, V) \\ \qquad -\dot{G}_1(u_1, \dot{U}, V) + \dot{G}_2(u_1, \dot{U}, V). \end{cases} \tag{23}$$

This is a gradient flow for the function

$$(\dot{U}, V) \;\to\; J(U,V) = J(u_1\varphi_1 + \sum_{i=2}^{M} u_i\varphi_i, \sum_{i=1}^{M} v_i\varphi_i) \in \mathbb{R},$$

$$\dot{U} = (u_2, \cdots, u_M),\; V = (v_1, \cdots, v_M), \tag{24}$$

with respect to the scalar product in $\mathbb{R}^{M-1} \times \mathbb{R}^M$

$$\langle\,\cdot\,,\,\cdot\,\rangle^h = \Big((\dot{A} + \delta \dot{A}_\Gamma)^{-1}\,\cdot\,,\,\cdot\,\Big) + \Big(\,\cdot\,,\,\cdot\,\Big). \tag{25}$$

Since f and g are assumed to be real analytic and $W^h \subset \mathcal{C}^0(\bar{\Omega})$, the function defined by (24) is also analytic on $\mathbb{R}^{M-1} \times \mathbb{R}^M$. Then the Lojasiewicz inequality [15] implies the convergence to a steady state as $t \to +\infty$. □

5. Convergence as $h \to 0$

In this section (and only this section), we have to assume that f and g have a subcritical growth (i.e., f and g satisfy respectively (6) and (7)). We will first introduce the following definition, which will be used in the following

Definition 5.1. *We set the operator $\mathcal{A}_\delta : W \to W'$ defined by*

$$(\mathcal{A}_\delta u, \Psi) = (\nabla u, \nabla \Psi)_\Omega + \delta(\nabla_\Gamma u, \nabla_\Gamma \Psi)_\Gamma,$$

where W' is the dual of W. We also define $\mathcal{A}_\delta^{-1} : W' \to \dot{W},\; f \to \mathcal{A}_\delta^{-1} f$, where $\mathcal{A}_\delta^{-1} f$ is the unique solution of the problem

$$(\nabla \mathcal{A}_\delta^{-1} f, \nabla \chi)_\Omega + \delta(\nabla_\Gamma \mathcal{A}_\delta^{-1} f, \nabla_\Gamma \chi)_\Gamma = (f, \chi)_H, \quad \forall \chi \in \dot{W}. \tag{26}$$

Theorem 5.1. *Assume that $f,\; g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ satisfy (4), (6) and (7). Let $u_0, v_0 \in W$ and let $u_0^h, v_0^h \in W^h$ such that $u_0^h \to u_0$ and $v_0^h \to v_0$ in W as $h \to 0$. Then, for all $T > 0$,*

$$\begin{cases} (u^h, v^h) \to (u, v) \;\; \textit{weak star in} \;\; L^\infty(0, T; W \times W) \\ \qquad\qquad \textit{and strongly in} \;\; \mathcal{C}^0([0, T]; L^2(\Omega) \times L^2(\Omega)), \\ (u_t^h, v_t^h) \to (u_t, v_t) \;\; \textit{weakly in} \;\; L^2(0, T; W' \times H), \\ (u_{|\Gamma}^h, v_{|\Gamma}^h) \to (u_{|\Gamma}, v_{|\Gamma}) \;\; \textit{strongly in} \;\; \mathcal{C}^0([0, T]; L^2(\Gamma) \times L^2(\Gamma)), \\ \mu^h \to \mu \;\; \textit{weakly in} \;\; L^2(0, T; W), \end{cases}$$

where (u, μ, v) is the unique solution of the continuous problem (10) such that $u(0) = u_0$ and $v(0) = v_0$.

Proof. In order to handle the nonlinear terms in (21), we use the assumptions (6) and (7), which imply the existence of some positive constants $c_5, \cdots, c_8$ such that

$$\forall s \in \mathbb{R},\; |F(s)| \;\leqslant\; c_5|s|^p + c_6 \quad \text{and} \quad |G(s)| \leqslant c_7|s|^q + c_8. \tag{27}$$

Using the Sobolev embeddings $H_p^1(\Omega) \hookrightarrow L^p(\Omega)$, $H_{per}^1(\Gamma) \hookrightarrow L^q(\Gamma)$ and $(u_0^h, v_0^h) \to (u_0, v_0)$ in W as $h \to 0$, we obtain that the energy $J(u_0^h, v_0^h)$ is bounded by a constant independent of h. From the discrete energy estimate (15) and (5), we conclude the following bounds: $(u^h)_h$ and $(v^h)_h$ are bounded in $L^\infty(0, T; W)$, $(\nabla \mu^h)_h$ is bounded in $L^2(0, T; L^2(\Omega))$ and $(\nabla_\Gamma \mu_{|\Gamma}^h)_h$ is bounded in $L^2(0, T; L^2(\Gamma))$. Now, taking $\Psi = 1$ in $(13)_2$, we get

$$(\mu^h, 1)_H = (f(u^h + v^h) + f(u^h - v^h), 1)_\Omega + (g(u^h + v^h) + g(u^h - v^h), 1)_\Gamma,$$

from which we deduce that $(\mu^h, 1)_H$ is bounded in $L^2(0, T)$. Subtracting eventually a subsequence we obtain the following convergences: $u^h \to u$ weak star in $L^\infty(0, T; W)$, $v^h \to v$ weak star in $L^\infty(0, T; W)$ and $\mu^h \to \mu$ weakly in $L^2(0, T; W)$.

It remains to prove the strong convergence of $(u^h, u^h|_\Gamma)$ in $\mathcal{C}^0\left([0,T], L^2(\Omega)\right) \times \mathcal{C}^0\left([0,T], L^2(\Gamma)\right)$. For that we use the Ascoli theorem. Indeed,

$$\|u^h(t)-u^h(s)\|_\Omega^2 + \|u^h(t)-u^h(s)\|_\Gamma^2$$

$$= 2\int_s^t (u_t^h(\sigma), u^h(\sigma)-u^h(s))_\Omega \, d\sigma + 2\int_s^t (u_t^h(\sigma), u^h(\sigma)-u^h(s))_\Gamma \, d\sigma$$

$$= -2\int_s^t (\nabla\mu^h(\sigma), \nabla(u^h(\sigma)-u^h(s)))_\Omega \, d\sigma - 2\delta\int_s^t (\nabla_\Gamma\mu^h(\sigma), \nabla_\Gamma(u^h(\sigma)-u^h(s)))_\Gamma \, d\sigma$$

$$\leqslant c\|\mu^h\|_{L^2(0,T;W)}\|u^h\|_{L^\infty(0,T;W)}|t-s|^{1/2}, \quad \text{for all} \quad 0 \leqslant s \leqslant t \leqslant T.$$

We can thus deduce that $(u^h)_h$ is uniformly equicontinuous in $\mathcal{C}^0([0,T]; L^2(\Omega))$ and $(u^h)_{|\Gamma}$ is uniformly equicontinuous in $\mathcal{C}^0([0,T]; L^2(\Gamma))$. Since $(u^h)_h$ is bounded in $L^\infty(0,T;W)$, the Ascoli theorem implies that $u^h \to u$ strongly in $\mathcal{C}^0([0,T]; L^2(\Omega))$ and $u^h_{|\Gamma} \to u_{|\Gamma}$ strongly in $\mathcal{C}^0([0,T], L^2(\Gamma))$.

Now, for the strong convergence of $(v^h)_h$, we have from (15) that $(v_t^h)_h$ is bounded in $L^2(0,T;H)$ and we get, up to a subsequence that

$$v_t^h \to v_t \quad \text{weakly in} \quad L^2([0,T];H),$$

so, by the Aubin-Lions compacteness Lemma, we have the strong convergence for $(v^h)_h$ such that

$$v^h \to v \quad \text{strongly} \quad \mathcal{C}^0([0,T];H).$$

Uniqueness of (u,μ,v):

Let (u_1,μ_1,v_1) and (u_2,μ_2,v_2) be two solutions to (10) with initial data $(u_{0,1},\mu_{0,1}, v_{0,1})$ and $(u_{0,2},\mu_{0,2},v_{0,2})$, respectively. We set

$$(u,\mu,v) = (u_1-u_2, \mu_1-\mu_2, v_1-v_2)$$

and

$$(u_0,\mu_0,v_0) = (u_{0,1}-u_{0,2}, \mu_{0,1}-\mu_{0,2}, v_{0,1}-v_{0,2}).$$

Then, (u,μ,v) satisfies

$$\tag{28}\begin{cases} (u_t,\Psi)_H + (\nabla\mu,\nabla\Psi)_\Omega + \delta(\nabla_\Gamma\mu,\nabla_\Gamma\Psi)_\Gamma = 0, \\ (\mu,\Phi)_H = (\nabla u,\nabla\Phi)_\Omega + \sigma(\nabla_\Gamma u,\nabla_\Gamma\Phi)_\Gamma + (f(u_1+v_1)-f(u_2+v_2),\Phi)_\Omega \\ \qquad +(f(u_1-v_1)-f(u_2-v_2),\Phi)_\Omega + (g(u_1+v_1)-g(u_2+v_2),\Phi)_\Gamma \\ \qquad\quad +(g(u_1-v_1)-g(u_2-v_2),\Phi)_\Gamma, \\ (v_t,\varphi)_H + (\nabla v,\nabla\varphi) + \kappa(\nabla_\Gamma v,\nabla_\Gamma\varphi)_\Gamma + \alpha(v,\varphi)_\Omega \\ \qquad = -(f(u_1+v_1)-f(u_2+v_2),\varphi)_\Omega + (f(u_1-v_1)-f(u_2-v_2),\varphi)_\Omega \\ \qquad\quad -(g(u_1+v_1)-g(u_2+v_2),\varphi)_\Gamma + (g(u_1-v_1)-g(u_2-v_2),\varphi)_\Gamma. \end{cases}$$

Taking $\Psi = \mathcal{A}_\delta^{-1}u$, $\Phi = u$ and $\varphi = v$ in (28) and summing the resulting equations, in view of Definition 5.1, we obtain

$$\tag{29}\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\{|u|_{-1,H}^2 + \|v\|_H^2\right\} + \|\nabla u\|_\Omega^2 + \sigma\|\nabla_\Gamma u\|_\Gamma^2 + \|\nabla v\|_\Omega^2 \\ &+ \kappa\|\nabla_\Gamma v\|_\Gamma^2 + \alpha\|v\|_\Omega^2 \\ &+ (f(u_1+v_1)-f(u_2+v_2), u+v)_\Omega \\ &+ (f(u_1-v_1)-f(u_2-v_2), u-v)_\Omega + (g(u_1+v_1)-g(u_2+v_2), u+v)_\Gamma \\ &+ (g(u_1-v_1)-g(u_2-v_2), u-v)_\Gamma = 0. \end{aligned}$$

