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ENERGY STABLE TIME DOMAIN FINITE ELEMENT METHODS FOR NONLINEAR MODELS IN OPTICS AND PHOTONICS

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Abstract. Novel time domain finite element methods are proposed to numerically solve the system of Maxwell's equations with a cubic nonlinearity in the spatial 3D case. The effects of linear and nonlinear electric polarization are precisely modeled in this approach. In order to achieve an energy stable discretization at the semi-discrete and the fully discrete levels, a novel technique is developed to handle the discrete nonlinearity, with spatial discretization either using edge and face elements (Nédélec-Raviart-Thomas) or discontinuous spaces and edge elements (Lee-Madsen). In particular, the proposed time discretization scheme is unconditionally stable with respect to the electromagnetic energy and is free of any Courant-Friedrichs-Lewy-type condition. Optimal error estimates are presented at semi-discrete and fully discrete levels for the nonlinear problem. The methods are robust and allow for discretization of complicated geometries and nonlinearities of spatially 3D problems that can be directly derived from the full system of nonlinear Maxwell's equations.

Key words. Finite element analysis, nonlinear Maxwell's equations, energy stability, convergence analysis, error estimate, time domain analysis.

1. Introduction

In nonlinear Optics and Photonics, the presence, behaviour and application of light or photons in nonlinear media, in which the polarization density $\mathbf{P}$ depends nonlinearly on the electric field $\mathbf{E}$, are investigated. Nonlinear effects are observed and used in many real-world applications, e.g., in lasers because of their high light intensities. Nonlinear optical phenomena, in which the optical fields are not considered to be too large, e.g., parametric and instantaneous nonlinear optical phenomena (i.e., lossless and dispersion-free materials) are often described mathematically by means of a power series expansion of the dielectric polarization density $\mathbf{P}$ with respect to the electric field $\mathbf{E}$. Frequently, the behaviour of light waves in a material is modeled by means of a third-order polarization response, that is the polarization $\mathbf{P} = \mathbf{P}(\mathbf{E})$ is a cubic polynomial in the electric field intensity $\mathbf{E}$. The fundamental concepts of nonlinear Optics can be found in details in [11, 8, 38, 2]. Due to the wide range of the nonlinear Optics applications, numerous numerical techniques for approximating the solutions of the mathematical models are employed, for instance slowly-varying envelope approximations (SVEA), beam propagation (BP), finite difference time domain (FDTD), time domain finite element (TDFE), time domain discontinuous Galerkin (TDDG) methods, – among them pseudo-spectral–, finite volume (FV) methods, and many more. The development of efficient and accurate productive numerical techniques plays an essential role for many real-world applications.

The method of SVEA is normally used for the approximate solution of nonlinear problems that are close to linear ones, and oscillations that are close to harmonic

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ones [11]. Since it is based on the assumption that the amplitude of the wave changes slowly in time and space compared to the wave period, it is thus quite restrictive for many applications. The BP method with second-order indices of refraction was employed for modeling of nonlinear optical devices exhibiting on-axis behaviour [17]. Classical FDTD methods are considered as robust numerical schemes for linear and nonlinear models in Optics and Photonics [48, 26, 50, 27, 46, 16, 21, 13, 34, 24]. However, they exhibit considerable limitations in their application, for example with regard to their applicability to complex geometries, less smooth data (e.g., due to material interfaces), etc. In particular, the spatial domain is discretized by regular, structured (quadrilateral or hexahedral) and staggered grids. The difference scheme presented in [48] served as the basis for one of the most frequently used methods for solving the linear Maxwell's equations. This scheme is of second order in time and shows a significant numerical spread over long time intervals of the simulation of wave propagation[13]. FDTD simulations for the full system of nonlinear Maxwell's equations have been presented in [27, 50]. Among other things, interacting waves of different frequencies could be treated directly [27]. The auxiliary differential equation (ADE) method along with finite difference time domain (FDTD) schemes has been originally employed for linear dispersive materials [26], and for the coupling between the polarization vector and the electric field intensity [20, 50]. This scheme was applied to second- and third-order nonlinear phenomena including spatial soliton propagation [20, 25], linear and nonlinear interface scattering [49], and pulse propagation through nonlinear wave guides [51]. In the paper [9], a higher-order discontinuous Galerkin method for spatial 1D discretization in conjunction with the ADE approach for the treatment of nonlinearity was investigated, where the energy stability of the proposed methods could be proven. In the latter respect, this work is very closely related to our results.

A lot of interesting modeling and simulation results for linear and nonlinear Lorentz dispersion with nonlinear Kerr response in case of 1D, 2D and 3D can be found in [19, 23, 10, 43, 25, 41, 36]. Among nonstandard difference methods, pseudospectral spatial domain schemes have been employed for optical carrier shock [28] and linear Lorentz dispersion with nonlinear response [47] simulation.

In this paper, based on the semi-discrete mixed finite element method [3], [4] and the fully discrete finite element method [5], [6], we provide the detailed proofs of our results to the fully time-dependent Maxwell's equations with cubic nonlinearities as a supplement to [7].

Let Ω be a simply connected domain in $\mathbb{R}^3$ with Lipschitz boundary Γ and unit outward normal $\mathbf{n}$ on Γ. Let $\mathbf{D} = \mathbf{D}(\mathbf{x}, t)$, $\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$, $\mathbf{E} = \mathbf{E}(\mathbf{x}, t)$ and $\mathbf{H} = \mathbf{H}(\mathbf{x}, t)$ represent the electric displacement field, magnetic induction, electric and magnetic field intensities, respectively, where $\mathbf{x} \in \Omega$ and the time variable t ranges in some interval $(0, T)$, $T > 0$. Given an electric current density $\mathbf{J} = \mathbf{J}(\mathbf{x}, t)$, Maxwell's curl-equations in SI units read as

$$
\begin{aligned}
\partial_t \mathbf{D} - \nabla \times \mathbf{H} &= \mathbf{J} \quad \text{in } \Omega \times (0, T), \\
\partial_t \mathbf{B} + \nabla \times \mathbf{E} &= 0 \quad \text{in } \Omega \times (0, T).
\end{aligned} \tag{1}
$$

The electric displacement $\mathbf{D}$ and magnetic induction $\mathbf{B}$ are related to the electric and magnetic fields, respectively, through the following constitutive laws:

$$
\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}(\mathbf{E}), \quad \mathbf{B} = \mu_0 \mathbf{H}. \tag{2}
$$

The vacuum permittivity and permeability are denoted by the constants $\varepsilon_0 > 0$ and $\mu_0 > 0$, respectively. Often the polarization $\mathbf{P} = \mathbf{P}(\mathbf{E})$ is approximated by a truncated Taylor series [11]. For an inhomogeneous, isotropic material, it takes the form

$$\mathbf{P}(\mathbf{E}) := \varepsilon_0\Big(\chi^{(1)}\mathbf{E} + \chi^{(3)}|\mathbf{E}|^2\mathbf{E}\Big) \tag{3}$$

with the susceptibility functions $\chi^{(i)} : \Omega \to \mathbb{R}$ for $i = 1, 3$, where $\chi^{(1)}$ takes positive and $\chi^{(3)}$ nonnegative values. For $\chi^{(3)} = 0$, we recover the standard linear Maxwell's equations. Thus the nonlinear Maxwell's problem (1)–(3) can be rewritten as

$$\partial_t\mathbf{D} - \nabla\times\mathbf{H} = \mathbf{J} \quad \text{in } \Omega\times(0,T), \tag{4}$$

$$\mu_0\partial_t\mathbf{H} + \nabla\times\mathbf{E} = 0 \quad \text{in } \Omega\times(0,T) \tag{5}$$

with

$$\partial_t\mathbf{D} = \varepsilon_0\Big((1+\chi^{(1)})\partial_t\mathbf{E} + \chi^{(3)}\partial_t\big[|\mathbf{E}|^2\,\mathbf{E}\big]\Big). \tag{6}$$

We assume to have a perfect electric conductor (PEC) boundary condition on the cylindrical surface:

$$\mathbf{n}\times\mathbf{E} = 0 \quad \text{on } \Gamma\times(0,T).$$

In addition, the following initial conditions are prescribed:

$$\mathbf{E}(\mathbf{x},0) = \mathbf{E}_0(\mathbf{x}) \quad \text{and} \quad \mathbf{H}(\mathbf{x},0) = \mathbf{H}_0(\mathbf{x}) \quad \text{for all } \mathbf{x}\in\Omega, \tag{7}$$

where $\mathbf{E}_0, \mathbf{H}_0 : \Omega \to \mathbb{R}^3$ are given, and $\mathbf{H}_0$ satisfies

$$\nabla\cdot(\mu_0\mathbf{H}_0) = 0 \quad \text{in } \Omega, \qquad \mathbf{H}_0\cdot\mathbf{n} = 0 \quad \text{on } \Gamma. \tag{8}$$

The divergence-free condition in (8) together with (5) implies that

$$\nabla\cdot(\mu_0\mathbf{H}) = 0 \quad \text{in } \Omega\times(0,T). \tag{9}$$

Regarding the notation for function spaces, norms etc., we will follow [6, Section II] closely, more detailed explanations can be found in [1], [18], [12], and [33], only to mention a small selection of literature.

The paper is structured as follows. Sections 2 and 3 describe the weak formulations and aspects of the spatial discretization of the nonlinear problem including considerations of an energy functional adapted to the nonlinear situation. An error estimate at the semi-discrete level is demonstrated in Section 4. The time discretization, energy and error estimates at the fully discrete level are discussed in Section 5 and Section 6, where in both sections we only discuss the case of the so-called Lee-Madsen formulation in detail for reasons of space. Error estimates for the Nédélec-Raviart-Thomas formulation of the linear problem ($\chi^{(3)} = 0$) can be found in [6]. Also for reasons of space we do without numerical results and refer to [7] for some descriptive computational examples.

2. Weak Formulation of the Nonlinear Electromagnetic Problem

We multiply both eqs. (4), (6) by test functions $\mathbf{\Psi}\in\mathbf{L}^2(\Omega)$ and integrate over Ω. Similarly we multiply eq. (5) by a test function $\mathbf{\Phi}\in\mathbf{H}(\mathrm{curl},\Omega)$, integrate the result over Ω and integrate by parts the second term of eq. (5). We look for a weak solution $(\mathbf{D},\mathbf{E},\mathbf{H}) \in C^1(0,T,\mathbf{L}^2(\Omega)) \times \big(C^1(0,T,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega)) \cap C(0,T,\mathbf{H}_0(\mathrm{curl},\Omega))\big) \times \big(C^1(0,T,\mathbf{L}^2(\Omega)) \cap C(0,T,\mathbf{H}(\mathrm{curl},\Omega))\big)$ of (4)–(6) such that

(10)

$$(\partial_t\mathbf{D},\mathbf{\Psi}) - (\nabla\times\mathbf{H},\mathbf{\Psi}) = (\mathbf{J},\mathbf{\Psi}) \qquad \forall\,\mathbf{\Psi}\in\mathbf{L}^2(\Omega),$$

(11)
$$(\partial_t \mathbf{D}, \mathbf{\Psi}) = (\varepsilon_0(1+\chi^{(1)})\partial_t \mathbf{E}, \mathbf{\Psi}) + \big(\varepsilon_0\chi^{(3)}\partial_t\big(|\mathbf{E}|^2\,\mathbf{E}\big), \mathbf{\Psi}\big) \qquad \forall\, \mathbf{\Psi} \in \mathbf{L}^2(\Omega),$$
(12)
$$(\mu_0 \partial_t \mathbf{H}, \mathbf{\Phi}) + (\mathbf{E}, \nabla\times\mathbf{\Phi}) = 0 \qquad \forall\, \mathbf{\Phi} \in \mathbf{H}(\mathrm{curl}, \Omega).$$

Alternatively, the test functions can be chosen as $\mathbf{\Psi} \in \mathbf{H}_0(\mathrm{curl}, \Omega)$, $\mathbf{\Phi} \in \mathbf{H}(\mathrm{div}, \Omega)$, and integration by parts in eq. (4) leads to a weak solution $(\mathbf{D}, \mathbf{E}, \mathbf{H}) \in C^1(0, T, \mathbf{H}_0(\mathrm{curl}, \Omega)) \times \big(C^1(0, T, \mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega)) \cap C(0, T, \mathbf{H}_0(\mathrm{curl}, \Omega))\big) \times \big(C^1(0, T, \mathbf{L}^2(\Omega)) \cap C(0, T, \mathbf{H}(\mathrm{div}, \Omega))\big)$ of (4)–(6) such that

(13)
$$(\partial_t \mathbf{D}, \mathbf{\Psi}) - (\mathbf{H}, \nabla\times\mathbf{\Psi}) = (\mathbf{J}, \mathbf{\Psi}) \qquad \forall\, \mathbf{\Psi} \in \mathbf{H}_0(\mathrm{curl}, \Omega),$$
(14)
$$(\partial_t \mathbf{D}, \mathbf{\Psi}) = (\varepsilon_0(1+\chi^{(1)})\partial_t \mathbf{E}, \mathbf{\Psi}) + \big(\varepsilon_0\chi^{(3)}\partial_t\big(|\mathbf{E}|^2\,\mathbf{E}\big), \mathbf{\Psi}\big) \qquad \forall\, \mathbf{\Psi} \in \mathbf{H}_0(\mathrm{curl}, \Omega),$$
(15)
$$(\mu_0 \partial_t \mathbf{H}, \mathbf{\Phi}) + (\nabla\times\mathbf{E}, \mathbf{\Phi}) = 0 \qquad \forall\, \mathbf{\Phi} \in \mathbf{H}(\mathrm{div}, \Omega).$$

In both cases, the initial conditions (7)–(8) are to be satisfied.

Remark 1. *(i) The first formulation* (10)*–*(12) *is often referred to as* Lee-Madsen *formulation* [30]*, while the second* (13)*–*(15) *is called* Nédélec-Raviart-Thomas *formulation* [37]*,* [42]*.*

(ii) As a consequence of the embedding (as sets)
$$[C_0^\infty(\Omega)]^3 \subset [C^\infty(\Omega) \cap \mathbf{H}_1(\Omega)]^3 \subset [\mathbf{H}_1(\Omega)]^3 \subset \mathbf{H}(\mathrm{div}, \Omega) \subset \mathbf{L}^2(\Omega),$$
and of the fact that $C_0^\infty(\Omega)$ *is dense in* $\mathbf{L}^2(\Omega)$ [1]*, we see that* $\mathbf{H}(\mathrm{div}, \Omega)$ *is a dense subset of* $\mathbf{L}^2(\Omega)$*. Therefore the test space* $\mathbf{H}(\mathrm{div}, \Omega)$ *in* (15) *can be replaced by* $\mathbf{L}^2(\Omega)$*.*

(iii) In case if μ_0 *is a highly variable function* $\mu = \mu(\mathbf{x})$*, it is more appropriate to use a* $(\mathbf{D}, \mathbf{E}, \mathbf{B})$ *formulation instead of* $(\mathbf{D}, \mathbf{E}, \mathbf{H})$ [31]*. Then problem* (13)*–*(15) *is substituted by*
$$(\partial_t \mathbf{D}, \mathbf{\Psi}) - (\mu^{-1}\mathbf{B}, \nabla\times\mathbf{\Psi}) = (\mathbf{J}, \mathbf{\Psi}) \qquad \forall\, \mathbf{\Psi} \in \mathbf{H}_0(\mathrm{curl}, \Omega),$$
$$(\partial_t \mathbf{D}, \mathbf{\Psi}) = (\varepsilon_0(1+\chi^{(1)})\partial_t \mathbf{E}, \mathbf{\Psi}) + \big(\varepsilon_0\chi^{(3)}\partial_t\big(|\mathbf{E}|^2\,\mathbf{E}\big), \mathbf{\Psi}\big) \qquad \forall\, \mathbf{\Psi} \in \mathbf{H}_0(\mathrm{curl}, \Omega),$$
$$(\partial_t \mathbf{B}, \mathbf{\Phi}) + (\nabla\times\mathbf{E}, \mathbf{\Phi}) = 0 \qquad \forall\, \mathbf{\Phi} \in \mathbf{L}^2(\Omega).$$

In what follows we assume that a unique weak solution to the problem (10)–(12) and (13)–(15), resp., in the above sense exists. To the authors' knowledge there are only a very few theoretical results about existence, uniqueness and regularity of solution(s) to problems of the type (4)–(8) with nontrivial susceptibility coefficient $\chi^{(3)}$. We mention [39], [15], [29], [40], the PhD theses [35] as well as [44], and the paper [45] (including the references cited therein), where in particular [35, Proposition 4.8] and [44, Theorem 7.23] (or [45, Theorem 7.23]) are closely related to our situation (but not completely matching).

The Energy of the Nonlinear Problem at the Continuous Level. The energy of nonlinear electromagnetic systems (10)–(12) and (13)–(15) at time $t \in [0, T]$ can be defined by
$$\mathcal{E}(t) := \|\mathbf{E}(t)\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|\mathbf{E}^2(t)\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}(t)\|^2_{\mu_0}.$$
Next we will prove a stability result of the solution of (13)–(15) w.r.t. this functional.

