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# CONVERGENCE ANALYSIS OF NITSCHE EXTENDED FINITE ELEMENT METHODS FOR H(CURL)-ELLIPTIC INTERFACE PROBLEMS

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**Abstract.** An H(curl)-conforming Nitsche extended finite element method is proposed for H(curl)-elliptic interface problems in three dimensional Lipschitz domains with smooth interfaces. Under interface-unfitted meshes, the continuous problems are discretized by an H(curl)-conforming extended finite element space, which is constructed based on the the lowest order of second family Nédélec edge elements (Whitney elements). A stabilization term defined on transmission faces is added to the standard discrete bilinear form. Stability results and the optimal error estimate in the parameter-dependent H(curl)-norm are derived, which are both uniform with respect to not only the mesh size and the interface position but also the physical parameters. Numerical experiments are carried out to validate theoretical results.

**Key words.** Nitsche extended finite element method, H(curl)-elliptic interface problems, interface-unfitted meshes, the lowest order of second family Nédélec edge elements.

## 1. Introduction

A motivation for considering H(curl)-elliptic interface problems comes from the modeling of electromagnetic fields. In some electric machine applications, engineers need to solve an H(curl)-elliptic interface problem at each time step. Due to the large variety of applications in scientific computing and engineering, there have been a lot of work about the numerical approximations and convergence analyses for time-dependent Maxwell interface equations, stationary Maxwell interface equations and also other related models, such as [10], [13], [11], [33], [26], [28], [14], [30], [29], [19], [34], [3], [4], [22] and so on.

Among these papers, there are fitted-mesh methods ([28], [30], [29]), extended finite element methods with unfitted-meshes ([33]), adaptive immersed finite element methods with unfitted-meshes ([13]), Lagrange multiplier methods ([11], [3], [4] )and so on. The optimal error estimates were obtained under interface-fitted meshes in [28], [30], [29]. Unfortunately, it is usually a time-consuming and non-trivial task to construct a good fitted-mesh for problems with moving interface or geometrically complicated interface. To avoid the expensive remeshing requirements, researchers pay more attention to unfitted-mesh methods. In this paper, we focus on one kind of interface-unfitted mesh methods−the extended finite element method.

The extended finite element method (XFEM) was first proposed by T. Belytschko and T. Black in [1] to deal with elastic equations in a cracked domain. In [23], A. Hansbo and P. Hansbo combined this method with Nitsche's method together, introduced a new method named Nitsche-XFEM. They successfully applied this new method to elliptic interface problems and obtained optimal error estimates independently of the interface position with respect to the mesh. Later, Nitsche-XFEM was taken to solve other elasticity and Stokes interface problems, such as

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[24], [31], [15], [32] and so on. As for the time-harmonic Maxwell equations, authors ([33]) study XFEM in two dimensional domains.

In this paper, we propose an H(curl)-conforming Nitsche extended finite element method for the H(curl)-elliptic interface problems in three dimensions. The extended finite element space is based on the lowest order of second family Nédélec edge elements. The discrete approximation scheme is formed by the standard bilinear formulation and a stabilization term defined on the transmission faces. By the help of the stabilization term, stable results and the optimal convergent order are derived, independent of not only the mesh size but also the interface position. Harmonic weights (see [41]) are applied in this paper, which make sure that all results are robust with respect to the physical parameters. In addition, comparing with the Lagrange multiplier method, we also have fewer degrees of freedom.

The layout of this paper is organized as follows. In Section 2, we define some notations, give the weak form of the original H(curl)-elliptic interface problem and construct its discrete formulation. Section 3 introduces some necessary assumptions and auxiliary lemmas. The stability properties containing the continuity and the coercivity are analyzed in Section 4. Section 5 shows the optimal error estimation under a parameter-dependent H(curl)-norm. Numerical experiments are presented in Section 6 to validate the theoretical results. Section 7 discusses the final conclusion.