Let $p = u_1 + v_1,\quad q = u_2 + v_2,\quad h = u_1 - v_1$ and $l = u_2 - v_2$.We have

$$
\begin{aligned}
(f(u_1+v_1) - f(u_2+v_2), u+v)_\Omega &= (f(p) - f(q), p-q)_\Omega \\
&= (f'(\xi)(p-q), p-q)_\Omega \\
&\geq -C_f\|p-q\|_\Omega^2 \\
&\geq -C_f\|u+v\|_\Omega^2,
\end{aligned}
\tag{30}
$$

and

$$
(f(u_1-v_1) - f(u_2-v_2), u-v)_\Omega = (f(h) - f(l), h-l)_\Omega \geqslant -C_f\|u-v\|_\Omega^2. \tag{31}
$$

Similarly, we obtain

$$
(g(u_1+v_1) - g(u_2+v_2), u+v)_\Gamma \geqslant -C_g\|u+v\|_\Gamma^2, \tag{32}
$$

and

$$
(g(u_1-v_1) - g(u_2-v_2), u-v)_\Omega \geqslant -C_g\|u-v\|_\Gamma^2. \tag{33}
$$

Here, we have used the fact that the dissipative property (4) imply the existence of some positive constant $C_f \geqslant 0$ and $C_g \geqslant 0$ such that

$$
f'(s) \geqslant -C_f, \quad g'(s) \geqslant -C_g \quad \forall s \in \mathbb{R}. \tag{34}
$$

Therefore, using the previous inequalities, equation (29) yields

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left\{|u|_{-1,H}^2 + \|v\|_H^2\right\} &+ \|\nabla u\|_\Omega^2 + \sigma\|\nabla_\Gamma u\|_\Gamma^2 + \|\nabla v\|_\Omega^2 + \kappa\|\nabla_\Gamma v\|_\Gamma^2 + \alpha\|v\|_\Omega^2 \\
&\leqslant C_f(\|u+v\|_\Omega^2 + \|u-v\|_\Omega^2) + C_g(\|u+v\|_\Gamma^2 + \|u-v\|_\Gamma^2) \\
&\leqslant C_f(\|u\|_\Omega^2 + \|v\|_\Omega^2) + C_g(\|u\|_\Gamma^2 + \|v\|_\Gamma^2) \\
&\leqslant 2\max(C_f, C_g)(\|u\|_H^2 + \|v\|_H^2).
\end{aligned}
\tag{35}
$$

Employing the interpolation inequality

$$
\|u\|_H^2 \leqslant c\ |u|_{-1,H}\|\nabla u\|_H \leqslant \varepsilon\|\nabla u\|_H^2 + c|u|_{-1,H}^2,
$$

we deduce that

$$
\begin{aligned}
\frac{d}{dt}\left\{|u|_{-1,H}^2 + \|v\|_H^2\right\} &+ \|\nabla u\|_\Omega^2 + \sigma\|\nabla_\Gamma u\|_\Gamma^2 + \|\nabla v\|_\Omega^2 + \kappa\|\nabla_\Gamma v\|_\Gamma^2 + \alpha\|v\|_\Omega^2 \\
&\leqslant c(|u|_{-1,H}^2 + \|v\|_H^2).
\end{aligned}
\tag{36}
$$

It finally follows from Gronwall's lemma that

$$
|u(t)|_{-1,H}^2 + \|v(t)\|_H^2 \leqslant e^{c't}(|u_0|_{-1,H}^2 + \|v_0\|_H^2). \tag{37}
$$

Hence the uniqueness of (u, μ, v) follows. □

6. Error estimates for the space semidicrete scheme

We are now ready to derive error estimates on a finite time interval $[0, T],\ T > 0$. But first, we have the following standard approximation results (see also [6]),

$$
\forall \rho \in V_2,\ \|\rho - I^h\rho\|_\Omega\ +\ h|\rho - I^h\rho|_{1,\Omega}\ \leqslant\ Ch^2|\rho|_{2,\Omega}, \tag{38}
$$

$$
\forall \varphi \in H_{per}^2(\Gamma),\quad \|\varphi - I^h\varphi\|_\Gamma + h|\varphi - I^h\varphi|_{1,\Gamma} \leqslant Ch^2|\varphi|_{2,\Gamma}, \tag{39}
$$

where C is a strictly positive constant which depends only on the family Ω^h, $|\rho|_{m,\Omega}$ and $|\rho|_{m,\Gamma}$ are the seminorms associated with respectively the $H_p^m(\Omega)$ and $H_{per}^m(\Gamma)$ norms, for $m \in \mathbb{N}^*$.

Moreover, since the family $\{\Omega^h\}_{h>0}$ is quasiuniform, we have the inverse estimates (see for instance [12]):

$$\forall \omega^h \in W^h, \quad \|\omega^h\|_{\mathcal{C}^0(\bar{\Omega})} \leqslant Ch^{-d/2}\|\omega^h\|_{\Omega}, \tag{40}$$

where $d = 2$ or 3 is the dimension of the space Ω.

In order to estimate the errors $u^h - u$, $\mu^h - \mu$ and $v^h - v$, we write, following a standard approach (see for instance [7, 8, 11, 14, 23]):

$$\begin{aligned} u^h(t) - u(t) &= \theta^u(t) + w^u(t) \quad \text{with} \quad \theta^u = u^h - \tilde{u}^h, \quad w^u = \tilde{u}^h - u, \\ \mu^h(t) - \mu(t) &= \theta^\mu(t) + w^\mu(t) \quad \text{with} \quad \theta^\mu = \mu^h - \tilde{\mu}^h, \quad w^\mu = \tilde{\mu}^h - \mu, \\ v^h(t) - v(t) &= \theta^v(t) + w^v(t) \quad \text{with} \quad \theta^v = v^h - \tilde{v}^h, \quad w^v = \tilde{v}^h - v, \end{aligned} \tag{41}$$

with $\tilde{u}^h, \tilde{\mu}^h, \tilde{v}^h$ the elliptic projections of u, μ, v in W^h, defined by

$$\begin{cases} (\nabla \tilde{u}^h, \nabla \Phi)_\Omega + \sigma(\nabla_\Gamma \tilde{u}^h, \nabla_\Gamma \Phi)_\Gamma = (\nabla u, \nabla \Phi)_\Omega + \sigma(\nabla_\Gamma u, \nabla_\Gamma \Phi)_\Gamma, \\ (\nabla \tilde{\mu}^h, \nabla \Phi)_\Omega + \delta(\nabla_\Gamma \tilde{\mu}^h, \nabla_\Gamma \Phi)_\Gamma = (\nabla \mu, \nabla \Phi)_\Omega + \delta(\nabla_\Gamma \mu, \nabla_\Gamma \Phi)_\Gamma, \\ (\nabla \tilde{v}^h, \nabla \Phi)_\Omega + \kappa(\nabla_\Gamma \tilde{v}^h, \nabla_\Gamma \Phi)_\Gamma + \alpha(\tilde{v}^h, \Phi)_\Omega \\ \qquad = (\nabla v, \nabla \Phi)_\Omega + \kappa(\nabla_\Gamma v, \nabla_\Gamma \Phi)_\Gamma + \alpha(v, \Phi)_\Omega, \\ (\tilde{\mu}^h, \mathbf{1})_H = (\mu, \mathbf{1})_H, \end{cases} \tag{42}$$

for all $\Phi \in W^h$.

We first start by estimating the error between the elliptic projections $\tilde{u}^h, \tilde{\mu}^h, \tilde{v}^h$ and the projected quantities (u, μ, v). Thus,

Lemma 6.1. *Given $u, \mu, v \in V_2$, the functions $\tilde{u}^h, \tilde{\mu}^h, \tilde{v}^h$ defined by (42) satisfy the following inequalities:*

$$\begin{aligned} \|\tilde{u}^h - u\|_H \; + \; h|\tilde{u}^h - u|_{1,W} &\leqslant ch^2(|u|_{2,\Omega} + |u|_{2,\Gamma}), \\ \|\tilde{\mu}^h - \mu\|_H \; + \; h|\tilde{\mu}^h - \mu|_{1,W} &\leqslant ch^2(|\mu|_{2,\Omega} + |u|_{2,\Gamma}), \\ \|\tilde{v}^h - v\|_H \; + \; h|\tilde{v}^h - v|_{1,W} &\leqslant ch^2(|v|_{2,\Omega} + |v|_{2,\Gamma}), \end{aligned} \tag{43}$$

where c is a positive constant independent of h.

Proof. The proof of this result classical and follows closely the same arguments as in [7, 8, 11], we thus omit it. □

Definition 6.1. *Let us define the operator $T^h : \dot{H} \to \dot{W}^h, \quad f \mapsto T^h f$, where $T^h f$ is the unique solution of the problem*

$$(\nabla T^h f, \nabla \chi^h)_\Omega + \delta(\nabla_\Gamma T^h f, \nabla_\Gamma \chi^h)_\Gamma = (f, \chi^h)_H, \quad \forall \chi^h \in \dot{W}^h. \tag{44}$$

We also introduce the discrete negative seminorm:

$$|f|_{-1,h} = (T^h f, f)^{1/2} = \left(\|\nabla T^h f\|_\Omega^2 + \delta\|\nabla_\Gamma T^h f\|_\Gamma^2\right)^{1/2}, \quad \forall f \in \dot{H}.$$

We remark that (44) is still valid for $\chi^h \in W^h$, thanks to the fact that $f \in \dot{H}$.

Concerning the operator T^h we have the following result (see [7] for details on the proof):

Lemma 6.2. *The operator T^h is self-adjoint, positive, semi-definite on $\dot{H}$. Moreover, the following interpolation inequality holds, for all $\rho^h \in \dot{W}^h$,*

$$\|\rho^h\|_H^2 \leqslant c|\rho^h|_{-1,h}\|\rho^h\|_{1,W} \quad \forall \rho^h \in \dot{W}^h. \tag{45}$$

and

$$|f|_{-1,h} \leqslant \; c\|f\|_H \quad \forall f \in \dot{H}, \tag{46}$$

where c is a positive constant independent of h.

We have now all the tools in order to estimate $\theta^u = u^h - \tilde{u}^h, \theta^\mu = \mu^h - \tilde{\mu}^h$ and $\theta^v = v^h - \tilde{v}^h$. Using (10), $(13)_2$ and $(42)_1$, we obtain that $\theta^u = u^h - \tilde{u}^h$ satisfies the following equation:

$$\begin{aligned}(47)\qquad &(\nabla\theta^u, \nabla\Phi)_\Omega + \sigma(\nabla_\Gamma\theta^u, \nabla_\Gamma\Phi)_\Gamma + (f(u^h+v^h) - f(u+v), \Phi)_\Omega \\ &\quad + (f(u^h-v^h) - f(u-v), \Phi)_\Omega + (g(u^h+v^h) - g(u+v), \Phi)_\Gamma \\ &\quad + (g(u^h-v^h) - g(u-v), \Phi)_\Gamma - (\theta^\mu, \Phi)_H = (w^\mu, \Phi)_H,\end{aligned}$$

for all $\Phi \in W^h$.