Theorem 1. *Let* $\mathbf{J} \in C(0,T,\mathbf{L}^2_{(\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}}(\Omega))$.
If $(\mathbf{E},\mathbf{H}) \in \big(C^1(0,T,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega)) \cap C(0,T,\mathbf{H}_0(\mathrm{curl},\Omega))\big) \times \big(C^1(0,T,\mathbf{L}^2(\Omega)) \cap C(0,T,\mathbf{H}(\mathrm{div},\Omega))\big)$ *is the weak solution of the system* (13)–(15)*, then its nonlinear electromagnetic energy at any time* $t \in [0,T]$ *satisfies*

$$\mathcal{E}^{\frac{1}{2}}(t) \le \Big(\|\mathbf{E}_0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|\mathbf{E}_0^2\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}_0\|^2_{\mu_0}\Big)^{\frac{1}{2}} + \int_0^t \|\mathbf{J}(s)\|_{(\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}}\, ds.$$

Remark 2. *An analogous result is valid for the energy of* (10)–(12)*.*

Proof. Taking $\mathbf{\Psi} := \mathbf{E}$ in (13) and (14), we have

$$(\partial_t \mathbf{D},\mathbf{E}) - (\mathbf{H},\nabla\times\mathbf{E}) = (\mathbf{J},\mathbf{E}), \tag{16}$$

$$(\partial_t \mathbf{D},\mathbf{E}) = (\varepsilon_0(1+\chi^{(1)})\partial_t\mathbf{E},\mathbf{E}) + \big(\varepsilon_0\chi^{(3)}\partial_t\big(|\mathbf{E}|^2\,\mathbf{E}\big),\mathbf{E}\big). \tag{17}$$

The choice $\mathbf{\Phi} := \mathbf{H}$ in (15) leads to

$$(\mu_0\partial_t\mathbf{H},\mathbf{H}) + (\nabla\times\mathbf{E},\mathbf{H}) = 0. \tag{18}$$

Substituting (17) into (16) and adding the result to (18), we obtain

$$(\varepsilon_0(1+\chi^{(1)})\partial_t\mathbf{E},\mathbf{E}) + \big(\varepsilon_0\chi^{(3)}\partial_t\big(|\mathbf{E}|^2\,\mathbf{E}\big),\mathbf{E}\big) + (\mu_0\partial_t\mathbf{H},\mathbf{H}) = (\mathbf{J},\mathbf{E}).$$

This can be written as

$$\frac{1}{2}\frac{d}{dt}\,\mathcal{E}(t) = (\mathbf{J},\mathbf{E}). \tag{19}$$

The right-hand side of eq. (19) is estimated by means of the Cauchy-Schwarz inequality:

$$(\mathbf{J},\mathbf{E}) = ((\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}\mathbf{J},\varepsilon_0(1+\chi^{(1)})\mathbf{E}) \le \|\mathbf{J}\|_{(\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}}\,\|\mathbf{E}\|_{\varepsilon_0(1+\chi^{(1)})}.$$

Then we get from eq. (19)

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}(t) \le |\mathbf{J}\|_{(\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}}\Big(\|\mathbf{E}\|^2_{\varepsilon_0(1+\chi^{(1)})}\Big)^{\frac{1}{2}} \le \|\mathbf{J}\|_{(\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}}\,\mathcal{E}^{\frac{1}{2}}(t).$$

This implies, by the chain rule,

$$\frac{d}{dt}\,\mathcal{E}^{\frac{1}{2}}(t) = \frac{1}{2}\,\mathcal{E}^{-\frac{1}{2}}(t)\,\frac{d}{dt}\,\mathcal{E}(t) \le \|\mathbf{J}\|_{(\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}}.$$

Integrating this inequality from 0 to t, we obtain

$$\mathcal{E}^{\frac{1}{2}}(t) - \mathcal{E}^{\frac{1}{2}}(0) \le \int_0^t \|\mathbf{J}(s)\|_{(\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}}\, ds,$$

and the use of the initial conditions (7) completes the proof. □

3. Semi-Discretization in Space

Let $\mathbf{W}_h \subset \mathbf{L}^2(\Omega)$, $\mathbf{U}_h \subset \mathbf{H}(\mathrm{curl},\Omega)$, $\mathbf{U}_{0h} \subset \mathbf{H}_0(\mathrm{curl},\Omega)$, and $\mathbf{V}_h \subset \mathbf{H}(\mathrm{div},\Omega)$ be finite dimensional subspaces.

The semi-discrete (in space) problem for the system (10)–(12) consists in finding elements $(\mathbf{D}_h,\mathbf{E}_h,\mathbf{H}_h) \in C^1(0,T,\mathbf{W}_h) \times C^1(0,T,\mathbf{W}_h) \times C^1(0,T,\mathbf{U}_h)$ such that

$$(\partial_t\mathbf{D}_h,\mathbf{\Psi}_h) - (\nabla\times\mathbf{H}_h,\mathbf{\Psi}_h) = (\mathbf{J}_h,\mathbf{\Psi}_h) \qquad \forall\mathbf{\Psi}_h \in \mathbf{W}_h, \tag{20}$$

$$(\partial_t\mathbf{D}_h,\mathbf{\Psi}) = (\varepsilon_0(1+\chi^{(1)})\partial_t\mathbf{E}_h,\mathbf{\Psi}_h) + \big(\varepsilon_0\chi^{(3)}\partial_t\big[|\mathbf{E}_h|^2\,\mathbf{E}_h\big],\mathbf{\Psi}_h\big) \quad \forall\,\mathbf{\Psi}_h \in \mathbf{W}_h, \tag{21}$$

(22)

$$(\mu_0 \partial_t \mathbf{H}_h, \mathbf{\Phi}_h) + (\mathbf{E}_h, \nabla \times \mathbf{\Phi}_h) = 0 \qquad \forall \mathbf{\Phi}_h \in \mathbf{U}_h.$$

For the equations (13)–(15), the semi-discrete problem involves the determination of elements $(\mathbf{D}_h, \mathbf{E}_h, \mathbf{H}_h) \in C^1(0,T,\mathbf{U}_{0h}) \times C^1(0,T,\mathbf{U}_{0h}) \times C^1(0,T,\mathbf{V}_h)$ satisfying

(23)

$$(\partial_t \mathbf{D}_h, \mathbf{\Psi}_h) - (\mathbf{H}_h, \nabla \times \mathbf{\Psi}_h) = (\mathbf{J}_h, \mathbf{\Psi}_h) \qquad \forall \mathbf{\Psi}_h \in \mathbf{U}_{0h},$$

(24)

$$(\partial_t \mathbf{D}_h, \mathbf{\Psi}) = (\varepsilon_0(1+\chi^{(1)})\partial_t \mathbf{E}_h, \mathbf{\Psi}_h) + \left(\varepsilon_0 \chi^{(3)} \partial_t \left[|\mathbf{E}_h|^2 \, \mathbf{E}_h\right], \mathbf{\Psi}_h\right) \quad \forall\, \mathbf{\Psi}_h \in \mathbf{U}_{0h},$$

(25)

$$(\mu_0 \partial_t \mathbf{H}_h, \mathbf{\Phi}_h) + (\nabla \times \mathbf{E}_h, \mathbf{\Phi}_h) = 0 \qquad \forall \mathbf{\Phi}_h \in \mathbf{V}_h.$$

In both cases, the initial conditions read formally as

$$\mathbf{E}_h(\mathbf{x},0) = \mathbf{E}_{0h}(\mathbf{x}) \quad \text{and} \quad \mathbf{H}_h(\mathbf{x},0) = \mathbf{H}_{0h}(\mathbf{x}),$$

where the particular choice of the discrete initial data $(\mathbf{E}_{0h}, \mathbf{H}_{0h}) \in \mathbf{W}_h \times \mathbf{U}_h$ or $(\mathbf{E}_{0h}, \mathbf{H}_{0h}) \in \mathbf{U}_{0h} \times \mathbf{V}_h$ will be given later.

The Energy of the Nonlinear Problem at the Semi-Discrete Level. The nonlinear electromagnetic energy of the semi-discrete systems (20)–(22) and (23)–(25) at time $t \in [0,T]$ is defined analogously to the continuous case by

$$\mathcal{E}_h(t) := \|\mathbf{E}_h(t)\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|\mathbf{E}_h^2(t)\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}_h(t)\|^2_{\mu_0}.$$

Since we are in a conforming setting, it is not surprising that the stability result of Theorem 1 can be transferred to the semi-discretization.

Theorem 2. *If* $\mathbf{J} \in C(0,T,\mathbf{L}^2_{(\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}}(\Omega))$ *and* $\left(\mathbf{E}_h, \mathbf{H}_h\right) \in C^1(0,T,\mathbf{U}_{0h}) \times C^1(0,T,\mathbf{V}_h)$ *is the semi-discrete finite element solution of the system* (23)–(25), *then its electromagnetic energy at any time* $t \in [0,T]$ *satisfies*

$$\mathcal{E}_h^{\frac{1}{2}}(t) \le \left(\|\mathbf{E}_{0h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|\mathbf{E}_{0h}^2\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}_{0h}\|^2_{\mu_0}\right)^{\frac{1}{2}} + \int_0^t \|\mathbf{J}_h(s)\|_{(\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}} \, ds.$$

The proof of Theorem 2 runs analogously to the proof of Theorem 1. Furthermore, an analogous result can be demonstrated for the system (20)–(22).

The above results show that the semi-discrete system maintains the energy stability, either in the implementation of the spatial discretization (20)–(22) or (23)–(25). In contrast to the linear case, stability estimates for nonlinear problems can only be seen as an intermediate step in investigating the question of the continuous dependence of the solution on the data in the context of a well-posedness discussion. Nevertheless, it is important that discretizations have the same or at least similar stability properties as the original problem.

4. Error Estimates for the Semi-Discrete Problem

To obtain error estimates, more precise information about the properties of the finite element spaces is required, such as approximation or interpolation properties. For reasons of space, we will forego the introduction of the finite element spaces used and only list the required properties. In the paper [6, Sect. IV] we described the so-called first family of Nédélec edge elements as a concrete implementation; it can serve as a reference. The references given under the properties relate to this particular case. For details we refer to [37], [33, Ch. 5], and [31].

The importance of such semi-discrete estimates is to be seen, among other things, in the fact that they are the starting point for the investigation of various time discretizations (and not just Euler-like ones).

We assume that there exist, for (moderate) $k \in \mathbb{N}$, interpolation operators $\mathbf{r}_h : \mathbf{H}^{k+1}(\Omega) \to \mathbf{U}_h$ and $\mathbf{w}_h : \mathbf{H}^k(\Omega) \to \mathbf{V}_h$ with the following properties:

$$\|\mathbf{u} - \mathbf{r}_h\mathbf{u}\|_{\mathbf{H}(\mathrm{curl},\Omega)} \le Ch^k\|\mathbf{u}\|_{\mathbf{H}^{k+1}(\Omega)} \qquad [37, \text{Theorem } 2], \tag{26}$$

$$\|\mathbf{v} - \mathbf{w}_h\mathbf{v}\| \le Ch^k\|\mathbf{v}\|_{\mathbf{H}^{k}(\Omega)} \qquad [37, \text{Theorem } 4], [31, \text{eq. } (19)], \tag{27}$$

where $C > 0$ are (possibly different) constants independent of the discretization parameter $h > 0$ (typically a characteristic mesh width).

Moreover we assume that the spaces $\mathbf{U}_h$ and $\mathbf{W}_h$ are related via

$$\nabla \times \mathbf{U}_h \subset \mathbf{W}_h \tag{28}$$

and the interpolation operators $\mathbf{r}_h$ and $\mathbf{w}_h$ are linked together as follows: $\nabla\times\mathbf{r}_h\mathbf{v} = \mathbf{w}_h(\nabla\times\mathbf{v})$ for all $\mathbf{v}$ such that both the interpolants $\mathbf{r}_h\mathbf{v}$ and $\mathbf{w}_h(\nabla\times\mathbf{v})$ are defined [33, Lemma 5.40].

Next we introduce the standard $\mathbf{L}^2$-projection $\mathbf{P}_{LM} : \mathbf{L}^2(\Omega) \to \mathbf{W}_h$ defined via

$$(\mathbf{P}_{LM}\mathbf{w}, \mathbf{\Psi}_h) = (\mathbf{w}, \mathbf{\Psi}_h) \quad \forall \mathbf{\Psi} \in \mathbf{W}_h \tag{29}$$

and assume that, for $\mathbf{w} \in \mathbf{H}^k(\Omega)$, the following error estimate can be derived:

$$\|\mathbf{w} - \mathbf{P}_{LM}\mathbf{w}\| \le Ch^k\|\mathbf{u}\|_{\mathbf{H}^{k}(\Omega)} \qquad [32, \text{eq. } (3.10)]. \tag{30}$$

We will also need a further projection operator $\Pi_{LM} : \mathbf{H}(\mathrm{curl},\Omega) \to \mathbf{U}_h$ with the properties

$$\big(\nabla \times \Pi_{LM}\mathbf{u}, \mathbf{\Psi}_h\big) = \big(\nabla \times \mathbf{u}, \mathbf{\Psi}_h\big) \quad \forall \mathbf{\Psi}_h \in \mathbf{W}_h \tag{31}$$

and, for $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$,

$$\|\mathbf{u} - \Pi_{LM}\mathbf{u}\|_{\mathbf{H}(\mathrm{curl},\Omega)} \le Ch^k\|\mathbf{u}\|_{\mathbf{H}^{k+1}(\Omega)} \qquad [32, \text{Theorem } 4.6]. \tag{32}$$

Now we prove an error estimate for the semi-discrete problem without a source term. The latter is not a substantial restriction but rather reduces the technicalities.

Theorem 3. *Assume $k \in \mathbb{N}$, $\chi^{(1)}, \chi^{(3)} \in L^\infty(\Omega)$, $\mathbf{E}_0 \in \mathbf{L}^\infty(\Omega)$, $\mathbf{H}_0 \in \mathbf{H}(\mathrm{div},\Omega)$ satisfying (8). Let the weak solution*

$$(\mathbf{E},\mathbf{H}) \in \big(C^1(0,T,\mathbf{H}^k(\Omega)\cap\mathbf{L}^\infty(\Omega)) \cap C(0,T,\mathbf{H}_0(\mathrm{curl},\Omega))\big) \times C^1(0,T,\mathbf{H}^{k+1}(\Omega))$$

of the system (10)–(12), and the finite element solution

$$(\mathbf{E}_h,\mathbf{H}_h) \in C^1(0,T,\mathbf{L}^\infty(\Omega)) \cap C(0,T,\mathbf{W}_h) \times C(0,T,\mathbf{U}_h) \tag{33}$$

of the system (20)–(22) with $\mathbf{J}_h := 0$, respectively, exist, where the inclusion in (33) is to be understood uniformly w.r.t. the discretization parameter h in the sense that $\|\mathbf{E}_h\|_{C^1(0,T,\mathbf{L}^\infty(\Omega))}$ is bounded by a constant independent of h. Then the following error estimate holds with a factor $C > 0$ independent of h (but dependent on t, in general):

$$\|\mathbf{E}_h(t) - \mathbf{E}(t)\|_{\varepsilon_0} + \|\mathbf{H}_h(t) - \mathbf{H}(t)\|_{\mu_0} \le Ch^k.$$

Remark 3. *An analogous result can be obtained for Nédélec-Raviart-Thomas formulation at the semi-discrete level.*

Proof. We set $\mathbf{\Psi} := \mathbf{\Psi}_h \in \mathbf{W}_h$ in (10)–(11) and $\mathbf{\Phi} := \mathbf{\Phi}_h \in \mathbf{U}_h$ in (12):

$$\begin{aligned}
&(\partial_t \mathbf{D}, \mathbf{\Psi}_h) - (\nabla \times \mathbf{H}, \mathbf{\Psi}_h) = 0,\\
&\qquad (\partial_t \mathbf{D}, \mathbf{\Psi}_h) = (\varepsilon_0(1+\chi^{(1)})\partial_t \mathbf{E}, \mathbf{\Psi}_h) + \big(\varepsilon_0 \chi^{(3)} \partial_t \big[|\mathbf{E}|^2\, \mathbf{E}\big], \mathbf{\Psi}_h\big),\\
&(\mu_0 \partial_t \mathbf{H}, \mathbf{\Phi}_h) + (\mathbf{E}, \nabla \times \mathbf{\Phi}_h) = 0.
\end{aligned}$$

By means of the projection operators $\mathbf{P}_{LM}$ and Π_{LM} defined in (29) and (31), respectively, from this we get

$$(\partial_t \mathbf{D}, \mathbf{\Psi}_h) - (\nabla \times \Pi_{LM}\mathbf{H}, \mathbf{\Psi}_h) = (\nabla \times (\mathbf{H} - \Pi_{LM}\mathbf{H}), \mathbf{\Psi}_h), \tag{34}$$

$$\begin{aligned}
(\partial_t \mathbf{D}, \mathbf{\Psi}_h) &= (\varepsilon_0(1+\chi^{(1)})\partial_t \mathbf{E}, \mathbf{\Psi}_h)\\
&\quad + \big(\varepsilon_0 \chi^{(3)} \partial_t \big[|\mathbf{E}|^2\, \mathbf{E}\big], \mathbf{\Psi}_h\big),
\end{aligned}$$

$$\begin{aligned}
(\mu_0 \partial_t \Pi_{LM}\mathbf{H}, \mathbf{\Phi}_h) + (\mathbf{P}_{LM}\mathbf{E}, \nabla \times \mathbf{\Phi}_h) &= \mu_0(\Pi_{LM}\partial_t \mathbf{H} - \partial_t \mathbf{H}, \mathbf{\Phi}_h)\\
&\quad + \mu_0(\partial_t \Pi_{LM}\mathbf{H} - \Pi_{LM}\partial_t \mathbf{H}, \mathbf{\Phi}_h)\\
&\quad + (\mathbf{P}_{LM}\mathbf{E} - \mathbf{E}, \nabla \times \mathbf{\Phi}_h).
\end{aligned} \tag{35}$$

The right-hand side of (34) vanishes thanks to (31) (see also [32, eq. (2.4)]). The second term on the right-hand side of (35) can be omitted because of the commutation property $\partial_t \Pi_{LM}\mathbf{H} = \Pi_{LM}\partial_t \mathbf{H}$. The last term on the right-hand side vanishes thanks to $\nabla \times \mathbf{U}_h \subset \mathbf{W}_h$ and the property (29) of $\mathbf{P}_{LM}$.

Therefore the equations (34)–(35) simplify to

$$(\partial_t \mathbf{D}, \mathbf{\Psi}_h) - (\nabla \times \Pi_{LM}\mathbf{H}, \mathbf{\Psi}_h) = 0, \tag{36}$$

$$(\partial_t \mathbf{D}, \mathbf{\Psi}_h) = (\varepsilon_0(1+\chi^{(1)})\partial_t \mathbf{E}, \mathbf{\Psi}_h) + \big(\varepsilon_0 \chi^{(3)} \partial_t \big[|\mathbf{E}|^2\, \mathbf{E}\big], \mathbf{\Psi}_h\big), \tag{37}$$

$$(\mu_0 \partial_t \Pi_{LM}\mathbf{H}, \mathbf{\Phi}_h) + (\mathbf{P}_{LM}\mathbf{E}, \nabla \times \mathbf{\Phi}_h) = \mu_0(\Pi_{LM}\partial_t \mathbf{H} - \partial_t \mathbf{H}, \mathbf{\Phi}_h). \tag{38}$$

Now, subtracting (36)–(38) from the system (20)–(22) and taking into consideration that μ_0 is constant, we obtain:

$$\begin{aligned}
&(\varepsilon_0(1+\chi^{(1)})(\partial_t \mathbf{E}_h - \partial_t \mathbf{E}), \mathbf{\Psi}_h) + \big(\varepsilon_0 \chi^{(3)} \partial_t \big[|\mathbf{E}_h|^2\, \mathbf{E}_h - |\mathbf{E}|^2\, \mathbf{E}\big], \mathbf{\Psi}_h\big)\\
&- (\nabla \times (\mathbf{H}_h - \Pi_{LM}\mathbf{H}), \mathbf{\Psi}_h) = 0,
\end{aligned} \tag{39}$$

$$\mu_0(\partial_t(\mathbf{H}_h - \Pi_{LM}\mathbf{H}), \mathbf{\Phi}_h) + (\mathbf{E}_h - \mathbf{P}_{LM}\mathbf{E}, \nabla \times \mathbf{\Phi}_h) = \mu_0(\partial_t \mathbf{H} - \Pi_{LM}\partial_t \mathbf{H}, \mathbf{\Phi}_h). \tag{40}$$

Now we will deal with the first two terms of (39), where we have in mind the choice $\mathbf{\Psi}_h := \mathbf{E}_h - \mathbf{P}_{LM}\mathbf{E}$:

$$\begin{aligned}
&\varepsilon_0(1+\chi^{(1)})(\partial_t \mathbf{E}_h - \partial_t \mathbf{E}) + \varepsilon_0 \chi^{(3)} \partial_t \big[|\mathbf{E}_h|^2\, \mathbf{E}_h - |\mathbf{E}|^2\, \mathbf{E}\big]\\
&= \varepsilon_0(1+\chi^{(1)})\partial_t(\mathbf{E}_h - \mathbf{E}) + \varepsilon_0\, \chi^{(3)}\, \big[|\mathbf{E}_h|^2 \partial_t \mathbf{E}_h - |\mathbf{E}|^2 \partial_t \mathbf{E}\big]\\
&\quad + 2\, \varepsilon_0 \chi^{(3)}\, \big[\mathbf{E}_h \mathbf{E}_h^\top \partial_t \mathbf{E}_h - \mathbf{E}\mathbf{E}^\top \partial_t \mathbf{E}\big] =: \varepsilon_0\, [\delta_1 + \delta_2 + \delta_3].
\end{aligned}$$

