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A MULTISCALE PARALLEL ALGORITHM FOR PARABOLIC INTEGRO-DIFFERENTIAL EQUATION IN COMPOSITE MEDIA

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Abstract. This paper studies the multiscale algorithm for parabolic integro-differential equations in composite media combining with Laplace transformation. The new contributions reported in this study are threefold: the convergence estimates with an explicit rate for the multiscale solutions of the equations in general domains are proved, the boundary layer solution is defined and the multiscale finite element algorithm which is suitable for parallel computation is presented. Numerical simulations are then carried out to validate the theoretical results.

Key words. Parabolic integro-differential equation, the multiscale asymptotic method, Laplace transformation, composite media.

1. Introduction

In this paper, we consider the parabolic integro-differential equations with rapidly oscillating coefficients as follows:

$$(1)\quad \begin{cases} \dfrac{\partial u^\varepsilon(x,t)}{\partial t} - \dfrac{\partial}{\partial x_i}\Big(a_{ij}(\frac{x}{\varepsilon})\dfrac{\partial u^\varepsilon(x,t)}{\partial x_j}\Big) - \dfrac{\partial}{\partial x_i}\displaystyle\int_0^t \beta(t-s)a_{ij}(\frac{x}{\varepsilon})\dfrac{\partial u^\varepsilon(x,s)}{\partial x_j}\,\mathrm{d}s \\ \qquad = f(x,t), \quad (x,t)\in\Omega\times(0,T), \\ u^\varepsilon(x,t) = g(x,t), \quad (x,t)\in\partial\Omega\times(0,T), \\ u^\varepsilon(x,0) = \bar{u}_0(x), \quad x\in\Omega, \end{cases}$$

where $\Omega\subset\mathbb{R}^n$, $n\geq 1$ is a bounded convex polygonal domain or a bounded smooth domain with a periodic microstructure. Here ε is a small periodic parameter. $a_{ij}(\frac{x}{\varepsilon})$, $\beta(t)$, $f(x,t)$, $g(x,t)$ and $\bar{u}_0(x)$ are given functions. We note that here and in the sequel the Einstein summation convention is adopted on repeated indices.

Let $\xi=\varepsilon^{-1}x$ and we make the following assumptions:

(A_1) $a_{ij}(\xi)$, $i,j=1,2,\cdots,n$ are 1-periodic in ξ.

(A_2) $a_{ij}=a_{ji}$, $\gamma_0|\eta|^2\leq a_{ij}(\xi)\eta_i\eta_j\leq\gamma_1|\eta|^2$, $\gamma_0,\gamma_1>0$, $\forall(\eta_1,\eta_2,\cdots,\eta_n)\in\mathbb{R}^n$, where γ_0,γ_1 are constants independent of ε.

(A_3) $a_{ij}\in L^\infty(\Omega), \beta\in L^1(0,T), f\in L^2(0,T;L^2(\Omega)), g\in L^\infty(0,T;H^{\frac{1}{2}}(\partial\Omega))$, $\bar{u}_0\in H^1(\Omega)$.

(A_4) Let $Q=(0,1)^n$ be the reference cell and let Q' be a bounded domain in $\mathbb{R}^n$ with a $C^{1,\mu}$ boundary, $0<\mu<1$. Let $Q\subset\subset Q'$, $\bar{Q}'=\bigcup_{m=1}^L\big(\overline{D}_m\big)$ be a union of some subdomains, where for each D_m, $\partial D_m\in C^{1,\mu}$ and $D_m\cap D_k=\emptyset$ for $m\neq k$. Assume $a_{ij}\in C^\gamma(\overline{D}_m), i,j=1,2,\cdots,n$, for some $0<\gamma<1, m=1,2,\cdots,L$, where L, μ, γ are constants independent of ε (cf. [16]).

Remark 1.1. *Under assumptions (A_2)-(A_3), the well-posedness for the problem* (1) *can be established (see, e.g.,* [10, 12, 27]*).*

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The problem (1) has wide applications in heat conduction with memory effects, nuclear reactor dynamics, blow-up problems in composite material or in porous media(see,e.g.,[20, 22, 33, 35] and the references therein). For a special choice of the kernel $\beta(t) = t^{\alpha-1}/\Gamma(\alpha)$ $(0 < \alpha < 1)$, the integral term of (1) is actually the Riemann-Liouville fractional integral of the function $\frac{\partial}{\partial x_i}\Big(a_{ij}(\frac{x}{\varepsilon})\frac{\partial u^{\varepsilon}(x,t)}{\partial x_j}\Big)$ (see, e.g., [25, 28]). In this case, (1) is a linear integro-differential equation of fractional order. These kinds of equations describe anomalous diffusion processes and the wave propagation in viscoelastic materials (cf. [13, 17, 19, 26]), which have attracted considerable attention of researchers in recent years (see, e.g.,[7, 8]). As a parameter $\varepsilon > 0$ is small enough, the direct numerical simulation for the problem (1) is a hard work because it would require a very fine mesh, a very small time step and the massive storage of the numerical solutions at all time steps.

All kinds of homogenization methods are utilized to solve the partial differential equations and the integro-differential equations with rapidly oscillating coefficients (e.g., periodic, almost periodic, quasi-periodic and non-periodic). For instance, about the homogenization methods concerning linear parabolic equations, we refer to Bensoussan et al. [2] and Sanchez-Palencia [29] for periodic cases and to Colombini and Spagnolo [6] for general non-periodic cases. Zhikov et al. [37] studied parabolic operators with almost periodic coefficients and derived convergence results for the asymptotic homogenization. The homogenization method for the nonlinear parabolic equations can be found in [24, 30].

Numerous simulation results have shown that the numerical accuracy of the homogenization methods may not be satisfactory when ε is not small enough (see, e.g. [3, 32]). So we need to seek the multiscale methods to improve the numerical accuracy. Bensoussan et al. [2] investigated the first-order multiscale asymptotic method for the linear parabolic equations with oscillating periodic coefficients. For the higher-order multiscale method for the linear parabolic equations, we refer to Allegretto et al. [1]. Huang et al. [15] studied the multiscale method and obtained the strong convergence results with an explicit rate for a kind of nonlinear parabolic equations. On the other hand, various multiscale numerical approaches are also available. Hou [14] and his collaborators first presented the multiscale finite element method (MsFEM) for elliptic equations in composite materials or in porous media. Efendiev et al. [4, 5, 9] developed the multiscale finite element methods, for instance, GMsFEM, CEM-GMsFEM, NLMC and so forth. Ming and Zhang [21] proposed the heterogeneous multiscale method(HMM) for parabolic problems.

To the best of our knowledge, few results of the multiscale methods for the problem (1) have been reported. In this paper, we will study the multiscale analysis and computation for the problem (1). We notice that the classical multiscale asymptotic method fails in the study of the problem (1), due to the integro-differential term in the equation. Bensoussan et al. [2] first employed the Laplace transformation to investigate the homogenized method for an integro-differential equation of hyperbolic type. Wang et al. [32] combined the Laplace transform method with the multiscale method to study the coupled thermoelastic system in composite materials and derived the first strong convergence results with an explicit rate of the second-order multiscale solutions for the coupled thermoelastic system. In our recent work [36], we used the Laplace transformation to discuss the multiscale analysis and computation for the dual-phase-lagging equation in composite materials. Inspired by the above ideas, in this paper we use the Laplace transform method to discuss the multiscale analysis and algorithm for the problem (1). The procedure of our method is briefly described. First, we employ the Laplace transform to transfer the original problem (1) into a steady state problem. Second, we present the multiscale asymptotic expansions of the solution for the steady state problem. Finally, we acquire the multiscale approximate solutions for the original problem by the numerical inversion of Laplace transform. It should be emphasized that the problems resulting from the first step and second step could be solved simultaneously. So our method is suitable for parallel computation.

The new contributions in this study are threefold: to derive the convergence estimates with an explicit rate of the multiscale approximate solutions for the problem (1), to define the boundary layer solution and then to present a multiscale finite element method which is suitable for parallel computation.

The remainder of this paper is organized as follows. In section 2, we first convert the original problem (1) to a steady state problem by applying Laplace transformation, meanwhile the stability analysis of the weak solution for the steady state problem is established. Then we present the multiscale asymptotic expansions of the solution for the steady state problem with rapidly oscillating coefficients and derive their convergence analysis. In section 3, we obtain the multiscale approximate solutions of the original problem (1) by employing the Riemann-sum formula for the inversion of Laplace transformation. The error estimates of the multiscale approximate solutions are proved. Furthermore, the boundary layer solution is defined and the convergence results for the multiscale method in a bounded convex polygonal domain are obtained. Section 4 is devoted to the finite element computations for the related problems and the multiscale finite element algorithm is proposed for the problem (1) in section 4. Finally, numerical test studies are carried out to validate the theoretical results.

Throughout the paper, by C we shall denote a positive constant independent of ε. For each integer $m \geq 0$ and real p with $1 \leq p \leq \infty$, $W^{m,p}(D)$ denotes the standard Sobolev space of real scalar functions with their weak derivatives of order up to m in the Lebesgue space $L^p(D)$, where D is any domain in $\mathbb{R}^d$. When $p = 2$, we use $H^m(D)$ to stand for $W^{m,2}(D)$.

2. Laplace transformation and the multiscale method for the steady state problem

In this section, we first transfer the original problem (1) into a steady state problem by means of the Laplace transformation and give the stability analysis of the weak solution for the steady state problem. Then the multiscale asymptotic method for the steady state problem and their convergence results are presented.

2.1. Laplace transformation. For any $p \in \mathbb{C}$ with a positive real part, i.e. $\Re(p) > 0$, we use the Laplace transformation to the problem (1) and have

$$(2) \quad \begin{cases} -(1+\hat{\beta}(p))\dfrac{\partial}{\partial x_i}\Big(a_{ij}(\dfrac{x}{\varepsilon})\dfrac{\partial \hat{u}^\varepsilon(x,p)}{\partial x_j}\Big) + p\hat{u}^\varepsilon(x,p) = F(x,p), \quad x \in \Omega, \\ \hat{u}^\varepsilon(x,p) = \hat{g}(x,p), \quad x \in \partial\Omega, \end{cases}$$

here $F(x,p) = \hat{f}(x,p) + \bar{u}_0(x)$ and $\hat{u}^\varepsilon$, $\hat{\beta}$, $\hat{f}$ and $\hat{g}$ are the Laplace transformations of u^ε, β, f and g, respectively.

Applying Propositions 4.1 and 4.3 in [34], one can prove that there exists a unique solution $\hat{u}^\varepsilon \in H^1(\Omega)$ for the problem (2) for $\widetilde{p} \in \Sigma_\theta$, where $\widetilde{p} = \frac{p}{1+\hat{\beta}(p)}$ and $\Sigma_\theta = \{z \in \mathbb{C} : z = 0 \text{ or } |\arg z| < \pi - \theta\}$ for $\theta \in (0, \pi/2)$. For the stability estimate of the weak solution for the problem (2), we have the following lemma:

Lemma 2.1. *If assumptions* (A_2)*-*(A_4) *are satisfied, there exists a positive constant* C *depending on* θ*,* Ω*,* γ_0 *and* γ_1 *such that*

$$\int\limits_{\Omega} |\hat{u}^{\varepsilon}|^2 \,\mathrm{d}x \le C\Big\{ \frac{\|F\|^2_{L^2(\Omega)}}{|p|^2} + \big(1 + \frac{|1+\hat{\beta}(p)|}{|p|}\big)\|\hat{g}\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \Big\}, \tag{3}$$

$$\int\limits_{\Omega} |\nabla\hat{u}^{\varepsilon}|^2 \,\mathrm{d}x \le C\Big\{ \frac{\|F\|^2_{L^2(\Omega)}}{|p||1+\hat{\beta}(p)|} + \big(1 + \frac{|p|}{|1+\hat{\beta}(p)|}\big)\|\hat{g}\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \Big\}, \tag{4}$$

for all $\widetilde{p} = \frac{p}{1+\hat{\beta}(p)} \in \Sigma_\theta \setminus \{0\}$.

Proof. Suppose that $\hat{u}^{\varepsilon}$ is the solution of (2). From (2), we have

$$\int_{\Omega} a_{ij}\frac{\partial \hat{u}^{\varepsilon}}{\partial x_j}\frac{\partial \bar{v}}{\partial x_i}\,\mathrm{d}x + \widetilde{p}\int_{\Omega}\hat{u}^{\varepsilon}\bar{v}\,\mathrm{d}x = \int_{\Omega}\widetilde{F}\bar{v}\,\mathrm{d}x, \quad \forall v \in H^1_0(\Omega), \tag{5}$$

where $\widetilde{p} = \frac{p}{1+\hat{\beta}(p)}$, $\widetilde{F} = \frac{F}{1+\hat{\beta}(p)}$, and $\bar{v}(x)$ is the conjugate of a complex function $v(x)$.