Throughout this paper, we use bold typefaces to distinguish vectors from scalars, such as $\mathbf{E}$ and $\mathbf{H^2}(\Omega)$, denoting a vector function $\mathbf{E} = (E_1,\ E_2,\ E_3)$ and a vector space $\mathbf{H^2}(\Omega) = [H^2(\Omega)]^3$, respectively. $\mathbf{x} = (x_1,\ x_2,\ x_3) \in \mathbb{R}^3$ denotes the position of one point in the three dimensional space. Constants $c$ or $C$ with or without subscripts will be used to denote different positive constants which are independent of the mesh size, the physical parameters, and the interface location relative to the mesh.

## 2. H(curl)-elliptic interface problem

**2.1. Weak formulation.** Consider the following H(curl)-elliptic interface problem in the domain $\Omega \subseteq \mathbb{R}^3$

$$
\begin{aligned}
\mathbf{curl}(\alpha \mathbf{curl}\ \mathbf{u}) + \beta \mathbf{u} &= \mathbf{f} \text{ in } \Omega_1 \cup \Omega_2,\\
[\mathbf{n}_\Gamma \times \mathbf{u}] &= \mathbf{0} \text{ on } \Gamma,\\
[\mathbf{n}_\Gamma \times (\alpha \mathbf{curl}\ \mathbf{u})] &= \mathbf{0} \text{ on } \Gamma,\\
\mathbf{n} \times \mathbf{u} &= \mathbf{0} \text{ on } \partial\Omega,
\end{aligned}
\tag{1}
$$

where $\Gamma$ is a $C^2$-smooth boundary of a simple connected Lipschitz polyhedral domain $\Omega_1$ with $\overline{\Omega}_1 \subseteq \Omega$ and $\Omega_2 = \Omega \setminus \overline{\Omega}_1$, vectors $\mathbf{n}_\Gamma$, $\mathbf{n}$ represent the unit normal vector on $\Gamma$ pointing from $\Omega_1$ to $\Omega_2$ and the unit outward normal vector of $\partial\Omega$ respectively, see Figure 1. For a suitable scalar function $v$, its jump across the interface is defined by $[v] = v|_{\Omega_1} - v|_{\Omega_2}$ , and a component-wise application to a vector function. $\alpha,\ \beta$ are related physical parameters. For simplicity, we only concern about the case with $\beta$ being a strictly positive constant and $\alpha$ being a piecewise constant in the domain $\Omega$, namely

$$
\alpha = \begin{cases} \alpha_1 \text{ in } \Omega_1,\\ \alpha_2 \text{ in } \Omega_2.\end{cases}
$$

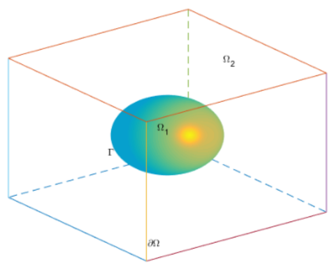



FIGURE 1. The sketch of a domain with interface.

We introduce the following Sobolev spaces

$$\mathbf{H}(\mathbf{curl};\Omega) = \{\mathbf{v} \in \mathbf{L^2}(\Omega); \mathbf{curl}\ \mathbf{v} \in \mathbf{L^2}(\Omega)\},$$

$$\mathbf{H_0}(\mathbf{curl};\Omega) = \{\mathbf{v} \in \mathbf{H}(\mathbf{curl};\Omega); \mathbf{n} \times \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega\},$$

$$\mathbf{H^1}(\mathbf{curl};\Omega) = \{\mathbf{v} \in \mathbf{H^1}(\Omega); \mathbf{curl}\ \mathbf{v} \in \mathbf{H^1}(\Omega)\},$$

$$\mathbf{H^1}(\mathbf{curl};\Omega_1 \cup \Omega_2) = \{\mathbf{v} \in \mathbf{L^2}(\Omega); \mathbf{v}|_{\Omega_i} \in \mathbf{H^1}(\mathbf{curl};\Omega_i),\ i = 1,2\},$$

$$\mathbf{H^2}(\Omega_1 \cup \Omega_2) = \{\mathbf{v} \in \mathbf{L^2}(\Omega); \mathbf{v}|_{\Omega_i} \in \mathbf{H^2}(\Omega_i),\ i = 1,2\}.$$