The treatment of δ_1 is quite obvious. With $\mathbf{E}_h - \mathbf{E} = \mathbf{\Psi}_h + \mathbf{P}_{LM}\mathbf{E} - \mathbf{E}$ we get

$$\delta_1 = (1+\chi^{(1)})\partial_t \mathbf{\Psi}_h + (1+\chi^{(1)})\partial_t(\mathbf{P}_{LM}\mathbf{E} - \mathbf{E}) =: \delta_{11} + \delta_{12}.$$

The term δ_2 is decomposed as follows:

$$\begin{aligned}
\delta_2 &= \chi^{(3)}(\mathbf{E}_h + \mathbf{E})^\top(\mathbf{E}_h - \mathbf{E})\partial_t \mathbf{E}_h + \chi^{(3)}|\mathbf{E}|^2 \partial_t \mathbf{\Psi}_h + \chi^{(3)}|\mathbf{E}|^2 \partial_t(\mathbf{P}_{LM}\mathbf{E} - \mathbf{E})\\
&=: \delta_{21} + \delta_{22} + \delta_{23}.
\end{aligned}$$

For δ_3, we use the following decomposition:

$$\delta_3 = 2\chi^{(3)}(\mathbf{E}_h - \mathbf{E})\mathbf{E}_h^\top \partial_t \mathbf{E}_h + 2\chi^{(3)}\mathbf{E}(\mathbf{E}_h - \mathbf{E})^\top \partial_t \mathbf{E}_h$$

$$
\begin{aligned}
&+ 2\chi^{(3)}\mathbf{E}\mathbf{E}^\top \partial_t \mathbf{\Psi}_h + 2\chi^{(3)}\mathbf{E}\mathbf{E}^\top \partial_t(\mathbf{P}_{LM}\mathbf{E} - \mathbf{E})\\
&=: \delta_{31} + \delta_{32} + \delta_{33} + \delta_{34}.
\end{aligned}
$$

With these decompositions, eq. (39) takes the form

$$
\begin{aligned}
&(\varepsilon_0(1+\chi^{(1)})(\partial_t \mathbf{E}_h - \partial_t \mathbf{E}), \mathbf{\Psi}_h) + \left(\varepsilon_0 \chi^{(3)} \partial_t \left[|\mathbf{E}_h|^2 \mathbf{E}_h - |\mathbf{E}|^2 \mathbf{E}\right], \mathbf{\Psi}_h\right)\\
&\quad - (\nabla \times (\mathbf{H}_h - \Pi_{LM}\mathbf{H}), \mathbf{\Psi}_h)\\
&= \varepsilon_0 \int_\Omega [\delta_{11} + \delta_{22} + \delta_{33}]^\top \mathbf{\Psi}_h d\mathbf{x} + \varepsilon_0 \int_\Omega [\delta_{12} + \delta_{21} + \delta_{23} + \delta_{31} + \delta_{32} + \delta_{34}]^\top \mathbf{\Psi}_h d\mathbf{x}\\
&\quad - (\nabla \times (\mathbf{H}_h - \Pi_{LM}\mathbf{H}), \mathbf{\Psi}_h) = 0,
\end{aligned}
$$

or, after some rearrangement,

$$
\begin{aligned}
&\varepsilon_0 \int_\Omega [\delta_{11} + \delta_{22} + \delta_{33}]^\top \mathbf{\Psi}_h d\mathbf{x} - (\nabla \times (\mathbf{H}_h - \Pi_{LM}\mathbf{H}), \mathbf{\Psi}_h)\\
&= -\varepsilon_0 \int_\Omega [\delta_{12} + \delta_{21} + \delta_{23} + \delta_{31} + \delta_{32} + \delta_{34}]^\top \mathbf{\Psi}_h d\mathbf{x}.
\end{aligned}
\tag{41}
$$

Then:

$$
\begin{aligned}
&\varepsilon_0 \int_\Omega [\delta_{11} + \delta_{22} + \delta_{33}]^\top \mathbf{\Psi}_h d\mathbf{x}\\
&= \frac{\varepsilon_0}{2} \int_\Omega \left[(1+\chi^{(1)})\partial_t |\mathbf{\Psi}_h|^2 + \chi^{(3)}|\mathbf{E}|^2 \partial_t |\mathbf{\Psi}_h|^2 + 4\chi^{(3)}\mathbf{E}^\top \partial_t \mathbf{\Psi}_h \mathbf{E}^\top \mathbf{\Psi}_h\right] d\mathbf{x}\,.
\end{aligned}
$$

Since

$|\mathbf{E}|^2 \partial_t |\mathbf{\Psi}_h|^2 = \partial_t(|\mathbf{E}|^2 |\mathbf{\Psi}_h|^2) - \partial_t(|\mathbf{E}|^2)|\mathbf{\Psi}_h|^2$ and $\mathbf{E}^\top \partial_t \mathbf{\Psi}_h = \partial_t(\mathbf{E}^\top \mathbf{\Psi}_h) - \partial_t \mathbf{E}^\top \mathbf{\Psi}_h$,

it follows that

$$
\begin{aligned}
&\varepsilon_0 \int_\Omega [\delta_{11} + \delta_{22} + \delta_{33}]^\top \mathbf{\Psi}_h d\mathbf{x}\\
&= \frac{\varepsilon_0}{2} \int_\Omega \left[(1+\chi^{(1)})\partial_t |\mathbf{\Psi}_h|^2 + \chi^{(3)} \partial_t(|\mathbf{E}|^2 |\mathbf{\Psi}_h|^2) + 2\chi^{(3)} \partial_t |\mathbf{E}^\top \mathbf{\Psi}_h|^2\right] d\mathbf{x}\\
&\quad - \frac{\varepsilon_0}{2} \int_\Omega \chi^{(3)} \partial_t(|\mathbf{E}|^2)|\mathbf{\Psi}_h|^2 d\mathbf{x} - 2\varepsilon_0 \int_\Omega \chi^{(3)} \partial_t \mathbf{E}^\top \mathbf{\Psi}_h \mathbf{E}^\top \mathbf{\Psi}_h d\mathbf{x}\,.
\end{aligned}
$$

From the estimates

$$
\begin{aligned}
&\left|\frac{\varepsilon_0}{2} \int_\Omega \chi^{(3)} \partial_t(|\mathbf{E}|^2)|\mathbf{\Psi}_h|^2 d\mathbf{x}\right| = \left|\varepsilon_0 \int_\Omega \chi^{(3)} \partial_t \mathbf{E}^\top \mathbf{E} |\mathbf{\Psi}_h|^2 d\mathbf{x}\right|\\
&\le \|\chi^{(3)}\|_{L^\infty(\Omega)} \|\mathbf{E}\|^2_{C^1(0,T,L^\infty(\Omega))} \|\mathbf{\Psi}_h\|^2_{\varepsilon_0}
\end{aligned}
$$

and, analogously,

$$
\left|\varepsilon_0 \int_\Omega \chi^{(3)} \partial_t \mathbf{E}^\top \mathbf{\Psi}_h \mathbf{E}^\top \mathbf{\Psi}_h d\mathbf{x}\right| \le \|\chi^{(3)}\|_{L^\infty(\Omega)} \|\mathbf{E}\|^2_{C^1(0,T,L^\infty(\Omega))} \|\mathbf{\Psi}_h\|^2_{\varepsilon_0},
$$

we conclude that

$$
\begin{aligned}
&\varepsilon_0 \int_\Omega [\delta_{11} + \delta_{22} + \delta_{33}]^\top \mathbf{\Psi}_h d\mathbf{x}\\
&\ge \frac{\varepsilon_0}{2} \int_\Omega \left[(1+\chi^{(1)})\partial_t |\mathbf{\Psi}_h|^2 + \chi^{(3)} \partial_t(|\mathbf{E}|^2 |\mathbf{\Psi}_h|^2) + 2\chi^{(3)} \partial_t |\mathbf{E}^\top \mathbf{\Psi}_h|^2\right] d\mathbf{x}\\
&\quad - 2\|\chi^{(3)}\|_{L^\infty(\Omega)} \|\mathbf{E}\|^2_{C^1(0,T,L^\infty(\Omega))} \|\mathbf{\Psi}_h\|^2_{\varepsilon_0}\\
&= \frac{1}{2}\partial_t \|\mathbf{\Psi}_h\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{\varepsilon_0}{2}\partial_t \int_\Omega \chi^{(3)} \left[|\mathbf{E}|^2 |\mathbf{\Psi}_h|^2 + 2|\mathbf{E}^\top \mathbf{\Psi}_h|^2\right] d\mathbf{x}
\end{aligned}
$$

$$- 2\|\chi^{(3)}\|_{L^\infty(\Omega)}\|\mathbf{E}\|^2_{C^1(0,T,L^\infty(\Omega))}\|\mathbf{\Psi}_h\|^2_{\varepsilon_0}. \tag{42}$$

For the right-hand side, we have:

$$\begin{aligned}
&- \varepsilon_0 \int_\Omega [\delta_{12} + \delta_{21} + \delta_{23} + \delta_{31} + \delta_{32} + \delta_{34}]^\top \mathbf{\Psi}_h d\mathbf{x} \\
&= -\varepsilon_0 \int_\Omega \Big[(1+\chi^{(1)})\partial_t(\mathbf{P}_{LM}\mathbf{E} - \mathbf{E})^\top \mathbf{\Psi}_h \\
&\quad + \chi^{(3)}(\mathbf{E}_h + \mathbf{E})^\top(\mathbf{E}_h - \mathbf{E})\partial_t \mathbf{E}_h^\top \mathbf{\Psi}_h + \chi^{(3)}|\mathbf{E}|^2\partial_t(\mathbf{P}_{LM}\mathbf{E} - \mathbf{E})^\top \mathbf{\Psi}_h \\
&\quad + 2\chi^{(3)}\left((\mathbf{E}_h - \mathbf{E})\mathbf{E}_h^\top \partial_t \mathbf{E}_h\right)^\top \mathbf{\Psi}_h + 2\chi^{(3)}\left(\mathbf{E}(\mathbf{E}_h - \mathbf{E})^\top \partial_t \mathbf{E}_h\right)^\top \mathbf{\Psi}_h \\
&\quad + 2\chi^{(3)}\left(\mathbf{E}\mathbf{E}^\top \partial_t(\mathbf{P}_{LM}\mathbf{E} - \mathbf{E})\right)^\top \mathbf{\Psi}_h\Big]\, d\mathbf{x} \\
&\le \varepsilon_0 \int_\Omega \Big[(1+\chi^{(1)} + 3\chi^{(3)}|\mathbf{E}|^2)|\partial_t(\mathbf{P}_{LM}\mathbf{E} - \mathbf{E})||\mathbf{\Psi}_h| \\
&\quad + 3\chi^{(3)}|\mathbf{E}_h||\partial_t\mathbf{E}_h||\mathbf{P}_{LM}\mathbf{E} - \mathbf{E}||\mathbf{\Psi}_h| + 3\chi^{(3)}|\mathbf{E}||\partial_t\mathbf{E}_h||\mathbf{P}_{LM}\mathbf{E} - \mathbf{E}||\mathbf{\Psi}_h| \\
&\quad + 3\chi^{(3)}|\mathbf{E}_h||\partial_t\mathbf{E}_h||\mathbf{\Psi}_h|^2 + 3\chi^{(3)}|\mathbf{E}||\partial_t\mathbf{E}_h||\mathbf{\Psi}_h|^2\Big]\, d\mathbf{x} \\
&\le \Big[\|1+\chi^{(1)}\|_{L^\infty(\Omega)} + 3\|\chi^{(3)}\|_{L^\infty(\Omega)}\|\mathbf{E}\|^2_{C(0,T,L^\infty(\Omega))}\Big] \|\partial_t(\mathbf{P}_{LM}\mathbf{E} - \mathbf{E})\|_{\varepsilon_0}\|\mathbf{\Psi}_h\|_{\varepsilon_0} \\
&\quad + 3\|\chi^{(3)}\|_{L^\infty(\Omega)}\left[\|\mathbf{E}_h\|_{C(0,T,L^\infty(\Omega))} + \|\mathbf{E}\|_{C(0,T,L^\infty(\Omega))}\right] \\
&\qquad \times \|\partial_t\mathbf{E}_h\|_{C(0,T,L^\infty(\Omega))}\|\mathbf{P}_{LM}\mathbf{E} - \mathbf{E}\|_{\varepsilon_0}\|\mathbf{\Psi}_h\|_{\varepsilon_0} \\
&\quad + 3\|\chi^{(3)}\|_{L^\infty(\Omega)}\left[\|\mathbf{E}_h\|_{C(0,T,L^\infty(\Omega))} + \|\mathbf{E}\|_{C(0,T,L^\infty(\Omega))}\right] \\
&\qquad \times \|\partial_t\mathbf{E}_h\|_{C(0,T,L^\infty(\Omega))}\|\mathbf{\Psi}_h\|^2_{\varepsilon_0}
\end{aligned}$$

(43)

$$=: C_1\|\partial_t(\mathbf{P}_{LM}\mathbf{E} - \mathbf{E})\|_{\varepsilon_0}\|\mathbf{\Psi}_h\|_{\varepsilon_0} + C_2\|\mathbf{P}_{LM}\mathbf{E} - \mathbf{E}\|_{\varepsilon_0}\|\mathbf{\Psi}_h\|_{\varepsilon_0} + C_3\|\mathbf{\Psi}_h\|^2_{\varepsilon_0},$$

where the positive constants C_1, C_2, C_3 depend on certain norms of $\chi^{(1)}$, $\chi^{(3)}$, $\mathbf{E}$, and $\mathbf{E}_h$. Combining the estimates (42) and (43) with (41), we get

$$\begin{aligned}
&\frac{1}{2}\partial_t\|\mathbf{\Psi}_h\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{\varepsilon_0}{2}\partial_t\int_\Omega \chi^{(3)}\left[|\mathbf{E}|^2|\mathbf{\Psi}_h|^2 + 2|\mathbf{E}^\top\mathbf{\Psi}_h|^2\right] d\mathbf{x} \\
&\quad - 2\|\chi^{(3)}\|_{L^\infty(\Omega)}\|\mathbf{E}\|^2_{C^1(0,T,L^\infty(\Omega))}\|\mathbf{\Psi}_h\|^2_{\varepsilon_0} - (\nabla\times(\mathbf{H}_h - \Pi_{LM}\mathbf{H}), \mathbf{\Psi}_h) \\
&\le \varepsilon_0\int_\Omega [\delta_{11} + \delta_{22} + \delta_{33}]^\top \mathbf{\Psi}_h d\mathbf{x} - (\nabla\times(\mathbf{H}_h - \Pi_{LM}\mathbf{H}), \mathbf{\Psi}_h) \\
&= -\varepsilon_0\int_\Omega [\delta_{12} + \delta_{21} + \delta_{23} + \delta_{31} + \delta_{32} + \delta_{34}]^\top \mathbf{\Psi}_h d\mathbf{x} \\
&\le C_1\|\partial_t(\mathbf{P}_{LM}\mathbf{E} - \mathbf{E})\|_{\varepsilon_0}\|\mathbf{\Psi}_h\|_{\varepsilon_0} + C_2\|\mathbf{P}_{LM}\mathbf{E} - \mathbf{E}\|_{\varepsilon_0}\|\mathbf{\Psi}_h\|_{\varepsilon_0} + C_3\|\mathbf{\Psi}_h\|^2_{\varepsilon_0}.
\end{aligned}$$

This finally leads to

$$\begin{aligned}
&\frac{1}{2}\partial_t\|\mathbf{\Psi}_h\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{\varepsilon_0}{2}\partial_t\int_\Omega \chi^{(3)}\left[|\mathbf{E}|^2|\mathbf{\Psi}_h|^2 + 2|\mathbf{E}^\top\mathbf{\Psi}_h|^2\right] d\mathbf{x} \\
&\quad - (\nabla\times(\mathbf{H}_h - \Pi_{LM}\mathbf{H}), \mathbf{\Psi}_h) \\
&\le \left[C_1\|\partial_t(\mathbf{P}_{LM}\mathbf{E} - \mathbf{E})\|_{\varepsilon_0} + C_2\|\mathbf{P}_{LM}\mathbf{E} - \mathbf{E}\|_{\varepsilon_0}\right]\|\mathbf{\Psi}_h\|_{\varepsilon_0} + C_4\|\mathbf{\Psi}_h\|^2_{\varepsilon_0},
\end{aligned}$$

where

$$C_4 := C_3 + 2\|\chi^{(3)}\|_{L^\infty(\Omega)}\|\mathbf{E}\|^2_{C^1(0,T,L^\infty(\Omega))}.$$

Now we consider (40) with $\mathbf{\Phi}_h := \mathbf{H}_h - \Pi_{LM}\mathbf{H}$ and get

$$\frac{1}{2}\partial_t\|\mathbf{\Phi}_h\|_{\mu_0}^2 + (\mathbf{E}_h - \mathbf{P}_{LM}\mathbf{E}, \nabla\times\mathbf{\Phi}_h) = \mu_0(\partial_t\mathbf{H} - \Pi_{LM}\partial_t\mathbf{H}, \mathbf{\Phi}_h) \\ \leq \|\partial_t\mathbf{H} - \Pi_{LM}\partial_t\mathbf{H}\|_{\mu_0}\|\mathbf{\Phi}_h\|_{\mu_0}.$$

Adding both inequalities and making use of the commutation property of $\mathbf{P}_{LM}$, we arrive at

$$\frac{1}{2}\partial_t\|\mathbf{\Psi}_h\|_{\varepsilon_0(1+\chi^{(1)})}^2 + \frac{1}{2}\partial_t\|\mathbf{\Phi}_h\|_{\mu_0}^2 + \frac{\varepsilon_0}{2}\partial_t\int_\Omega \chi^{(3)}\left[|\mathbf{E}|^2|\mathbf{\Psi}_h|^2 + 2|\mathbf{E}^\top\mathbf{\Psi}_h|^2\right]d\mathbf{x} \\ \leq \left[C_1\|\partial_t\mathbf{E} - \mathbf{P}_{LM}\partial_t\mathbf{E}\|_{\varepsilon_0} + C_2\|\mathbf{E} - \mathbf{P}_{LM}\mathbf{E}\|_{\varepsilon_0}\right]\|\mathbf{\Psi}_h\|_{\varepsilon_0} \\ + \|\partial_t\mathbf{H} - \Pi_{LM}\partial_t\mathbf{H}\|_{\mu_0}\|\mathbf{\Phi}_h\|_{\mu_0} + C_4\|\mathbf{\Psi}_h\|_{\varepsilon_0}^2.$$