Given $\hat{g} \in H^{\frac{1}{2}}(\partial\Omega)$, it follows from the trace theorem that there exists $\hat{G} \in H^1(\Omega)$ such that $\gamma(\hat{G}) = \hat{g}$ and

$$\|\hat{G}\|_{H^1(\Omega)} \le C\|\hat{g}\|_{H^{\frac{1}{2}}(\partial\Omega)}. \tag{6}$$

Let $v = \hat{u}^{\varepsilon} - \hat{G} \in H^1_0(\Omega)$ in (5) and we obtain

$$\begin{aligned} \int_{\Omega} a_{ij}\frac{\partial \hat{u}^{\varepsilon}}{\partial x_j}\frac{\partial \bar{\hat{u}}^{\varepsilon}}{\partial x_i}\,\mathrm{d}x + \widetilde{p}\int_{\Omega}\hat{u}^{\varepsilon}\bar{\hat{u}}^{\varepsilon}\,\mathrm{d}x &= \int_{\Omega} a_{ij}\frac{\partial \hat{u}^{\varepsilon}}{\partial x_j}\frac{\partial \bar{\hat{G}}}{\partial x_i}\,\mathrm{d}x + \widetilde{p}\int_{\Omega}\hat{u}^{\varepsilon}\bar{\hat{G}}\,\mathrm{d}x \\ &+ \int_{\Omega}\widetilde{F}\bar{\hat{u}}^{\varepsilon}\,\mathrm{d}x - \int_{\Omega}\widetilde{F}\bar{G}\,\mathrm{d}x. \end{aligned} \tag{7}$$

We observe that

$$(1 + 2\cot\theta)\Im(\widetilde{p}) + \Re(\widetilde{p}) \ge |\widetilde{p}|, \quad \forall \widetilde{p} \in \Sigma_\theta, \tag{8}$$

and take the imaginary part and the real part of (7), respectively. It follows from (A_2), (6) and (8) that

$$\begin{aligned} \int\limits_{\Omega}|\nabla\hat{u}^{\varepsilon}|^2\,\mathrm{d}x + |\widetilde{p}|\int\limits_{\Omega}|\hat{u}^{\varepsilon}|^2\,\mathrm{d}x &\le C\Big\{\|\nabla\hat{u}^{\varepsilon}\|_{L^2(\Omega)}\|\hat{g}\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &+ |\widetilde{p}|\|\hat{u}^{\varepsilon}\|_{L^2(\Omega)}\|\hat{g}\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|\hat{u}^{\varepsilon}\|_{L^2(\Omega)}\|\widetilde{F}\|_{L^2(\Omega)} + \|\widetilde{F}\|_{L^2(\Omega)}\|\hat{g}\|_{H^{\frac{1}{2}}(\partial\Omega)}\Big\}. \end{aligned} \tag{9}$$

For $\widetilde{p} \in \Sigma_\theta \setminus \{0\}$, using the Young's inequality, we have

$$C\|\nabla\hat{u}^{\varepsilon}\|_{L^2(\Omega)}\|\hat{g}\|_{H^{\frac{1}{2}}(\partial\Omega)} \le \frac{1}{4}\|\nabla\hat{u}^{\varepsilon}\|^2_{L^2(\Omega)} + C^2\|\hat{g}\|^2_{H^{\frac{1}{2}}(\partial\Omega)}, \tag{10}$$

$$C|\widetilde{p}|\|\hat{u}^{\varepsilon}\|_{L^2(\Omega)}\|\hat{g}\|_{H^{\frac{1}{2}}(\partial\Omega)} \le \frac{|\widetilde{p}|}{4}\|\hat{u}^{\varepsilon}\|^2_{L^2(\Omega)} + C^2|\widetilde{p}|\|\hat{g}\|^2_{H^{\frac{1}{2}}(\partial\Omega)}, \tag{11}$$

$$C\|\hat{u}^{\varepsilon}\|_{L^2(\Omega)}\|\widetilde{F}\|_{L^2(\Omega)} \le C^2\frac{1}{|\widetilde{p}|}\|\widetilde{F}\|^2_{L^2(\Omega)} + \frac{|\widetilde{p}|}{4}\|\hat{u}^{\varepsilon}\|^2_{L^2(\Omega)}, \tag{12}$$

$$C\|\widetilde{F}\|_{L^2(\Omega)}\|\hat{g}\|_{H^{\frac{1}{2}}(\partial\Omega)} \le \frac{C}{4|\widetilde{p}|}\|\widetilde{F}\|^2_{L^2(\Omega)} + C|\widetilde{p}|\|\hat{g}\|^2_{H^{\frac{1}{2}}(\partial\Omega)}. \tag{13}$$

Combined with (10)-(13), then (9) gives the following estimate,

$$\int\limits_{\Omega}|\nabla\hat{u}^{\varepsilon}|^2\,\mathrm{d}x + |\widetilde{p}|\int\limits_{\Omega}|\hat{u}^{\varepsilon}|^2\,\mathrm{d}x \le C\frac{1}{|\widetilde{p}|}\|\widetilde{F}\|^2_{L^2(\Omega)} + C(|\widetilde{p}|+1)\|\hat{g}\|^2_{H^{\frac{1}{2}}(\partial\Omega)}. \tag{14}$$

Therefore, substituting $\widetilde{p}=\frac{p}{1+\hat{\beta}(p)}$ and $\widetilde{F}=\frac{F}{1+\hat{\beta}(p)}$ into (14), we complete the proof of Lemma 2.1. □

2.2. Multiscale asymptotic method for the steady state problem. To begin, we define the formal multiscale asymptotic expansions of the solution for the steady state problem (2) as follows:

$$\hat{u}_1^\varepsilon(x,p)=\hat{u}^0(x,p)+\varepsilon N_{\alpha_1}(\xi)\frac{\partial \hat{u}^0(x,p)}{\partial x_{\alpha_1}}, \tag{15}$$

$$\hat{u}_2^\varepsilon(x,p)=\hat{u}^0(x,p)+\varepsilon N_{\alpha_1}(\xi)\frac{\partial \hat{u}^0(x,p)}{\partial x_{\alpha_1}}+\varepsilon^2 N_{\alpha_1\alpha_2}(\xi)\frac{\partial^2 \hat{u}^0(x,p)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}, \tag{16}$$

where $\xi=\varepsilon^{-1}x$, cell functions $N_{\alpha_1}(\xi)$, $N_{\alpha_1\alpha_2}(\xi)$, $\alpha_1,\alpha_2=1,2,\cdots,n$ are defined on the reference cell $Q=(0,1)^n$ and satisfy the following equations in turn:

$$\begin{cases} \dfrac{\partial}{\partial \xi_i}\Big(a_{ij}(\xi)\dfrac{\partial N_{\alpha_1}(\xi)}{\partial \xi_j}\Big)=-\dfrac{\partial}{\partial \xi_i}(a_{i\alpha_1}(\xi)), \quad \xi\in Q,\\ N_{\alpha_1}(\xi)=0,\ \xi\in\partial Q. \end{cases} \tag{17}$$

$$\begin{cases} \dfrac{\partial}{\partial \xi_i}\Big(a_{ij}(\xi)\dfrac{\partial N_{\alpha_1\alpha_2}(\xi)}{\partial \xi_j}\Big)=-\dfrac{\partial}{\partial \xi_i}\Big(a_{i\alpha_1}(\xi)N_{\alpha_2}(\xi)\Big)\\ \qquad -a_{\alpha_1 j}(\xi)\dfrac{\partial N_{\alpha_2}(\xi)}{\partial \xi_j}-a_{\alpha_1\alpha_2}(\xi)+a^*_{\alpha_1\alpha_2}, \quad \xi\in Q,\\ N_{\alpha_1\alpha_2}(\xi)=0,\ \xi\in\partial Q. \end{cases} \tag{18}$$

The homogenized equation associated with equation (2) is as follows:

$$\begin{cases} -(1+\hat{\beta}(p))\dfrac{\partial}{\partial x_i}\Big(a^*_{ij}\dfrac{\partial \hat{u}^0(x,p)}{\partial x_j}\Big)+p\hat{u}^0=F(x,p),\ x\in\Omega,\ \Re(p)>0,\\ \hat{u}^0(x,p)=\hat{g}(x,p), \quad x\in\partial\Omega. \end{cases} \tag{19}$$

Here the homogenized coefficients a^*_{ij} are calculated by

$$a^*_{ij}=\frac{1}{|Q|}\int_Q\Big[a_{ij}(\xi)+a_{ik}(\xi)\frac{\partial N_j(\xi)}{\partial \xi_k}\Big]\,\mathrm{d}\xi. \tag{20}$$

Remark 2.2. *The existence and uniqueness of the solution $\hat{u}^0\in H^1(\Omega)$ for the homogenized steady state problem* (19) *can be proved for any $\frac{p}{1+\hat{\beta}(p)}\in\Sigma_\theta\setminus\{0\}$ (see, e.g.,* [34]*).*

Remark 2.3. *As usual, $\hat{u}_1^\varepsilon(x,p)$ and $\hat{u}_2^\varepsilon(x,p)$ are called the first-order and the second-order multiscale asymptotic solutions for the steady state problem* (2)*.*

Carrying out the inverse Laplace transform gives rise to the homogenized equation of the problem (2) in a space-time domain as follows:

$$\begin{cases} \dfrac{\partial u^0(x,t)}{\partial t}-\dfrac{\partial}{\partial x_i}\Big(a^*_{ij}\dfrac{\partial u^0(x,t)}{\partial x_j}\Big)-\dfrac{\partial}{\partial x_i}\displaystyle\int_0^t a^*_{ij}\beta(t-s)\dfrac{\partial u^0(x,s)}{\partial x_j}\,\mathrm{d}s\\ \quad =f(x,t), \quad (x,t)\in\Omega\times(0,T),\\ u^0(x,t)=g(x,t), \quad (x,t)\in\partial\Omega\times(0,T),\\ u^0(x,0)=\bar{u}_0(x), \quad x\in\Omega. \end{cases} \tag{21}$$

Next we give the convergence results of the multiscale asymptotic method for the problem (2). Note that the boundary conditions of cell functions $N_{\alpha_1}(\xi), N_{\alpha_1\alpha_2}(\xi)$ defined in (17) and (18) are taken as the homogeneous Dirichlet boundary conditions instead of the periodic conditions. Generally speaking, the normal derivatives of cell functions with the homogeneous Dirichlet boundary conditions are not continuous on the boundary ∂Q of the unit cell Q. To overcome this difficulty, we need to assume the coefficient matrix (a_{ij}) satisfies the following additional conditions:

(B_1) $a_{ij} = 0,\ i \neq j,\ i,j = 1,2,\cdots,n.$

(B_2) $a_{ii},\ i = 1,2,\cdots,n$, are symmetric with respect to the middle superplanes $\Delta_1, \Delta_2, \cdots, \Delta_n$ of the reference cell $Q = (0,1)^n$.

Under assumptions $(A_1) - (A_3)$ and $(B_1) - (B_2)$, one can verify that the normal derivatives of cell functions N_{α_1}, $N_{\alpha_1\alpha_2}$, $\alpha_1,\ \alpha_2 = 1,2,\cdots,n$ are continuous on the boundary ∂Q. For more details, we refer readers to [1, 3].

Proposition 2.4. *Let $\hat{u}^\varepsilon(x,p)$ and $\hat{u}^0(x,p)$ be the solutions to the problem* (2) *and* (19)*, respectively. Let $\hat{u}_2^\varepsilon(x,p)$ be the second-order multiscale asymptotic solution defined in* (16)*. Under assumptions (A_1)-(A_4) and (B_1)-(B_2), if $\hat{u}^0 \in H^4(\Omega)$, $\hat{f} \in H^2(\Omega)$ and $\widetilde{p} = \frac{p}{1+\hat{\beta}(p)} \in \Sigma_\theta \setminus \{0\}$ for some $\theta \in (0, \pi/2)$, we have the following error estimates:*

$$\|\hat{u}^\varepsilon - \hat{u}_2^\varepsilon\|_{L^2(\Omega)} \le C\varepsilon^{\frac{1}{2}}\Big(\frac{|1+\hat{\beta}(p)|}{|p|} + 1\Big)\|\hat{u}^0\|_{H^4(\Omega)}, \tag{22}$$

$$\|\nabla(\hat{u}^\varepsilon - \hat{u}_2^\varepsilon)\|_{L^2(\Omega)} \le C\varepsilon^{\frac{1}{2}}\Big(\frac{|1+\hat{\beta}(p)|^{\frac{1}{2}}}{|p|^{\frac{1}{2}}} + \frac{(|p| + |1+\hat{\beta}(p)|)^{\frac{1}{2}}}{|1+\hat{\beta}(p)|^{\frac{1}{2}}}\Big)\|\hat{u}^0\|_{H^4(\Omega)}, \tag{23}$$

where C is a constant independent of ε and p.