The spaces $\mathbf{H}(\mathbf{curl};\Omega)$, $\mathbf{H^1}(\mathbf{curl};\Omega)$, $\mathbf{H^1}(\mathbf{curl};\Omega_1 \cup \Omega_2)$ and $\mathbf{H^2}(\Omega_1 \cup \Omega_2)$ are equipped with the following norms respectively

$$||\mathbf{v}||_{\mathbf{H}(\mathbf{curl};\Omega)} = (||\mathbf{v}||^2_{0,\Omega} + ||\mathbf{curl}\ \mathbf{v}||^2_{0,\Omega})^{\frac{1}{2}},$$

$$||\mathbf{v}||_{\mathbf{H^1}(\mathbf{curl};\Omega)} = (||\mathbf{v}||^2_{1,\Omega} + ||\mathbf{curl}\ \mathbf{v}||^2_{1,\Omega})^{\frac{1}{2}},$$

$$||\mathbf{v}||_{\mathbf{H^1}(\mathbf{curl};\Omega_1\cup\Omega_2)} = (||\mathbf{v}||^2_{\mathbf{H^1}(\mathbf{curl};\Omega_1)} + ||\mathbf{v}||^2_{\mathbf{H^1}(\mathbf{curl};\Omega_2)})^{\frac{1}{2}},$$

$$||\mathbf{v}||_{\mathbf{H^2}(\Omega_1\cup\Omega_2)} = (||\mathbf{v}||^2_{2,\Omega_1} + ||\mathbf{v}||^2_{2,\Omega_2})^{\frac{1}{2}},$$

where the norm notations $||v||_{0,\Omega}$, $||v||_{1,\Omega}$ and $||v||_{2,\Omega}$ used in this paper are all standard, refer to [2] for their precise definitions.

The weak formulation of the problem (1) is: find $\mathbf{u} \in \mathbf{H_0}(\mathbf{curl};\Omega) \triangleq \mathbf{V}$ such that (throughout this paper, we use $(\cdot,\cdot)_\omega$ to denote the $L^2$-inner product in the domain $\omega$)

$$a(\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v})_\Omega,\ \forall\ \mathbf{v} \in \mathbf{V}, \tag{2}$$

where

$$a(\mathbf{u},\mathbf{v}) = (\alpha\mathbf{curl}\ \mathbf{u}, \mathbf{curl}\ \mathbf{v})_\Omega + (\beta\mathbf{u},\mathbf{v})_\Omega.$$

Define a parameter-dependent norm for any $\mathbf{v} \in \mathbf{V}$,

$$||\mathbf{v}||_{\alpha,\beta,\mathrm{curl},\Omega} = (||\beta^{\frac{1}{2}}\mathbf{v}||^2_{0,\Omega} + ||\alpha^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}||^2_{0,\Omega})^{\frac{1}{2}}. \tag{3}$$

By Cauchy-Schwarz inequality, for any $\mathbf{u},\mathbf{v} \in \mathbf{V}$, we derive

$$a(\mathbf{u},\mathbf{v}) \le ||\mathbf{u}||_{\alpha,\beta,\mathrm{curl},\Omega}||\mathbf{v}||_{\alpha,\beta,\mathrm{curl},\Omega} \tag{4}$$

and

$$a(\mathbf{v},\mathbf{v}) = ||\mathbf{v}||^2_{\alpha,\beta,\mathrm{curl},\Omega}. \tag{5}$$

According to the Lax-Milgram theorem, the weak formulation (2) admits a unique solution.