The projection errors can be estimated by means of (30) and (32), that is, for $\mathbf{E}, \partial_t\mathbf{E} \in \mathbf{H}^k(\Omega)$ and $\partial_t\mathbf{H} \in \mathbf{H}^{k+1}(\Omega)$, we have that

$$\|\mathbf{E} - \mathbf{P}_{LM}\mathbf{E}\|_{\varepsilon_0} \leq C\sqrt{\varepsilon_0}\, h^k\|\mathbf{E}\|_{\mathbf{H}^k(\Omega)} \leq C\sqrt{\varepsilon_0}\, h^k\|\mathbf{E}\|_{C(0,T,\mathbf{H}^k(\Omega))},$$

(44)

$$\|\partial_t\mathbf{E} - \mathbf{P}_{LM}\partial_t\mathbf{E}\|_{\varepsilon_0} \leq C\sqrt{\varepsilon_0}\, h^k\|\partial_t\mathbf{E}\|_{\mathbf{H}^k(\Omega)} \leq C\sqrt{\varepsilon_0}\, h^k\|\partial_t\mathbf{E}\|_{C(0,T,\mathbf{H}^k(\Omega))},$$

(45)

$$\|\partial_t\mathbf{H} - \Pi_{LM}\partial_t\mathbf{H}\|_{\mu_0} \leq C\sqrt{\mu_0}\, h^k\|\partial_t\mathbf{H}\|_{\mathbf{H}^{k+1}(\Omega)} \leq C\sqrt{\mu_0}\, h^k\|\partial_t\mathbf{H}\|_{C(0,T,\mathbf{H}^{k+1}(\Omega))}.$$

In this way the above estimate can be written as

$$\frac{1}{2}\partial_t\|\mathbf{\Psi}_h\|_{\varepsilon_0(1+\chi^{(1)})}^2 + \frac{1}{2}\partial_t\|\mathbf{\Phi}_h\|_{\mu_0}^2 + \frac{\varepsilon_0}{2}\partial_t\int_\Omega \chi^{(3)}\left[|\mathbf{E}|^2|\mathbf{\Psi}_h|^2 + 2|\mathbf{E}^\top\mathbf{\Psi}_h|^2\right]d\mathbf{x} \\ \leq C_5 h^k\left[\|\mathbf{\Psi}_h\|_{\varepsilon_0} + \|\mathbf{\Phi}_h\|_{\mu_0}\right] + C_4\|\mathbf{\Psi}_h\|_{\varepsilon_0}^2.$$

Setting

$$w_h(t) := \sqrt{\|\mathbf{\Psi}_h\|_{\varepsilon_0}^2 + \|\mathbf{\Phi}_h\|_{\mu_0}^2},$$

we get

$$\frac{1}{2}\partial_t\|\mathbf{\Psi}_h\|_{\varepsilon_0(1+\chi^{(1)})}^2 + \frac{1}{2}\partial_t\|\mathbf{\Phi}_h\|_{\mu_0}^2 + \frac{\varepsilon_0}{2}\partial_t\int_\Omega \chi^{(3)}\left[|\mathbf{E}|^2|\mathbf{\Psi}_h|^2 + 2|\mathbf{E}^\top\mathbf{\Psi}_h|^2\right]d\mathbf{x} \\ \leq C_5\sqrt{2}h^k w_h(t) + C_4\|\mathbf{\Psi}_h\|_{\varepsilon_0}^2 \leq C_5\sqrt{2}h^k w_h(t) + C_4 w_h^2(t).$$

Integrating this inequality, we obtain

$$\frac{1}{2}\|\mathbf{\Psi}_h(t)\|_{\varepsilon_0(1+\chi^{(1)})}^2 + \frac{1}{2}\|\mathbf{\Phi}_h(t)\|_{\mu_0}^2 + \frac{\varepsilon_0}{2}\int_\Omega \chi^{(3)}\left[|\mathbf{E}(t)|^2|\mathbf{\Psi}_h(t)|^2 + 2|\mathbf{E}(t)^\top\mathbf{\Psi}_h(t)|^2\right]d\mathbf{x} \\ \leq \frac{1}{2}\|\mathbf{\Psi}_h(0)\|_{\varepsilon_0(1+\chi^{(1)})}^2 + \frac{1}{2}\|\mathbf{\Phi}_h(0)\|_{\mu_0}^2 \\ + \frac{\varepsilon_0}{2}\int_\Omega \chi^{(3)}\left[|\mathbf{E}(0)|^2|\mathbf{\Psi}_h(0)|^2 + 2|\mathbf{E}(0)^\top\mathbf{\Psi}_h(0)|^2\right]d\mathbf{x}$$

(46)

$$+ \int_0^t \left[C_5\sqrt{2}h^k w_h(s) + C_4 w_h^2(s)\right]ds.$$

By the monotonicity of the weighted norms w.r.t. the weight and the nonnegativity of the integral term on the left-hand side, we see that

$$\frac{1}{2}w_h^2(t) \leq \frac{1}{2}\|\mathbf{\Psi}_h(t)\|_{\varepsilon_0(1+\chi^{(1)})}^2 + \frac{1}{2}\|\mathbf{\Phi}_h(t)\|_{\mu_0}^2$$

$$+\frac{\varepsilon_0}{2}\int_\Omega \chi^{(3)}\left[|\mathbf{E}(t)|^2|\mathbf{\Psi}_h(t)|^2+2|\mathbf{E}(t)^\top\mathbf{\Psi}_h(t)|^2\right]d\mathbf{x}. \tag{47}$$

On the other hand, we have the estimates

$$\|\mathbf{\Psi}_h(0)\|^2_{\varepsilon_0(1+\chi^{(1)})}\le\|1+\chi^{(1)}\|_{L^\infty(\Omega)}\|\mathbf{\Psi}_h(0)\|^2_{\varepsilon_0}\le\|1+\chi^{(1)}\|_{L^\infty(\Omega)}w_h^2(0) \tag{48}$$

and

$$\begin{aligned}&\varepsilon_0\int_\Omega \chi^{(3)}\left[|\mathbf{E}(0)|^2|\mathbf{\Psi}_h(0)|^2+2|\mathbf{E}(0)^\top\mathbf{\Psi}_h(0)|^2\right]d\mathbf{x}\\ &\le 3\|\chi^{(3)}\|_{L^\infty(\Omega)}\|\mathbf{E}(0)\|^2_{L^\infty(\Omega)}\|\mathbf{\Psi}_h(0)\|^2_{\varepsilon_0}\le 3\|\chi^{(3)}\|_{L^\infty(\Omega)}\|\mathbf{E}(0)\|^2_{L^\infty(\Omega)}w_h^2(0).\end{aligned} \tag{49}$$

Combining (47), (48), (49) with (46), we get

$$\begin{aligned}\frac{1}{2}w_h^2(t)\le{}&\frac{1}{2}\|1+\chi^{(1)}\|_{L^\infty(\Omega)}w_h^2(0)+\frac{3}{2}\|\chi^{(3)}\|_{L^\infty(\Omega)}\|\mathbf{E}(0)\|^2_{L^\infty(\Omega)}w_h^2(0)\\ &+\int_0^t\left[C_5\sqrt{2}h^kw_h(s)+C_4w_h^2(s)\right]ds,\end{aligned}$$

or, equivalently,

$$w_h^2(t)\le C_6^2w_h^2(0)+\int_0^t\left[2C_5\sqrt{2}h^kw_h(s)+2C_4w_h^2(s)\right]ds, \tag{50}$$

where $C_6^2:=\|1+\chi^{(1)}\|_{L^\infty(\Omega)}+3\|\chi^{(3)}\|_{L^\infty(\Omega)}\|\mathbf{E}(0)\|^2_{L^\infty(\Omega)}$.

In the paper [14], a Gronwall-type lemma (Lemma 4.1) is specified which extracts a bound for the value $w(T)$ if an inequality like (50) is satisfied:

$$w_h(T)\le C_6e^{C_4T}w_h(0)+C_5\sqrt{2}h^kTe^{C_4T}.$$

From this and the triangle inequality in conjunction with (30) and (32) the statement follows. □

5. Time Discretization for the Nonlinear Maxwell's Equations

In this section, we investigate the novel fully discrete scheme for the nonlinear Maxwell's equations presented in [7]. We intend to demonstrate that the time discretization of the systems (20)–(22) and (23)–(25) by means of the classical backward Euler-type method satisfies a discrete energy estimate, is unconditionally stable and convergent even in the presence of cubic nonlinearities. Analogous investigations for the linear case (that is $\chi^{(3)}=0$) have been presented in [6].

The time discretization considered here can be used not only in conjunction with the Lee-Madsen scheme or the Nédélec-Raviart-Thomas spatial discretizations, but also with other types of spatial discretizations. The Newton's method is often employed to obtain the unknown values $\mathbf{E}_h^n$ and $\mathbf{H}_h^n$ from the nonlinear equations (51)–(53) or (54)–(56).

We divide the time interval $(0,T)$ into $N\in\mathbb{N}$ equally spaced subintervals by using the nodal points $t^n:=n\Delta t$, $n=0,1,2,\ldots,N$, with $\Delta t:=T/N$.

Given initial values $(\mathbf{E}_h^0,\mathbf{H}_h^0)\in\mathbf{W}_h\times\mathbf{U}_h$ of the approximate electric and magnetic field intensities, the fully discrete electric and magnetic field intensities $(\mathbf{E}_h^n,\mathbf{H}_h^n)\in\mathbf{W}_h\times\mathbf{U}_h$, $n=1,2,\ldots,N$, satisfy the system

(51)

$$\begin{aligned}&\Big(\frac{\mathbf{D}_h^n-\mathbf{D}_h^{n-1}}{\Delta t},\mathbf{\Psi}_h\Big)-(\nabla\times\mathbf{H}_h^n,\mathbf{\Psi}_h)=(\mathbf{J}_h^n,\mathbf{\Psi}_h),\\ &\big(\mathbf{D}_h^n-\mathbf{D}_h^{n-1},\mathbf{\Psi}_h\big)=(\varepsilon_0(1+\chi^{(1)})\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{\Psi}_h)\end{aligned}$$

$$+\frac{1}{2}(\varepsilon_0\chi^{(3)}((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2)\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{\Psi}_h) \tag{52}$$
$$+\Big(\varepsilon_0\chi^{(3)}\Big[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T+\mathbf{E}_h^{n-1}\,\big[\mathbf{E}_h^{n-1}\big]^T\Big]\,\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{\Psi}_h\Big),$$

(53)
$$\Big(\mu_0\frac{\mathbf{H}_h^n-\mathbf{H}_h^{n-1}}{\Delta t},\mathbf{\Phi}_h\Big)+(\mathbf{E}_h^n,\nabla\times\mathbf{\Phi}_h)=0$$

for all $(\mathbf{\Psi}_h,\mathbf{\Phi}_h)\in\mathbf{W}_h\times\mathbf{U}_h$. Note that in this scheme the differences $\mathbf{D}_h^n-\mathbf{D}_h^{n-1}$ of the displacement approximations only play the role of auxiliary variables.

For the full discretization of the second formulation, we prescribe initial values $(\mathbf{E}_h^0,\mathbf{H}_h^0)\in\mathbf{U}_{0h}\times\mathbf{V}_h$ of the approximate electric and magnetic field intensities and determine the fully discrete electric and magnetic field intensities $(\mathbf{E}_h^n,\mathbf{H}_h^n)\in\mathbf{U}_{0h}\times\mathbf{V}_h$, $n=1,2,\ldots,N$, such that the following system is satisfied:

(54)
$$\Big(\frac{\mathbf{D}_h^n-\mathbf{D}_h^{n-1}}{\Delta t},\mathbf{\Psi}_h\Big)-(\mathbf{H}_h^n,\nabla\times\mathbf{\Psi}_h)=(\mathbf{J}_h^n,\mathbf{\Psi}_h),$$

$$\big(\mathbf{D}_h^n-\mathbf{D}_h^{n-1},\mathbf{\Psi}_h\big)=(\varepsilon_0(1+\chi^{(1)})\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{\Psi}_h)$$
$$+\frac{1}{2}(\varepsilon_0\chi^{(3)}((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2)\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{\Psi}_h) \tag{55}$$
$$+\big(\varepsilon_0\chi^{(3)}\Big[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T+\mathbf{E}_h^{n-1}\,\big[\mathbf{E}_h^{n-1}\big]^T\Big]\,\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{\Psi}_h\big),$$

(56)
$$\Big(\mu_0\frac{\mathbf{H}_h^n-\mathbf{H}_h^{n-1}}{\Delta t},\mathbf{\Phi}_h\Big)+(\nabla\times\mathbf{E}_h^n,\mathbf{\Phi}_h)=0$$

for all $(\mathbf{\Psi}_h,\mathbf{\Phi}_h)\in\mathbf{U}_{0h}\times\mathbf{V}_h$. As above, the differences $\mathbf{D}_h^n-\mathbf{D}_h^{n-1}$ play the role of auxiliary variables.

The Energy of the Nonlinear Problem at the Fully Discrete Level. The nonlinear electromagnetic energy for the fully discrete approximation (i.e. both in space and time) of the systems (51)–(53) and (54)–(56) at t^n, $n=0,1,2,\ldots,N$, is defined by

$$\mathcal{E}_h^n:=\|\mathbf{E}_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}+\frac{3}{2}\|(\mathbf{E}_h^n)^2\|^2_{\varepsilon_0\chi^{(3)}}+\|\mathbf{H}_h^n\|^2_{\mu_0}. \tag{57}$$

In analogy to the boundedness results for the continuous and semi-discrete nonlinear electromagnetic energy (Theorems 1, 2), in this section we will show that the fully discrete nonlinear electromagnetic energy of the systems (51)–(53) and (54)–(56) at the final time step N is bounded, too.

Theorem 4. *Let $(\mathbf{E}_h^n,\mathbf{H}_h^n)$ be the fully discrete solution of (54)–(56). Then, for sufficiently small Δt and h, there exists a constant $C>0$ independent of Δt and h such that*

$$\mathcal{E}_h^N\le C.$$

Remark 4. *An analogous result can be obtained for the system (51)–(53).*

Proof. Taking $\mathbf{\Psi}_h:=2\,\mathbf{E}_h^n$ in eq. (55), we have

$$\big(\mathbf{D}_h^n-\mathbf{D}_h^{n-1},2\,\mathbf{E}_h^n\big)=2\Big[(\varepsilon_0(1+\chi^{(1)})\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n)$$
$$+\frac{1}{2}(\varepsilon_0\chi^{(3)}((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2)\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n)$$

$$+\big(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\left[\mathbf{E}_h^n\right]^T+\mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right]\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n\big)\Big]. \tag{58}$$

Setting $\mathbf{\Psi}_h := 2\Delta t\,\mathbf{E}_h^n$ in the eq. (54), we get

$$\big(\mathbf{D}_h^n-\mathbf{D}_h^{n-1},2\,\mathbf{E}_h^n\big)=2\Delta t(\mathbf{H}_h^n,\nabla\times\mathbf{E}_h^n)+2\Delta t(\mathbf{J}_h^n,\mathbf{E}_h^n). \tag{59}$$

Replacing the left-hand side of eq. (58) by (59), we arrive at

$$\begin{aligned}
&2\Big[(\varepsilon_0(1+\chi^{(1)})\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n)\\
&+\frac{1}{2}(\varepsilon_0\chi^{(3)}((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2)\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n)\\
&+\big(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\left[\mathbf{E}_h^n\right]^T+\mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right]\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n\big)\Big]\\
&-2\Delta t(\mathbf{H}_h^n,\nabla\times\mathbf{E}_h^n)=2\Delta t(\mathbf{J}_h^n,\mathbf{E}_h^n).
\end{aligned} \tag{60}$$

Taking $\mathbf{\Phi}_h := 2\Delta t\mathbf{H}_h^n$ in eq. (56), we obtain

$$2(\mu_0(\mathbf{H}_h^n-\mathbf{H}_h^{n-1}),\mathbf{H}_h^n)+2\Delta t\,(\nabla\times\mathbf{E}_h^n,\mathbf{H}_h^n)=0. \tag{61}$$

Adding the equations (60) and (61), we get

$$\begin{aligned}
&2\Big[(\varepsilon_0(1+\chi^{(1)})\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n)+(\mu_0(\mathbf{H}_h^n-\mathbf{H}_h^{n-1}),\mathbf{H}_h^n)\\
&+\frac{1}{2}(\varepsilon_0\chi^{(3)}((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2)\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n)\\
&+\big(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\left[\mathbf{E}_h^n\right]^T+\mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right]\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n\big)\Big]\\
&-2\Delta t(\mathbf{H}_h^n,\nabla\times\mathbf{E}_h^n)+2\Delta t\,(\nabla\times\mathbf{E}_h^n,\mathbf{H}_h^n)=2\Delta t(\mathbf{J}_h^n,\mathbf{E}_h^n).
\end{aligned}$$

This implies

$$\begin{aligned}
&2\Big[(\varepsilon_0(1+\chi^{(1)})\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n)+(\mu_0(\mathbf{H}_h^n-\mathbf{H}_h^{n-1}),\mathbf{H}_h^n)\\
&+\frac{1}{2}(\varepsilon_0\chi^{(3)}((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2)\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n)\\
&+\big(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\left[\mathbf{E}_h^n\right]^T+\mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right]\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n\big)\Big]=2\Delta t(\mathbf{J}_h^n,\mathbf{E}_h^n).
\end{aligned} \tag{62}$$

Now we apply a well-known identity from Hilbert space theory (see e.g. [6, Lemma 1, 1)]) to the first and second terms on the left-hand side. Then, the first term from the left-hand side of eq. (62) can be written and estimated as

$$\begin{aligned}
&2(\varepsilon_0(1+\chi^{(1)})\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n)\\
&=\|\mathbf{E}_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\|^2_{\varepsilon_0(1+\chi^{(1)})}-\|\mathbf{E}_h^{n-1}\|^2_{\varepsilon_0(1+\chi^{(1)})}\\
&\ge\|\mathbf{E}_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}-\|\mathbf{E}_h^{n-1}\|^2_{\varepsilon_0(1+\chi^{(1)})}.
\end{aligned}$$

The second term from the left-hand side of eq. (62) is estimated in a similar way:

$$\begin{aligned}
2(\mu_0(\mathbf{H}_h^n-\mathbf{H}_h^{n-1}),\mathbf{H}_h^n)=&\|\mathbf{H}_h^n\|^2_{\mu_0}+\|\mathbf{H}_h^n-\mathbf{H}_h^{n-1}\|^2_{\mu_0}-\|\mathbf{H}_h^{n-1}\|^2_{\mu_0}\\
\ge&\|\mathbf{H}_h^n\|^2_{\mu_0}-\|\mathbf{H}_h^{n-1}\|^2_{\mu_0}.
\end{aligned}$$

The third and the fourth terms from the left-hand side of eq. (62) can be treated as follows. Writing the test function $\mathbf{E}_h^n$ in the form

$$\mathbf{E}_h^n=\frac{1}{2}\big(\mathbf{E}_h^n+\mathbf{E}_h^{n-1}\big)+\frac{1}{2}\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),$$

we have that

$$\big(\varepsilon_0\chi^{(3)}\frac{1}{2}\big((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big)\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n\big)$$

$$
\begin{aligned}
&= \frac{1}{4}\big(\varepsilon_0\chi^{(3)}\big((\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2\big)\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{E}_h^n + \mathbf{E}_h^{n-1}\big) \\
&\quad + \frac{1}{4}\big(\varepsilon_0\chi^{(3)}\big((\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2\big)\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big) \\
&\geq \frac{1}{4}\big\|(\mathbf{E}_h^n)^2\big\|^2_{\varepsilon_0\chi^{(3)}} - \frac{1}{4}\big\|(\mathbf{E}_h^{n-1})^2\big\|^2_{\varepsilon_0\chi^{(3)}}.
\end{aligned}
$$