From $(10)_1$, $(13)_1$ and $(42)_2$, we get

$$(48)\qquad (\theta^u_t, \Psi)_H + (\nabla\theta^\mu, \nabla\Psi)_\Omega + \delta(\nabla_\Gamma\theta^\mu, \nabla_\Gamma\Psi)_\Gamma = -(w^u_t, \Psi)_H, \quad \Psi \in W^h.$$

Taking $\Psi = 1$ in (48) we obtain

$$(49)\qquad m(\theta^u_t(t)) = -m(w^u_t(t)) \quad \forall t \geqslant 0.$$

From $(13)_3$ and $(42)_3$, we also get that $\theta^v = v^h - \tilde{v}^h$ satisfies the following equation

$$\begin{aligned}(50)\qquad &(\theta^v_t, \varphi)_H + (\nabla\theta^v, \nabla\varphi)_\Omega + \kappa(\nabla_\Gamma\theta^v, \nabla_\Gamma\varphi)_\Gamma + \alpha(\theta^v, \varphi)_\Omega = -(w^v_t, \varphi)_H \\ &\quad -(f(u^h+v^h) - f(u+v), \varphi)_\Omega + (f(u^h-v^h) - f(u-v), \varphi)_\Omega \\ &\quad -(g(u^h+v^h) - g(u+v), \varphi)_\Gamma + (g(u^h-v^h) - g(u-v), \varphi)_\Gamma.\end{aligned}$$

Appropriate estimates on $(\theta^u, \theta^\mu, \theta^v)$ will allow us to prove the following result:

Lemma 6.3. *Let (u, μ, v) be a solution of the continuous problem* (10) *supposed here to be regular enough and (u^h, μ^h, v^h) a solution of* (13). *Assume that*

$$(51)\qquad \sup_{t\in[0,T]} \|u(t)\|_{\mathcal{C}^0(\bar\Omega)} \leqslant R, \quad \sup_{t\in[0,T]} \|u_t(t)\|_{\mathcal{C}^0(\bar\Omega)} \leqslant R, \quad \|u^h(0)\|_{\mathcal{C}^0(\bar\Omega)} < R$$

and

$$(52)\qquad \sup_{t\in[0,T]} \|v(t)\|_{\mathcal{C}^0(\bar\Omega)} \leqslant R, \quad \sup_{t\in[0,T]} \|v_t(t)\|_{\mathcal{C}^0(\bar\Omega)} \leqslant R, \quad \|v^h(0)\|_{\mathcal{C}^0(\bar\Omega)} < R,$$

for some constant $R < +\infty$ and let $T^h \in (0, T]$ be the maximal time such that $\|u^h(t)\|_{L^\infty(\Omega)} \leqslant R$ and $\|v^h(t)\|_{L^\infty(\Omega)} \leqslant R$ for all $t \in [0, T^h]$. Then, for all $t \in [0, T^h]$

$$\begin{aligned}(53)\qquad &\mathcal{N}(t) + \int_0^t \Big\{ \|\nabla\theta^u_t\|^2_\Omega + \sigma\|\nabla_\Gamma\theta^u_t\|^2_\Gamma + \|\nabla\theta^\mu\|^2_\Omega + \delta\|\nabla_\Gamma\theta^\mu\|^2_\Gamma \\ &\quad + \|\theta^v_t\|^2_H + \|\nabla\theta^v_t\|^2_\Omega + \kappa\|\nabla_\Gamma\theta^v_t\|^2_\Gamma + \alpha\|\theta^v_t\|^2_\Omega \Big\}\, ds \\ &\leqslant \mathcal{N}(0) + c\int_0^t \Big\{ \|w^u\|^2_H + \|w^v\|^2_H + \|w^u_t\|^2_H + \|w^v_t\|^2_H \\ &\quad + \|w^u_{tt}\|^2_H + \|w^v_{tt}\|^2_H + \|w^\mu\|^2_H + \|w^\mu_t\|^2_H \Big\}\, ds,\end{aligned}$$

where

$$\begin{aligned}(54)\qquad \mathcal{N}(t) = &|\theta^u_t - m(\theta^u_t)|^2_{-1,h} + \|\nabla\theta^u\|^2_\Omega + \sigma\|\nabla_\Gamma\theta^u\|^2_\Gamma + \|\nabla\theta^v\|^2_\Omega \\ &+ \kappa\|\nabla_\Gamma\theta^v\|^2_\Gamma + \alpha\|\theta^v\|^2_\Omega + \|\theta^v_t\|^2_H.\end{aligned}$$

Furthermore,

$$(55)\qquad |(\theta^\mu, 1)| \leqslant C(\mathcal{N}(t)^{1/2} + \|w^u\|^2_H + \|w^v\|^2_H), \quad \forall t \in [0, T^h].$$

Proof. Testing (47) by $\Phi = \theta_t^u$ and (48) by $\Psi = \theta^\mu$, we get

(56)
$$\frac{1}{2}\frac{d}{dt}\{\|\nabla\theta^u\|_\Omega^2 + \sigma\|\nabla_\Gamma\theta^u\|_\Gamma^2\} + \|\nabla\theta^\mu\|_\Omega^2 + \delta\|\nabla_\Gamma\theta^\mu\|_\Gamma^2 + (f(u^h+v^h) - f(u+v), \theta_t^u)_\Omega$$
$$+ (f(u^h-v^h) - f(u-v), \theta_t^h)_\Omega + (g(u^h+v^h) - g(u+v), \theta_t^u)_\Gamma$$
$$+ (g(u^h-v^h) - g(u-v), \theta_t^h)_\Gamma = -(w_t^u, \theta^\mu)_H + (w^\mu, \theta_t^u)_H.$$

We have the following estimates on the nonlinear terms f and g:

$$\left\{\begin{array}{ll} \|f(u^h+v^h) - f(u+v)\|_\Omega & \leqslant L_f\|(u^h-u) + (v^h-v)\|_\Omega \\ & \leqslant L_f(\|u^h-u\|_\Omega + \|v^h-v\|_\Omega), \\ \|f(u^h-v^h) - f(u-v)\|_\Omega & \leqslant L_f\|(u^h-u) - (v^h-v)\|_\Omega \\ & \leqslant L_f(\|u^h-u\|_\Omega + \|v^h-v\|_\Omega), \\ \|g(u^h+v^h) - g(u+v)\|_\Gamma & \leqslant L_g\|(u^h-u) + (v^h-v)\|_\Gamma \\ & \leqslant L_g(\|u^h-u\|_\Gamma + \|v^h-v\|_\Gamma), \\ \|g(u^h-v^h) - g(u-v)\|_\Gamma & \leqslant L_g\|(u^h-u) - (v^h-v)\|_\Gamma \\ & \leqslant L_g(\|u^h-u\|_\Gamma + \|v^h-v\|_\Gamma), \end{array}\right. \tag{57}$$

for all $t \in [0, T^h]$, where L_f and L_g are respectively the Lipschitz constant of f and g on $[-R, R]$.

Combining (56) and (57), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\{\|\nabla\theta^u\|_\Omega^2 + \sigma\|\nabla_\Gamma\theta^u\|_\Gamma^2\} + \|\nabla\theta^\mu\|_\Omega^2 + \delta\|\nabla_\Gamma\theta^\mu\|_\Gamma^2 \\ \leqslant& L_f\left(\|\theta^u\|_\Omega + \|w^u\|_\Omega + \|\theta^v\|_\Omega + \|w^v\|_\Omega\right)\|\theta_t^u\|_\Omega \\ &+ L_g\left(\|\theta^u\|_\Gamma + \|w^u\|_\Gamma + \|\theta^v\|_\Gamma + \|w^v\|_\Gamma\right)\|\theta_t^u\|_\Gamma \\ &+ \|w_t^u\|_H\|\theta^\mu\|_H + \|w^\mu\|_H\|\theta_t^u\|_H, \end{aligned} \tag{58}$$

which further implies

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\{\|\nabla\theta^u\|_\Omega^2 + \sigma\|\nabla_\Gamma\theta^u\|_\Gamma^2\} + \|\nabla\theta^\mu\|_\Omega^2 + \delta\|\nabla_\Gamma\theta^\mu\|_\Gamma^2 \\ \leqslant& c\left(\|\theta^u\|_H\|\theta_t^u\|_H + \|w^u\|_H\|\theta_t^u\|_H + \|\theta^v\|_H\|\theta_t^u\|_H + \|w^v\|_H\|\theta_t^u\|_H\right) \\ &+ \|w_t^u\|_H\|\theta^\mu\|_H + \|w^\mu\|_H\|\theta_t^u\|_H. \end{aligned} \tag{59}$$

The mass of θ^μ is estimated by testing (47) by 1 and using (57) and $(42)_4$:

$$|(\theta^\mu, \mathbf{1})_H| \leqslant c(\|\theta^u\|_H + \|w^u\|_H + \|\theta^v\|_H + \|w^v\|_H), \tag{60}$$

and by the Poincaré inequality (8), we obtain

$$\|\theta^\mu\|_H \leqslant c(\|\nabla\theta^\mu\|_\Omega^2 + \delta\|\nabla_\Gamma\theta^\mu\|_\Gamma^2)^{\frac{1}{2}} + c|(\theta^\mu, \mathbf{1})_H|. \tag{61}$$

From (59) and (61), we finally get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\{\|\nabla\theta^u\|_\Omega^2 + \sigma\|\nabla_\Gamma\theta^u\|_\Gamma^2\} + \frac{1}{2}\|\nabla\theta^\mu\|_\Omega^2 + \frac{\delta}{2}\|\nabla_\Gamma\theta^\mu\|_\Gamma^2 \\ &\leqslant c(\|\theta^u\|_H^2 + \|\theta^v\|_H^2 + \|\theta_t^u\|_H^2 + \|w^u\|_H^2 + \|w^v\|_H^2 + \|w_t^u\|_H^2 + \|w^\mu\|_H^2). \end{aligned} \tag{62}$$