**2.2. Notation.** Let $\mathcal{T}_h$ be a quasi-uniform tetrahedral mesh of $\Omega$. For any element $K \in \mathcal{T}_h$, $h_K = \mathrm{diam}(K)$ denotes the diameter of $K$, and the maximum value of all diameters $h = \max\limits_{K \in \mathcal{T}_h} h_K$ is called mesh size. Notice the quasi-uniformity of the mesh, there always holds $h \simeq h_K$. Note that any element $K \in \mathcal{T}_h$ is considered as closed. Define the set of interface elements by $\mathcal{T}_h^\Gamma = \{K \in \mathcal{T}_h : K \cap \Gamma \neq \emptyset\}$, see Figure 2, in a two dimensional setting for an illustration. For any interface element $K \in \mathcal{T}_h^\Gamma$, $K_i = K \cap \Omega_i$ means the part of $K$ located in $\Omega_i$ $(i = 1, 2)$ and $\Gamma_K = \Gamma \cap K$ represents the restriction of the interface $\Gamma$ in $K$.

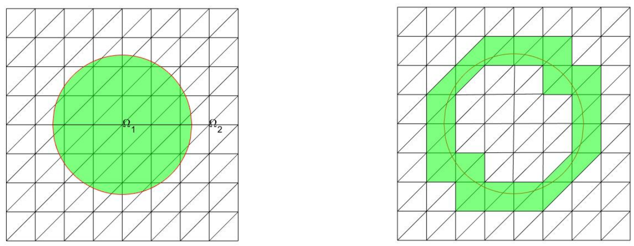



FIGURE 2. Left panel: $\Omega_1$ (the dark color area) and $\Omega_2$; Right panel: $\mathcal{T}_h^\Gamma$ (the set of dark color triangles).

For $i = 1, 2$, denote the non-interface domain by $\omega_{h,i} = \bigcup\limits_{K \in \mathcal{T}_{h,i}^-} K$ and the extended domain containing interface by $\Omega_{h,i} = \bigcup\limits_{K \in \mathcal{T}_{h,i}^+} K$, where $\mathcal{T}_{h,i}^- = \{K \in \mathcal{T}_h; K \subseteq \Omega_i\}$ and $\mathcal{T}_{h,i}^+ = \{K \in \mathcal{T}_h; K \subseteq \Omega_i \text{ or } K \in \mathcal{T}_h^\Gamma\}$, see Figure 3 (in a two dimensional setting) for an illustration. For a scalar function $\varphi$, we define its weighted average on the interface $\{\varphi\} = \kappa_1 \varphi_1 + \kappa_2 \varphi_2$, where $\kappa_i = \dfrac{1/\alpha_i}{1/\alpha_1 + 1/\alpha_2}$ and $\varphi_i = \varphi|_{\Omega_i}$, $i = 1, 2$. For a vector function, the weighted average definition holds for every component. To ensure the uniformity of our results with respect to the interface position, we need to introduce a stabilization term defined on transmission faces. The transmission face sets are noted as $\mathcal{F}_h^{\Gamma,i} = \{f \subseteq \partial K; K \in \mathcal{T}_h^\Gamma \text{ and } f \nsubseteq \partial\Omega_{h,i}\}$, $i = 1, 2$, see Figure 4 (in a two dimensional setting) for an illustration. Define the jump across the transmission face $f \in \mathcal{F}_h^{\Gamma,i}$: $[\mathbf{v}]_f = \mathbf{v}|_{K_l} - \mathbf{v}|_{K_r}$, where $K_l$ and $K_r$ denote different tetrahedral elements located at the different sides of the transmission face $f$.

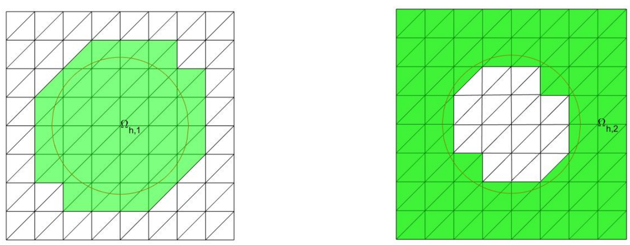



FIGURE 3. Left panel: $\Omega_{h,1}$ (the dark color area); Right panel: $\Omega_{h,2}$ (the dark color area).