Analogously,

$$
\begin{aligned}
&\big(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\left[\mathbf{E}_h^n\right]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right]\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{E}_h^n\big) \\
&= \frac{1}{2}\big(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\left[\mathbf{E}_h^n\right]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right]\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{E}_h^n + \mathbf{E}_h^{n-1}\big) \\
&\quad + \frac{1}{2}\big(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\left[\mathbf{E}_h^n\right]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right]\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big) \\
&\geq \frac{1}{2}\big\|(\mathbf{E}_h^n)^2\big\|^2_{\varepsilon_0\chi^{(3)}} - \frac{1}{2}\big\|(\mathbf{E}_h^{n-1})^2\big\|^2_{\varepsilon_0\chi^{(3)}}.
\end{aligned}
$$

So the left-hand side of eq. (62) can be estimated as follows:

$$
\begin{aligned}
&\mathcal{E}_h^n - \mathcal{E}_h^{n-1} \\
&\leq 2\Big[(\varepsilon_0(1+\chi^{(1)})\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{E}_h^n) + \frac{1}{2}(\varepsilon_0\chi^{(3)}((\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2)\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{E}_h^n) \\
&\quad + \big(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\left[\mathbf{E}_h^n\right]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right]\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{E}_h^n) + (\mu_0(\mathbf{H}_h^n - \mathbf{H}_h^{n-1}), \mathbf{H}_h^n)\Big].
\end{aligned} \tag{63}
$$

The right-hand side of eq. (62) is estimated by means of Young's inequality (see e.g. [6, Lemma 1, 2)]). This results in

$$
\begin{aligned}
2\Delta t\,(\mathbf{J}_h^n, \mathbf{E}_h^n) &= \Delta t\,([\varepsilon_0(1+\chi^{(1)})]^{-1/2}\mathbf{J}_h^n, [\varepsilon_0(1+\chi^{(1)})]^{1/2}\mathbf{E}_h^n) \\
&\leq \Delta t\,\|\mathbf{J}_h^n\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} + \Delta t\,\|\mathbf{E}_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}.
\end{aligned}
$$

Finally, using this estimate together with (63) in (62), we get

$$
\mathcal{E}_h^n - \mathcal{E}_h^{n-1} \leq \Delta t\,\|\mathbf{J}_h^n\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} + \Delta t\,\|\mathbf{E}_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}.
$$

Summing up from $n = 1$ to N, we arrive at

$$
\mathcal{E}_h^N - \mathcal{E}_h^0 \leq \sum_{n=1}^{N} \Delta t\,\|\mathbf{J}_h^n\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} + \sum_{n=1}^{N} \Delta t\,\|\mathbf{E}_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}. \tag{64}
$$

Therefore, we also have

$$
\mathcal{E}_h^N \leq \Delta t \sum_{n=1}^{N} \mathcal{E}_h^n + \Delta t \sum_{n=1}^{N} \|\mathbf{J}_h^n\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} + \mathcal{E}_h^0.
$$

Now we employ a discrete Gronwall's inequality [22, Lemma 5.1] (also cited in [6, Lemma 2]) with
$\delta := \Delta t \geq 0$,
$g_0 := \mathcal{E}_h^0$,
$a_n := \mathcal{E}_h^n$,
$b_n := 0$,
$c_0 := 0$, $c_n := \|\mathbf{J}_h^n\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} \geq 0$ for $n \in \mathbb{N}$, and
$\gamma_0 := 0$, $\gamma_n := 1 \geq 0$ for $n \in \mathbb{N}$.

If we only allow $\Delta t \leq \frac{1}{2}$, then the condition $\gamma_n \delta < 1$ from the cited Gronwall's lemma is clearly satisfied, and thus we get

$$\mathcal{E}_h^N \leq \left(\Delta t \sum_{n=1}^{N} \|\mathbf{J}_h^n\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} + \mathcal{E}_h^0 \right) \exp(2T).$$

Since the term $\Delta t \sum_{n=1}^{N} \|\mathbf{J}_h^n\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}}$ is an approximation to

$$\int_0^T \|\mathbf{J}_h\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} d\tau = \|\mathbf{J}_h\|^2_{\mathbf{L}^2(0,T,\mathbf{L}^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}}(\Omega))},$$

it is bounded. $\square$

In what follows we will make use of different variants for the representation of terms like $\mathbf{E}(t^n) - \mathbf{E}(t^{n-1})$. To this purpose we remember the Newton-Leibniz formula:

$$\mathbf{u}(t) = \mathbf{u}(t^{n-1}) + \int_{t^{n-1}}^{t} \partial_t \mathbf{u}(s) ds \quad \text{for all } \mathbf{u} \in C^1(0, T, X),$$

where X is a Banach space. In particular, for $t := t^n$ it holds that

$$\mathbf{u}(t^n) - \mathbf{u}(t^{n-1}) = \Delta t\, \mathbf{r}_{\mathbf{u}}^n \quad \text{with} \quad \mathbf{r}_{\mathbf{u}}^n := \frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} \partial_t \mathbf{u}(s) ds. \tag{65}$$

Furthermore, from Taylor's formula with integral remainder it follows that

$$\mathbf{u}(t) = \mathbf{u}(t^n) + \mathbf{u}_t(t^n)(t - t^n) + \int_{t^n}^{t} (t - s) \partial_{tt} \mathbf{u}(s) ds \quad \text{for all } \mathbf{u} \in C^2(0, T, X).$$

Hence, with $t = t^{n-1}$ we have:

$$\frac{\mathbf{u}(t^n) - \mathbf{u}(t^{n-1})}{\Delta t} = \partial_t \mathbf{u}(t^n) + \mathbf{R}_{\mathbf{u}}^n, \tag{66}$$

where

$$\mathbf{R}_{\mathbf{u}}^n := \frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} (t^{n-1} - s) \partial_{tt} \mathbf{u}(s) ds. \tag{67}$$

The remainder terms $\mathbf{r}_{\mathbf{u}}^n$, $\mathbf{R}_{\mathbf{u}}^n$ allow the following estimates.

Lemma 1. *Let X be a Banach space with the norm $\|\cdot\|_X$. The following estimates hold:*

(i) $\|\mathbf{r}_{\mathbf{u}}^n\|_X \leq \sqrt{\frac{1}{\Delta t}}\, \|\partial_t \mathbf{u}\|_{\mathbf{L}^2(t^{n-1}, t^n, X)}, \quad \mathbf{u} \in C^1(t^{n-1}, t^n, X),$

(ii) $\sum_{n=1}^{N} \|\mathbf{r}_{\mathbf{u}}^n\|_X^2 \leq \frac{1}{\Delta t} \|\partial_t \mathbf{u}\|^2_{\mathbf{L}^2(0,T,X)}, \quad \mathbf{u} \in C^1(0, T, X),$

(iii) $\|\mathbf{R}_{\mathbf{u}}^n\|_X \leq \sqrt{\frac{\Delta t}{3}}\, \|\partial_{tt} \mathbf{u}\|_{\mathbf{L}^2(t^{n-1}, t^n, X)}, \quad \mathbf{u} \in C^2(t^{n-1}, t^n, X),$

(iv) $\sum_{n=1}^{N} \|\mathbf{R}_{\mathbf{u}}^n\|_X^2 \leq \frac{\Delta t}{3} \|\partial_{tt} \mathbf{u}\|^2_{\mathbf{L}^2(0,T,X)}, \quad \mathbf{u} \in C^2(0, T, X).$

Proof. (i) By the definition (65), we have

$$\begin{aligned}
\|\mathbf{r}_{\mathbf{u}}^n\|_X &= \frac{1}{\Delta t} \left\| \int_{t^{n-1}}^{t^n} \partial_t \mathbf{u}(s) ds \right\|_X \leq \frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} \|\partial_t \mathbf{u}(s)\|_X ds \\
&\leq \frac{1}{\Delta t} \left(\int_{t^{n-1}}^{t^n} 1\, ds \right)^{1/2} \left(\int_{t^{n-1}}^{t^n} \|\partial_t \mathbf{u}(s)\|_X^2 ds \right)^{1/2} = \sqrt{\frac{1}{\Delta t}}\, \|\partial_t \mathbf{u}\|_{\mathbf{L}^2(t^{n-1}, t^n, X)}.
\end{aligned}$$

(ii) is a simple consequence of (i) and an elementar integral property:

$$\sum_{n=1}^{N} \|\mathbf{r}_{\mathbf{u}}^n\|_X^2 \leq \frac{1}{\Delta t} \sum_{n=1}^{N} \|\partial_t \mathbf{u}\|_{\mathbf{L}^2(t^{n-1},t^n,X)}^2 = \frac{1}{\Delta t} \|\partial_t \mathbf{u}\|_{\mathbf{L}^2(0,T,X)}^2.$$

(iii) From (67), we have

$$\begin{aligned}
\|\mathbf{R}_{\mathbf{u}}^n\|_X &= \frac{1}{\Delta t} \left\| \int_{t^{n-1}}^{t^n} (t^{n-1} - s)\partial_{tt}\mathbf{u}(s) ds \right\|_X \\
&\leq \frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} (s - t^{n-1}) \|\partial_{tt}\mathbf{u}(s)\|_X ds \\
&\leq \frac{1}{\Delta t} \left(\int_{t^{n-1}}^{t^n} (s - t^{n-1})^2 ds \right)^{1/2} \left(\int_{t^{n-1}}^{t^n} \|\partial_{tt}\mathbf{u}(s)\|_X^2 ds \right)^{1/2} \\
&= \sqrt{\frac{\Delta t}{3}} \|\partial_{tt}\mathbf{u}\|_{\mathbf{L}^2(t^{n-1},t^n,X)}.
\end{aligned}$$

(iv) This proof is analogous to the proof of (ii). □

6. Error Estimates for the Fully Discrete Nonlinear Problem

Before formulating the fully discrete theorem for the Lee-Madsen formulation, we introduce the error terms for the electric field as

$$\zeta^n := \mathbf{E}(t^n) - \mathbf{E}_h^n = \eta^n - \eta_h^n, \tag{68}$$

where

$$\eta^n := \mathbf{E}(t^n) - \mathbf{P}_{LM}\mathbf{E}(t^n), \quad \eta_h^n := \mathbf{E}_h^n - \mathbf{P}_{LM}\mathbf{E}(t^n). \tag{69}$$

Analogously, for the magnetic field we set

$$\xi^n := \mathbf{H}(t^n) - \mathbf{H}_h^n = \theta^n - \theta_h^n, \tag{70}$$

where

$$\theta^n := \mathbf{H}(t^n) - \Pi_{LM}\mathbf{H}(t^n), \quad \theta_h^n := \mathbf{H}_h^n - \Pi_{LM}\mathbf{H}(t^n). \tag{71}$$

Finally, we denote the discrete time derivative on the sequence $(\mathbf{E}_h^n)$ at t^n by

$$\partial_{\Delta t}\mathbf{E}_h^n := \frac{1}{\Delta t}\left[\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\right]. \tag{72}$$

Theorem 5. *Assume $\chi^{(1)}, \chi^{(3)} \in L^\infty(\Omega)$. Let $(\mathbf{E}, \mathbf{H})$ be the solution of (10)–(12) with $\mathbf{J} := 0$ such that, for some $k \in \mathbb{N}$,*

$$\begin{aligned}
&\mathbf{E} \in C^1(0,T,\mathbf{L}^\infty(\Omega) \cap \mathbf{H}^k(\Omega)),\ \partial_{tt}\mathbf{E} \in L^2(0,T,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega)), \\
&\mathbf{H} \in C^1(0,T,\mathbf{H}^{k+1}(\Omega)),\ \partial_{tt}\mathbf{H} \in L^2(0,T,\mathbf{L}^2_{\mu_0}(\Omega)),
\end{aligned}$$

and let $(\mathbf{E}_h^n, \mathbf{H}_h^n)$ be the fully discrete solution of (51)–(53) such that there is a constant $C^* > 0$ independent of Δt and h such that $\|\mathbf{E}_h^n\|_{\mathbf{L}^\infty(\Omega))} \leq C^*$ and $\|\partial_{\Delta t}\mathbf{E}_h^n\|_{\mathbf{L}^\infty(\Omega))} \leq C^*$ for all $n = 1, 2, \dots, N$. Then, for sufficiently small Δt and h, the following error estimate holds:

$$\|\mathbf{E}(T) - \mathbf{E}_h^N\|_{\varepsilon_0(1+\chi^{(1)})} + \|\mathbf{H}(T) - \mathbf{H}_h^N\|_{\mu_0} \leq C\left[h^k + \Delta t\right],$$

where the constant $C > 0$ does not depend on Δt and h (the concrete structure of C will be seen from the proof).

Proof. Eliminating in the equations (51)–(52) the difference term $\mathbf{D}_h^n - \mathbf{D}_h^{n-1}$, we obtain

$$\begin{aligned}
&(\varepsilon_0(1+\chi^{(1)})\Big(\frac{\mathbf{E}_h^n - \mathbf{E}_h^{n-1}}{\Delta t}\Big), \mathbf{\Psi}_h) - (\nabla\times\mathbf{H}_h^n, \mathbf{\Psi}_h)\\
&\quad + (\frac{1}{2}\varepsilon_0\chi^{(3)}((\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2)\Big(\frac{\mathbf{E}_h^n - \mathbf{E}_h^{n-1}}{\Delta t}\Big), \mathbf{\Psi}_h)\\
&\quad + \big(\varepsilon_0\chi^{(3)}\left(\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right)\Big(\frac{\mathbf{E}_h^n - \mathbf{E}_h^{n-1}}{\Delta t}\Big), \mathbf{\Psi}_h) = 0.
\end{aligned} \tag{73}$$

Taking $\mathbf{\Psi} := \mathbf{\Psi}_h$ and $t := t^n$ in the equations (10)–(11) and replacing the term $\partial_t\mathbf{E}(t^n)$ by means of (66), we have

$$\begin{aligned}
&(\varepsilon_0(1+\chi^{(1)})\Big(\frac{\mathbf{E}(t^n) - \mathbf{E}(t^{n-1})}{\Delta t}\Big), \mathbf{\Psi}_h) - (\nabla\times\mathbf{H}(t^n), \mathbf{\Psi}_h)\\
&\quad + \big(\varepsilon_0\chi^{(3)}|\mathbf{E}(t^n)|^2\Big(\frac{\mathbf{E}(t^n) - \mathbf{E}(t^{n-1})}{\Delta t}\Big), \mathbf{\Psi}_h\big)\\
&\quad + \big(\varepsilon_0\chi^{(3)}2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T\Big(\frac{\mathbf{E}(t^n) - \mathbf{E}(t^{n-1})}{\Delta t}\Big), \mathbf{\Psi}_h\big)\\
&= (\varepsilon_0(1+\chi^{(1)})\mathbf{R}_{\mathbf{E}}^n, \mathbf{\Psi}_h) + \big(\varepsilon_0\chi^{(3)}|\mathbf{E}(t^n)|^2\mathbf{R}_{\mathbf{E}}^n, \mathbf{\Psi}_h\big)\\
&\quad + \big(\varepsilon_0\chi^{(3)}2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T\,\mathbf{R}_{\mathbf{E}}^n, \mathbf{\Psi}_h\big).
\end{aligned} \tag{74}$$

Subtracting eq. (73) from eq. (74), adding to both sides the two terms

$$\begin{aligned}
&(\frac{1}{2}\varepsilon_0\chi^{(3)}\Big[(\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2\Big]\Big(\frac{\mathbf{E}(t^n) - \mathbf{E}(t^{n-1})}{\Delta t}\Big), \mathbf{\Psi}_h),\\
&\big(\varepsilon_0\chi^{(3)}\Big[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\Big]\Big(\frac{\mathbf{E}(t^n) - \mathbf{E}(t^{n-1})}{\Delta t}\Big), \mathbf{\Psi}_h),
\end{aligned}$$

and remembering the error terms (68), (70), we obtain

$$\begin{aligned}
&(\varepsilon_0(1+\chi^{(1)})\Big(\frac{\zeta^n - \zeta^{n-1}}{\Delta t}\Big), \mathbf{\Psi}_h) - (\nabla\times\xi^n, \mathbf{\Psi}_h)\\
&\quad + \big(\varepsilon_0\chi^{(3)}\Big[\frac{1}{2}\big((\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2\big)\Big]\Big(\frac{\zeta^n - \zeta^{n-1}}{\Delta t}\Big), \mathbf{\Psi}_h\big)\\
&\quad + \big(\varepsilon_0\chi^{(3)}\big[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\big]\Big(\frac{\zeta^n - \zeta^{n-1}}{\Delta t}\Big), \mathbf{\Psi}_h\big)\\
&= (\varepsilon_0(1+\chi^{(1)})\mathbf{R}_{\mathbf{E}}^n, \mathbf{\Psi}_h) + \big(\varepsilon_0\chi^{(3)}|\mathbf{E}(t^n)|^2\mathbf{R}_{\mathbf{E}}^n, \mathbf{\Psi}_h\big)\\
&\quad + \big(\varepsilon_0\chi^{(3)}2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T\,\mathbf{R}_{\mathbf{E}}^n, \mathbf{\Psi}_h\big)\\
&\quad - \varepsilon_0\chi^{(3)}\Big[\Big(\Big[|\mathbf{E}(t^n)|^2 - \frac{1}{2}[(\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2]\Big]\Big(\frac{\mathbf{E}(t^n) - \mathbf{E}(t^{n-1})}{\Delta t}\Big), \mathbf{\Psi}_h\Big)\\
&\quad - \big(\big[2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T - \big(\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\big)\big]\Big(\frac{\mathbf{E}(t^n) - \mathbf{E}(t^{n-1})}{\Delta t}\Big), \mathbf{\Psi}_h\big)\Big].
\end{aligned} \tag{75}$$

Next we take $\mathbf{\Phi} := \mathbf{\Phi}_h$ and $t := t^n$ in eq. (12), subtract the (53) from the result, and make use of (67). In terms of the quantities defined in (68) and (70), we obtain

$$\Big(\mu_0\frac{\xi^n - \xi^{n-1}}{\Delta t}, \mathbf{\Phi}_h\Big) + (\zeta^n, \nabla\times\mathbf{\Phi}_h) = (\mu_0\mathbf{R}_{\mathbf{H}}^n, \mathbf{\Phi}_h). \tag{76}$$