In (62) we need an estimate on $\|\theta_t^u\|_H$. Differentiating (47) and (48) with respect to time, we get

$$(\theta_{tt}^u, \Psi)_H + (\nabla\theta_t^\mu, \nabla\Psi)_\Omega + \delta(\nabla_\Gamma\theta_t^\mu, \nabla_\Gamma\Psi)_\Gamma + (w_{tt}^u, \Psi)_H = 0, \quad \forall\Psi \in W^h \tag{63}$$

and

(64)

$$\begin{aligned}&(\nabla\theta_t^u,\nabla\Phi)_\Omega+\sigma(\nabla_\Gamma\theta_t^u,\nabla_\Gamma\Phi)_\Gamma+(f'(u^h+v^h)(u_t^h+v_t^h)-f'(u+v)(u_t+v_t),\Phi)_\Omega\\&-(\theta_t^\mu,\Phi)+(f'(u^h-v^h)(u_t^h+v_t^h)-f'(u-v)(u_t-v_t),\Phi)_\Omega\\&\quad+(g'(u^h+v^h)(u_t^h+v_t^h)-g'(u+v)(u_t+v_t),\Phi)_\Gamma\\&\qquad+(g'(u^h-v^h)(u_t^h+v_t^h)-g'(u-v)(u_t-v_t),\Phi)_\Gamma=(w_t^\mu,\Phi)_H,\quad\forall\Phi\in W^h.\end{aligned}$$

Testing (63) by $\Psi=T^h(\theta_t^u-m(\theta_t^u))$ and using (44), we get

$$\begin{aligned}(\theta_{tt}^u-m(\theta_{tt}^u),T^h(\theta_t^u-m(\theta_t^u)))_H+(m(\theta_{tt}^u),T^h(\theta_t^u-m(\theta_t^u)))_H\\+(\theta_t^u-m(\theta_t^u),\theta_t^\mu)_H=-(w_{tt}^u,T^h(\theta_t^u-m(\theta_t^u)))_H.\end{aligned}\tag{65}$$

Since $m(w_{tt}^u)=-m(\theta_{tt}^u)$, we obtain

$$\frac12\frac{d}{dt}|\theta_t^u-m(\theta_t^u)|_{-1,h}^2+(\theta_t^u-m(\theta_t^u),\theta_t^\mu)_H=-(w_{tt}^u-m(w_{tt}^u),T^h(\theta_t^u-m(\theta_t^u)))_H.\tag{66}$$

Testing (64) by $\Phi=\theta_t^u-m(\theta_t^u)$, we have

(67)

$$\begin{aligned}&\|\nabla\theta_t^u\|_\Omega^2+\sigma\|\nabla_\Gamma\theta_t^u\|_\Gamma^2+(f'(u^h+v^h)(u_t^h+v_t^h)-f'(u+v)(u_t+v_t),\theta_t^u-m(\theta_t^u))_\Omega\\&\quad-(\theta_t^\mu,\theta_t^u-m(\theta_t^u))_H+(f'(u^h-v^h)(u_t^h+v_t^h)-f'(u-v)(u_t-v_t),\theta_t^u-m(\theta_t^u))_\Omega\\&\quad+(g'(u^h+v^h)(u_t^h+v_t^h)-g'(u+v)(u_t+v_t),\theta_t^u-m(\theta_t^u))_\Gamma\\&\quad+(g'(u^h-v^h)(u_t^h+v_t^h)-g'(u-v)(u_t-v_t),\theta_t^u-m(\theta_t^u))_\Gamma\\&=(w_t^\mu,\theta_t^u-m(\theta_t^u))_H.\end{aligned}$$

Since

$$\begin{aligned}&f'(u^h+v^h)(u_t^h+v_t^h)-f'(u+v)(u_t+v_t)\\=&f'(u^h+v^h)(u_t^h-u_t)+f'(u^h+v^h)(v_t^h-v_t)\\&+(f'(u^h+v^h)-f'(u+v))(u_t+v_t),\end{aligned}$$

the following inequality holds

(68)

$$\begin{aligned}&|(f'(u^h+v^h)(u_t^h+v_t^h)-f'(u+v)(u_t+v_t),\theta_t^u-m(\theta_t^u))_\Omega|\\&\quad\leqslant\sup_{[-R,R]}|f'|\|u_t^h-u_t\|_\Omega\|\theta_t^u-m(\theta_t^u)\|_\Omega+\sup_{[-R,R]}|f'|\|v_t^h-v_t\|_\Omega\|\theta_t^u-m(\theta_t^u)\|_\Omega\\&\qquad+2L_{f'}R(\|u^h-u\|_\Omega+\|v^h-v\|_\Omega)\|\theta_t^u-m(\theta_t^u)\|_\Omega\end{aligned}$$

where $L_{f'}$ is the Lipschitz constant of f' on $[-R,R]$ and we argue similarly for the second term in (67) containing f' as well as for the term g'.

Using these estimates on the terms in f' and g' from (67), we obtain

(69)

$$\begin{aligned}&\|\nabla\theta_t^u\|_\Omega^2+\sigma\|\nabla_\Gamma\theta_t^u\|_\Gamma^2-(\theta_t^\mu,\theta_t^u-m(\theta_t^u))_H\\&\leqslant\sup_{[-R,R]}|f'|\|u_t^h-u_t\|_\Omega\|\theta_t^u-m(\theta_t^u)\|_\Omega+\sup_{[-R,R]}|f'|\|v_t^h-v_t\|_\Omega\|\theta_t^u-m(\theta_t^u)\|_\Omega\\&\quad+L_{f'}R(\|u^h-u\|_\Omega+\|v^h-v\|_\Omega)\|\theta_t^u-m(\theta_t^u)\|_\Omega\\&\quad+\sup_{[-R,R]}|g'|\|u_t^h-u_t\|_\Gamma\|\theta_t^u-m(\theta_t^u)\|_\Gamma+\sup_{[-R,R]}|g'|\|v_t^h-v_t\|_\Gamma\|\theta_t^u-m(\theta_t^u)\|_\Gamma\\&\quad+L_{g'}R(\|u^h-u\|_\Gamma+\|v^h-v\|_\Gamma)\|\theta_t^u-m(\theta_t^u)\|_\Gamma+|(w_t^\mu,\theta_t^u-m(\theta_t^u))_H|,\end{aligned}$$

which combined to (66) leads to:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\theta_t^u - m(\theta_t^u)|_{-1,h}^2 + \|\nabla\theta_t^u\|_\Omega^2 + \sigma\|\nabla_\Gamma\theta_t^u\|_\Gamma^2 \\
&\leqslant |(w_{tt}^u - m(w_{tt}^u), T^h(\theta_t^u - m(\theta_t^u)))_H| + |(w_t^\mu, \theta_t^u - m(\theta_t^u))_H| \\
&\quad + c\left(\|u_t^h - u_t\|_\Omega + \|v_t^h - v_t\|_\Omega + \|u^h - u\|_\Omega + \|v^h - v\|_\Omega\right)\|\theta_t^u - m(\theta_t^u)\|_\Omega \\
&\quad + c\left(\|u_t^h - u_t\|_\Gamma + \|v_t^h - v_t\|_\Gamma + \|u^h - u\|_\Gamma + \|v^h - v\|_\Gamma\right)\|\theta_t^u - m(\theta_t^u)\|_\Gamma,
\end{aligned} \tag{70}
$$

where the constant c depends on $R, L_{f'}$ and $L_{g'}$. Using $m(w_t^u) = -m(\theta_t^u)$, we can bound each term from the right hand side of (70) and finally obtain:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\theta_t^u - m(\theta_t^u)|_{-1,h}^2 + \|\nabla\theta_t^u\|_\Omega^2 + \sigma\|\nabla_\Gamma\theta_t^u\|_\Gamma^2 \\
&\leqslant c(|w_{tt}^u - m(w_{tt}^u)|_{-1,h}^2 + |\theta_t^u - m(\theta_t^u)|_{-1,h}^2 + \|\theta_t^u - m(\theta_t^u)\|_H^2) \\
&\quad + c(\|\theta^u\|_H^2 + \|\theta^v\|_H^2 + \|\theta_t^v\|_H^2 + \|w^u\|_H^2 + \|w_t^u\|_H^2 + \|w^v\|_H^2 + \|w_t^v\|_H^2 + \|w_t^\mu\|_H^2).
\end{aligned} \tag{71}
$$

Thanks to (45) and (46), we find

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\theta_t^u - m(\theta_t^u)|_{-1,h}^2 + \|\nabla\theta_t^u\|_\Omega^2 + \sigma\|\nabla_\Gamma\theta_t^u\|_\Gamma^2 \leqslant c|\theta_t^u - m(\theta_t^u)|_{-1,h}^2 \\
&\quad + c(\|\theta^u\|_H^2 + \|\theta^v\|_H^2 + \|\theta_t^v\|_H^2 + \|w^u\|_H^2 \\
&\quad + \|w_t^u\|_H^2 + \|w^v\|_H^2 + \|w_t^v\|_H^2 + \|w_t^\mu\|_H^2 + \|w_{tt}^u\|_H^2).
\end{aligned} \tag{72}
$$

Summing (62) and (72) and using the generalized Poincaré inequality (8), we have:

$$
\begin{aligned}
&\frac{d}{dt}\left\{|\theta_t^u - m(\theta_t^u)|_{-1,h}^2 + \|\nabla\theta^u\|_\Omega^2 + \sigma\|\nabla_\Gamma\theta^u\|_\Gamma^2\right\} \\
&\quad + \frac{1}{2}\|\nabla\theta_t^u\|_\Omega^2 + \frac{\sigma}{2}\|\nabla_\Gamma\theta_t^u\|_\Gamma^2 + \|\nabla\theta^\mu\|_\Omega^2 + \delta\|\nabla_\Gamma\theta^\mu\|_\Gamma^2 \\
&\quad \leqslant c(|\theta_t^u - m(\theta_t^u)|_{-1,h}^2 + \|\nabla\theta^u\|_\Omega^2 + \sigma\|\nabla_\Gamma\theta^u\|_\Gamma^2) + c(\|\theta^v\|_H^2 + \|\theta_t^v\|_H^2) \\
&\quad\quad + c(\|w^u\|_H^2 + \|w_t^u\|_H^2 + \|w^v\|_H^2 + \|w_t^v\|_H^2 + \|w^\mu\|_H^2 + \|w_t^\mu\|_H^2 + \|w_{tt}^u\|_H^2).
\end{aligned} \tag{73}
$$

Taking $\varphi = \theta_t^v$ in (50) we get:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{\|\nabla\theta^v\|_\Omega^2 + \kappa\|\nabla_\Gamma\theta^v\|_\Gamma^2 + \alpha\|\theta^v\|_\Omega^2\} + \|\theta_t^v\|_H^2 = -(w_t^v, \theta_t^v)_H \\
&\quad - (f(u^h + v^h) - f(u + v), \theta_t^v)_\Omega + (f(u^h - v^h) - f(u - v), \theta_t^v)_\Omega \\
&\quad - (g(u^h + v^h) - g(u + v), \theta_t^v)_\Gamma + (g(u^h - v^h) - g(u - v), \theta_t^v)_\Gamma.
\end{aligned} \tag{74}
$$