Proof. From (2), (16)-(19), we obtain the following equation which holds in the sense of distributions:

$$-(1+\hat{\beta}(p))\frac{\partial}{\partial x_i}\Big(a_{ij}(\frac{x}{\varepsilon})\frac{\partial(\hat{u}^\varepsilon - \hat{u}_2^\varepsilon)}{\partial x_j}\Big) + p(\hat{u}^\varepsilon - \hat{u}_2^\varepsilon) = F_2^\varepsilon(x,\xi,p), \quad x \in \Omega, \tag{24}$$

where

$$\begin{aligned} F_2^\varepsilon(x,\xi,p) =& \varepsilon(1+\hat{\beta}(p))\Big[\frac{\partial}{\partial \xi_i}(a_{ij}N_{\alpha_1\alpha_2})\frac{\partial^3 \hat{u}^0}{\partial x_j \partial x_{\alpha_1}\partial x_{\alpha_2}} + a_{ij}N_{\alpha_1}\frac{\partial^3 \hat{u}^0}{\partial x_{\alpha_1}\partial x_i \partial x_j} \\ &+ a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial \xi_j}\frac{\partial^3 \hat{u}^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_i} + \varepsilon a_{ij}N_{\alpha_1\alpha_2}\frac{\partial^4 \hat{u}^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_i \partial x_j}\Big] \\ &- p\varepsilon\Big(N_{\alpha_1}\frac{\partial \hat{u}^0}{\partial x_{\alpha_1}} + \varepsilon N_{\alpha_1\alpha_2}\frac{\partial^2 \hat{u}^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\Big). \end{aligned}$$

Under assumptions (A_1)-(A_4), applying Theorem 1.2 of [16] yields

$$\|N_{\alpha_1}\|_{W^{1,\infty}(Q)} \le C, \quad \|N_{\alpha_1\alpha_2}\|_{W^{1,\infty}(Q)} \le C, \quad \alpha_1, \alpha_2 = 1,\cdots,n, \tag{25}$$

where C is a constant independent of ε. We thus can prove

$$\|F_2^\varepsilon\|_{L^2(\Omega)} \le C\varepsilon\Big(|1+\hat{\beta}(p)| + |p|\Big)\|\hat{u}^0\|_{H^4(\Omega)}, \tag{26}$$

where C is a constant independent of ε and p.

For $x \in \partial\Omega$, we have

$$\hat{u}^\varepsilon(x,p) - \hat{u}_2^\varepsilon(x,p) = -\varepsilon N_{\alpha_1}\frac{\partial \hat{u}^0}{\partial x_{\alpha_1}} - \varepsilon^2 N_{\alpha_1\alpha_2}\frac{\partial^2 \hat{u}^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}} \equiv \Psi_\varepsilon(x,p). \tag{27}$$

For any fixed p, following the lines of the proof of Theorem 1.2 ([23],p.124), we get

$$\|\Psi_\varepsilon\|_{H^{\frac{1}{2}}(\partial\Omega)} \le C\varepsilon^{\frac{1}{2}}\|\hat{u}^0\|_{H^4(\Omega)}, \tag{28}$$

where C is constant independent of ε and p.

Using the stability estimates (3) and (4), together with (26) and (28), the estimates (22) and (23) are proved. □

Proposition 2.5. *Let $\hat{u}^\varepsilon(x,p)$ and $\hat{u}^0(x,p)$ be the solutions of the problem* (2) *and the homogenized equation* (19)*, respectively. Suppose that $\hat{u}_1^\varepsilon(x,p)$ is the first-order multiscale asymptotic solution defined in* (15)*. Under the assumptions of Proposition 2.4, we have the following error estimates:*

$$\|\hat{u}^\varepsilon - \hat{u}_1^\varepsilon\|_{L^2(\Omega)} \le C\varepsilon^{\frac{1}{2}} \frac{\Big(|1+\hat{\beta}(p)| + |p|\Big)^{\frac{3}{2}}}{|p||1+\hat{\beta}(p)|^{\frac{1}{2}}} \|\hat{u}^0\|_{H^3(\Omega)}, \tag{29}$$

$$\|\nabla(\hat{u}^\varepsilon - \hat{u}_1^\varepsilon)\|_{L^2(\Omega)} \le C\varepsilon^{\frac{1}{2}} \frac{\Big(|1+\hat{\beta}(p)| + |p|\Big)^{\frac{3}{2}}}{|p|^{\frac{1}{2}}|1+\hat{\beta}(p)|} \|\hat{u}^0\|_{H^3(\Omega)}, \tag{30}$$

where C is a constant independent of ε and p.

Proof. On the basis of (2), (15) and (17)-(19), the following equality holds in the sense of distributions:

$$-(1+\hat{\beta}(p))\frac{\partial}{\partial x_i}\Big(a_{ij}(\frac{x}{\varepsilon})\frac{\partial(\hat{u}^\varepsilon - \hat{u}_1^\varepsilon)}{\partial x_j}\Big) + p(\hat{u}^\varepsilon - \hat{u}_1^\varepsilon) = F_1^\varepsilon(x,\xi,p), \quad x \in \Omega, \tag{31}$$

where

$$\begin{aligned} F_1^\varepsilon(x,\xi,p) =& (1+\hat{\beta}(p))\Big[(a_{ij} + a_{ik}\frac{\partial N_j}{\partial \xi_k} + \frac{\partial}{\partial \xi_k}(a_{ki}N_j) - a_{ij}^*\Big]\frac{\partial^2 \hat{u}^0}{\partial x_i \partial x_j} \\ &+ (1+\hat{\beta}(p))\varepsilon a_{ij}(\xi)N_{\alpha_1}(\xi)\frac{\partial^3 \hat{u}^0}{\partial x_i \partial x_j \partial x_{\alpha_1}} - p\varepsilon N_{\alpha_1}\frac{\partial \hat{u}^0}{\partial x_{\alpha_1}}. \end{aligned}$$

For $x \in \partial\Omega$, we have

$$\hat{u}^\varepsilon(x,p) - \hat{u}_1^\varepsilon(x,p) = -\varepsilon N_{\alpha_1}\frac{\partial \hat{u}^0}{\partial x_{\alpha_1}} \equiv \varphi_\varepsilon(x,p), \quad \text{on } \partial\Omega. \tag{32}$$

Let $w^\varepsilon(x,p) = \hat{u}^\varepsilon(x,p) - \hat{u}_1^\varepsilon(x,p)$, following the lines of the proof of Lemma 2.1, it is not difficult to verify that

$$\begin{aligned} &\int_\Omega |\nabla w^\varepsilon|^2\,\mathrm{d}x + |\widetilde{p}|\int_\Omega |w^\varepsilon|^2\,\mathrm{d}x \le C\|\nabla w^\varepsilon\|_{L^2(\Omega)}\|\varphi_\varepsilon\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\quad + C|\widetilde{p}|\|w^\varepsilon\|_{L^2(\Omega)}\|\varphi_\varepsilon\|_{H^{\frac{1}{2}}(\partial\Omega)} + \tfrac{C}{|1+\hat{\beta}(p)|}\Big|\int_\Omega F_1^\varepsilon w^\varepsilon\,\mathrm{d}x\Big| \\ &\quad + \tfrac{C}{|1+\hat{\beta}(p)|}\Big|\int_\Omega F_1^\varepsilon \varphi_\varepsilon\,\mathrm{d}x\Big|, \quad \widetilde{p} = \tfrac{p}{1+\hat{\beta}(p)}. \end{aligned} \tag{33}$$

Similar to (10) and (11), taking into account the estimate of φ_ε on $\partial\Omega$ ([23],p.127), we find

$$C\|\nabla w^\varepsilon\|_{L^2(\Omega)}\|\varphi_\varepsilon\|_{H^{\frac{1}{2}}(\partial\Omega)} \le \tfrac{1}{4}\|\nabla w^\varepsilon\|^2_{L^2(\Omega)} + C^2\varepsilon\|\hat{u}^0\|^2_{H^3(\Omega)}, \tag{34}$$

$$C|\widetilde{p}|\|w^\varepsilon\|_{L^2(\Omega)}\|\varphi_\varepsilon\|_{H^{\frac{1}{2}}(\partial\Omega)} \le \tfrac{|\widetilde{p}|}{4}\|w^\varepsilon\|^2_{L^2(\Omega)} + C^2|\widetilde{p}|\varepsilon\|\hat{u}^0\|^2_{H^3(\Omega)}. \tag{35}$$

From the definition (2) of a_{ij}^*, it is obvious that

$$\int_Q \Big(a_{ij} + a_{ik}\frac{\partial N_j}{\partial \xi_k} + \frac{\partial}{\partial \xi_k}(a_{ki}N_j) - a_{ij}^*\Big)\,\mathrm{d}\xi = 0.$$

It follows from Lemma 1.6 of [23] and the Young's inequality that

$$\begin{aligned}(36)\quad &\frac{1}{|1+\hat{\beta}(p)|}\Big|\int_\Omega F_1^\varepsilon w^\varepsilon\,\mathrm{d}x\Big| \le C\varepsilon(1+|\widetilde{p}|)\|\hat{u}^0\|_{H^3(\Omega)}\|w^\varepsilon\|_{H^1(\Omega)}\\ &\le C\varepsilon(1+|\widetilde{p}|)\|\hat{u}^0\|_{H^3(\Omega)}\Big(\|w^\varepsilon\|_{L^2(\Omega)}+\|\nabla w^\varepsilon\|_{L^2(\Omega)}\Big)\\ &\le C\varepsilon^2\frac{\big(1+|\widetilde{p}|\big)^2}{|\widetilde{p}|}\|\hat{u}^0\|^2_{H^3(\Omega)}+\frac{|\widetilde{p}|}{4}\|w^\varepsilon\|^2_{L^2(\Omega)}\\ &+C\varepsilon^2\big(1+|\widetilde{p}|\big)^2\|\hat{u}^0\|^2_{H^3(\Omega)}+\frac{1}{4}\|\nabla w^\varepsilon\|^2_{L^2(\Omega)}.\end{aligned}$$

Using Lemma 1.6 of [23] and the property of φ_ε on $\partial\Omega$ ([23],p.127) again, we deduce that

$$(37)\qquad \frac{1}{|1+\hat{\beta}(p)|}\Big|\int_\Omega F_1^\varepsilon\varphi_\varepsilon\,\mathrm{d}x\Big| \le C\varepsilon^{\frac{3}{2}}(1+|\widetilde{p}|)\|\hat{u}^0\|^2_{H^3(\Omega)}.$$

This inequality together with (34)- (36) yields

$$(38)\qquad \int_\Omega|\nabla w^\varepsilon|^2\,\mathrm{d}x+|\widetilde{p}|\int_\Omega|w^\varepsilon|^2\,\mathrm{d}x \le C\varepsilon\frac{(1+|\widetilde{p}|)^2}{|\widetilde{p}|}\|\hat{u}^0\|^2_{H^3(\Omega)}.$$

Since $\widetilde{p}=\frac{p}{1+\hat{\beta}(p)}$ in (38), we finally acquire (29) and (30), which completes the proof of Proposition 2.5. □

3. The multiscale asymptotic solutions for the original problem (1)

In this section, we use the inverse transform to $\hat{u}_1^\varepsilon(x,p)$ and $\hat{u}_2^\varepsilon(x,p)$ defined in (15) and (16) to give the multiscale asymptotic solutions for the original problem (1). To begin, we recall that the inverse Laplace transform of a function $\hat{u}(x,p)$ given by

$$(39)\qquad u(x,t)=\mathcal{L}^{-1}(\hat{u}(x,p))=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{tp}\hat{u}(x,p)\,\mathrm{d}p,$$

where $p=\gamma+i\beta$, $\gamma=\Re(p)$, $\beta=\Im(p)$, $\Re(p)>0$.

Since it is very difficult to obtain the analytical formula of (39), we compute (39) numerically by the Riemann-sum approximate formula (see, e.g., [31]):

$$(40)\qquad u(x,t)=\frac{e^{\gamma t}}{t}\Big[\frac{1}{2}\hat{u}^\varepsilon(x,\gamma)+\Re\sum_{n=1}^{\infty}\hat{u}(x,\gamma+\frac{in\pi}{t})(-1)^n\Big],\quad \gamma=\frac{4.7}{t}.$$

In the real computation, we take a truncated function for the Riemann-sum formula as follows:

$$(41)\qquad u_N(x,t)=\frac{e^{\gamma t}}{t}\Big[\frac{1}{2}\hat{u}(x,\gamma)+\Re\sum_{n=1}^{N}\hat{u}(x,\gamma+\frac{in\pi}{t})(-1)^n\Big].$$

Hence the first-order and the second-order multiscale approximate solutions for the original problem (1) are defined as

$$\begin{aligned}(42)\quad u^\varepsilon_{1,N}(x,t)=\frac{e^{\gamma t}}{t}\Big\{&\frac{1}{2}\Big[\hat{u}^0(x,p_0)+\varepsilon N_{\alpha_1}(\xi)\frac{\partial\hat{u}^0(x,p_0)}{\partial x_{\alpha_1}}\Big]\\ &+\Re\sum_{j=1}^{N}[\hat{u}^0(x,p_j)+\varepsilon N_{\alpha_1}(\xi)\frac{\partial\hat{u}^0(x,p_j)}{\partial x_{\alpha_1}}](-1)^j\Big\},\end{aligned}$$

and

$$\begin{aligned}(43)\quad u^\varepsilon_{2,N}(x,t)=\frac{e^{\gamma t}}{t}\Big\{&\frac{1}{2}\Big[\hat{u}^0(x,p_0)+\varepsilon N_{\alpha_1}(\xi)\frac{\partial\hat{u}^0(x,p_0)}{\partial x_{\alpha_1}}+\varepsilon^2N_{\alpha_1\alpha_2}(\xi)\frac{\partial^2\hat{u}^0(x,p_0)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\Big]\\ &+\Re\sum_{j=1}^{N}[\hat{u}^0(x,p_j)+\varepsilon N_{\alpha_1}(\xi)\frac{\partial\hat{u}^0(x,p_j)}{\partial x_{\alpha_1}}+\varepsilon^2N_{\alpha_1\alpha_2}(\xi)\frac{\partial^2\hat{u}^0(x,p_j)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}](-1)^j\Big\},\end{aligned}$$

where $p_j=\frac{4.7}{t}+i\frac{j\pi}{t}$, $j=0,1,\cdots,N$.