Using the decompositions $\zeta^n = \eta^n - \eta_h^n$, $\xi^n = \theta^n - \theta_h^n$ from (69) and (71), after a little rearrangement in the equations (75)–(76) we arrive at

$$
\begin{aligned}
&(\varepsilon_0(1+\chi^{(1)})\Big(\frac{(\eta^n-\eta^{n-1})-(\eta_h^n-\eta_h^{n-1})}{\Delta t}\Big), \mathbf{\Psi}_h) - (\nabla\times(\theta^n-\theta_h^n), \mathbf{\Psi}_h)\\
&\quad + \big(\varepsilon_0\chi^{(3)}\Big[\frac{1}{2}\big((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big)\Big]\Big(\frac{(\eta^n-\eta^{n-1})-(\eta_h^n-\eta_h^{n-1})}{\Delta t}\Big), \mathbf{\Psi}_h\big)\\
&\quad + \big(\varepsilon_0\chi^{(3)}\big[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\,\big[\mathbf{E}_h^{n-1}\big]^T\big]\Big(\frac{(\eta^n-\eta^{n-1})-(\eta_h^n-\eta_h^{n-1})}{\Delta t}\Big), \mathbf{\Psi}_h\big)\\
&= (\varepsilon_0(1+\chi^{(1)})\mathbf{R}_{\mathbf{E}}^n, \mathbf{\Psi}_h) + \big(\varepsilon_0\chi^{(3)}|\mathbf{E}(t^n)|^2\mathbf{R}_{\mathbf{E}}^n, \mathbf{\Psi}_h\big)\\
&\quad + \big(\varepsilon_0\chi^{(3)}2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T\,\mathbf{R}_{\mathbf{E}}^n, \mathbf{\Psi}_h\big)\\
&\quad - \big(\varepsilon_0\chi^{(3)}\Big[|\mathbf{E}(t^n)|^2 - \frac{1}{2}[(\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2]\Big]\Big(\frac{\mathbf{E}(t^n)-\mathbf{E}(t^{n-1})}{\Delta t}\Big), \mathbf{\Psi}_h\big)\\
&\quad - \big(\varepsilon_0\chi^{(3)}\big[2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T - \big(\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\,\big[\mathbf{E}_h^{n-1}\big]^T\big)\big]\Big(\frac{\mathbf{E}(t^n)-\mathbf{E}(t^{n-1})}{\Delta t}\Big), \mathbf{\Psi}_h\big)
\end{aligned}
$$

and

$$
\Big(\mu_0\frac{(\theta^n-\theta^{n-1})-(\theta_h^n-\theta_h^{n-1})}{\Delta t}, \mathbf{\Phi}_h\Big) + ((\eta^n-\eta_h^n), \nabla\times\mathbf{\Phi}_h) = (\mu_0\mathbf{R}_{\mathbf{H}}^n, \mathbf{\Phi}_h).
$$

Setting $\mathbf{\Psi}_h := 2\Delta t\,\eta_h^n$ and $\mathbf{\Phi}_h := 2\Delta t\theta_h^n$ in the above equations, we have

$$
\begin{aligned}
&2\,(\varepsilon_0(1+\chi^{(1)})(\eta_h^n-\eta_h^{n-1}), \eta_h^n) - 2\Delta t(\nabla\times\theta_h^n, \eta_h^n)\\
&\quad + 2\,\big(\varepsilon_0\chi^{(3)}\Big[\frac{1}{2}\big((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big)\Big](\eta_h^n-\eta_h^{n-1}), \eta_h^n\big)\\
&\quad + 2\,\big(\varepsilon_0\chi^{(3)}\big[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\,\big[\mathbf{E}_h^{n-1}\big]^T\big](\eta_h^n-\eta_h^{n-1}), \eta_h^n\big)\\
&= 2\,(\varepsilon_0(1+\chi^{(1)})(\eta^n-\eta^{n-1}), \eta_h^n) - 2\Delta t(\nabla\times\theta^n, \eta_h^n)\\
&\quad - 2\Delta t(\varepsilon_0(1+\chi^{(1)})\mathbf{R}_{\mathbf{E}}^n,\, \eta_h^n)\\
&\quad + 2\,\big(\varepsilon_0\chi^{(3)}\Big[\frac{1}{2}\big((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big)\Big](\eta^n-\eta^{n-1}), \eta_h^n\big)\\
&\quad + 2\,\big(\varepsilon_0\chi^{(3)}\big[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\,\big[\mathbf{E}_h^{n-1}\big]^T\big](\eta^n-\eta^{n-1}), \eta_h^n\big)\\
&\quad - 2\Delta t\,\big(\varepsilon_0\chi^{(3)}|\mathbf{E}(t^n)|^2\mathbf{R}_{\mathbf{E}}^n, \eta_h^n\big) - 2\Delta t\,\big(\varepsilon_0\chi^{(3)}2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T\,\mathbf{R}_{\mathbf{E}}^n, \eta_h^n\big)\\
&\quad + 2\big(\varepsilon_0\chi^{(3)}\Big[|\mathbf{E}(t^n)|^2 - \frac{1}{2}[(\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2]\Big]\big(\mathbf{E}(t^n)-\mathbf{E}(t^{n-1})\big), \eta_h^n\big)
\end{aligned}
$$

(77)

$$
\quad + 2\big(\varepsilon_0\chi^{(3)}\big[2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T - \big(\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\,\big[\mathbf{E}_h^{n-1}\big]^T\big)\big]\big(\mathbf{E}(t^n)-\mathbf{E}(t^{n-1})\big), \eta_h^n\big)
$$

and

$$
\begin{aligned}
&2\,\big(\mu_0(\theta_h^n-\theta_h^{n-1}), \theta_h^n\big) + 2\Delta t(\eta_h^n, \nabla\times\theta_h^n)\\
&= 2\,\big(\mu_0(\theta^n-\theta^{n-1}), \theta_h^n\big) + 2\Delta t(\eta^n, \nabla\times\theta_h^n) - (\mu_0\mathbf{R}_{\mathbf{H}}^n, 2\Delta t\theta_h^n).
\end{aligned}
\tag{78}
$$

The second terms from the left-hand sides of equations (77) and (78) vanish due to (31) and (28)–(29), respectively. Adding the equations (77) and (78), we obtain

$$
\begin{aligned}
&2(\varepsilon_0(1+\chi^{(1)})(\eta_h^n-\eta_h^{n-1}), \eta_h^n) + 2\,\big(\mu_0(\theta_h^n-\theta_h^{n-1}), \theta_h^n\big)\\
&\quad + 2\,\big(\varepsilon_0\chi^{(3)}\Big[\frac{1}{2}\big((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big)\Big](\eta_h^n-\eta_h^{n-1}), \eta_h^n\big)\\
&\quad + 2\,\big(\varepsilon_0\chi^{(3)}\big[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\,\big[\mathbf{E}_h^{n-1}\big]^T\big](\eta_h^n-\eta_h^{n-1}), \eta_h^n\big)
\end{aligned}
$$

$$
\begin{aligned}
&= 2(\varepsilon_0(1+\chi^{(1)})(\eta^n-\eta^{n-1}),\eta_h^n) + 2\big(\mu_0(\theta^n-\theta^{n-1}),\theta_h^n\big)\\
&\quad + 2\left(\varepsilon_0\chi^{(3)}\Big[\frac{1}{2}\big((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big)\Big](\eta^n-\eta^{n-1}),\eta_h^n\right)\\
&\quad + 2\left(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\left[\mathbf{E}_h^n\right]^T+\mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right](\eta^n-\eta^{n-1}),\eta_h^n\right)\\
&\quad - 2\Delta t(\varepsilon_0(1+\chi^{(1)})\mathbf{R}_{\mathbf{E}}^n,\,\eta_h^n) - 2\Delta t(\mu_0\mathbf{R}_{\mathbf{H}}^n,\theta_h^n)\\
&\quad - 2\Delta t\left(\varepsilon_0\chi^{(3)}|\mathbf{E}(t^n)|^2\mathbf{R}_{\mathbf{E}}^n,\eta_h^n\right) - 2\Delta t\left(\varepsilon_0\chi^{(3)}2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T\,\mathbf{R}_{\mathbf{E}}^n,\eta_h^n\right)\\
&\quad + 2\Big(\varepsilon_0\chi^{(3)}\Big[|\mathbf{E}(t^n)|^2-\frac{1}{2}[(\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2]\Big]\big(\mathbf{E}(t^n)-\mathbf{E}(t^{n-1})\big),\eta_h^n\Big)\\
&\quad + \Big(\big[2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T-\big(\mathbf{E}_h^n\left[\mathbf{E}_h^n\right]^T+\mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\big)\big]\big(\mathbf{E}(t^n)-\mathbf{E}(t^{n-1})\big),\eta_h^n\Big)\Big].
\end{aligned}
\tag{79}
$$

An Estimate of the Left-Hand Side at Level n. A well-known identity from Hilbert space theory (see e.g. [6, Lemma 1, 1)]) allows us to rewrite and estimate the first four terms on the left-hand side of (79) in the following way:

$$
2(\varepsilon_0(1+\chi^{(1)})(\eta_h^n-\eta_h^{n-1}),\eta_h^n) \ge \|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})} - \|\eta_h^{n-1}\|^2_{\varepsilon_0(1+\chi^{(1)})},
$$

and

$$
2\left(\mu_0(\theta_h^n-\theta_h^{n-1}),\theta_h^n\right) \ge \|\theta_h^n\|^2_{\mu_0} - \|\theta_h^{n-1}\|^2_{\mu_0}.
$$

In order to simplify the treatment of the third and fourth terms, we introduce the abbreviations

$$
\mathbf{C}_1^{n-\frac{1}{2}} := \frac{1}{2}\big[(\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big],\quad \mathbf{C}_2^{n-\frac{1}{2}} := \mathbf{E}_h^n\left[\mathbf{E}_h^n\right]^T+\mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T.
\tag{80}
$$

Then we have that

$$
\begin{aligned}
&2\left(\varepsilon_0\chi^{(3)}\Big[\frac{1}{2}\big((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big)\Big](\eta_h^n-\eta_h^{n-1}),\eta_h^n\right)\\
&\ge \left(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{n-\frac{1}{2}}\eta_h^n,\eta_h^n\right) - \left(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{n-\frac{1}{2}}\eta_h^{n-1},\eta_h^{n-1}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&2\left(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\left[\mathbf{E}_h^n\right]^T+\mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right](\eta_h^n-\eta_h^{n-1}),\eta_h^n\right)\\
&\ge \left(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{n-\frac{1}{2}}\eta_h^n,\eta_h^n\right) - \left(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{n-\frac{1}{2}}\eta_h^{n-1},\eta_h^{n-1}\right).
\end{aligned}
$$

Here we have used the fact that the matrices $\mathbf{C}_2^{n-\frac{1}{2}}$, $n=1,2,\ldots,N$, are positively semidefinite. In summary, the left-hand side of eq. (79) can be estimated from below as follows:

$$
\begin{aligned}
&\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})} - \|\eta_h^{n-1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^n\|^2_{\mu_0} - \|\theta_h^{n-1}\|^2_{\mu_0} + \left(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{n-\frac{1}{2}}\eta_h^n,\eta_h^n\right)\\
&\quad - \left(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{n-\frac{1}{2}}\eta_h^{n-1},\eta_h^{n-1}\right) + \left(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{n-\frac{1}{2}}\eta_h^n,\eta_h^n\right) - \left(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{n-\frac{1}{2}}\eta_h^{n-1},\eta_h^{n-1}\right)\\
&\le 2(\varepsilon_0(1+\chi^{(1)})(\eta_h^n-\eta_h^{n-1}),\eta_h^n) + 2\left(\mu_0(\theta_h^n-\theta_h^{n-1}),\theta_h^n\right)\\
&\quad + 2\left(\varepsilon_0\chi^{(3)}\big[\frac{1}{2}\big((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big)\big](\eta_h^n-\eta_h^{n-1}),\eta_h^n\right)\\
&\quad + 2\left(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\left[\mathbf{E}_h^n\right]^T+\mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right](\eta_h^n-\eta_h^{n-1}),\eta_h^n\right).
\end{aligned}
$$

So from eq. (79) we get the inequality

$$
\begin{aligned}
&\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})} - \|\eta_h^{n-1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^n\|^2_{\mu_0} - \|\theta_h^{n-1}\|^2_{\mu_0}\\
&\quad + \left(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{n-\frac{1}{2}}\eta_h^n,\eta_h^n\right) - \left(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{n-\frac{1}{2}}\eta_h^{n-1},\eta_h^{n-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \big(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{n-\frac{1}{2}}\eta_h^n, \eta_h^n\big) - \big(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{n-\frac{1}{2}}\eta_h^{n-1}, \eta_h^{n-1}\big)\\
\le\ & 2(\varepsilon_0(1+\chi^{(1)})(\eta^n - \eta^{n-1}), \eta_h^n) + 2\big(\mu_0(\theta^n - \theta^{n-1}), \theta_h^n\big)\\
&+ 2\left(\varepsilon_0\chi^{(3)}\left[\frac{1}{2}\big((\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2\big)\right](\eta^n - \eta^{n-1}), \eta_h^n\right)\\
&+ 2\left(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right](\eta^n - \eta^{n-1}), \eta_h^n\right)\\
&- 2\Delta t(\varepsilon_0(1+\chi^{(1)})\mathbf{R}_{\mathbf{E}}^n,\, \eta_h^n) - 2\Delta t(\mu_0\mathbf{R}_{\mathbf{H}}^n, \theta_h^n)\\
&- 2\Delta t\left(\varepsilon_0\chi^{(3)}|\mathbf{E}(t^n)|^2\mathbf{R}_{\mathbf{E}}^n, \eta_h^n\right) - 2\Delta t\left(\varepsilon_0\chi^{(3)}2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T\,\mathbf{R}_{\mathbf{E}}^n, \eta_h^n\right)\\
&+ 2\Big(\varepsilon_0\chi^{(3)}\Big[|\mathbf{E}(t^n)|^2 - \frac{1}{2}[(\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2]\Big]\big(\mathbf{E}(t^n) - \mathbf{E}(t^{n-1})\big), \eta_h^n\Big)\\
&+ \Big(\big[2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T - \big(\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\big)\big]\,\big(\mathbf{E}(t^n) - \mathbf{E}(t^{n-1})\big), \eta_h^n\Big)\Big].
\end{aligned}
$$

In order to simplify the further presentation, we denote the ten summands of the right-hand side in the specified order by $\tilde{\delta}_j^n$, $j = 1, \ldots, 10$ (the detailed definitions will be repeated later).

Now we sum up these inequalities from $n = 1$ to N:

$$
\begin{aligned}
&\|\eta_h^N\|^2_{\varepsilon_0(1+\chi^{(1)})} - \|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^N\|^2_{\mu_0} - \|\theta_h^0\|^2_{\mu_0}\\
&\quad+ \sum_{n=1}^{N}\Big[\big(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{n-\frac{1}{2}}\eta_h^n, \eta_h^n\big) - \big(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{n-\frac{1}{2}}\eta_h^{n-1}, \eta_h^{n-1}\big)\Big]\\
&\quad+ \sum_{n=1}^{N}\Big[\big(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{n-\frac{1}{2}}\eta_h^n, \eta_h^n\big) - \big(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{n-\frac{1}{2}}\eta_h^{n-1}, \eta_h^{n-1}\big)\Big] \le \sum_{j=1}^{10}\tilde{\delta}_j,
\end{aligned}
$$

where

$$
\tilde{\delta}_j := \sum_{n=1}^{N}\tilde{\delta}_j^n, \quad j = 1, \ldots, 10. \tag{81}
$$

Rewriting the two sums on the left-hand side ("discrete partial integration") we obtain

$$
\begin{aligned}
&\|\eta_h^N\|^2_{\varepsilon_0(1+\chi^{(1)})} - \|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^N\|^2_{\mu_0} - \|\theta_h^0\|^2_{\mu_0} + \big(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{N-\frac{1}{2}}\eta_h^N, \eta_h^N\big)\\
&\quad- \big(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{\frac{1}{2}}\eta_h^0, \eta_h^0\big) + \sum_{n=1}^{N-1}\big(\varepsilon_0\chi^{(3)}\big[\mathbf{C}_1^{n-\frac{1}{2}} - \mathbf{C}_1^{n+\frac{1}{2}}\big]\eta_h^n, \eta_h^n\big) + \big(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{N-\frac{1}{2}}\eta_h^N, \eta_h^N\big)\\
&\quad- \big(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{\frac{1}{2}}\eta_h^0, \eta_h^0\big) + \sum_{n=1}^{N-1}\big(\varepsilon_0\chi^{(3)}\big[\mathbf{C}_2^{n-\frac{1}{2}} - \mathbf{C}_2^{n+\frac{1}{2}}\big]\eta_h^n, \eta_h^n\big) \le \sum_{j=1}^{10}\tilde{\delta}_j.
\end{aligned}
$$

Setting

$$
\tilde{\delta}_{11}^n := \big(\varepsilon_0\chi^{(3)}\big[\mathbf{C}_1^{n+\frac{1}{2}} - \mathbf{C}_1^{n-\frac{1}{2}}\big]\eta_h^n, \eta_h^n\big), \quad \tilde{\delta}_{12}^n := \big(\varepsilon_0\chi^{(3)}\big[\mathbf{C}_2^{n+\frac{1}{2}} - \mathbf{C}_2^{n-\frac{1}{2}}\big]\eta_h^n, \eta_h^n\big),
$$

we get

$$
\begin{aligned}
&\|\eta_h^N\|^2_{\varepsilon_0(1+\chi^{(1)})} - \|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^N\|^2_{\mu_0} - \|\theta_h^0\|^2_{\mu_0} + \big(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{N-\frac{1}{2}}\eta_h^N, \eta_h^N\big)\\
&\quad- \big(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{\frac{1}{2}}\eta_h^0, \eta_h^0\big) + \big(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{N-\frac{1}{2}}\eta_h^N, \eta_h^N\big) - \big(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{\frac{1}{2}}\eta_h^0, \eta_h^0\big) \le \sum_{j=1}^{12}\tilde{\delta}_j,
\end{aligned} \tag{82}
$$

where $\tilde{\delta}_{11}$, $\tilde{\delta}_{12}$ are defined in analogy to (81).