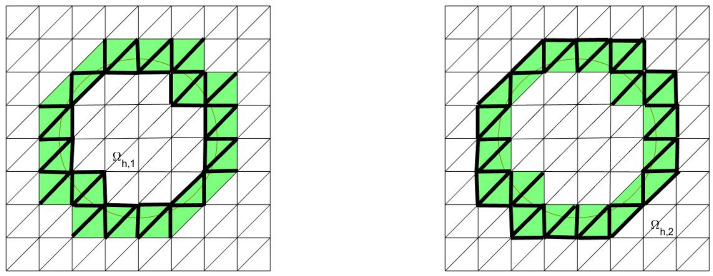



FIGURE 4. Left panel: $\mathcal{F}_h^{\Gamma,1}$ (the set of bold edges); Right panel: $\mathcal{F}_h^{\Gamma,2}$ (the set of bold edges).

**2.3. Discrete formulation.** Let $\mathbf{V}_{h,i}$ denote the lowest order of second family Nédélec edge element space defined in the extended domain $\Omega_{h,i},\ i = 1, 2$, i.e.,

$$\begin{aligned}\mathbf{V}_{h,i} =& \{\mathbf{v}_h \in \mathbf{H}(\mathbf{curl}; \Omega_{h,i}); \mathbf{v}_h|_K \in [P_1(K)]^3, \forall\ K \in \mathcal{T}_{h,i}^+,\\ & \mathbf{n} \times \mathbf{v}_h = \mathbf{0} \text{ on } \partial\Omega\} \text{ for } i = 1, 2.\end{aligned}$$

The extended finite element space is given by

$$\mathbf{V}_h = \{\mathbf{v}_h = \mathbf{v}_{h,1}\chi_1 + \mathbf{v}_{h,2}\chi_2;\ \mathbf{v}_{h,i} \in \mathbf{V}_{h,i},\ i = 1, 2\} \tag{6}$$

with characteristic functions $\chi_i$ $(i = 1, 2)$ being

$$\chi_i(x) = \begin{cases} 1 \text{ in } \Omega_i, \\ 0 \text{ else.} \end{cases}$$

The discrete approximation of the problem (1) is: find $\mathbf{u}_h \in \mathbf{V}_h$ such that

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_\Omega,\ \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{7}$$

where $a_h(\mathbf{u}_h, \mathbf{v}_h)$ is a bilinear form defined by

$$\begin{aligned}a_h(\mathbf{u}_h, \mathbf{v}_h) =& \sum_{i=1}^{2}(\alpha_i \mathbf{curl}\ \mathbf{u}_{h,i}, \mathbf{curl}\ \mathbf{v}_{h,i})_{\Omega_i} + \sum_{i=1}^{2}(\beta \mathbf{u}_{h,i}, \mathbf{v}_{h,i})_{\Omega_i}\\ & + (\{\mathbf{n}_\Gamma \times (\alpha \mathbf{curl}\ \mathbf{u}_h)\}, [\mathbf{n}_\Gamma \times (\mathbf{v}_h \times \mathbf{n}_\Gamma)])_\Gamma\\ & + (\{\mathbf{n}_\Gamma \times (\alpha \mathbf{curl}\ \mathbf{v}_h)\}, [\mathbf{n}_\Gamma \times (\mathbf{u}_h \times \mathbf{n}_\Gamma)])_\Gamma\\ & + \gamma_1 \sum_{K \in \mathcal{T}_h^\Gamma} h^{-1}\{\alpha\}([\mathbf{u}_h \times \mathbf{n}_\Gamma], [\mathbf{v}_h \times \mathbf{n}_\Gamma])_{\Gamma_K}\\ & + \gamma_2 \sum_{i=1}^{2} \sum_{f \in \mathcal{F}_h^{\Gamma,i}} h\alpha_i([\mathbf{curl}\ \mathbf{u}_{h,i}]_f, [\mathbf{curl}\ \mathbf{v}_{h,i}]_f)_f.\end{aligned} \tag{8}$$