Next we derive the convergence results of the first-order and the second-order multiscale approximate solutions (42) and (43) for the problem (1). To this end, we first introduce the following lemma:

Lemma 3.1. *(see Lemma 3.2 of [36]) If $u \in W^{m,1}(0,t^*;H^s(\Omega))$ and $m, s \in \mathbb{N}$, then it holds*

$$\|\hat{u}(x,p)\|_{H^s(\Omega)} \le \frac{C}{|\Im(p)|^m}, \tag{44}$$

where C is a constant independent of p, $\Im(p) \neq 0$.

Theorem 3.2. *Let $u^\varepsilon(x,t)$ be the solution of the problem (1), and let $\hat{u}^0(x,p)$ and $u^0(x,t)$ be the solutions of the homogenized equation (19) and (21), respectively. Suppose that $u^\varepsilon_{2,N}(x,t)$ is the second-order multiscale asymptotic solution defined in (43). If $u^0 \in W^{2,1}(0,t^*;H^4(\Omega)) \cap W^{1,\infty}(0,t^*;H^1(\Omega))$, $u^\varepsilon \in W^{2,1}(0,t^*;H^1(\Omega)) \cap W^{1,\infty}(0,t^*;H^1(\Omega))$, $\beta \in W^{1,1}(0,t^*)$ and $1+\hat{\beta}(p) \neq 0$ for $\Re(p) > 0$, assumptions (A_1)-(A_4) and (B_1)-(B_2) are satisfied, then we have the following estimates:*

$$\|u^\varepsilon(x,t^*) - u^\varepsilon_{2,N}(x,t^*)\|_{L^2(\Omega)} \le C\big(\varepsilon^{\frac{1}{2}} + \frac{1}{N}\big), \tag{45}$$

$$\|\nabla(u^\varepsilon(x,t^*) - u^\varepsilon_{2,N}(x,t^*))\|_{L^2(\Omega)} \le C\big(\varepsilon^{\frac{1}{2}} + \frac{1}{N}\big), \tag{46}$$

where C is a constant independent of ε and N, but dependent on t^*.

Proof. It follows from (40) that

$$u^\varepsilon(x,t^*) = \frac{e^{\gamma t^*}}{t^*}\Big[\frac{1}{2}\hat{u}^\varepsilon(x,\gamma) + \Re\sum_{n=1}^{\infty}\hat{u}^\varepsilon(x,\gamma+\frac{in\pi}{t^*})(-1)^n\Big], \quad \gamma = \frac{4.7}{t^*}. \tag{47}$$

Combining (16) and (43), we have

$$\begin{aligned} u^\varepsilon(x,t^*) - u^\varepsilon_{2,N}(x,t^*) = \tfrac{e^{\gamma t^*}}{t^*}\Big\{\tfrac{1}{2}[\hat{u}^\varepsilon(x,\gamma) - \hat{u}^\varepsilon_2(x,\gamma)] + \Re\sum_{n=1}^{N}[\hat{u}^\varepsilon(x,\gamma+\tfrac{in\pi}{t^*}) \\ -\hat{u}^\varepsilon_2(x,\gamma+\tfrac{in\pi}{t^*})](-1)^n + \Re\sum_{n=N+1}^{\infty}\hat{u}^\varepsilon(x,\gamma+\tfrac{in\pi}{t^*})(-1)^n\Big\}, \quad \gamma = \tfrac{4.7}{t^*}. \end{aligned} \tag{48}$$

Given $u^0 \in W^{2,1}(0,t^*;H^4(\Omega))$, $u^\varepsilon \in W^{2,1}(0,t^*;H^1(\Omega))$, $\beta \in W^{1,1}(0,t^*)$, it follows from Lemma 3.1 that

$$\|\hat{u}(x,p)\|_{H^4(\Omega)} \le \tfrac{C}{|\Im(p)|^2}, \quad \|\hat{u}^\varepsilon(x,p)\|_{H^1(\Omega)} \le \tfrac{C}{|\Im(p)|^2}, \quad |\hat{\beta}(p)| \le \tfrac{C}{|\Im(p)|}. \tag{49}$$

Together with (22) of Proposition 2.4, (49) gives

$$\begin{aligned} &\|\hat{u}^\varepsilon(x,\gamma) - \hat{u}^\varepsilon_2(x,\gamma)\|_{L^2(\Omega)} \le C\varepsilon^{\frac{1}{2}}, \\ &\Big\|\Re\sum_{n=1}^{N}[\hat{u}^\varepsilon(x,\gamma+\tfrac{in\pi}{t^*}) - \hat{u}^\varepsilon_2(x,\gamma+\tfrac{in\pi}{t^*})](-1)^n\Big\|_{L^2(\Omega)} \le C\varepsilon^{\frac{1}{2}}, \\ &\Big\|\Re\sum_{n=N+1}^{\infty}\hat{u}^\varepsilon(x,\gamma+\tfrac{in\pi}{t^*})(-1)^n\Big\|_{L^2(\Omega)} \le \tfrac{C}{N}. \end{aligned} \tag{50}$$

We thus have

$$\|u^\varepsilon(x,t^*) - u^\varepsilon_{2,N}(x,t^*)\|_{L^2(\Omega)} \le C\big(\varepsilon^{\frac{1}{2}} + \frac{1}{N}\big).$$

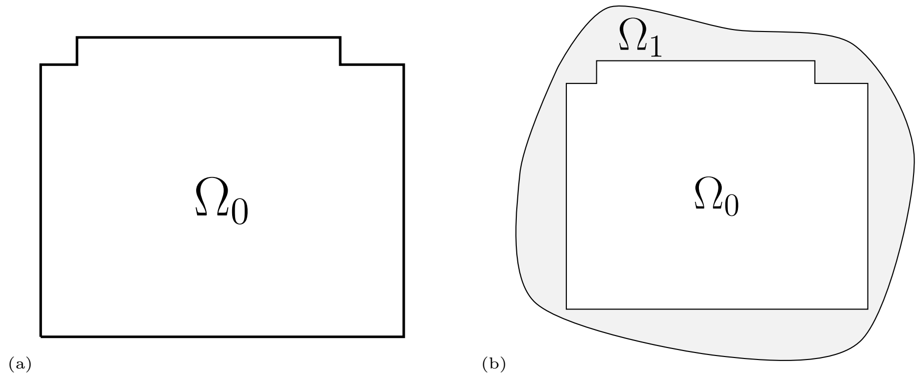

FIGURE 1. (a) Interior subdomain Ω_0 of a whole domain Ω. (b) The boundary layer Ω_1.

As $1+\hat{\beta}(p)\neq 0$ for $\Re(p)>0$, then $(1+\hat{\beta}(p))^{-1}$ is uniformly bounded for $\Re(p)>0$. It follows from (49) and (23) of Proposition 2.4 that

$$\begin{aligned}&\|\nabla(\hat{u}^\varepsilon(x,\gamma)-\hat{u}_2^\varepsilon(x,\gamma))\|_{L^2(\Omega}\leq C\varepsilon^{\frac{1}{2}},\\ &\left\|\nabla\Big\{\Re\sum_{n=1}^{N}[\hat{u}^\varepsilon(x,\gamma+\tfrac{in\pi}{t^*})-\hat{u}_2^\varepsilon(x,\gamma+\tfrac{in\pi}{t^*})](-1)^n\Big\}\right\|_{L^2(\Omega)}\leq C\varepsilon^{\frac{1}{2}},\\ &\left\|\nabla[\Re\sum_{n=N}^{\infty}\hat{u}^\varepsilon(x,\gamma+\tfrac{in\pi}{t^*})(-1)^n]\right\|_{L^2(\Omega)}\leq\tfrac{C}{N}.\end{aligned}\tag{51}$$

Therefore, using (51), we complete the proof of Theorem 3.2. Throughout the proof, C is a constant independent of ε and N, but may depend on t^*. □

Following the lines of the proof of (45) and (46), we get the following convergence results for the first-order multiscale approximate solution:

Theorem 3.3. *Let $u_{1,N}^\varepsilon(x,t)$ be the first-order multiscale approximate solution defined in (42). Under the assumptions of Theorem 3.2, if $u^0\in W^{3,1}(0,t^*;H^3(\Omega))\cap W^{1,\infty}(0,t^*;H^1(\Omega))$, $u^\varepsilon\in W^{2,1}(0,t^*;H^1(\Omega))\cap W^{1,\infty}(0,t^*;H^1(\Omega))$, we have*

$$\begin{aligned}&\|u^\varepsilon(x,t^*)-u_{1,N}^\varepsilon(x,t^*)\|_{L^2(\Omega)}\leq C\big(\varepsilon^{\frac{1}{2}}+\tfrac{1}{N}\big),\\ &\|\nabla(u^\varepsilon(x,t^*)-u_{1,N}^\varepsilon(x,t^*))\|_{L^2(\Omega)}\leq C\big(\varepsilon^{\frac{1}{2}}+\tfrac{1}{N}\big),\end{aligned}\tag{52}$$

where C is a constant independent of ε and N, but dependent on t^*.

We have to state that, in order to prove Theorems 3.2 and 3.3, one needs the assumption $\hat{u}^0\in H^{s+2}(\Omega)$, $s=1,2$, where $\hat{u}^0$ is the solution of the homogenized equation (19). However, for a general bounded convex polygonal domain, the condition $\hat{u}^0\in H^{s+2}(\Omega)$, $s=1,2$ may not be satisfied. To overcome this difficulty, we resort to the boundary layer solutions [1]. To this end, we first introduce some notations. Let $\bar{\Omega}_0=\cup_{z\in I\varepsilon}\varepsilon(z+\bar{Q})\subset\Omega$, where $I_\varepsilon=\{z\in Z^n,\varepsilon(z+\bar{Q})\subset\Omega\}$, $dist(\partial\Omega_0,\partial\Omega)>2\varepsilon$, and the boundary layer $\Omega_1=\Omega\setminus\overline{\Omega}_0$. They are illustrated in Figure 1:(a) and (b).

We define the boundary layer solutions for the problem (1) given by

$$
(53)\qquad \begin{cases} \dfrac{\partial u^{\varepsilon,b}_{s,N}}{\partial t} - \dfrac{\partial}{\partial x_i}\Big(a_{ij}(\frac{x}{\varepsilon})\dfrac{\partial u^{\varepsilon,b}_{s,N}}{\partial x_j}\Big) - \dfrac{\partial}{\partial x_i}\displaystyle\int_0^t \beta(t-s)a_{ij}(\frac{x}{\varepsilon})\dfrac{\partial u^{\varepsilon,b}_{s,N}}{\partial x_j}\,\mathrm{d}s \\ \quad = f(x,t), \quad (x,t)\in \Omega_1\times(0,T), \\ u^{\varepsilon,b}_{s,N}(x,t) = g(x,t), \quad (x,t)\in\partial\Omega\times(0,T), \\ u^{\varepsilon,b}_{s,N}(x,t) = u^{\varepsilon}_{s,N}(x,t), \quad (x,t)\in(\partial\Omega_0\cap\partial\Omega_1)\times(0,T), \\ u^{\varepsilon,b}_{s,N}(x,0) = \bar{u}_0(x), \quad x\in\Omega_1, \end{cases}
$$

where $u^{\varepsilon}_{s,N}(x,t)$, $s=1,2$ are given in (42) and (43), respectively.

Remark 3.4. *The existence and uniqueness of the boundary layer solutions can be established under assumptions* (A_2)*-*(A_4) *(see, e.g.,*[10, 12, 27]*).*

We define the multiscale asymptotic solutions for the problem (1) in the following way:

$$
(54)\qquad U^{\varepsilon}_{s,N}(x,t) = \begin{cases} u^{\varepsilon}_{s,N}(x,t), & x\in\overline{\Omega}_0, \\ u^{\varepsilon,b}_{s,N}(x,t), & x\in\Omega_1. \end{cases}
$$

Next we give the convergence theorem for the multiscale asymptotic solutions $U^{\varepsilon}_{s,N}(x,p)(s=1,2)$ for the problem (1).