Using (57), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\{\|\nabla\theta^v\|_\Omega^2 + \kappa\|\nabla_\Gamma\theta^v\|_\Gamma^2 + \alpha\|\theta^v\|_\Omega^2\} + \|\theta_t^v\|_H^2 \\
&\leqslant c(\|\theta^u\|_H^2 + \|\theta^v\|_H^2) + c(\|w^u\|_H^2 + \|w^v\|_H^2 + \|w_t^v\|_H^2).
\end{aligned} \tag{75}
$$

Now, we differentiate (50) with respect to time. We get

$$
\begin{aligned}
&(\theta^v_{tt},\varphi)_H + (\nabla\theta^v_t, \nabla\varphi)_\Omega + \kappa(\nabla_\Gamma\theta^v_t, \nabla_\Gamma\varphi)_\Gamma + \alpha(\theta^v_t, \varphi)_\Omega \\
&\quad = -(w^v_{tt}, \varphi)_H - (f'(u^h + v^h)(u^h_t + v^h_t) - f'(u + v)(u_t + v_t), \varphi)_\Omega \\
&\quad + (f'(u^h - v^h)(u^h_t - v^h_t) - f'(u - v)(u_t - v_t), \varphi)_\Omega \\
&\quad - (g'(u^h + v^h)(u^h_t + v^h_t) - g'(u + v)(u_t + v_t), \varphi)_\Gamma \\
&\quad + (g'(u^h - v^h)(u^h_t - v^h_t) - g'(u - v)(u_t - v^t), \varphi)_\Gamma.
\end{aligned} \tag{76}
$$

We take the test function $\varphi = \theta^v_t$ in (76) and obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\theta^v_t\|^2_H + \|\nabla\theta^v_t\|^2_\Omega + \kappa\|\nabla_\Gamma\theta^v_t\|^2_\Gamma + \alpha\|\theta^v_t\|^2_\Omega = -(w^v_{tt}, \theta^v_t)_H \\
&\quad - (f'(u^h + v^h)(u^h_t + v^h_t) - f'(u + v)(u_t + v_t), \theta^v_t)_\Omega \\
&\quad + (f'(u^h - v^h)(u^h_t - v^h_t) - f'(u - v)(u_t - v_t), \theta^v_t)_\Omega \\
&\quad - (g'(u^h + v^h)(u^h_t + v^h_t) - g'(u + v)(u_t + v_t), \theta^v_t)_\Gamma \\
&\quad + (g'(u^h - v^h)(u^h_t - v^h_t) - g'(u - v)(u_t - v_t), \theta^v_t)_\Gamma.
\end{aligned} \tag{77}
$$

Repeating the same type of computations as for (68), we obtain, using the fact that $m(\theta^u) = -m(w^u)$,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\theta^v_t\|^2_H + \|\nabla\theta^v_t\|^2_\Omega + \kappa\|\nabla_\Gamma\theta^v_t\|^2_\Gamma + \alpha\|\theta^v_t\|^2_\Omega \\
&\quad \leqslant c(\|\theta^u\|^2 + \|\theta^v\|^2 + \|\theta^u_t\|^2 + \|\theta^v_t\|^2) \\
&\qquad + c(\|w^u\|^2_H + \|w^v\|^2_H + \|w^u_t\|^2_H + \|w^v_t\|^2_H + \|w^v_{tt}\|^2_H),
\end{aligned} \tag{78}
$$

Combining (73), (75) and (78), since $m(\theta^u_t) = -m(w^u_t)$ and using the Poincaré inequality, we obtain:

$$
\begin{aligned}
&\frac{d}{dt}\left\{|\theta^u_t - m(\theta^u_t)|^2_{-1,h} + \|\nabla\theta^u\|^2_\Omega + \sigma\|\nabla_\Gamma\theta^u\|^2_\Gamma + \|\nabla\theta^v\|^2_\Omega \right.\\
&\quad \left. +\kappa\|\nabla_\Gamma\theta^v\|^2_\Gamma + \alpha\|\theta^v\|^2_\Omega + \|\theta^v_t\|^2_H\right\} \\
&\quad + \frac{1}{2}\|\nabla\theta^u_t\|^2_\Omega + \frac{\sigma}{2}\|\nabla_\Gamma\theta^u_t\|^2_\Gamma + \|\nabla\theta^\mu\|^2_\Omega + \delta\|\nabla_\Gamma\theta^\mu\|^2_\Gamma \\
&\quad + \|\theta^v_t\|^2_H + \|\nabla\theta^v_t\|^2_\Omega + \kappa\|\nabla_\Gamma\theta^v_t\|^2_\Gamma + \alpha\|\theta^v_t\|^2_\Omega \\
&\leqslant c(|\theta^u_t - m(\theta^u_t)|^2_{-1,h} + \|\nabla\theta^u\|^2_\Omega + \sigma\|\nabla_\Gamma\theta^u\|^2_\Gamma + \|\nabla\theta^v\|^2_\Omega \\
&\quad + \kappa\|\nabla_\Gamma\theta^v\|^2_\Gamma + \alpha\|\theta^v\|^2_\Omega + \|\theta^v_t\|^2_H) \\
&\quad + c(\|w^u\|^2_H + \|w^v\|^2_H + \|w^u_t\|^2_H + \|w^v_t\|^2_H \\
&\quad + \|w^u_{tt}\|^2_H + \|w^v_{tt}\|^2_H + \|w^\mu\|^2_H + \|w^\mu_t\|^2_H).
\end{aligned} \tag{79}
$$

Applying the Gronwall lemma to (79) and using (54), we derive (53) and (55) follows from the Generalized Poincaré inequality.

□

Theorem 6.1. *Let* (u, μ, v) *be the solution of* (10) *with* $u(0) = u_0$ *and* $v(0) = v_0$ *such that*

$$
u, u_t, u_{tt}, v, v_t, v_{tt}, \mu, \mu_t \in L^2(0, T; V_2) \tag{80}
$$

and let (u^h, μ^h, v^h) *be the solution of the discrete problem* (13) *with* $u^h(0) = u^h_0$ *and* $v^h(0) = v^h_0$. *If*

$$
\theta^u(0) = 0, \quad \theta^\mu(0) = 0 \quad \textit{and } \theta^v(0) = 0, \tag{81}
$$

then the following error estimates hold, for h small enough

$$\begin{aligned}
&\sup_{[0,T]}(\|u^h-u\|_H+|u^h_t-u_t|_{-1,h}+\|v^h-v\|_H+\|v^h_t-v_t\|_H)\\
&\quad+\left(\int_0^T\|\mu^h-\mu\|^2_H\,ds\right)^{\frac12}\leqslant ch^2,\\
&\sup_{[0,T]}\big(|u^h-u|_{1,W}+|v^h-v|_{1,W}\big)\\
&\quad+\left(\int_0^T|u^h_t-u_t|^2_{1,W}+|\mu^h-\mu|^2_{1,W}+|v^h_t-v_t|^2_{1,W}\,ds\right)^{\frac12}\leqslant ch,
\end{aligned}$$

with c a positive constant independent of h.

Proof. Applying Lemma 6.1 to u,v replaced by u_t,v_t or u_{tt},v_{tt} and μ replaced by μ_t, we have the following estimates

$$\begin{aligned}
&\|w^u\|_H\leqslant ch^2(|u|_{2,\Omega}+|u|_{2,\Gamma}),\quad \|w^\mu\|_H\leqslant ch^2(|\mu|_{2,\Omega}+|\mu|_{2,\Gamma}),\\
&\|w^v\|_H\leqslant ch^2(|v|_{2,\Omega}+|v|_{2,\Gamma}),\quad \|w^u_t\|_H\leqslant ch^2(|u_t|_{2,\Omega}+|u_t|_{2,\Gamma}),\\
&\|w^\mu_t\|_H\leqslant ch^2(|\mu_t|_{2,\Omega}+|\mu_t|_{2,\Gamma}),\quad \|w^v_t\|_H\leqslant ch^2(|v_t|_{2,\Omega}+|v_t|_{2,\Gamma}),\\
&\|w^u_{tt}\|_H\leqslant ch^2(|u_{tt}|_{2,\Omega}+|u_{tt}|_{2,\Gamma}),\quad \|w^v_{tt}\|_H\leqslant ch^2(|v_{tt}|_{2,\Omega}+|v_{tt}|_{2,\Gamma}).
\end{aligned}\tag{82}$$

The regularity required on u,v implies that $u,v\in\mathcal{C}^1([0,T];V_2)$ and by the Sobolev continuous injection $H^2(\Omega)\subset\mathcal{C}^0(\bar\Omega)$, we have $u,u_t,v,v_t\in\mathcal{C}^0([0,T];\mathcal{C}^0(\bar\Omega))$. Thus

$$\sup_{t\in[0,T]}\|u(t)\|_{\mathcal{C}^0(\bar\Omega)}<R,\quad \sup_{t\in[0,T]}\|u_t(t)\|_{\mathcal{C}^0(\bar\Omega)}\leqslant R\tag{83}$$

and

$$\sup_{t\in[0,T]}\|v(t)\|_{\mathcal{C}^0(\bar\Omega)}<R,\quad \sup_{t\in[0,T]}\|v_t(t)\|_{\mathcal{C}^0(\bar\Omega)}\leqslant R,\tag{84}$$

for some fixed $R>0$. By a standard argument using the inverse estimate (13) (see for instance the of Theorem 4.6 in [14]), we also have

$$\begin{aligned}
\|u^h(0)-u(0)\|_{\mathcal{C}^0(\bar\Omega)}\leqslant&\|u^h(0)-I^hu(0)\|_{\mathcal{C}^0(\bar\Omega)}+\|I^hu(0)-u(0)\|_{\mathcal{C}^0(\bar\Omega)}\\
\leqslant&ch^{-\frac d2}\left\{\|u^h(0)-u(0)\|_\Omega+\|u(0)-I^hu(0)\|_\Omega\right\}\\
&+ch^\gamma\|u(0)\|_{H^2_p(\Omega)}
\end{aligned}$$

and

$$\begin{aligned}
\|v^h(0)-v(0)\|_{\mathcal{C}^0(\bar\Omega)}\leqslant&\|v^h(0)-I^hv(0)\|_{\mathcal{C}^0(\bar\Omega)}+\|I^hv(0)-v(0)\|_{\mathcal{C}^0(\bar\Omega)}\\
\leqslant&ch^{-\frac d2}\left\{\|v^h(0)-v(0)\|_\Omega+\|v(0)-I^hv(0)\|_\Omega\right\}\\
&+ch^\gamma\|v(0)\|_{H^2_p(\Omega)},
\end{aligned}$$

with $\gamma\in(0,1)$ is such that $H^2_p(\Omega)\subset\mathcal{C}^\gamma(\bar\Omega)$. Thus by Lemma 6.1, (38) and (81),

$$\|u^h(0)-u(0)\|_{\mathcal{C}^0(\bar\Omega)}\leqslant ch^{2-\frac d2}(|u(0)|_{2,\Omega}+|u(0)|_{2,\Gamma})+ch^\gamma\|u(0)\|_{H^2_p(\Omega)}\tag{85}$$

and

$$\|v^h(0)-v(0)\|_{\mathcal{C}^0(\bar\Omega)}\leqslant ch^{2-\frac d2}(|v(0)|_{2,\Omega}+|v(0)|_{2,\Gamma})+ch^\gamma\|v(0)\|_{H^2_p(\Omega)},\tag{86}$$

for h small enough,

$$\|u^h(0)\|_{\mathcal{C}^0(\bar{\Omega})} < R \quad \text{and} \quad \|v^h(0)\|_{\mathcal{C}^0(\bar{\Omega})} < R$$

and we may apply Lemma 6.3.