**Remark 2.1.** *In the bilinear form* (8)*, the last two terms are added to ensure the coercivity of the system* (7)*. The last one, a stabilization term defined on transmission faces, guarantees the uniformity of results with respect to the interface position. Both terms vanish in the continuous case. Constants $\gamma_1$ and $\gamma_2$ are called stabilization parameters, which are independent of the mesh size $h$, the physical parameters $\alpha$, $\beta$ and the position of the interface with respect to the mesh.*

## 3. Preliminary lemma

**Assumption 3.1.** *The tetrahedral triangulation $\mathcal{T}_h$ is shape-regular, i.e., for any element $K \in \mathcal{T}_h$, there exists a positive constant $C$ such that*

$$\frac{h_K}{\rho_K} \le C, \tag{9}$$

*where $\rho_K$ denotes the diameter of the biggest ball contained in $K$.*

**Assumption 3.2.** *For all $K \in \mathcal{T}_h^\Gamma$, there exists $K^i \in \mathcal{T}_{h,i}^-$ such that $K \cap K^i \neq \emptyset$ for $i = 1, 2$.*

In the error estimation, the trace inequality on the interface segment $\Gamma_K$ is crucial. We state the trace inequality in three dimensions in the following (see [24]).

**Lemma 3.3.** *For any interface element $K \in \mathcal{T}_h^\Gamma$, there exists a positive constant $C$, depending on the interface $\Gamma$ but independent of its position with respect to the mesh, such that for any $\mathbf{v} \in [H^1(K)]^3$, there holds*

$$||\mathbf{v}||_{0,\Gamma_K} \le C(h_K^{-\frac{1}{2}}||\mathbf{v}||_{0,K} + h_K^{\frac{1}{2}}||\nabla \mathbf{v}||_{0,K}). \tag{10}$$

*Further, for any $\mathbf{v}_h \in [P_1(K)]^3$, by an inverse inequality $||\nabla \mathbf{v}_h||_{0,K} \le h_K^{-1}||\mathbf{v}_h||_{0,K}$,*

$$||\mathbf{v}_h||_{0,\Gamma_K} \le C h_K^{-\frac{1}{2}}||\mathbf{v}_h||_{0,K}. \tag{11}$$

We need the following technical lemma.

**Lemma 3.4.** *There exists a positive constant $C$ such that*

$$\begin{aligned}\sum_{K\in\mathcal{T}_{h,i}^+} ||\alpha_i^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_{h,i}||_{0,K}^2 \le C(&\sum_{K\in\mathcal{T}_{h,i}^-} ||\alpha_i^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_{h,i}||_{0,K}^2 \\ &+ \sum_{f\in\mathcal{F}_h^{\Gamma,i}} h\alpha_i||[\mathbf{curl}\ \mathbf{v}_{h,i}]_f||_{0,f}^2), i = 1, 2.\end{aligned} \tag{12}$$

*Proof.* Note that

$$\begin{aligned}\sum_{K\in\mathcal{T}_{h,i}^+} ||\alpha_i^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_{h,i}||_{0,K}^2 = &\sum_{K\in\mathcal{T}_{h,i}^-} ||\alpha_i^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_{h,i}||_{0,K}^2 \\ &+ \sum_{K\in\mathcal{T}_h^\Gamma} ||\alpha_i^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_{h,i}||_{0,K}^2, i = 1, 2.\end{aligned} \tag{13}$$

We just show

$$\sum_{K\in\mathcal{T}_h^\Gamma} ||\mathbf{curl}\ \mathbf{v}_{h,1}||_{0,K}^2 \le C(\sum_{K\subseteq\Omega_1} ||\mathbf{curl}\ \mathbf{v}_{h,1}||_{0,K}^2 + \sum_{f\in\mathcal{F}_h^{\Gamma,1}} h||[\mathbf{curl}\ \mathbf{v}_{h,1}]_f||_{0,f}^2). \tag{14}$$