Theorem 3.5. *Suppose that* $\Omega\subset R^n$*,* $n\geq 1$ *is a bounded convex polygonal domain. Let* $u^{\varepsilon}(x,t)$ *be the weak solution of the problem* (1) *and let* $U^{\varepsilon}_{s,N}(x,p)$ *be the multiscale asymptotic solutions given in* (54)*. If* $f\in L^2(0,T;L^2(\Omega))\cap H^1(0,T;H^s(\Omega))$*,* $g\in L^{\infty}(0,T;H^{\frac{1}{2}}(\partial\Omega))$*,* $\bar{u}_0\in H^{s+1}(\Omega)$*,* $s=1,2$*, where* $\Omega_0\subset\subset\Omega$*,* $\Omega_1=\Omega\setminus\overline{\Omega}_0$*,* $dist(\partial\Omega_0,\partial\Omega)>2\varepsilon$*, then it holds*

$$
(55)\qquad \|u^{\varepsilon}(x,t^*) - U^{\varepsilon}_{s,N}(x,t^*)\|_{H^1(\Omega)} \leq C\big(\varepsilon^{\frac{1}{2}} + \frac{1}{N}\big), \quad s=1,2,
$$

where C *is a constant independent of* ε *and* N*, but dependent on* t^**.*

Proof. We divide our proof into two parts. First, we derive the error estimates for the multiscale asymptotic solutions $U^{\varepsilon}_{s,N}(x,p), s=1,2$ in a subdomain Ω_0. Second, we give the error estimates of the boundary layer solutions using the trace theorem and then get the error estimates of $U^{\varepsilon}_{s,N}(x,p)$, $s=1,2$ in a whole domain Ω. To begin, we introduce the following subdomains:

$$
\Omega' = \{x\in\Omega : \text{if } \operatorname{dist}(x,\partial\Omega)\geq\varepsilon/2\}, \quad K_{\varepsilon} = \{x\in\Omega : \text{if } \operatorname{dist}(x,\partial\Omega)\leq 2\varepsilon\},
$$
$$
K'_{\varepsilon} = \{x\in\Omega : \text{if } \varepsilon\leq\operatorname{dist}(x,\partial\Omega)\leq 2\varepsilon\}.
$$

Since $\Omega_0\subset\subset\Omega'\subset\subset\Omega$, under the assumptions of this theorem, using the interior regularity of elliptic equations(see, e.g.,[18]), we deduce that $\hat{u}^0\in H^{s+2}(\Omega')$, $s=1,2$, where $\hat{u}^0$ is the solution of the homogenized equation (19). Define the cutoff function $m_{\varepsilon}(x)$ as follows:

$$
(56)\qquad \begin{aligned} & m_{\varepsilon}\in\mathcal{D}(\Omega), \quad m_{\varepsilon}=0, \text{ if dist } (x,\partial\Omega)\leq\varepsilon, \quad m_{\varepsilon}=1, \text{ if dist } (x,\partial\Omega)\geq 2\varepsilon, \\ & \varepsilon\big|\tfrac{\partial m_{\varepsilon}}{\partial x_i}\big|\leq C, \ i=1,2,\cdots,n. \end{aligned}
$$

Let

$$\hat{\theta}_1^\varepsilon(x,p) = \hat{u}^0(x,p) + \varepsilon m_\varepsilon(x) N_{\alpha_1}(\xi)\frac{\partial \hat{u}^0(x,p)}{\partial x_{\alpha_1}},$$
$$\hat{\theta}_2^\varepsilon(x,p) = \hat{u}^0(x,p) + \varepsilon m_\varepsilon(x) N_{\alpha_1}(\xi)\frac{\partial \hat{u}^0(x,p)}{\partial x_{\alpha_1}} + \varepsilon^2 m_\varepsilon(x) N_{\alpha_1\alpha_2}(\xi)\frac{\partial^2 \hat{u}^0(x,p)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}. \tag{57}$$

Here we only prove Theorem 3.5 for the case $s=1$. The case $s=2$ can be proved similarly. From (17)-(19),(56)-(57), we have

$$\begin{aligned} &\int_\Omega a_{ij}(\tfrac{x}{\varepsilon})\tfrac{\partial(\hat{u}^\varepsilon(x,p)-\hat{\theta}_1^\varepsilon(x,p))}{\partial x_j}\tfrac{\partial \bar{v}(x)}{\partial x_i}\,\mathrm{d}x + \widetilde{p}\int_\Omega(\hat{u}^\varepsilon(x,p)-\hat{\theta}_1^\varepsilon(x,p))\bar{v}(x)\,\mathrm{d}x \\ &= \hat{J}_1^\varepsilon(v), \quad \forall v\in H_0^1(\Omega), \end{aligned} \tag{58}$$

where $\bar{v}(x)$ denotes the conjugate of the complex function $v(x)$, and

$$\begin{aligned} \hat{J}_1^\varepsilon(v) = &-\int_\Omega m_\varepsilon(x)\Big[(a_{ij}(\xi)+a_{ik}(\xi)\tfrac{\partial N_j(\xi)}{\partial \xi_k}-a_{ij}^*\Big]\tfrac{\partial \hat{u}^0(x,p)}{\partial x_j}\tfrac{\partial \bar{v}(x)}{\partial x_i}\,\mathrm{d}x \\ &-\int_\Omega(1-m_\varepsilon(x))\Big[(a_{ij}(\xi)+a_{ik}(\xi)\tfrac{\partial N_j(\xi)}{\partial \xi_k}-a_{ij}^*\Big]\tfrac{\partial \hat{u}^0(x,p)}{\partial x_j}\tfrac{\partial \bar{v}(x)}{\partial x_i}\,\mathrm{d}x \\ &-\int_\Omega(m_\varepsilon(x)-1)a_{ij}(\xi)\tfrac{\partial N_{\alpha_1}(\xi)}{\partial \xi_j}\tfrac{\partial \hat{u}^0(x,p)}{\partial x_{\alpha_1}}\tfrac{\partial \bar{v}(x)}{\partial x_i}\,\mathrm{d}x \\ &-\int_\Omega \varepsilon\tfrac{\partial m_\varepsilon(x)}{\partial x_j}a_{ij}(\xi)N_{\alpha_1}(\xi)\tfrac{\partial \hat{u}^0(x,p)}{\partial x_{\alpha_1}}\tfrac{\partial \bar{v}(x)}{\partial x_i}\,\mathrm{d}x \\ &-\int_\Omega \varepsilon m_\varepsilon(x)a_{ij}(\xi)N_{\alpha_1}(\xi)\tfrac{\partial^2 \hat{u}^0(x,p)}{\partial x_{\alpha_1}\partial x_j}\tfrac{\partial \bar{v}(x)}{\partial x_i}\,\mathrm{d}x \\ &-\widetilde{p}\int_\Omega \varepsilon m_\varepsilon(x)a_{ij}(\xi)N_{\alpha_1}(\xi)\tfrac{\partial \hat{u}^0(x,p)}{\partial x_{\alpha_1}}\bar{v}(x)\,\mathrm{d}x, \quad \widetilde{p}=\tfrac{p}{1+\hat{\beta}(p)}. \end{aligned} \tag{59}$$

Using (17), (20) and the Green formula, we have

$$\begin{aligned} &-\int_\Omega m_\varepsilon(x)\Big[a_{ij}(\xi)+a_{ik}(\xi)\tfrac{\partial N_j(\xi)}{\partial \xi_k}-a_{ij}^*\Big]\tfrac{\partial \hat{u}^0(x,p)}{\partial x_j}\tfrac{\partial \bar{v}(x)}{\partial x_i}\,\mathrm{d}x \\ &=\int_\Omega \tfrac{\partial m_\varepsilon(x)}{\partial x_i}\Big[a_{ij}(\xi)+a_{ik}\tfrac{\partial N_j(\xi)}{\partial \xi_k}-a_{ij}^*\Big]\tfrac{\partial \hat{u}^0(x,p)}{\partial x_j}\bar{v}(x)\,\mathrm{d}x \\ &+\varepsilon^{-1}\int_\Omega m_\varepsilon(x)\tfrac{\partial}{\partial \xi_i}\Big[a_{ij}(\xi)+a_{ik}(\xi)\tfrac{\partial N_j(\xi)}{\partial \xi_k}-a_{ij}^*\Big]\tfrac{\partial \hat{u}^0(x,p)}{\partial x_j}\bar{v}(x)\,\mathrm{d}x \\ &+\int_\Omega m_\varepsilon(x)\Big[a_{ij}(\xi)+a_{ik}(\xi)\tfrac{\partial N_j(\xi)}{\partial \xi_k}-a_{ij}^*\Big]\tfrac{\partial^2 \hat{u}^0(x,p)}{\partial x_i\partial x_j}\bar{v}(x)\,\mathrm{d}x \\ &=\int_\Omega \tfrac{\partial m_\varepsilon(x)}{\partial x_i}\Big[a_{ij}(\xi)+a_{ik}(\xi)\tfrac{\partial N_j(\xi)}{\partial \xi_k}-a_{ij}^*\Big]\tfrac{\partial \hat{u}^0(x,p)}{\partial x_j}\bar{v}(x)\,\mathrm{d}x \\ &+\int_\Omega m_\varepsilon(x)\Big[a_{ij}(\xi)+a_{ik}(\xi)\tfrac{\partial N_j(\xi)}{\partial \xi_k}-a_{ij}^*\Big]\tfrac{\partial^2 \hat{u}^0(x,p)}{\partial x_i\partial x_j}\bar{v}(x)\,\mathrm{d}x. \end{aligned} \tag{60}$$

By setting $h(x,\xi)=\varepsilon\frac{\partial m_\varepsilon(x)}{\partial x_i}\Big(a_{ij}(\xi)+a_{ik}(\xi)\frac{\partial N_j(\xi)}{\partial \xi_k}-a_{ij}^*\Big)$, from (20) and (56), one can verify that $h(x,\xi)$ satisfies all conditions of Lemmas 1.5 and 1.6 of [23]. It follows from the above Lemmas that

$$\begin{aligned} &\Big|\varepsilon^{-1}\int_\Omega \varepsilon\tfrac{\partial m_\varepsilon(x)}{\partial x_i}\Big[(a_{ij}+a_{ik}\tfrac{\partial N_j}{\partial \xi_k}-a_{ij}^*\Big]\tfrac{\partial \hat{u}^0}{\partial x_j}\bar{v}\,\mathrm{d}x\Big| \\ &\le C\varepsilon^{-1}\varepsilon\|\hat{u}^0\|_{H^2(K'_\varepsilon)}\|v\|_{H^1(K'_\varepsilon)} \le C\varepsilon^{\frac12}\|\hat{u}^0\|_{H^3(\Omega')}\|v\|_{H_0^1(\Omega)}. \end{aligned} \tag{61}$$

Using Lemma 1.6 of [23] again, we get

$$\begin{aligned} &\Big|\int_\Omega m_\varepsilon(x)\Big[(a_{ij}+a_{ik}\tfrac{\partial N_j}{\partial \xi_k}-a_{ij}^*\Big]\tfrac{\partial^2 \hat{u}^0}{\partial x_i\partial x_j}\bar{v}\,\mathrm{d}x\Big| \\ &\le C\varepsilon\|\hat{u}^0\|_{H^3(\Omega')}\|v\|_{H_0^1(\Omega)}. \end{aligned} \tag{62}$$

From (25) and (56), applying Lemma 1.5 of [23] gives

$$\begin{aligned} &\Big|\int_\Omega(1-m_\varepsilon(x))\Big[(a_{ij}(\xi)+a_{ik}(\xi)\tfrac{\partial N_j(\xi)}{\partial \xi_k}-a_{ij}^*\Big]\tfrac{\partial \hat{u}^0(x,p)}{\partial x_j}\tfrac{\partial \bar{v}(x)}{\partial x_i}\,\mathrm{d}x\Big| \\ &=\Big|\int_{K_\varepsilon}(1-m_\varepsilon(x))\Big[(a_{ij}(\xi)+a_{ik}(\xi)\tfrac{\partial N_j(\xi)}{\partial \xi_k}-a_{ij}^*\Big]\tfrac{\partial \hat{u}^0(x,p)}{\partial x_j}\tfrac{\partial \bar{v}(x)}{\partial x_i}\,\mathrm{d}x\Big| \\ &\le C\|\hat{u}^0\|_{H^1(K_\varepsilon)}\|v\|_{H^1(K_\varepsilon)} \le C\varepsilon^{\frac12}\|\hat{u}^0\|_{H^2(\Omega)}\|v\|_{H_0^1(\Omega)}, \end{aligned} \tag{63}$$

$$\begin{aligned}&\Big|\int_\Omega (m_\varepsilon(x)-1)a_{ij}(\xi)\tfrac{\partial N_{\alpha_1}(\xi)}{\partial \xi_j}\tfrac{\partial \hat{u}^0(x,p)}{\partial x_{\alpha_1}}\tfrac{\partial \bar{v}(x)}{\partial x_i}\,\mathrm{d}x\Big|\\&\le C\|\hat{u}^0\|_{H^1(K_\varepsilon)}\|v\|_{H^1(K_\varepsilon)}\le C\varepsilon^{\frac12}\|\hat{u}^0\|_{H^2(\Omega)}\|v\|_{H_0^1(\Omega)},\end{aligned}\tag{64}$$

and

$$\begin{aligned}&\Big|\int_\Omega \varepsilon\tfrac{\partial m_\varepsilon(x)}{\partial x_j}a_{ij}N_{\alpha_1}\tfrac{\partial \hat{u}^0}{\partial x_{\alpha_1}}\tfrac{\partial \bar{v}}{\partial x_i}\,\mathrm{d}x\Big|\le C\|\hat{u}^0\|_{H^1(K'_\varepsilon)}\|v\|_{H^1(K'_\varepsilon)}\\&\le C\varepsilon^{\frac12}\|\hat{u}^0\|_{H^2(\Omega')}\|v\|_{H_0^1(\Omega)}.\end{aligned}\tag{65}$$

Thanks to (25) and (56), we have

$$\Big|\int_\Omega \varepsilon m_\varepsilon(x)a_{ij}(\xi)N_{\alpha_1}(\xi)\frac{\partial^2\hat{u}^0(x,p)}{\partial x_{\alpha_1}\partial x_j}\frac{\partial\bar{v}(x)}{\partial x_i}\,\mathrm{d}x\Big|\le C\varepsilon\|\hat{u}^0\|_{H^2(\Omega')}\|v\|_{H_0^1(\Omega)},\tag{66}$$

$$\Big|\widetilde{p}\int_\Omega \varepsilon m_\varepsilon(x)a_{ij}(\xi)N_{\alpha_1}(\xi)\frac{\partial\hat{u}^0(x,p)}{\partial x_{\alpha_1}}\bar{v}(x)\,\mathrm{d}x\Big|\le C|\widetilde{p}|\varepsilon\|\hat{u}^0\|_{H^1(\Omega')}\|v\|_{L^2(\Omega)}.\tag{67}$$

It follows from (59) to (67) that

$$|\hat{J}_1^\varepsilon(v)|\le C\Big\{\varepsilon^{\frac12}(\|\hat{u}^0\|_{H^3(\Omega')}+\|\hat{u}^0\|_{H^2(\Omega)})\|v\|_{H_0^1(\Omega)}+|\widetilde{p}|\varepsilon\|\hat{u}^0\|_{H^2(\Omega)}\|v\|_{L^2(\Omega)}\Big\},\tag{68}$$

where C is a constant independent of ε and p.