It remains to prove that $\mathcal{N}(0) \leqslant ch^4$. We first notice that, by (54) and (81), $\mathcal{N}(0)$ reduces to

$$\mathcal{N}(0) = |\theta_t^u(0) - m(\theta_t^u(0))|^2_{-1,h} + \|\theta_t^v(0)\|^2_H.$$

We also infer from (48) and (81) that

$$(\theta_t^u(0), \Psi) = -(w_t^u(0), \Psi) \qquad \forall \Psi \in W^h,$$

or, equivalently,

$$(\theta_t^u - m(\theta_t^u)(0), \Psi) = -(w_t^u(0) + m(\theta_t^u)(0), \Psi) \qquad \forall \Psi \in W^h, \tag{87}$$

the terms $\theta_t^u(0) - m(\theta_t^u)(0)$ and $w_t^u(0) + m(\theta_t^u)(0)$ having null average by (49). Then, choosing $\Psi = T^h(\theta_t^u(0) - m(\theta_t^u)(0))$ in (87), we obtain

$$|\theta_t^u(0) - m(\theta_t^u)(0)|^2_{-1,h} \leqslant |w_t^u(0) + m(\theta_t^u)(0)|_{-1,h} |\theta_t^u(0) - m(\theta_t^u)(0)|_{-1,h}.$$

Hence, we deduce (using (46) and (82))

$$\begin{aligned} |\theta_t^u(0) - m(\theta_t^u)(0)|_{-1,h} &\leqslant |w_t^u(0) + m(\theta_t^u)(0)|_{-1,h} \leqslant \|w_t^u(0) + m(w_t^u)(0)\|_H \\ &\leqslant c\|w_t^u(0)\|_H \leqslant ch^2(|u_t(0)|_{2,\Omega} + |u_t(0)|_{2,\Gamma}). \end{aligned} \tag{88}$$

Now, we infer from (50) and (81) that

$$(\theta_t^v(0), \varphi) = -(w_t^v(0), \varphi) \quad \forall \varphi \in W^h. \tag{89}$$

Taking $\varphi = \theta_t^v(0)$ in (89) and have

$$\|\theta_t^v(0)\|^2_H \leqslant \|\theta_t^v(0)\|_H \|w_t^v(0)\|_H,$$

which yields

$$\|\theta_t^v(0)\|_H \leqslant \|w_t^v(0)\|_H \leqslant ch^2(|v_t(0)|_{2,\Omega} + |v_t(0)|_{2,\Gamma}) \tag{90}$$

and, finally, from (88) and (90), we get that $\mathcal{N}(0) \leqslant ch^4$ as claimed. Hence, we conclude from (53) and (82) that $\mathcal{N}(t) \leqslant ch^4 \ \ \forall t \in [0, T^h]$.

We also deduce that

$$\sup_{t \in [0,T^h]} \{\|u^h(t) - u(t)\|_{\mathcal{C}^0(\bar{\Omega})} + \|v^h(t) - v(t)\|_{\mathcal{C}^0(\bar{\Omega})}\} \to 0 \quad \text{as} \quad h \to 0.$$

Thus, for h small enough, $T^h = T$. The conclusion follows from Lemma 6.1, Lemma 6.3 and (61).

□

Remark 6.1. *We remark here that the regularity required in* (80) *is a strong one, this is due to the fact that we need strong regularity results in order to estimate the term* θ_t^u *and* θ_t^v. *Taking into account the parabolic nature of system* (1), *we expect that the solutions are regular enough provided that the initial data* u_0 *and* v_0 *are also regular enough.*

7. Stability of the backward Euler scheme

In what follows, we study the backward Euler scheme applied to the space semidiscrete problem (13). Considering the fixed time step function $\tau > 0$ the fully discrete problem reads as follows:

Let $u_h^0, v_h^0 \in W^h$*. For* $n \geqslant 1$*, find* $(u_h^n, \mu_h^n, v_h^n) \in W^h \times W^h \times W^h$ *such that*

$$\left\{\begin{array}{l} \dfrac{1}{\tau}(u_h^n, \Psi)_H + (\nabla \mu_h^n, \nabla \Psi)_\Omega + \delta(\nabla_\Gamma \mu_h^n, \nabla_\Gamma \Psi)_\Gamma = \dfrac{1}{\tau}(u_h^{n-1}, \Psi)_H, \\ (\mu_h^n, \Phi)_H = (\nabla u_h^n, \nabla \Phi)_\Omega + (f(u_h^n + v_h^n) + f(u_h^n - v_h^n), \Phi)_\Omega \\ \quad + \sigma(\nabla_\Gamma u_h^n, \nabla_\Gamma \Phi)_\Gamma + (g(u_h^n + v_h^n) + g(u_h^n - v_h^n), \Phi)_\Gamma, \\ \dfrac{1}{\tau}(v_h^n, \varphi)_H + \alpha(v_h^n, \varphi)_\Omega + (\nabla v_h^n, \nabla \varphi)_\Omega + \kappa(\nabla_\Gamma v_h^n, \nabla_\Gamma \varphi)_\Gamma \\ \quad = -(f(u_h^n + v_h^n) - f(u_h^n - v_h^n), \varphi)_\Omega \\ \quad -(g(u_h^n + v_h^n) - g(u_h^n - v_h^n), \varphi)_\Gamma + \dfrac{1}{\tau}(v_h^{n-1}, \varphi), \end{array}\right. \tag{91}$$

for all $\Psi, \Phi, \varphi \in W^h$.

In what follows, we prove the existence, uniqueness and stability of the sequence $(u_h^n, \mu_h^n, v_h^n)_n$ constructed by (91).

Theorem 7.1. *If* $u_h^0, v_h^0 \in W^h$*, there exists a sequence* $(u_h^n, \mu_h^n, v_h^n)_{n\geqslant 1}$ *generated by* (91) *and satisfying the following discrete energy inequality*

$$J(u_h^n, v_h^n) + \frac{1}{2\tau}|u_h^n - u_h^{n-1}|_{-1,h}^2 + \frac{1}{2\tau}\|v_h^n - v_h^{n-1}\|_H^2 \leqslant J(u_h^{n-1}, v_h^{n-1}), \quad \forall n \geqslant 1. \tag{92}$$

Furthermore, if the time step τ *is small enough, more exactly if*

$$\tau < \min\left\{\frac{1}{\max(C_f, C_g)^2}, \frac{\sigma}{\delta \max(C_f, C_g)^2}, \frac{1}{2\max(C_f, C_g)}\right\},$$

then the sequence is uniquely defined.

Proof. Let us start by considering the following variational problem

$$\mathcal{E}^h(\rho, \eta) = \inf_{(\tilde{u}, \tilde{v}) \in \mathcal{K}^h} \mathcal{E}(\tilde{u}, \tilde{v}), \tag{93}$$

where

$$\mathcal{E}^h(\rho, \eta) = J(\rho, \eta) + \frac{1}{2\tau}|\rho - u_h^{n-1}|_{-1,h}^2 + \frac{1}{2\tau}\|\eta - v_h^{n-1}\|_H^2$$

and

$$\mathcal{K}^h = \left\{(\rho, \eta) \in W^h \times W^h; (\rho - u_h^{n-1}, \mathbf{1})_H = 0\right\}. \tag{94}$$

Using (2) and (5), we can bound from below $\mathcal{E}^h$ as follows:

(95)

$$\mathcal{E}^h(\tilde{u}, \tilde{v}) \geqslant \frac{1}{2}\left\{(\min(1, \sigma)|u|_{1,W}^2 + \min(1, \kappa)|v|_{1,W}^2 + 2c_1(\|u\|_H^2 + \|v\|_H^2) + \alpha\|v\|_\Omega^2\right\}$$
$$- c_2(|\Omega| + |\Gamma|), \quad \forall (\tilde{u}, \tilde{v}) \in W^h \times W^h$$

and using the fact that $\mathcal{E}^h$ is continuous, we deduce that problem (93) has a solution for (93) satisfies the following Euler-Lagrange equations

$$\begin{aligned} &(\nabla u, \nabla \varphi)_\Omega + \sigma(\nabla_\Gamma, \nabla_\Gamma \varphi)_\Gamma + (f(u+v) + f(u-v), \varphi)_\Omega \\ &\quad + (g(u+v) + g(u-v), \varphi)_\Gamma + \frac{1}{\tau}(T^h(u - u_h^{n-1}), \varphi)_H \\ &\quad - \beta(\mathbf{1}, \varphi)_H = 0, \quad \forall \varphi \in W^h, \end{aligned} \tag{96}$$

(97)
$$
\begin{aligned}
&(\nabla v,\nabla\Psi)_\Omega+\kappa(\nabla_\Gamma v,\nabla_\Gamma\Psi)_\Gamma+\alpha(v,\Psi)_\Omega+\frac{1}{\tau}(v-v_h^{n-1},\Psi)_H\\
&\qquad+(f(u+v)-f(u-v),\Psi)_\Omega+(g(u+v)-g(u-v),\Psi)_\Gamma=0,\quad\forall\Psi\in W^h,
\end{aligned}
$$
where β is the Lagrange multiplier for the constraint (94).

We take $u=u_h^n$, $\mu=\beta-\dfrac{1}{\tau}T^h(u_h^n-u_h^{n-1})$ in (96) and $v=v_h^n$ in (98) and obtain that (u_h^n,μ_h^n,v_h^n) is a solution of problem (91). By construction $\mathcal{E}^h(u_h^n,v_h^n)\leqslant\mathcal{E}^h(u_h^{n-1},v_h^{n-1})=J(u_h^{n-1},v_h^{n-1})$ and we thus deduce the energy inequality (92).