An Estimate of the Right-Hand Side. The first to fourth terms on the right-hand side of the inequality (82) are treated by means of the formula (65). Replacing there $\mathbf{u}$ by $(\mathbf{I}-\mathbf{P}_{LM})\mathbf{E}$ and $(\mathbf{I}-\Pi_{LM})\mathbf{H}$, respectively, we obtain for the the first term

$$\begin{aligned}\tilde{\delta}_1^n &= 2\big(\varepsilon_0(1+\chi^{(1)})(\eta^n-\eta^{n-1}),\eta_h^n\big) = 2\Delta t\big(\varepsilon_0(1+\chi^{(1)})(\mathbf{I}-\mathbf{P}_{LM})\mathbf{r}_{\mathbf{E}}^n,\eta_h^n\big)\\ &= 2\Delta t\big(\varepsilon_0(1+\chi^{(1)})\mathbf{r}^n_{(\mathbf{I}-\mathbf{P}_{LM})\mathbf{E}},\eta_h^n\big)\\ &\le \Delta t\left[\|\mathbf{r}^n_{(\mathbf{I}-\mathbf{P}_{LM})\mathbf{E}}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}\right]\\ &\le \|\partial_t\big((\mathbf{I}-\mathbf{P}_{LM})\mathbf{E}\big)\|^2_{\mathbf{L}^2(t^{n-1},t^n,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))}+\Delta t\,\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}\\ &\qquad\text{(by Lemma 1(i))}\\ &\le C\varepsilon_0\|1+\chi^{(1)}\|_{L^\infty(\Omega)}\,h^{2k}\|\partial_t\mathbf{E}\|^2_{\mathbf{L}^2(t^{n-1},t^n,\mathbf{H}^k(\Omega))}+\Delta t\,\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}\\ &\qquad\text{(cf. (44)).}\end{aligned}$$

Thus we get

$$\tilde{\delta}_1=\sum_{n=1}^N\tilde{\delta}_1^n\le C\varepsilon_0\|1+\chi^{(1)}\|_{L^\infty(\Omega)}\,h^{2k}\|\partial_t\mathbf{E}\|^2_{\mathbf{L}^2(0,T,\mathbf{H}^k(\Omega))}+\Delta t\,S_\eta^N, \tag{83}$$

where

$$S_\eta^N:=\sum_{n=1}^N\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}.$$

Analogously, the second term on the right-hand side of the (82) can be written and estimated as

$$\begin{aligned}\tilde{\delta}_2^n &= 2\big(\mu_0(\theta^n-\theta^{n-1}),\theta_h^n\big)=2\Delta t\big(\mu_0(\mathbf{I}-\Pi_{LM})\mathbf{r}^n_{\mathbf{H}},\theta_h^n\big)\\ &\le C\mu_0h^{2k}\|\partial_t\mathbf{H}\|^2_{\mathbf{L}^2(t^{n-1},t^n,\mathbf{H}^{k+1}(\Omega))}+\Delta t\,\|\theta_h^n\|^2_{\mu_0},\end{aligned}$$

where we have used (65), Lemma 1(i) and (45). Hence

$$\tilde{\delta}_2\le C\mu_0h^{2k}\|\partial_t\mathbf{H}\|^2_{\mathbf{L}^2(0,T,\mathbf{H}^{k+1}(\Omega))}+\Delta t\,S_\theta^N, \tag{84}$$

where

$$S_\theta^N:=\sum_{n=1}^N\|\theta_h^n\|^2_{\mu_0}.$$

The third term from the right-hand side of the inequality (82) is estimated as

$$\begin{aligned}\tilde{\delta}_3^n &= \big(\varepsilon_0\chi^{(3)}\Big[(\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\Big](\eta^n-\eta^{n-1}),\eta_h^n\big)\\ &\le \Delta t\|(\mathbf{E}_h^n)\|^2_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\left[\|\mathbf{r}^n_{(\mathbf{I}-\mathbf{P}_{LM})\mathbf{E}}\|^2_{\varepsilon_0\chi^{(3)}}+\|\eta_h^n\|^2_{\varepsilon_0\chi^{(3)})}\right],\end{aligned}$$

where we have used the notation

$$\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}:=\max_{n=0,1,\dots,N}\|\mathbf{E}_h^n\|_{\mathbf{L}^\infty(\Omega)}.$$

Since

$$\begin{aligned}\|\mathbf{r}^n_{(\mathbf{I}-\mathbf{P}_{LM})\mathbf{E}}\|^2_{\varepsilon_0\chi^{(3)}} &\le \frac{1}{\Delta t}\|\partial_t\big((\mathbf{I}-\mathbf{P}_{LM})\mathbf{E}\big)\|^2_{\mathbf{L}^2(t^{n-1},t^n,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))}\\ &\le \frac{C}{\Delta t}\varepsilon_0\|\chi^{(3)}\|_{L^\infty(\Omega)}\,h^{2k}\|\partial_t\mathbf{E}\|^2_{\mathbf{L}^2(t^{n-1},t^n,\mathbf{H}^k(\Omega))},\\ \|\eta_h^n\|^2_{\varepsilon_0\chi^{(3)})} &= (\varepsilon_0\chi^{(3)}\eta_h^n,\eta_h^n)=\left(\varepsilon_0(1+\chi^{(1)})\frac{\chi^{(3)}}{1+\chi^{(1)}}\eta_h^n,\eta_h^n\right)\end{aligned}$$

$$\leq \|\chi^{(3)}\|_{L^\infty(\Omega)} \|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})},$$

we arrive at

$$\begin{aligned}\tilde{\delta}_3^n \leq\, & C\varepsilon_0 \|\chi^{(3)}\|_{L^\infty(\Omega)} \|(\mathbf{E}_h^n)\|^2_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\, h^{2k} \|\partial_t \mathbf{E}\|^2_{\mathbf{L}^2(t^{n-1},t^n,\mathbf{H}^k(\Omega))} \\ & + \|\chi^{(3)}\|_{L^\infty(\Omega)} \|(\mathbf{E}_h^n)\|^2_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))} \Delta t\, \|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}.\end{aligned}$$

This leads to

$$\begin{aligned}\tilde{\delta}_3 \leq\, & C\varepsilon_0 \|\chi^{(3)}\|_{L^\infty(\Omega)} \|(\mathbf{E}_h^n)\|^2_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\, h^{2k} \|\partial_t \mathbf{E}\|^2_{\mathbf{L}^2(0,T,\mathbf{H}^k(\Omega))} \\ & + \|\chi^{(3)}\|_{L^\infty(\Omega)} \|(\mathbf{E}_h^n)\|^2_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))} \Delta t\, S_\eta^N. \end{aligned} \tag{85}$$

The fourth term from the right-hand side of (82) is treated in a similar manner:

$$\begin{aligned}\tilde{\delta}_4^n &= 2\big(\varepsilon_0 \chi^{(3)} \big[\mathbf{E}_h^n \left[\mathbf{E}_h^n\right]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\big]\,(\eta^n - \eta^{n-1}), \eta_h^n\big) \\ &\leq 2\Delta t \|(\mathbf{E}_h^n)\|^2_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))} \left[\|\mathbf{r}^n_{(\mathbf{I}-\mathbf{P}_{LM})\mathbf{E}}\|^2_{\varepsilon_0\chi^{(3)}} + \|\eta_h^n\|^2_{\varepsilon_0\chi^{(3)}}\right],\end{aligned}$$

and, as in the estimation for $\tilde{\delta}_3$, this results in

$$\begin{aligned}\tilde{\delta}_4 \leq\, & C\varepsilon_0 \|\chi^{(3)}\|_{L^\infty(\Omega)} \|(\mathbf{E}_h^n)\|^2_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\, h^{2k} \|\partial_t \mathbf{E}\|^2_{\mathbf{L}^2(0,T,\mathbf{H}^k(\Omega))} \\ & + 2\|\chi^{(3)}\|_{L^\infty(\Omega)} \|(\mathbf{E}_h^n)\|^2_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))} \Delta t\, S_\eta^N. \end{aligned} \tag{86}$$

Now we turn to the consideration of the terms $\tilde{\delta}_5^n$ to $\tilde{\delta}_8^n$ containing the remainders $\mathbf{R}_\mathbf{E}^n$, $\mathbf{R}_\mathbf{H}^n$. For $\tilde{\delta}_5^n$ we have:

$$\tilde{\delta}_5^n = -2\Delta t(\varepsilon_0(1+\chi^{(1)})\mathbf{R}_\mathbf{E}^n,\, \eta_h^n) \leq \Delta t\, \|\mathbf{R}_\mathbf{E}^n\|^2_{\varepsilon_0(1+\chi^{(1)})} + \Delta t\, \|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}.$$

Then Lemma 1(iv) implies that

$$\tilde{\delta}_5 \leq \frac{(\Delta t)^2}{3} \|\partial_{tt}\mathbf{E}\|^2_{\mathbf{L}^2(0,T,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))} + \Delta t\, S_\eta^N. \tag{87}$$

A completely analogous argument shows that

$$\tilde{\delta}_6 \leq \frac{(\Delta t)^2}{3} \|\partial_{tt}\mathbf{H}\|^2_{\mathbf{L}^2(0,T,\mathbf{L}^2_{\mu_0}(\Omega))} + \Delta t\, S_\theta^N. \tag{88}$$

The estimate of $\tilde{\delta}_7^n$ runs as follows:

$$\begin{aligned}\tilde{\delta}_7^n &= -2\Delta t\left(\varepsilon_0\chi^{(3)}|\mathbf{E}(t^n)|^2 \mathbf{R}_\mathbf{E}^n, \eta_h^n\right) \\ &\leq \|\chi^{(3)}\|_{L^\infty(\Omega)} \|\mathbf{E}_h^n\|^2_{\mathbf{L}^\infty(\Omega)} \left[\Delta t\, \|\mathbf{R}_\mathbf{E}^n\|^2_{\varepsilon_0(1+\chi^{(3)})} + \Delta t\, \|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}\right].\end{aligned}$$

Then we get, using Lemma 1(iv) again, that

$$\begin{aligned}\tilde{\delta}_7 \leq\, & \frac{(\Delta t)^2}{3} \|\chi^{(3)}\|_{L^\infty(\Omega)} \|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))} \|\partial_{tt}\mathbf{E}\|^2_{\mathbf{L}^2(0,T,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))} \\ & + \|\chi^{(3)}\|_{L^\infty(\Omega)} \|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))} \Delta t\, S_\eta^N. \end{aligned} \tag{89}$$

For $\tilde{\delta}_8^n = -4\Delta t\left(\varepsilon_0\chi^{(3)}\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T\, \mathbf{R}_\mathbf{E}^n, \eta_h^n\right)$, it is easy to see that

$$\begin{aligned}\tilde{\delta}_8 \leq\, & 2\frac{(\Delta t)^2}{3} \|\chi^{(3)}\|_{L^\infty(\Omega)} \|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))} \|\partial_{tt}\mathbf{E}\|^2_{\mathbf{L}^2(0,T,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))} \\ & + 2\|\chi^{(3)}\|_{L^\infty(\Omega)} \|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))} \Delta t\, S_\eta^N \end{aligned} \tag{90}$$

holds. The estimation technique for $\tilde{\delta}_9^n$ and $\tilde{\delta}_{10}^n$ is similar to that for $\tilde{\delta}_3^n$ and $\tilde{\delta}_4^n$ in the sense that it is based on the remainders $\mathbf{r}_{\mathbf{E}}^n$, $\mathbf{r}_{\mathbf{H}}^n$. Namely, for $\tilde{\delta}_9^n$ we have, by (65), that

$$\begin{aligned}\tilde{\delta}_9^n &= 2\big(\varepsilon_0\chi^{(3)}\Big[|\mathbf{E}(t^n)|^2 - \frac{1}{2}[(\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2]\Big]\big(\mathbf{E}(t^n) - \mathbf{E}(t^{n-1})\big), \eta_h^n\big)\\ &= \Delta t\,\big(\varepsilon_0\chi^{(3)}\Big[2|\mathbf{E}(t^n)|^2 - [(\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2]\Big]\mathbf{r}_{\mathbf{E}}^n, \eta_h^n\big).\end{aligned}$$

Next we consider the term in the big square brackets (cf. (68)):

$$\begin{aligned}&2|\mathbf{E}(t^n)|^2 - [(\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2]\\ &= [\mathbf{E}(t^n) + \mathbf{E}_h^n]^T\eta^n - [\mathbf{E}(t^n) + \mathbf{E}_h^n]^T\eta_h^n + \Delta t\,[\mathbf{E}(t^n) + \mathbf{E}_h^{n-1}]^T\mathbf{r}_{\mathbf{E}}^n\\ &\quad + [\mathbf{E}(t^n) + \mathbf{E}_h^{n-1}]^T\eta^{n-1} - [\mathbf{E}(t^n) + \mathbf{E}_h^{n-1}]^T\eta_h^{n-1}.\end{aligned}$$

These five summands generate in a straightforward way a decomposition of $\tilde{\delta}_9^n$:

$$\tilde{\delta}_9^n = \sum_{j=1}^{5}\tilde{\delta}_{9j}^n.$$

The subsequent steps are devoted to the estimation of the five terms $\tilde{\delta}_{9j}^n$. We have that

$$\begin{aligned}\tilde{\delta}_{91}^n &= \Delta t\,\big(\varepsilon_0\chi^{(3)}[\mathbf{E}(t^n) + \mathbf{E}_h^n]^T\eta^n\mathbf{r}_{\mathbf{E}}^n, \eta_h^n\big)\\ &\le \Delta t\,\|\chi^{(3)}\|_{L^\infty(\Omega)}\big[\|\mathbf{E}(t^n)\|_{\mathbf{L}^\infty(\Omega)} + \|\mathbf{E}_h^n\|_{\mathbf{L}^\infty(\Omega)}\big]\\ &\quad\times\|\mathbf{r}_{\mathbf{E}}^n\|_{\mathbf{L}^\infty(\Omega)}\|\eta^n\|_{\varepsilon_0(1+\chi^{(1)})}\|\eta_h^n\|_{\varepsilon_0(1+\chi^{(1)})}.\end{aligned}$$

Since

$$\|\mathbf{E}(t^n)\|_{\mathbf{L}^\infty(\Omega)} + \|\mathbf{E}_h^n\|_{\mathbf{L}^\infty(\Omega)} \le \|\mathbf{E}\|_{C(0,T,\mathbf{L}^\infty(\Omega))} + \|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))},$$

$$\|\mathbf{r}_{\mathbf{E}}^n\|_{\mathbf{L}^\infty(\Omega)} \le \sqrt{\frac{1}{\Delta t}}\,\|\partial_t\mathbf{E}\|_{\mathbf{L}^2(t^{n-1},t^n,\mathbf{L}^\infty(\Omega))} \le \|\mathbf{E}\|_{C^1(0,T,\mathbf{L}^\infty(\Omega))},\tag{91}$$

we obtain

$$\begin{aligned}\tilde{\delta}_{91}^n &\le C\Delta t\,h^{2k}\,\varepsilon_0\|1+\chi^{(1)}\|_{L^\infty(\Omega)}\|\chi^{(3)}\|_{L^\infty(\Omega)}\Big[\|\mathbf{E}\|_{C(0,T,\mathbf{L}^\infty(\Omega))}\\ &\quad + \|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\Big]\|\mathbf{E}\|_{C^1(0,T,\mathbf{L}^\infty(\Omega))}\|\mathbf{E}\|^2_{C(0,T,\mathbf{H}^k(\Omega))}\\ &\quad + \frac{\Delta t}{2}\,\|\chi^{(3)}\|_{L^\infty(\Omega)}\big[\|\mathbf{E}\|_{C(0,T,\mathbf{L}^\infty(\Omega))} + \|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\big]\\ &\quad\times\|\mathbf{E}\|_{C^1(0,T,\mathbf{L}^\infty(\Omega))}\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}\\ &\qquad\qquad\text{(cf. (44))}.\end{aligned}$$

The treatment of $\tilde{\delta}_{92}^n$ is quite similar to $\tilde{\delta}_{91}^n$:

$$\begin{aligned}\tilde{\delta}_{92}^n &= -\Delta t\,\big(\varepsilon_0\chi^{(3)}[\mathbf{E}(t^n) + \mathbf{E}_h^n]^T\eta_h^n\mathbf{r}_{\mathbf{E}}^n, \eta_h^n\big)\\ &\le \Delta t\,\|\chi^{(3)}\|_{L^\infty(\Omega)}\big[\|\mathbf{E}\|_{C(0,T,\mathbf{L}^\infty(\Omega))} + \|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\big]\\ &\quad\times\|\mathbf{E}\|_{C^1(0,T,\mathbf{L}^\infty(\Omega))}\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}\\ &\qquad\qquad\text{(by (91))}.\end{aligned}$$

Next we see that

$$\begin{aligned}\tilde{\delta}_{93}^n &= (\Delta t)^2\big(\varepsilon_0\chi^{(3)}[\mathbf{E}(t^n) + \mathbf{E}_h^{n-1}]^T\mathbf{r}_{\mathbf{E}}^n\mathbf{r}_{\mathbf{E}}^n, \eta_h^n\big)\\ &\le (\Delta t)^2\|\chi^{(3)}\|_{L^\infty(\Omega)}\big[\|\mathbf{E}\|_{C(0,T,\mathbf{L}^\infty(\Omega))} + \|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\big]\end{aligned}$$

$$\times \|\mathbf{r}^n_{\mathbf{E}}\|_{\mathbf{L}^\infty(\Omega)} \|\mathbf{r}^n_{\mathbf{E}}\|_{\varepsilon_0(1+\chi^{(1)})} \|\eta^n_h\|_{\varepsilon_0(1+\chi^{(1)})}.$$

Hence it remains to observe that

$$\begin{aligned} \|\mathbf{r}^n_{\mathbf{E}}\|_{\mathbf{L}^\infty(\Omega)} &\leq \|\mathbf{E}\|_{C^1(0,T,\mathbf{L}^\infty(\Omega))} \quad \text{(by Lemma 1(i)) and} \\ \|\mathbf{r}^n_{\mathbf{E}}\|_{\varepsilon_0(1+\chi^{(1)})} \|\eta^n_h\|_{\varepsilon_0(1+\chi^{(1)})} &\leq \frac{1}{2}\Delta t\, \|\mathbf{r}^n_{\mathbf{E}}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{1}{2\Delta t}\, \|\eta^n_h\|^2_{\varepsilon_0(1+\chi^{(1)})} \\ &\quad \text{(by Young's inequality with } \alpha := \Delta t) \\ &\leq \frac{1}{2}\|\partial_t \mathbf{E}\|^2_{\mathbf{L}^2(t^{n-1},t^n,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))} + \frac{1}{2\Delta t}\, \|\eta^n_h\|^2_{\varepsilon_0(1+\chi^{(1)})} \\ &\quad \text{(by Lemma 1(i))}. \end{aligned}$$

So we get

$$\begin{aligned} \tilde{\delta}^n_{93} \leq{} & \frac{1}{2}(\Delta t)^2 \|\chi^{(3)}\|_{L^\infty(\Omega)} \left[\|\mathbf{E}\|_{C(0,T,\mathbf{L}^\infty(\Omega))} + \|(\mathbf{E}^n_h)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\right] \\ & \times \|\mathbf{E}\|_{C^1(0,T,\mathbf{L}^\infty(\Omega))} \|\partial_t \mathbf{E}\|^2_{\mathbf{L}^2(t^{n-1},t^n,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))} \\ & + \frac{1}{2}\|\chi^{(3)}\|_{L^\infty(\Omega)} \left[\|\mathbf{E}\|_{C(0,T,\mathbf{L}^\infty(\Omega))} + \|(\mathbf{E}^n_h)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\right] \\ & \times \|\mathbf{E}\|_{C^1(0,T,\mathbf{L}^\infty(\Omega))} \Delta t\, \|\eta^n_h\|^2_{\varepsilon_0(1+\chi^{(1)})}. \end{aligned}$$

The term

$$\tilde{\delta}^n_{94} = \Delta t \left(\varepsilon_0 \chi^{(3)} [\mathbf{E}(t^n) + \mathbf{E}^{n-1}_h]^T \eta^{n-1} \mathbf{r}^n_{\mathbf{E}}, \eta^n_h\right)$$

can be estimated as $\tilde{\delta}^n_{91}$ (with η^n replaced by η^{n-1}), thus

$$\begin{aligned} \tilde{\delta}^n_{94} \leq{} & C\Delta t\, h^{2k}\, \varepsilon_0 \|1+\chi^{(1)}\|_{L^\infty(\Omega)} \|\chi^{(3)}\|_{L^\infty(\Omega)} \Big[\|\mathbf{E}\|_{C(0,T,\mathbf{L}^\infty(\Omega))} \\ & + \|(\mathbf{E}^n_h)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\Big] \|\mathbf{E}\|_{C^1(0,T,\mathbf{L}^\infty(\Omega))} \|\mathbf{E}\|^2_{C(0,T,\mathbf{H}^k(\Omega))} \\ & + \frac{\Delta t}{2}\, \|\chi^{(3)}\|_{L^\infty(\Omega)} \left[\|\mathbf{E}\|_{C(0,T,\mathbf{L}^\infty(\Omega))} + \|(\mathbf{E}^n_h)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\right] \\ & \times \|\mathbf{E}\|_{C^1(0,T,\mathbf{L}^\infty(\Omega))} \|\eta^n_h\|^2_{\varepsilon_0(1+\chi^{(1)})}. \end{aligned}$$