It is obvious that $\hat{u}^\varepsilon-\hat{\theta}_1^\varepsilon\in H_0^1(\Omega)$ thanks to (56) and (57). Setting $v=\hat{u}^\varepsilon-\hat{\theta}_1^\varepsilon$ in (58) and following the lines of the proof of (9), we get

$$\begin{aligned}&\int_\Omega \nabla|\hat{u}^\varepsilon-\hat{\theta}_1^\varepsilon|^2\,\mathrm{d}x+|\widetilde{p}|\int_\Omega|\hat{u}^\varepsilon-\hat{\theta}_1^\varepsilon|^2\,\mathrm{d}x\le|\hat{J}_1^\varepsilon(\hat{u}^\varepsilon-\hat{\theta}_1^\varepsilon)|\\&\le C\varepsilon^{\frac12}(\|\hat{u}^0\|_{H^3(\Omega')}+\|\hat{u}^0\|_{H^2(\Omega)})\|\hat{u}^\varepsilon-\hat{\theta}_1^\varepsilon\|_{H_0^1(\Omega)}\\&+C|\widetilde{p}|\varepsilon\|\hat{u}^0\|_{H^2(\Omega)}\|\hat{u}^\varepsilon-\hat{\theta}_1^\varepsilon\|_{L^2(\Omega)},\quad \widetilde{p}\in\Sigma_\theta.\end{aligned}\tag{69}$$

Using the Poincaré inequality and the Young's inequality, we have

$$\begin{aligned}&\int_\Omega|\nabla(\hat{u}^\varepsilon-\hat{\theta}_1^\varepsilon)|^2\,\mathrm{d}x+|\widetilde{p}|\int_\Omega|\hat{u}^\varepsilon-\hat{\theta}_1^\varepsilon|^2\,\mathrm{d}x\\&\le C(\|\hat{u}^0\|_{H^3(\Omega')}+\|\hat{u}^0\|_{H^2(\Omega)})^2(1+|\widetilde{p}|)\varepsilon.\end{aligned}\tag{70}$$

Furthermore, we use the Sobolev imbedding inequality in (70) and obtain

$$\begin{aligned}&\int_\Omega|\nabla(\hat{u}^\varepsilon-\hat{\theta}_1^\varepsilon)|^2\,\mathrm{d}x+(|\widetilde{p}|+1)\int_\Omega|\hat{u}^\varepsilon-\hat{\theta}_1^\varepsilon|^2\,\mathrm{d}x\\&\le C(\|\hat{u}^0\|_{H^3(\Omega')}+\|\hat{u}^0\|_{H^2(\Omega)})^2(1+|\widetilde{p}|)\varepsilon,\end{aligned}\tag{71}$$

consequently,

$$\|\hat{u}^\varepsilon-\hat{\theta}_1^\varepsilon\|_{L^2(\Omega)}\le C(\|\hat{u}^0\|_{H^3(\Omega')}+\|\hat{u}^0\|_{H^2(\Omega)})\varepsilon^{\frac12},\tag{72}$$

$$\|\nabla(\hat{u}^\varepsilon-\hat{\theta}_1^\varepsilon)\|_{L^2(\Omega)}\le C(\|\hat{u}^0\|_{H^3(\Omega')}+\|\hat{u}^0\|_{H^2(\Omega)})\frac{\Big(|1+\hat{\beta}(p)|+|p|\Big)^{\frac12}}{|1+\hat{\beta}(p)|^{\frac12}}\varepsilon^{\frac12}.\tag{73}$$

Given $dist(\partial\Omega_0,\partial\Omega)>2\varepsilon$, combining (56) and (57) implies

$$\|\hat{u}^\varepsilon-\hat{u}_1^\varepsilon\|_{L^2(\Omega_0)}\le C(\|\hat{u}^0\|_{H^3(\Omega')}+\|\hat{u}^0\|_{H^2(\Omega)})\varepsilon^{\frac12},\tag{74}$$

$$\|\nabla(\hat{u}^\varepsilon-\hat{u}_1^\varepsilon)\|_{L^2(\Omega_0)}\le C(\|\hat{u}^0\|_{H^3(\Omega')}+\|\hat{u}^0\|_{H^2(\Omega)})\frac{\Big(|1+\hat{\beta}(p)|+|p|\Big)^{\frac12}}{|1+\hat{\beta}(p)|^{\frac12}}\varepsilon^{\frac12}.\tag{75}$$

Under the assumptions of this theorem, repeating the process of the proof of Theorem 3.2, we get

$$\|u^\varepsilon(x,t^*) - u^\varepsilon_{1,N}(x,t^*)\|_{L^2(\Omega_0)} \le C\big(\varepsilon^{\frac{1}{2}} + \frac{1}{N}\big), \tag{76}$$

$$\|\nabla(u^\varepsilon(x,t^*) - u^\varepsilon_{1,N}(x,t^*))\|_{L^2(\Omega_0)} \le C\big(\varepsilon^{\frac{1}{2}} + \frac{1}{N}\big). \tag{77}$$

Using the trace theorem gives

$$\begin{aligned}&\|u^\varepsilon(x,t^*) - u^{\varepsilon,b}_{1,N}(x,t^*)\|_{H^1(\Omega_1)} \le C\|u^\varepsilon(x,t^*) - u^\varepsilon_{1,N}(x,t^*)\|_{H^{\frac{1}{2}}(\partial\Omega_0\cap\partial\Omega_1)}\\ &\le \|u^\varepsilon(x,t^*) - u^\varepsilon_{1,N}(x,t^*)\|_{H^1(\Omega_0)} \le C\big(\varepsilon^{\frac{1}{2}} + \frac{1}{N}\big).\end{aligned} \tag{78}$$

This estimate together with (76), (77) and the triangle inequality leads to

$$\begin{aligned}&\|u^\varepsilon(x,t^*) - U^\varepsilon_{1,N}(x,t^*)\|_{H^1(\Omega)} \le \|u^\varepsilon(x,t^*) - u^\varepsilon_{1,N}(x,t^*)\|_{H^1(\Omega_0)}\\ &+\|u^\varepsilon(x,t^*) - u^{\varepsilon,b}_{1,N}(x,t^*)\|_{H^1(\Omega_1)} \le C(\varepsilon^{\frac{1}{2}} + \tfrac{1}{N}).\end{aligned} \tag{79}$$

Similar to the proof of (79), we have

$$\|u^\varepsilon(x,t^*) - U^\varepsilon_{2,N}(x,t^*)\|_{H^1(\Omega)} \le C\big(\varepsilon^{\frac{1}{2}} + \frac{1}{N}\big). \tag{80}$$

Therefore, it completes the proof of Theorem 3.5. Throughout the proof, C is a constant independent of ε and N, but may depend on t^*. □

Remark 3.6. *It should be mentioned that, if we take $\beta(t) = t^{\alpha-1}/\Gamma(\alpha)(0 < \alpha < 1)$ in (1), then the condition $\beta \in W^{1,1}(0,t^*)$ in Theorem 3.2 is not satisfied. However, since $\hat{\beta}(p) = \frac{1}{p^\alpha}$, we only need to change (49) to $|\hat{\beta}(p)| \le \frac{1}{|\Im(p)|^\alpha}$ in the proof of Theorem 3.2. So we can also prove (45) and (46). Furthermore, we can prove that Theorems 3.2, 3.3, 3.5 are also valid for the integro-differential equation of fractional order with rapidly oscillating coefficients.*

4. The finite element computations for the related problems

4.1. The finite element method for solving the homogenized equation. In this section, we will give the finite element method for solving the homogenized equation (19) with complex coefficients. The finite element method of cell functions $N_{\alpha_1}(\xi)$, $N_{\alpha_1\alpha_2}(\xi)$, $\alpha_1, \alpha_2 = 1, \cdots, n$ can be found in [1, 3]. Without loss of generality, we assume that $\hat{g}(x,p) \equiv 0$. The variational form of problem (19) is as follows:

$$a(\hat{u}^0, v) + p(\hat{u}^0, v) = (F, v), \quad \forall v \in H^1_0(\Omega), \tag{81}$$

where $a(w,v) = \int_\Omega a^*_{ij}(1+\hat{\beta}(p))\frac{\partial w}{\partial x_j}\frac{\partial \bar{v}}{\partial x_i}\,\mathrm{d}x$ and $(w,v) = \int_\Omega w\bar{v}\,\mathrm{d}x$.

Let $S_h \subset H^1_0(\Omega)$ be the linear Lagrangian finite element space. The finite element approximation of (81) is to find $\hat{u}^0_h \in S_h$ such that:

$$a(\hat{u}^0_h, v_h) + p(\hat{u}^0_h, v_h) = (F, v_h), \quad \forall v_h \in S_h. \tag{82}$$

Let $\{\phi_j\}_{j=1}^{N_h}$ be a set of basis of the finite element space S_h and let $\hat{u}^0_h = \sum\limits_{j=1}^{N_h} \alpha_j\phi_j(x)$. Then the discrete system for (82) is as follows:

$$\sum_{j=1}^{N_h} \alpha_j a(\phi_j, \phi_k) + p\sum_{j=1}^{N_h} \alpha_j(\phi_j, \phi_k) = (F, \phi_k),\ k = 1, \cdots, N_h. \tag{83}$$

Setting $p = \lambda_1+\lambda_2 i$, $\alpha_j = a_j+ib_j$, $a^*_{kl}(1+\hat{\beta}(p)) = \gamma_{kl}+i\beta_{kl}$, $k,l = 1,2,\cdots,n$, $j = 1,2,\cdots,N_h$, $F(x,p) = f_1(x) + if_2(x)$ and $i = \sqrt{-1}$, we get

$$\begin{aligned}&\sum_{j=1}^{N_h}(a_j + b_j i)(c(\phi_j,\phi_k) + id(\phi_j,\phi_k)) + \sum_{j=1}^{N_h}(\lambda_1 + \lambda_2 i)(a_j + b_j i)(\phi_j,\phi_k)\\ &= (f_1 + f_2 i,\phi_k), \quad k = 1,2,\cdots,N_h,\end{aligned} \tag{84}$$

where

$$\begin{aligned}&c(\phi_j,\phi_k) = \textstyle\int_\Omega \gamma_{lm}\frac{\partial\phi_j}{\partial x_m}\frac{\partial\phi_k}{\partial x_l}\,\mathrm{d}x, \quad d(\phi_j,\phi_k) = \int_\Omega \beta_{lm}\frac{\partial\phi_j}{\partial x_m}\frac{\partial\phi_k}{\partial x_l}\,\mathrm{d}x,\\ &\qquad j,k = 1,2,\cdots,N_h, \ \ l,m = 1,2,\cdots,n.\end{aligned}$$

Set

$$\begin{aligned}&C = (c(\phi_j,\phi_i))_{N_h\times N_h}, \ D = (d(\phi_j,\phi_i))_{N_h\times N_h}, \ M = ((\phi_j,\phi_i))_{N_h\times N_h},\\ &\alpha = (a_1,\cdots,a_{N_h})^T, \ \beta = (b_1,\cdots,b_{N_h})^T, \ F_1 = ((f_1,\phi_1),\cdots,(f_1,\phi_{N_h}))^T,\\ &F_2 = ((f_2,\phi_1),\cdots,(f_2,\phi_{N_h}))^T.\end{aligned}$$

Hence we need to solve the following linear discrete system:

$$\begin{pmatrix} C+\lambda_1 M & -D-\lambda_2 M \\ D+\lambda_2 M & C+\lambda_1 M\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}F_1\\ F_2\end{pmatrix}.$$

4.2. Multiscale finite element method for the original problem (1). Based on the above results, we now present the multiscale finite element method for solving the parabolic integro-differential equation with rapidly oscillating coefficients. It consists of the following steps:

Step 1. Compute numerically cell functions $N_{\alpha_1}(\xi)$, $N_{\alpha_1\alpha_2}(\xi)$, $\alpha_1,\alpha_2 = 1,2,\cdots,n$ defined in (17) and (18) on the reference cell Q.