It remains to prove the uniqueness of solutions to (91). Let us assume that there exist two solutions $(u_{h,1}^n,\mu_{h,1}^n,v_{h,1}^n)$ and $(u_{h,2}^n,\mu_{h,2}^n,v_{h,2}^n)$ originating from the same initial data $(u_h^{n-1},\mu_h^{n-1},v_h^{n-1})$ and set $z_h=u_{h,1}^n-u_{h,2}^n$, $\xi_h=\mu_{h,1}^n-\mu_{h,2}^n$ and $r_h=v_{h,1}^n-v_{h,2}^n$. Then (z_h,ξ_h,r_h) is solution to the following problem

(98)
$$
\begin{cases}
\dfrac{1}{\tau}(z_h,\Psi)_H+(\nabla\xi_h,\nabla\Psi)_\Omega+\delta(\nabla_\Gamma\xi_h,\nabla_\Gamma\Psi)_\Gamma=0,\\
(\xi_h,\Phi)_H=(\nabla z_h,\nabla\Phi)_\Omega+\sigma(\nabla_\Gamma z_h,\nabla\Phi)_\Gamma+(f(u_{h,1}^n+v_{h,1}^n)-f(u_{h,2}^n+v_{h,2}^n),\Phi)_\Omega\\
\quad+(f(u_{h,1}^n-v_{h,1}^n)-f(u_{h,2}^n-v_{h,2}^n),\Phi)_\Omega+(g(u_{h,1}^n+v_{h,1}^n)-g(u_{h,2}^n+v_{h,2}^n),\Phi)_\Gamma\\
\quad+(g(u_{h,1}^n-v_{h,1}^n)-g(u_{h,2}^n-v_{h,2}^n),\Phi)_\Gamma,\\
\dfrac{1}{\tau}(r_h,\varphi)_H+\alpha(r_h,\varphi)_\Omega+(\nabla r_h,\nabla\varphi)_\Omega+\kappa(\nabla_\Gamma r_h,\nabla_\Gamma\varphi)_\Gamma\\
\quad=-(f(u_{h,1}^n+v_{h,1}^n)-f(u_{h,2}^n+v_{h,2}^n),\varphi)_\Omega\\
\quad+(f(u_{h,1}^n-v_{h,1}^n)-f(u_{h,2}^n-v_{h,2}^n),\varphi)_\Omega-(g(u_{h,1}^n+v_{h,1}^n)-g(u_{h,2}^n+v_{h,2}^n),\varphi)_\Gamma\\
\quad+(g(u_{h,1}^n-v_{h,1}^n)-g(u_{h,2}^n-v_{h,2}^n),\varphi)_\Gamma,
\end{cases}
$$

Setting $\Psi=\xi_h$ and $\Phi=z_h$ in (98) and subtracting the two resulting equations, we get:

(99)
$$
\begin{aligned}
&\tau\|\nabla\xi_h\|_\Omega^2+\tau\delta\|\nabla_\Gamma\xi_h\|_\Gamma^2+\|\nabla z_h\|_\Omega^2+\sigma\|\nabla_\Gamma z_h\|_\Gamma^2\\
=&-(f(u_{h,1}^n+v_{h,1}^n)-f(u_{h,2}^n+v_{h,2}^n),z_h)_\Omega-(f(u_{h,1}^n-v_{h,1}^n)-f(u_{h,2}^n-v_{h,2}^n),z_h)_\Omega\\
&-(g(u_{h,1}^n+v_{h,1}^n)-g(u_{h,2}^n+v_{h,2}^n),z_h)_\Gamma-(g(u_{h,1}^n-v_{h,1}^n)-g(u_{h,2}^n-v_{h,2}^n),z_h)_\Gamma.
\end{aligned}
$$
Then, choosing $\varphi=r_h$ in (98), we obtain

(100)
$$
\begin{aligned}
&\frac{1}{\tau}\|r_h\|_H^2+\alpha\|r_h\|_\Omega^2+\|\nabla r_h\|_\Omega^2+\kappa\|\nabla_\Gamma r_h\|_\Gamma^2\\
=&-(f(u_{h,1}^n+v_{h,1}^n)-f(u_{h,2}^n+v_{h,2}^n),r_h)_\Omega+(f(u_{h,1}^n-v_{h,1}^n)-f(u_{h,2}^n-v_{h,2}^n),r_h)_\Omega\\
&-(g(u_{h,1}^n+v_{h,1}^n)-g(u_{h,2}^n+v_{h,2}^n),r_h)_\Gamma+(g(u_{h,1}^n-v_{h,1}^n)-g(u_{h,2}^n-v_{h,2}^n),r_h)_\Gamma.
\end{aligned}
$$

Summing (99)-(100), we get

$$
\begin{aligned}
&\tau\|\nabla\xi_h\|_\Omega^2+\tau\delta\|\nabla_\Gamma\xi_h\|_\Gamma^2+\|\nabla z_h\|_\Omega^2+\sigma\|\nabla_\Gamma z_h\|_\Gamma^2\\
&+\frac{1}{\tau}\|r_h\|_H^2+\alpha\|r_h\|_\Omega^2+\|\nabla r_h\|_\Omega^2+\kappa\|\nabla_\Gamma r_h\|_\Gamma^2\\
&=-(f(u_{h,1}^n+v_{h,1}^n)-f(u_{h,2}^n+v_{h,2}^n),z_h+r_h)_\Omega\\
&-(f(u_{h,1}^n-v_{h,1}^n)-f(u_{h,2}^n-v_{h,2}^n),z_h+r_h)_\Omega-(g(u_{h,1}^n+v_{h,1}^n)\\
&-g(u_{h,2}^n+v_{h,2}^n),z_h-r_h)_\Gamma-(g(u_{h,1}^n-v_{h,1}^n)-g(u_{h,2}^n-v_{h,2}^n),z_h-r_h)_\Gamma
\end{aligned}\tag{101}
$$
Thanks to (34), we have

$$
(f(u_{h,1}^n+v_{h,1}^n)-f(u_{h,2}^n+v_{h,2}^n),z_h+r_h)_\Omega\geqslant-C_f\|z_h+r_h\|_\Omega^2,\tag{102}
$$

$$(f(u^n_{h,1}-v^n_{h,1})-f(u^n_{h,2}-v^n_{h,2}),z_h+r_h)_\Omega \geqslant -C_f\|z_h-r_h\|^2_\Omega, \tag{103}$$

$$(g(u^n_{h,1}+v^n_{h,1})-g(u^n_{h,2}+v^n_{h,2}),z_h-r_h)_\Gamma \geqslant -C_g\|z_h+r_h\|^2_\Gamma, \tag{104}$$

and

$$(g(u^n_{h,1}-v^n_{h,1})-g(u^n_{h,2}-v^n_{h,2}),z_h-r_h)_\Gamma \geqslant -C_g\|z_h-r_h\|^2_\Gamma. \tag{105}$$

Thus, from (101)-(105), we obtain

$$\begin{aligned}
\tau\|\nabla\xi_h\|^2_\Omega &+ \tau\delta\|\nabla_\Gamma\xi_h\|^2_\Gamma+\|\nabla z_h\|^2_\Omega+\sigma\|\nabla_\Gamma z_h\|^2_\Gamma \\
&+\frac{1}{\tau}\|r_h\|^2_H+\alpha\|r_h\|^2_\Omega+\|\nabla r_h\|^2_\Omega+\kappa\|\nabla_\Gamma r_h\|^2_\Gamma \\
&\leqslant C_f(\|z_h+r_h\|^2_\Omega+\|z_h-r_h\|^2_\Omega)+C_g(\|z_h+r_h\|^2_\Gamma+\|z_h-r_h\|^2_\Gamma) \\
&\leqslant \max(C_f,C_g)(\|z_h+r_h\|^2_H+\|z_h-r_h\|^2_H) \\
&\leqslant 2\max(C_f,C_g)(\|z_h\|^2_H+\|r_h\|^2_H).
\end{aligned} \tag{106}$$

Setting $\Psi=z_h$ in equation (98), we obtain

$$\begin{aligned}
\|z_h\|^2_H &= -\tau\delta(\nabla_\Gamma\xi_h,\nabla_\Gamma z_h)_\Gamma-\tau(\nabla\xi_h,\nabla z_h)_\Omega \\
&\leqslant \tau\delta\|\nabla_\Gamma\xi_h\|_\Gamma\|\nabla_\Gamma z_h\|_\Gamma+\tau\|\nabla\xi_h\|_\Omega\|\nabla z_h\|_\Omega,
\end{aligned}$$

which implies that

$$\begin{aligned}
2\max(C_f,C_g)\|z_h\|^2_H \leqslant \tau\delta\|\nabla_\Gamma\xi_h\|^2_\Gamma &+\tau\delta\max(C_f,C_g)^2\|\nabla_\Gamma z_h\|^2_\Gamma \\
&+\tau\|\nabla\xi_h\|^2_\Omega+\tau\max(C_f,C_g)^2\|\nabla z_h\|^2_\Omega.
\end{aligned}$$

Returning to (107), we finally obtain

$$\begin{aligned}
&\left(1-\tau\max(C_f,C_g)^2\right)\|\nabla z_h\|^2_\Omega+\left(\sigma-\tau\delta\max(C_f,C_g)^2\right)\|\nabla_\Gamma z_h\|^2_\Gamma \\
&+\|\nabla r_h\|^2_\Omega+\kappa\|\nabla_\Gamma r_h\|^2_\Gamma+\left(\frac{1}{\tau}+\alpha-2\max(C_f,C_g)\right)\|r_h\|^2_\Omega \\
&+\left(\frac{1}{\tau}-2\max(C_f,C_g)\right)\|r_h\|^2_\Gamma\leqslant 0.
\end{aligned}$$

Choosing the time step τ small enough, more exactly

$$\tau<\min\left\{\frac{1}{\max(C_f,C_g)^2},\frac{\sigma}{\delta\max(C_f,C_g)^2},\frac{1}{2\max(C_f,C_g)}\right\},$$

we obtain that $z_h=r_h=0$ and, by (98) we also get that $\xi_h=0$. □

We can also prove the following fully discrete version of Corollary 4.1:

Corollary 7.1. *If f and g are real analytic, then for all $(u^0_h,v^0_h)\in W^h\times W^h$, any sequence $(u^n_h,\mu^n_h,v^n_h)_{n\geqslant 1}$ generated by (91) and which satisfies the energy estimate (92) converges to a steady state (u_h,μ_h,v_h) as $n\to+\infty$.*

Proof. The proof mimicks exactly the one made in [16, Theorem 3.1] or [8] for the Cahn-Hilliard equation but for the reader's convenience we give here the details. The main idea of the proof is to use the Lojasiewicz gradient inequality (see also [1]). Let $(u^0_h,v^0_h)\in W^h\times W^h$. By (92), the sequence $(J(u^n_h,v^n_h))_n$ is nonincreasing and since it is bounded from below by 0, we have $J(u^n_h,v^n_h)\to J^*$ for some $J^*\geqslant 0$. We assume without loss of generality that $J^*=0$. By (5), $J(u,v)\to+\infty$ as $\|u\|_H$ and $\|v\|_H$ goes to $+\infty$, so $(u^n_h,v^n_h)_n$ is bounded. We deduce that there exist $(u^\infty_h,v^\infty_h)\in W^h\times W^h$ and a subsequence $(u^{n'}_h,v^{n'}_h)_{n'}$ of $(u^n_h,v^n_h)_n$ such that $(u^{n'}_h,v^{n'}_h)_{n'}\longrightarrow(u^\infty_h,v^\infty_h)$ as $n'\to+\infty$.