Similarly

$$\tilde{\delta}^n_{95} = -\Delta t \left(\varepsilon_0 \chi^{(3)} [\mathbf{E}(t^n) + \mathbf{E}^{n-1}_h]^T \eta^{n-1}_h \mathbf{r}^n_{\mathbf{E}}, \eta^n_h\right)$$

is estimated as $\tilde{\delta}^n_{92}$ (with one of the terms η^n_h replaced by η^{n-1}_h):

$$\begin{aligned} \tilde{\delta}^n_{95} \leq{} & \frac{1}{2}\Delta t\, \|\chi^{(3)}\|_{L^\infty(\Omega)} \left[\|\mathbf{E}(t^n)\|_{\mathbf{L}^\infty(\Omega)} + \|\mathbf{E}^n_h\|_{\mathbf{L}^\infty(\Omega)}\right] \|\mathbf{E}\|_{C^1(0,T,\mathbf{L}^\infty(\Omega))} \\ & \times \left[\|\eta^{n-1}_h\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\eta^n_h\|^2_{\varepsilon_0(1+\chi^{(1)})}\right]. \end{aligned}$$

Summarizing the estimates of $\tilde{\delta}^n_{91}$ to $\tilde{\delta}^n_{95}$, we conclude that

$$\begin{aligned} \tilde{\delta}^n_9 \leq{} & C_1 \Delta t\, h^{2k} + C_2 (\Delta t)^2 \|\partial_t \mathbf{E}\|^2_{\mathbf{L}^2(t^{n-1},t^n,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))} \\ & + C_3 \Delta t \left[\|\eta^{n-1}_h\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\eta^n_h\|^2_{\varepsilon_0(1+\chi^{(1)})}\right], \end{aligned}$$

where the constant $C_1 > 0$ depends on ε_0, $\|1+\chi^{(1)}\|_{L^\infty(\Omega)}$, $\|\chi^{(3)}\|_{L^\infty(\Omega)}$, $\|(\mathbf{E}^n_h)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}$, $\|\mathbf{E}\|_{C^1(0,T,\mathbf{L}^\infty(\Omega))}$, $\|\mathbf{E}\|_{C(0,T,\mathbf{H}^k(\Omega))}$, the constant $C_2 > 0$ depends on $\|\chi^{(3)}\|_{L^\infty(\Omega)}$, $\|(\mathbf{E}^n_h)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}$, $\|\mathbf{E}\|_{C^1(0,T,\mathbf{L}^\infty(\Omega))}$, and the constant

$C_3 > 0$ depends on $\|\chi^{(3)}\|_{L^\infty(\Omega)}$, $\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}$, $\|\mathbf{E}\|_{C^1(0,T,\mathbf{L}^\infty(\Omega))}$. It follows that

$$\begin{aligned}\tilde{\delta}_9 = \sum_{n=1}^{N} \tilde{\delta}_9^n \leq C_1 T h^{2k} + C_2 (\Delta t)^2 \|\partial_t \mathbf{E}\|^2_{\mathbf{L}^2(0,T,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))} \\ + C_3 \Delta t\, \|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + 2C_3 \Delta t\, S_\eta^N. \end{aligned} \tag{92}$$

The term

$$\tilde{\delta}_{10}^n = 2\Delta t \left(\varepsilon_0 \chi^{(3)} \left[2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T - \left(\mathbf{E}_h^n \left[\mathbf{E}_h^n\right]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T \right)\right] \mathbf{r}_{\mathbf{E}}^n, \eta_h^n \right)$$

does not allow such a symmetric estimation argument as $\tilde{\delta}_9^n$. Here we start with

$$\begin{aligned} 2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T - \left(\mathbf{E}_h^n \left[\mathbf{E}_h^n\right]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T \right) = \mathbf{E}(t^n)[\mathbf{E}(t^n) - \mathbf{E}_h^n]^T \\ + [\mathbf{E}(t^n) - \mathbf{E}_h^n]\left[\mathbf{E}_h^n\right]^T + \mathbf{E}(t^n)[\mathbf{E}(t^n) - \mathbf{E}_h^{n-1}]^T + [\mathbf{E}(t^n) - \mathbf{E}_h^{n-1}]\left[\mathbf{E}_h^{n-1}\right]^T. \end{aligned}$$

From

$$\begin{aligned} \mathbf{E}(t^n) - \mathbf{E}_h^n &= \eta^n - \eta_h^n, \\ \mathbf{E}(t^n) - \mathbf{E}_h^{n-1} &= \mathbf{E}(t^n) - \mathbf{E}(t^{n-1}) + \mathbf{E}(t^{n-1}) - \mathbf{E}_h^{n-1} = \Delta t\, \mathbf{r}_{\mathbf{E}}^n + \eta^{n-1} - \eta_h^{n-1} \end{aligned}$$

we obtain:

$$\begin{aligned} 2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T - \left(\mathbf{E}_h^n \left[\mathbf{E}_h^n\right]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T \right) = \mathbf{E}(t^n)[\eta^n - \eta_h^n]^T + [\eta^n - \eta_h^n]\left[\mathbf{E}_h^n\right]^T \\ + \Delta t\, \mathbf{E}(t^n)[\mathbf{r}_{\mathbf{E}}^n]^T + \mathbf{E}(t^n)[\eta^{n-1} - \eta_h^{n-1}]^T + \Delta t\, [\mathbf{r}_{\mathbf{E}}^n]\left[\mathbf{E}_h^{n-1}\right]^T + [\eta^{n-1} - \eta_h^{n-1}]\left[\mathbf{E}_h^{n-1}\right]^T. \end{aligned}$$

This decomposition generates a decomposition of $\tilde{\delta}_{10}^n$ into ten terms in a natural way:

$$\tilde{\delta}_{10}^n = \sum_{j=1}^{10} \tilde{\delta}_{10j}^n,$$

where

$$\begin{aligned} \tilde{\delta}_{101}^n &:= 2\Delta t \left(\varepsilon_0 \chi^{(3)} \mathbf{E}(t^n)[\eta^n]^T\, \mathbf{r}_{\mathbf{E}}^n, \eta_h^n \right) \\ &\;\vdots \\ \tilde{\delta}_{1010}^n &:= -2\Delta t \left(\varepsilon_0 \chi^{(3)} \eta_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\, \mathbf{r}_{\mathbf{E}}^n, \eta_h^n \right). \end{aligned}$$

All these terms can be estimated similar to the terms $\tilde{\delta}_{9j}^n$ so that we get an analogous estimate:

$$\begin{aligned}\tilde{\delta}_{10} \leq 2C_1 T h^{2k} + 2C_2 (\Delta t)^2 \|\partial_t \mathbf{E}\|^2_{\mathbf{L}^2(0,T,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))} \\ + 2C_3 \Delta t\, \|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + 4C_3 \Delta t\, S_\eta^N. \end{aligned} \tag{93}$$

Finally we have to deal with the terms $\tilde{\delta}_{11}^n$ and $\tilde{\delta}_{12}^n$. Due to (80) it holds that

$$\mathbf{C}_1^{n+\frac{1}{2}} - \mathbf{C}_1^{n-\frac{1}{2}} = \frac{1}{2}\Delta t \big(\mathbf{E}_h^n + \mathbf{E}_h^{n-2}\big) \left(\frac{\mathbf{E}_h^n - \mathbf{E}_h^{n-1}}{\Delta t} + \frac{\mathbf{E}_h^{n-1} - \mathbf{E}_h^{n-2}}{\Delta t} \right).$$

By means of the discrete time derivative (72) this relation can be written as

$$\mathbf{C}_1^{n+\frac{1}{2}} - \mathbf{C}_1^{n-\frac{1}{2}} = \frac{1}{2}\Delta t \big(\mathbf{E}_h^n + \mathbf{E}_h^{n-2}\big)\big(\partial_{\Delta t}\mathbf{E}_h^n + \partial_{\Delta t}\mathbf{E}_h^{n-1}\big),$$

and it follows that

$$\|\mathbf{C}_1^{n+\frac{1}{2}} - \mathbf{C}_1^{n-\frac{1}{2}}\|_{\mathbf{L}^\infty(\Omega)} \leq \frac{1}{2}\Delta t\, \|\mathbf{E}_h^n + \mathbf{E}_h^{n-2}\|_{\mathbf{L}^\infty(\Omega)} \|\partial_{\Delta t}\mathbf{E}_h^n + \partial_{\Delta t}\mathbf{E}_h^{n-1}\|_{\mathbf{L}^\infty(\Omega)}$$

$$\leq 2\Delta t\,\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\|(\partial_{\Delta t}\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}.$$

Thus we get

$$\begin{aligned}\tilde{\delta}_{11}^n &= \left(\varepsilon_0\chi^{(3)}\left[\mathbf{C}_1^{n+\frac{1}{2}} - \mathbf{C}_1^{n-\frac{1}{2}}\right]\eta_h^n, \eta_h^n\right)\\ &\leq \|\chi^{(3)}\|_{L^\infty(\Omega)}\|\mathbf{C}_1^{n+\frac{1}{2}} - \mathbf{C}_1^{n-\frac{1}{2}}\|_{\mathbf{L}^\infty(\Omega)}\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}\\ &\leq 2\Delta t\,\|\chi^{(3)}\|_{L^\infty(\Omega)}\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\|(\partial_{\Delta t}\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}.\end{aligned}$$

The summation over n from 1 to $N-1$ gives

$$\tilde{\delta}_{11} = \sum_{n=1}^{N-1}\tilde{\delta}_{11}^n \leq 2\|\chi^{(3)}\|_{L^\infty(\Omega)}\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\|(\partial_{\Delta t}\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\Delta t\,S_\eta^N. \tag{94}$$

The estimate of

$$\tilde{\delta}_{12}^n = \left(\varepsilon_0\chi^{(3)}\left[\mathbf{C}_2^{n+\frac{1}{2}} - \mathbf{C}_2^{n-\frac{1}{2}}\right]\eta_h^n, \eta_h^n\right)$$

runs in the same way. By (80) we have that

$$\tilde{\delta}_{12}^n \leq 4\Delta t\,\|\chi^{(3)}\|_{L^\infty(\Omega)}\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\|(\partial_{\Delta t}\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}.$$

So we get

$$\tilde{\delta}_{12} = \sum_{n=1}^{N-1}\tilde{\delta}_{11}^n \leq 4\|\chi^{(3)}\|_{L^\infty(\Omega)}\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\|(\partial_{\Delta t}\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\Delta t\,S_\eta^N. \tag{95}$$

Now we are ready to summarize the right-hand side of the inequality (82):

$$\begin{aligned}\sum_{j=1}^{12}\tilde{\delta}_j \leq{}& 3C_3\Delta t\,\|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + C_4h^{2k} + C_5(\Delta t)^2\\ &+ C_6\Delta t\,S_\eta^N + 2\Delta t\,S_\theta^N,\end{aligned}$$

where the constant $C_4 > 0$ depends on T, ε_0, $\|1+\chi^{(1)}\|_{L^\infty(\Omega)}$, $\|\chi^{(3)}\|_{L^\infty(\Omega)}$, $\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}$, $\|\partial_t\mathbf{E}\|_{\mathbf{L}^2(0,T,\mathbf{H}^k(\Omega))}$, $\|\mathbf{E}\|_{C^1(0,T,\mathbf{L}^\infty(\Omega))}$, $\|\mathbf{E}\|_{C(0,T,\mathbf{H}^k(\Omega))}$, $\|\partial_t\mathbf{H}\|_{\mathbf{L}^2(0,T,\mathbf{H}^{k+1}(\Omega))}$, the constant $C_5 > 0$ depends on $\|\chi^{(3)}\|_{L^\infty(\Omega)}$, $\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}$, $\|\partial_t\mathbf{E}\|_{\mathbf{L}^2(0,T,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))}$, $\|\partial_{tt}\mathbf{E}\|_{\mathbf{L}^2(0,T,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))}$, $\|\mathbf{E}\|_{C^1(0,T,\mathbf{L}^\infty(\Omega))}$, $\|\partial_{tt}\mathbf{H}\|_{\mathbf{L}^2(0,T,\mathbf{L}^2_{\mu_0}(\Omega))}$, and the constant $C_6 > 0$ depends on $\|\chi^{(3)}\|_{L^\infty(\Omega)}$, $\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}$, $\|(\partial_{\Delta t}\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}$, $\|\mathbf{E}\|_{C^1(0,T,\mathbf{L}^\infty(\Omega))}$.

So we get from the inequality (82):

$$\begin{aligned}&\|\eta_h^N\|^2_{\varepsilon_0(1+\chi^{(1)})} + \left(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{N-\frac{1}{2}}\eta_h^N, \eta_h^N\right) + \left(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{N-\frac{1}{2}}\eta_h^N, \eta_h^N\right) + \|\theta_h^N\|^2_{\mu_0}\\ &\quad\leq 3C_3\Delta t\,\|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \left(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{\frac{1}{2}}\eta_h^0, \eta_h^0\right)\\ &\quad+ \left(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{\frac{1}{2}}\eta_h^0, \eta_h^0\right) + \|\theta_h^0\|^2_{\mu_0} + C_4h^{2k} + C_5(\Delta t)^2 + C_6\Delta t\,S_\eta^N + 2\Delta t\,S_\theta^N.\end{aligned} \tag{96}$$

Making use of the facts that

$$\left(\varepsilon_0\chi^{(3)}\mathbf{C}_j^{N-\frac{1}{2}}\eta_h^N, \eta_h^N\right) \geq 0,$$

$$\left(\varepsilon_0\chi^{(3)}\mathbf{C}_j^{\frac{1}{2}}\eta_h^0, \eta_h^0\right) \leq j\|\chi^{(3)}\|_{L^\infty(\Omega)}\|(\mathbf{E}_h^n)\|^2_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})}, \quad j = 1, 2,$$

(cf. the estimates of $\tilde{\delta}_{11}^n$ and $\tilde{\delta}_{12}^n$), we finally conclude from (96) that

$$\|\eta_h^N\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^N\|^2_{\mu_0}$$

$$
\begin{aligned}
&\le 3C_3\Delta t\, \|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} \\
&\quad + 3\|\chi^{(3)}\|_{L^\infty(\Omega)}\|(\mathbf{E}_h^n)\|^2_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^0\|^2_{\mu_0} \\
&\quad + C_4h^{2k} + C_5(\Delta t)^2 + C_6\Delta t\, S_\eta^N + 2\Delta t\, S_\theta^N \\
&\le C_7\|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^0\|^2_{\mu_0} + C_4h^{2k} + C_5(\Delta t)^2 + C_6\Delta t\, S_\eta^N + 2\Delta t\, S_\theta^N , \qquad (97)
\end{aligned}
$$

where $C_7 := 3C_3 + 1 + 3\|\chi^{(3)}\|_{L^\infty(\Omega)}\|(\mathbf{E}_h^n)\|^2_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}$. Here we have used that Δt can be bounded by 1, for instance, without loss of generality.

It remains to apply Gronwall's inequality [22, Lemma 5.1] (also cited in [6, Lemma 2]) with
$\delta := \Delta t \ge 0,$
$g_0 := C_7\|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^0\|^2_{\mu_0} + C_4h^{2k} + C_5(\Delta t)^2 \ge 0,$
$a_n := \|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^n\|^2_{\mu_0} \ge 0,$
$b_n := c_n := 0,$
$\gamma_0 := 0,\ \gamma_n := \gamma := \max\{C_6; 2\} \ge 0$ for $n \in \mathbb{N}$.

Then the condition $\gamma\delta < 1$ gives some (uniform) restriction to Δt. If we even require that $\Delta t < (2\max\{C_6; 2\})^{-1}$, then Gronwall's inequality leads to

$$
\begin{aligned}
&\|\eta_h^N\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^N\|^2_{\mu_0} \\
&\le \left[C_7\|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^0\|^2_{\mu_0} + C_4h^{2k} + C_5(\Delta t)^2\right] \exp\left(\gamma\Delta t \sum_{n=1}^{N}(1-\gamma\Delta t)^{-1}\right) \\
&\le \left[C_7\|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^0\|^2_{\mu_0} + C_4h^{2k} + C_5(\Delta t)^2\right] \exp\left(2\gamma T\right).
\end{aligned}
$$

If we take $\mathbf{E}_h^0 := \mathbf{P}_{LM}\mathbf{E}(0) = \mathbf{P}_{LM}\mathbf{E}_0$ and $\mathbf{H}_h^0 := \Pi_{LM}\mathbf{H}(0) = \Pi_{LM}\mathbf{H}_0$, we obtain

$$
\|\eta_h^N\|_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^N\|_{\mu_0} \le C\left[h^k + \Delta t\right]\exp\left(\gamma T\right),
$$

where the constant $C > 0$ involves all the dependencies of the above constants C_1 to C_7. Finally, by the triangle inequality, we see that

$$
\begin{aligned}
&\|\mathbf{E}(t^N) - \mathbf{E}_h^N\|_{\varepsilon_0(1+\chi^{(1)})} + \|\mathbf{H}(t^N) - \mathbf{H}_h^N\|_{\mu_0} \\
&\le \|(\mathbf{I} - \mathbf{P}_{LM})\mathbf{E}(t^N)\|_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_h^N\|_{\varepsilon_0(1+\chi^{(1)})} + \|(\mathbf{I} - \Pi_{LM})\mathbf{H}(t^N)\|_{\mu_0} + \|\theta_h^N\|_{\mu_0},
\end{aligned}
$$

so the estimates (30) and (32) imply that

$$
\begin{aligned}
&\|\mathbf{E}(t^N) - \mathbf{E}_h^N\|_{\varepsilon_0(1+\chi^{(1)})} + \|\mathbf{H}(t^N) - \mathbf{H}_h^N\|_{\mu_0} \\
&\le C\sqrt{\varepsilon_0}\, h^k\|\mathbf{E}\|_{C(0,T,\mathbf{H}^k(\Omega))} + C\sqrt{\mu_0}\, h^k\|\mathbf{H}\|_{C(0,T,\mathbf{H}^{k+1}(\Omega))} \\
&\quad + C\left[h^k + \Delta t\right]\exp\left(\gamma T\right).
\end{aligned}
$$

□

This theorem shows that the fully discrete (backward Euler-type) method for the nonlinear Maxwell's equations is unconditionally stable in the sense that there is no restriction to the relation between time step size and spatial grid size.

7. Conclusion

The paper summarizes investigations of time domain finite element methods which extend our results for the linear Maxwell's equations [6], [4], [3] and [5] to the case of a Kerr-type nonlinearity. Under reasonable assumptions, we could prove that the semi-discrete and the fully discrete finite element approximations possess bounded energies and converge to the weak solution of the system of nonlinear Maxwell's equations.

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