Step 2. For any fixed $t^* \in (0,T)$, solve numerically a set of homogenized equations (2) with different constant coefficients $p_j = \frac{4.7}{t^*} + i\frac{j\pi}{t^*}$, $j = 0,1,\cdots,N$ over a whole domain Ω in a coarse mesh. It should be mentioned that here they are solved in parallel.

Step 3. Let $\hat{u}^0_h$ be the finite element approximate solution of $\hat{u}^0$, where h is the mesh size in a whole domain Ω. Calculate the higher-order derivatives $\hat{u}^0_h$ by using the finite difference method. For more details, we refer to [3].

Step 4. Based on the multiscale asymptotic expansions (15) and (16) for the steady state problem (19), we get the first-order and the second-order multiscale asymptotic solutions $\hat{u}^\varepsilon_1(x,p_j)$ and $\hat{u}^\varepsilon_2(x,p_j)$, $j = 0,1,\cdots,N$, respectively.

Step 5. Using formulas (42) and (43) to compute the first-order and the second-order multiscale numerical solutions for the original problem (1).

Step 6. Solve numerically the boundary layer equation (53) in a subdomain $\Omega_1 \subset \Omega$ in a fine mesh.

Except for Step 6, the multiscale finite element scheme is given by

$$u_N^{0,h}(x,t^*) = \frac{e^{\gamma t^*}}{t^*}\Big[\frac{1}{2}\hat{u}_h^0(x,p_0) + \Re\sum_{j=1}^{N}\hat{u}_h^0(x,p_j)\Big], \tag{85}$$

$$\begin{aligned} u_{1,N}^{\varepsilon,h_0,h}(x,t^*) =& \frac{e^{\gamma t^*}}{t^*}\Big\{\frac{1}{2}[\hat{u}_h^0(x,p_0) + \varepsilon N_{\alpha_1}^{h_0}(\xi)\delta_{x_{\alpha_1}}\hat{u}_h^0(x,p_0)] \\ &+ \Re\sum_{j=1}^{N}[\hat{u}_h^0(x,p_j) + \varepsilon N_{\alpha_1}^{h_0}(\xi)\delta_{x_{\alpha_1}}\hat{u}_h^0(x,p_j)](-1)^j\Big\}, \end{aligned} \tag{86}$$

$$\begin{aligned} u_{2,N}^{\varepsilon,h_0,h}(x,t^*) =& \frac{e^{\gamma t^*}}{t^*}\Big\{\frac{1}{2}[\hat{u}_h^0(x,p_0) + \varepsilon N_{\alpha_1}^{h_0}(\xi)\delta_{x_{\alpha_1}}\hat{u}_h^0(x,p_0) \\ &+ \varepsilon^2 N_{\alpha_1\alpha_2}^{h_0}(\xi)\delta_{x_{\alpha_1}x_{\alpha_2}}\hat{u}_h^0(x,p_0)] \\ &+ \Re\sum_{j=1}^{N}[\hat{u}_h^0(x,p_j) + \varepsilon N_{\alpha_1}^{h_0}(\xi)\delta_{x_{\alpha_1}}\hat{u}_h^0(x,p_j) \\ &+ \varepsilon^2 N_{\alpha_1\alpha_2}^{h_0}(\xi)\delta_{x_{\alpha_1}x_{\alpha_2}}\hat{u}_h^0(x,p_j)](-1)^j\Big\}, \end{aligned} \tag{87}$$

where $u_N^{0,h}$, $u_{1,N}^{\varepsilon,h_0,h}$ and $u_{2,N}^{\varepsilon,h_0,h}$ are the homogenized numerical solution, the first-order and the second-order multiscale numerical solutions, respectively. $\hat{u}_h^0(x,p_j)$, $N_{\alpha_1}^{h_0}(\xi)$ and $N_{\alpha_1\alpha_2}^{h_0}(\xi)$ are respectively the finite element solutions of problems (19), (17) and (18) for different $p_j = \frac{4.7}{t^*} + i\frac{j\pi}{t^*}$, $j = 0,1,\cdots,N$. $\delta_{x_{\alpha_1}}\hat{u}_h^0(x,p_j)$ and $\delta_{x_{\alpha_1}x_{\alpha_2}}\hat{u}_h^0(x,p_j)$ denote the first-order and the second-order finite difference quotients of $\hat{u}_h^0(x,p_j)$, respectively. h_0 and h denote the mesh sizes of the reference cell Q and a whole domain Ω, respectively.

Remark 4.1. *If $N_{\alpha_1}(\xi), N_{\alpha_1\alpha_2}(\xi) \in H^2(Q)$, the standard finite element methods give the error estimates of $N_{\alpha_1}^{h_0}(\xi), N_{\alpha_1\alpha_2}^{h_0}(\xi)$ and $\hat{u}_h^0$ as follows:*

$$\|N_{\alpha_1} - N_{\alpha_1}^{h_0}\|_{L^2(Q)} \le Ch_0^2, \quad \|\nabla(N_{\alpha_1} - N_{\alpha_1}^{h_0})\|_{L^2(Q)} \le Ch_0, \tag{88}$$

$$\|N_{\alpha_1\alpha_2} - N_{\alpha_1\alpha_2}^{h_0}\|_{L^2(Q)} \le Ch_0^2, \quad \|\nabla(N_{\alpha_1\alpha_2} - N_{\alpha_1\alpha_2}^{h_0})\|_{L^2(Q)} \le Ch_0, \tag{89}$$

where C is a constant independent of h_0, $N_{\alpha_1}^{h_0}(\xi), N_{\alpha_1\alpha_2}^{h_0}(\xi) \in W_{h_0}(Q)$ and $W_{h_0}(Q)$ is the linear finite element space. Furthermore, if $\hat{u}^0 \in H^{k+1}(\Omega)$ we have

$$\|\hat{u}^0 - \hat{u}_h^0\|_{L^2(\Omega)} \le C\big(1 + |\widetilde{p}|\big)h^{k+1}\|\hat{u}^0\|_{H^{k+1}(\Omega)}, \tag{90}$$

$$\|\nabla(\hat{u}^0 - \hat{u}_h^0)\|_{L^2(\Omega)} \le C\big(1 + |\widetilde{p}|\big)h^{k}\|\hat{u}^0\|_{H^{k+1}(\Omega)}, \tag{91}$$

where C is a constant independent of h and p, $\widetilde{p} = \frac{p}{1+\hat{\beta}(p)} \in \Sigma_\theta \setminus \{0\}$, $\hat{u}^0 \in S_h(\Omega)$ and S_h is the finite element space consisting of kth-degree elements.

5. Numerical Examples

To validate the proposed multiscale algorithm, we do numerical simulations for the following case studies. We consider the 3-D integro-differential equations with rapidly oscillating coefficients as follows:

$$\left\{\begin{array}{l} \dfrac{\partial u^\varepsilon(x,t)}{\partial t} - \dfrac{\partial}{\partial x_i}\Big(a_{ij}(\frac{x}{\varepsilon})\dfrac{\partial u^\varepsilon(x,t)}{\partial x_j}\Big) - \dfrac{\partial}{\partial x_i}\displaystyle\int_0^t \beta(t-s)a_{ij}(\frac{x}{\varepsilon})\dfrac{\partial u^\varepsilon(x,s)}{\partial x_j}\,\mathrm{d}s \\ \quad = f(x,t), \quad (x,t) \in \Omega\times(0,T), \\ u^\varepsilon(x,t) = g(x,t), \quad (x,t)\in\partial\Omega\times(0,T), \\ u^\varepsilon(x,0) = \bar{u}_0(x), \quad x\in\Omega, \end{array}\right. \tag{92}$$

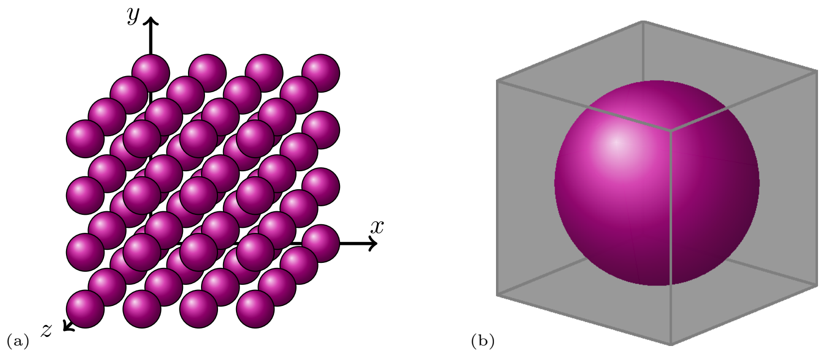

FIGURE 2. (a) A whole domain Ω. (b) The unit cell Q.

where a whole domain Ω is the union of periodic cells as shown in Figure 2:(a), the reference cell $Q = (0, 1)^3$ is as shown in Figure 2:(b). Here $\varepsilon = \frac{1}{4}$.

In the following numerical example, we choose $\beta(t) = e^{-t}$, $f(x, t) = e^{-t}$, $g(x, t) = 0$ and $\bar{u}_0(x) = 0$. Let a_{ij1} denote the value of a_{ij} in the inside sphere of Q, and let a_{ij0} denote the value of a_{ij} in the other part of Q, where δ_{ij} is a Kronecker symbol, $i, j = 1, 2, 3$.

Case 5.1 $a_{ij0} = 100\delta_{ij},\ a_{ij1} = 10\delta_{ij},\ t_* = 0.5.$

Case 5.2 $a_{ij0} = 100\delta_{ij},\ a_{ij1} = \delta_{ij},\ t_* = 0.5.$

To show the numerical accuracy of the proposed method, we need to seek the exact solution of the problem (92). However, it is extremely difficult to find out the exact solution of (92). To this end, we replace $u^\varepsilon(x, t)$ with its numerical solution in a fine mesh and at a small time step. We now implement the tetrahedron partition for Ω in a fine mesh, which is such that the discontinuities of the coefficients $a_{ij}(\frac{x}{\varepsilon})$ approximately coincide with faces of tetrahedron, and use linear Lagrangian elements to solve problem (92). We get the full-discrete system for (92) by using the backward Euler scheme and choose the time step $\Delta t = 0.001$ in this example. It should be emphasized that, in real applications, it is not necessary to solve the original problem in a very fine mesh and at a small time step.

For solving numerically cell problems (17), (18) and the homogenized equation (19), we implement respectively the tetrahedron partitions for Q and Ω, and use linear Lagrangian elements. We use the formulas (85),(86) and (87) to compute the homogenized solution and multiscale approximate solutions for the problem (92) with $N = 100$. The computational costs are listed in Table 1.

TABLE 1. Comparison of the numbers of elements and nodes.

	original equation	cell problem	homogenized equation
number of elements	129002	2006	24576
number of nodes	25044	461	4913

Figure 3-4 show the numerical results for $u^0(x, t)$, $u_1^\varepsilon(x, t)$, $u_2^\varepsilon(x, t)$ and $u^\varepsilon(x, t)$ at time $t^* = 0.5$ at the intersection $z = 0.625$ in Cases 5.1 and 5.2.

In Table 1, we can see that the proposed scheme requires much less computational cost compared to the finite element method. Moreover, since the direct finite element simulation is advanced step by step in time, the successive values of the solution have to be stored due to the existence of the integro-differential

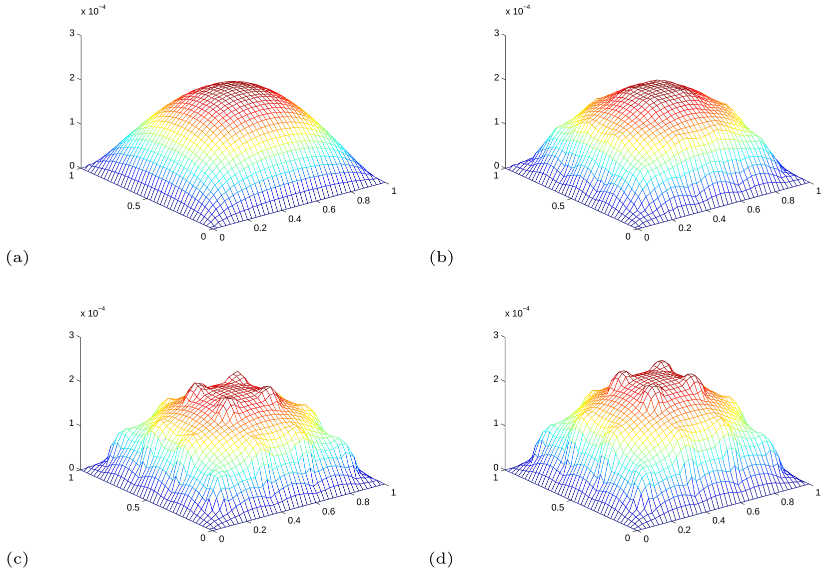

FIGURE 3. In Case 5.1: (a) the homogenized solution u^0 in a coarse mesh; (b) the first-order multiscale numerical solution u_1^ε; (c) the second-order multiscale numerical solution u_2^ε; (d) the solution u^ε in a fine mesh.