Since, $(u_h^n, 1)_H = (u_h^0, 1)$ for all $n \geqslant 1$ and eliminating μ_h^n from the scheme (91), we can write problem (91) under the following form:

$$\dot{C}\frac{W^n - W^{n-1}}{\tau} = -\nabla J^h(W^n), \tag{107}$$

where $\dot{C} = \begin{pmatrix} (\dot{A} + \delta \dot{A}_\Gamma)^{-1} & 0 \\ 0 & I_M \end{pmatrix}$ is a symmetric positive definite matrix of size $2M-1$ and

$$J^h(W) = J\left(u_1\varphi_1 + \sum_{i=2}^{M} u_i\varphi_i, \sum_{i=1}^{M} v_i\varphi_{M-1+i}\right), \ \forall W = (\dot{U}, V) \in \mathbb{R}^{M-1} \times \mathbb{R}^M.$$

This implies that

$$\lambda_1 \frac{\|W^n - W^{n-1}\|}{\tau} \leqslant \|\nabla J^h(W)\| \leqslant \lambda_{2M-1}\frac{\|W^n - W^{n-1}\|}{\tau}, \tag{108}$$

where $0 \leqslant \lambda_1 < \lambda_{2M-1} < +\infty$ are respectively the smallest and the largest eigenvalues of $\dot{C}$.

Since f and g are real analytic, the function J^h is real analytic on $\mathbb{R}^{2M-1}$ and it satisfies the Lojasiewicz inequality meaning that there exist $\varepsilon > 0$, $\xi > 0$ and $c_L > 0$ such that

$$\forall W \in \mathbb{R}^{2M-1}, \ \|W - W^\infty\| < \varepsilon \implies |J^h(W)|^{1-\xi} \leqslant c_L \|\nabla J^h(W)\|, \tag{109}$$

where here we have used that $J^h(W^\infty) = J^h(\dot{U}^\infty, V^\infty) = J(u_h^\infty, v_h^\infty) = J^* = 0$, and where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^{2M-1}$.

Let us consider n such that $\|W^n - W^\infty\| < \varepsilon$. We distinguish between the following two cases :

Case 1: $J^h(W^n) > J^h(W^{n-1})/2$. Then

$$\begin{aligned} J^h(W^{n-1})^\xi - J^h(W^n)^\xi &= \int_{J^h(W^n)}^{J^h(W^{n-1})} \xi x^{\xi-1}\,dx \\ &\overset{\text{case 1}}{\geqslant} 2^{\xi-1}\xi J^h(W^n)^{\xi-1}\left[J^h(W^{n-1}) - J^h(W^n)\right] \\ &\overset{(92)}{\geqslant} 2^{\xi-2}c_h \frac{\|W^n - W^{n-1}\|^2}{\tau J^h(W^n)^{1-\xi}}, \end{aligned}$$

where we used the equivalence of all norms on W^h and more exactly the existence of some positive constant $c_h > 0$, independent of τ such that

$$|\dot{u}^h|_{-1,h}^2 + \|v^h\|_H^2 \geqslant c_h \|\dot{u}^h\|_H^2 + \|v^h\|_H^2, \ \forall \dot{u}^h \in \dot{W}^h, \ v^h \in W^h, \tag{110}$$

From (108) and (109), we obtain

$$J^h(W^{n-1})^\xi - J^h(W^n)^\xi \geqslant \frac{2^{\xi-2}c_h}{\lambda_{2M-1}c_L}\|W^n - W^{n-1}\|. \tag{111}$$

Case 2: $J^h(W^n) \leqslant J^h(W^{n-1})/2$. Using (92) and (110), we have

$$\begin{aligned} \|W^n - W^{n-1}\| &\leqslant \left(\frac{2\tau}{c_h}\right)^{1/2}\left[J^h(W^{n-1}) - J^h(W^n)\right]^{1/2} \\ &\overset{\text{case 2}}{\leqslant} \left(1 - \frac{1}{\sqrt{2}}\right)^{1/2}\left(\frac{2\tau}{c_h}\right)^{1/2}\left[J^h(W^{n-1})^{1/2} - J^h(W^n)^{1/2}\right]. \end{aligned}$$

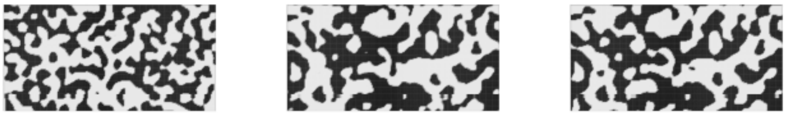

FIGURE 1. u (left), v (middle) and μ (right), after 200 iterations with $h_s = 0$

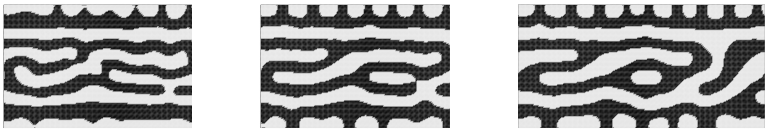

FIGURE 2. Solution u after 329 (left), 568 (middle) and 967 (right) iterations with $h_s = 0.1$.

Thus, in both cases

$$\begin{aligned}\|W^n - W^{n-1}\| \;&\leqslant \frac{2^{\xi-2}c_h}{\lambda_{2M-1}c_L}\left[J^h(W^{n-1})^\xi - J^h(W^n)^\xi\right]\\ &\quad+\left(1-\frac{1}{\sqrt{2}}\right)^{1/2}\left(\frac{2\tau}{c_h}\right)^{1/2}\left[J^h(W^{n-1})^{1/2} - J^h(W^n)^{1/2}\right]\end{aligned}$$

Arguing as in the proof of [16, Theorem 2.4], we conclude from this inequality that for N large enough

$$\begin{aligned}\sum_{n=N+1}^{+\infty}\|W^n - W^{n-1}\| \;&\leqslant\; \frac{2^{\xi-2}c_h}{\lambda_{2M-1}c_L}J^h(W^n)^\xi + \left(1-\frac{1}{\sqrt{2}}\right)^{1/2}\left(\frac{2\tau}{c_h}\right)^{1/2}J^h(W^n)^{1/2}\\ &< +\infty\end{aligned}$$

This implies that, the whole sequence $(W^n)_n$ converges to W^∞. Since μ_h^n, defined in (91) is a continuous function of (u_h^n, v_h^n) and (u_h^{n-1}, v_h^{n-1}), μ_h^n also has a limit μ_h^∞ as $n \to +\infty$. Passing to the limit in (91), we can conclude that (u_h^n, μ_h^n, v_h^n) converges to a steady state $(u_h^\infty, \mu_h^\infty, v_h^\infty)$. □

8. Numerical simulations

In this section we present some numerical simulations for the case when the system is considered in a two dimensional slab. The software used is FreeFem++[1]. The fully discrete scheme (91) requires at each time step the resolution of a nonlinear system, done here with the use of a Newton algorithm. The domain considered is the slab $[0, 80] \times [0, 40]$ and the triangulation Ω^h is obtained by dividing the domain into 200×100 identical rectangles, each rectangle being divided along the same diagonal into two triangles. The nonlinearities are given by the functions:

$$f(u) = u^3 - u \;\text{ and }\; g(u) = g_s u - h_s,\; u \in \mathbb{R},$$

and for these numerical simulations we take $g_s = 3$ and h_s either 0 or 0.1.

[1]FreeFem++ is a software freely available at http://www.freefem.org//ff++.

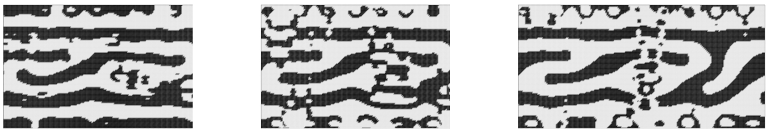

Figure 3. Solution v after 329 (left), 568 (middle) and 967 (right) iterations with $h_s = 0.1$.

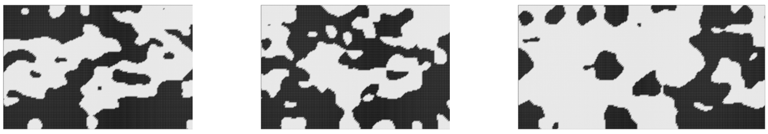

Figure 4. Solution μ after 329 (left), 568 (middle) and 967 (right) iterations with $h_s = 0.1$.

Table 1. L^2 error for u, v and μ with $h_s = 0.1$.

δt	$error(u)$	$error(v)$	$error(\mu)$
16.45	0.0878458	0.01432	0.0401734
28.4	0.0619785	0.0110204	0.0195772
37.7	0.0525436	0.0121907	0.340819
43.75	0.0389784	0.00696255	0.00997087
47.8	0.0335856	0.00581545	0.00907754

The time step considered is $\tau = 0.05$ and we take the parameters $\alpha = 4$ and $\delta = \sigma = \kappa = 1$. We consider the initial conditions for u and v to be uniformly distributed random fluctuations of amplitude ± 0.01.

In Figure 1, we represent the solution u on the left-hand side, the solution v in the middle and the solution μ on the right-hand side after 200 iterations with $h_s = 0$. The negative and the positive values of the solution are respectively represented in white and black. Since in Figure 1 we considered the parameter $h_s = 0$, we can see that, compared with the behavior of the Cahn-Hilliard model endowed with similar dynamic boundary conditions (see [7]), none of the components is preferentially attracted by the walls, which is visible on the fact that both white and black zones appear at the boundary. We also remark that away from the boundary, Figure 1 presents the same patterns. We notice that the choice of the boundary conditions significantly modifies the behavior of the solution near the boundary.

In Figures 2, 3 and 4, we represent the solutions u, v and μ respectively after 329, 568 and 967 iterations in the case $h_s = 0.1$. We see that one of the phases is preferentially attracted by the walls. We can also remark that away from the boundary, the patterns are the same for respectively u, v and μ. Also, we can see that we are close to the equilibrium for u, but that the convergence to equilibrium for $v,\ \mu$ is much slower as far as the observed patterns are considered.

In Table 1, we give the L^2-error between two consecutive iterates at several time steps for $u,\ v$ and μ with $h_s = 0.1$.

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