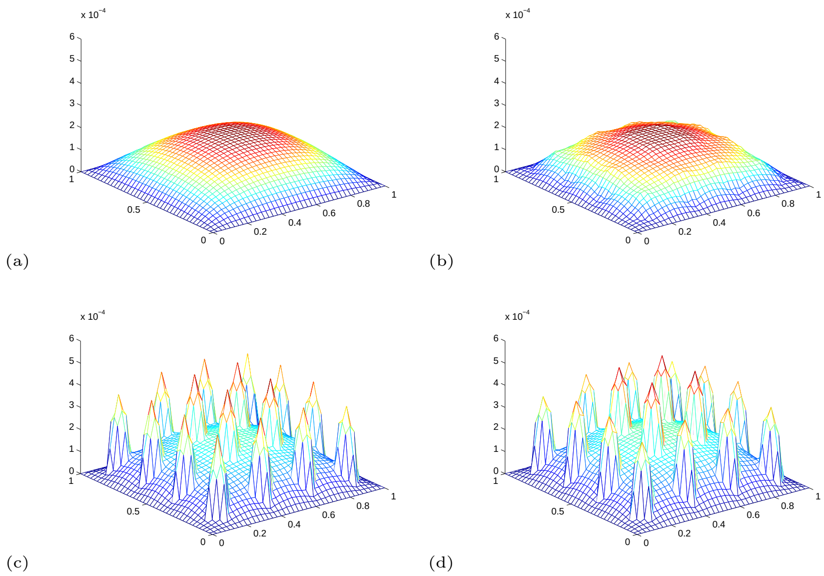

FIGURE 4. In Case 5.2: (a) the homogenized solution u^0 in a coarse mesh; (b) the first-order multiscale numerical solution u_1^ε; (c) the second-order multiscale numerical solution u_2^ε; (d) the solution u^ε in a fine mesh.

TABLE 2. Comparison of the computational errors.

	$\frac{\|e_0\|_{(0)}}{\|u^\varepsilon\|_{(0)}}$	$\frac{\|e_1\|_{(0)}}{\|u^\varepsilon\|_{(0)}}$	$\frac{\|e_2\|_{(0)}}{\|u^\varepsilon\|_{(0)}}$	$\frac{\|e_0\|_{(1)}}{\|u^\varepsilon\|_{(1)}}$	$\frac{\|e_1\|_{(1)}}{\|u^\varepsilon\|_{(1)}}$	$\frac{\|e_2\|_{(1)}}{\|u^\varepsilon\|_{(1)}}$
Case 5.1	0.12413	0.12250	0.10922	0.35390	0.29608	0.15661
Case 5.2	0.34468	0.34652	0.10612	0.94608	0.93626	0.19930

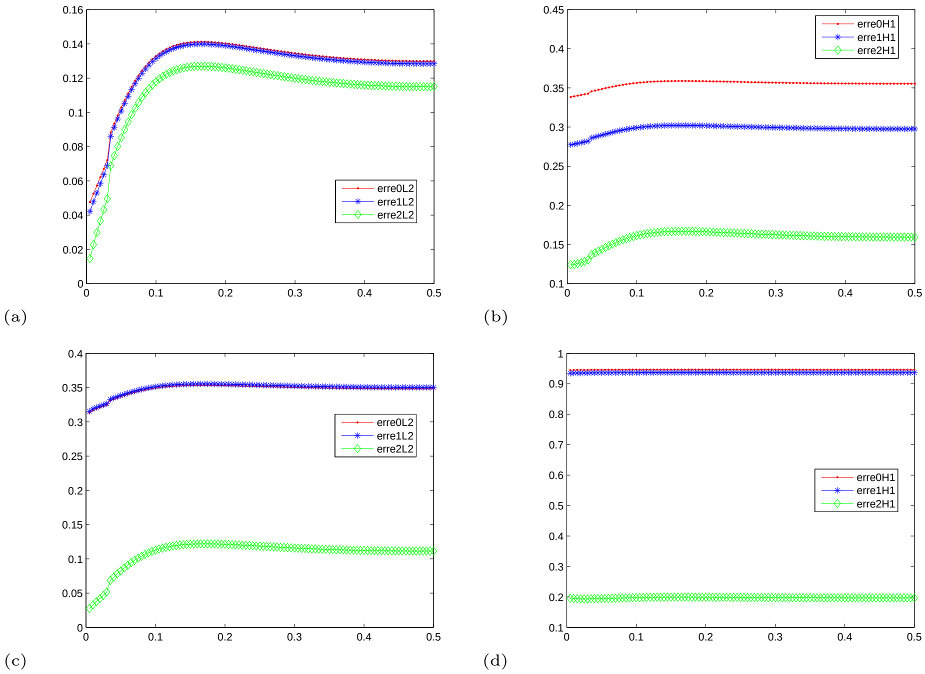

FIGURE 5. (a) Case 5.1, the evolution of relative errors in the $L^2(\Omega)$-norm; (b) Case 5.1, the evolution of relative errors in the $H^1(\Omega)$-norm.(c) Case 5.2, the evolution of relative errors in the $L^2(\Omega)$-norm; (d) Case 5.2, the evolution of relative errors in the $H^1(\Omega)$-norm.

term. However, the proposed method requires only the solution of a finite set of independent steady problems, which can be computated in parallel.

Without confusion we continue to use $u^\varepsilon(x,t)$ to denote its numerical solution in a fine mesh. For simplicity, let $u^0(x,t)$ denote the homogenized finite element solution based on (85) in a coarse mesh, and $u_1^\varepsilon(x,t), u_2^\varepsilon(x,t)$ be the first-order and the second-order multiscale finite element solutions based on (86) and (87) respectively. Set $e_0 = u^\varepsilon(x,t) - u^0(x,t)$, $e_1 = u^\varepsilon(x,t) - u_1^\varepsilon(x,t)$, $e_2 = u^\varepsilon(x,t) - u_2^\varepsilon(x,t)$. For convenience, we define:

$$\|u\|_{(0)} = \Big(\int_0^{t^*} \|u\|^2_{L^2(\Omega)}\,\mathrm{d}t\Big)^{1/2}, \qquad \|u\|_{(1)} = \Big(\int_0^{t^*} \|u\|^2_{H^1(\Omega)}\,\mathrm{d}t\Big)^{1/2}.$$

The relative errors of the homogenized numerical solution, the first-order and the second-order multiscale finite element solutions in $L^2(0,t^*;L^2(\Omega))$-norm and $L^2(0,t^*;H^1(\Omega))$-norm in Cases 5.1 and 5.2 are listed in Table 2. The numerical results reported in Table 2 demonstrate that the second-order multiscale finite element method yield more high-accuracy numerical results than the homogenization method and the first-order multiscale finite element method on the same mesh. Therefore, we conclude that the multiscale method not only save computing resources greatly but also provide satisfactory numerical accuracy for solving 3-D integro-differential equation with rapidly oscillating coefficients.

Figure 5 shows the evolution of the relative errors of approximate solutions in $L^2(\Omega)$-norm and $H^1(\Omega)$-norm with respect to time t in Cases 5.1 and 5.2, where abscissa axis is variable of time, ordinate axis is the relative error. We observe Figure 5 and conclude that our method is stable and highly accurate for solving parabolic integro-differential equations with rapidly oscillating coefficients.

References

[1] W. Allegretto, L.Q. Cao and Y.P. Lin, Multiscale asymptotic expansion for second order parabolic equations with rapidly oscillating coefficients, Discrete Contin. Dyn. Syst., 20 (2008), pp. 543-576.

[2] A. Bensoussan, J.L. Lions and G. Papanicolaou, Asymptotic Analysis of Periodic Structures, North-Holland,Amsterdan, 1978.

[3] L.Q. Cao, Multiscale asymptotic expansion and finite element methods for the mixed boundary value problems of second order elliptic equation in perforated domains, Numer. Math., 103 (2006), pp. 11-45.

[4] E.T. Chung, Y. Efendiev and W.T. Leung, Constraint energy minimizing generalized multiscale finite element method, Comput. Methods Appl. Mech. Engrg., 339 (2018), pp. 298-319.

[5] E.T. Chung, Y. Efendiev, W.T. Leung, Y. Wang and M. Vasilyeva, Non-local multi-continua upscaling for flows in heterogeneous fractured media, J. Comput. Phys., 372 (2018), pp. 22-34.

[6] F. Colombini and S. Spagnolo, Sur la convergence des solutions d'equations paraboliques, J. Math.Pures Appl., 56 (1977), pp. 263-306.

[7] E. Cuesta, Asymptotic behaviour of the solutions of fractional integro-differential equations and some time discretizations, Discrete Contin. Dyn. Syst.(Supplement), (2007), pp. 277-285.

[8] E. Cuesta, Ch. Lubich, and C. Palencia, Convolution quadrature time discretization of fractional diffusion-wave equations, Math. Comput., 75 (2006), pp. 673-696.

[9] Y. Efendiev, J. Galvis and T.Y. Hou, Generalized multiscale finite element methods, J. Comput. Phys., 251 (2013), pp. 116-135.

[10] H. Engler and S. Luckhaus, Weak solution classes for parabolic integrodifferential equations, MRC-TSR, Madison, WI, 1982.

[11] A. Friedman, Partial Differential Equations of Parbolic Type, Prentice-Hall, Inc., 1964.

[12] R.C. Grimmer and A.J. Pritchard, Analytic resolvent operators for integral equations in Banach space, J. Differential Equations, 50 (1983), pp. 234-259.

[13] A. Hanyga, Wave propagation in media with singular memory, Math. Comput. Model., 34 (2001), pp. 1399-1421.

[14] T.Y. Hou and X. Wu, A multiscale finite element method for elliptic problems in composite materials and porous media, J. Comput. Phys., 134 (1997), pp. 169-189.

[15] J.Z. Huang and L.Q. Cao, Global regularity and multiscale approach for thermalradiation heat transfer, Multiscale Model. Simul., 12 (2014), pp. 694-724.

[16] Y.Y. Li and M. Vogelius, Gradient estimates for solution to divergence form elliptic equations with discontinuous coefficients, Arch. Rational Mech. Anal., 153 (2000), pp. 91-151.

[17] F. Mainardi and P. Paradisi, Fractional diffusive waves, J. Comput. Acoustics, 9 (2001), pp. 1417-1436.

[18] W. McLean, Strongly Elliptic Systems and Boundary Integral Equations, Cambridge University Press, 2000.

[19] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, Physics Reports, 339 (2000), pp. 1-77.

[20] R. K. Miller, An integrodifferential equation for rigid heat conductors with memory, J. Math. Anal. Appl., 66 (1978), pp. 313-332.

[21] P.B. Ming and P.W. Zhang, Analysis of the heterogeneous multiscale mehtod for parabolic homogenization problems, Math. Comp., 1 (2007), pp.153-177.

[22] J.W. Nunziato, On heat conduction in materials with memory, Quart. Appl. Math., 29 (1971), pp. 187-204.

[23] O.A. Oleinik, A.S. Shamaev and G.A. Yosifian, Mathematical Problems in Elasticity and Homogenization, North-Holland, Amsterdam, 1992.
[24] A.A. Pankov, G-convergence and Homogenization of Nonlinear Parabolic Partial Differential Operators, Kluwer Academic Publishers, Dordrecht, 1997.
[25] I. Podlubny, Fractional Differential Equations. Academic Press, San Diego, 1999.
[26] J. Prüss, Evolutionary Integral Equations and Applications, Birkhäuser, Basel, 1993.
[27] C. Rennolet, Existence and boundedness of solutions of abstract nonlinear integrodifferential equations of nonconvolution type. J. Math. Anal. Appl., 70 (1979), pp. 42-60.
[28] B. Ross, Fractional Calculus, Math. Mag., 50 (1977), pp. 115-122.
[29] E. Sanchez-Palencia, Non Homogeneous Media and Vibration Theory, Lecture Notes in Physics, 127, Springer-Verlag, Berlin-New York, 1980.
[30] N. Svanstedt, G-convergence of parabolic operators, Nonlinear Anal., 36 (1999), pp. 807-842.
[31] D.Y. Tzou, Macro to Microscale Heat Transfer, The Lagging Behaviour, Taylor & Francis, Washington, 1997.
[32] X. Wang, L.Q. Cao and Y.S. Wong, Multiscale computation and convergence for coupled thermoelastic system in composite materials, Multiscale Model. Simul., 13 (2015), pp. 661-690.
[33] W.S. Wang and Q.G Hong, Two-grid economical algorithms for parabolic integro-differential equations with nonlinear memory, Appl. Numer. Math., 142 (2019), pp. 28-46.
[34] W. Wei and Z. Zhang, L^p resolvent estimates for variable coefficient elliptic systems on Lipschitz domains, Analysis and Applications, 13 (2015), pp. 591-609.
[35] E.G. Yanik and G. Fairweather, Finite element methods for parabolic and hyperbolic partial integro-differential equations, Nonlinear Anal., 12 (1988), pp. 785-809.
[36] F.M. Zhai and L.Q. Cao, A multiscale parallel algorithm for dual-phase-lagging heat conduction equation in composite materials, J. Comput. Appl. Math., 381 (2021), 113024.
[37] V.V. Zhikov, S.M. Kozlov and O.A. Oleinik, Averaging of parabolic operators with almost periodic coefficients, Mat. Sb. (N.S.), 117(159) (1982), pp. 69-85.

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