According to Assumption 3.2, for all $K \in \mathcal{T}_h^\Gamma$, there exists $K^1 \in \mathcal{T}_{h,1}^-$ satisfying $K \cap K^1 \neq \emptyset$. Denote $\mathcal{F}_{K,K^1}$ the set of faces passing from $K$ to $K^1$, the number of faces belonging to $\mathcal{F}_{K,K^1}$ is uniformly upper bounded by the shape regularity. We use the same idea as in [8], and write that

$$\mathbf{curl}\ \mathbf{v}_{h,1}|_K = \mathbf{curl}\ \mathbf{v}_{h,1}|_{K^1} + \sum_{f\in\mathcal{F}_{K,K^1}} \delta_f[\mathbf{curl}\ \mathbf{v}_{h,1}]_f, \tag{15}$$

with $\delta_f = 1$ or $-1$ denoting the modification of the direction of the jump across the transmission face $f$.

Since the subdivision of the domain is shape-regular, we have the fact that $|K| \simeq |K^1|$, $|f| \simeq O(h^2)$. There holds

$$||\mathbf{curl}\ \mathbf{v}_{h,1}||_{0,K}^2 \leq C(||\mathbf{curl}\ \mathbf{v}_{h,1}||_{0,K^1}^2 + \sum_{f\in\mathcal{F}_{K,K^1}} h||[\mathbf{curl}\ \mathbf{v}_{h,1}]_f||_{0,f}^2). \tag{16}$$

Summing over all interface elements on the left, we obtain the following result

$$\sum_{K\in\mathcal{T}_h^\Gamma} ||\mathbf{curl}\ \mathbf{v}_{h,1}||_{0,K}^2 \leq C(\sum_{K\in\mathcal{T}_{h,1}^-} ||\mathbf{curl}\ \mathbf{v}_{h,1}||_{0,K}^2 + \sum_{f\in\mathcal{F}_h^{\Gamma,1}} h||[\mathbf{curl}\ \mathbf{v}_{h,1}]_f||_{0,f}^2). \tag{17}$$

Similarly, we have

$$\sum_{K\in\mathcal{T}_h^\Gamma} ||\mathbf{curl}\ \mathbf{v}_{h,2}||_{0,K}^2 \leq C(\sum_{K\in\mathcal{T}_{h,2}^-} ||\mathbf{curl}\ \mathbf{v}_{h,2}||_{0,K}^2 + \sum_{f\in\mathcal{F}_h^{\Gamma,2}} h||[\mathbf{curl}\ \mathbf{v}_{h,2}]_f||_{0,f}^2). \tag{18}$$

The desired result follows from (13), (17) and (18). □

To obtain the extended finite element error estimates, we need to introduce an extension theorem. The standard extension theorem is for scalar functions, extension properties for vector fields $\mathbf{H^1(curl)}$ are firstly proposed in [28]. For the need in the error estimate, we make a slight modification and the results are stated in the following. The proof of first two inequalities can be found in [28], here we only prove the last inequality.

**Theorem 3.5.** *Assume that $U \subseteq \mathbb{R}^3$ is a connected bounded domain with $C^2$-smooth boundary $\partial U$. Then there exists a bounded linear extension operator*

$$\mathbf{E}:\ \mathbf{H^2}(U) \to \mathbf{H^2}(\mathbb{R}^3) \tag{19}$$

*such that for any $\mathbf{v} \in \mathbf{H^2}(U)$ there hold*

- $\mathbf{Ev} = \mathbf{v}$ *a.e. in $U$;*
- $||\mathbf{Ev}||_{\mathbf{H^1}(\mathbf{curl};\mathbb{R}^3)} \leq C||\mathbf{v}||_{\mathbf{H^1}(\mathbf{curl};U)}$ *with $C$ only depending on $U$;*
- $||\mathbf{Ev}||_{\mathbf{H^2}(\mathbb{R}^3)} \leq C||\mathbf{v}||_{\mathbf{H^2}(U)}$ *with $C$ only depending on $U$.*

*Proof.* For a fixed point $\mathbf{x}_0 \in \partial U$, we first suppose that $\partial U$ is flat near $\mathbf{x}_0$ which is lying in the plane $\{\mathbf{x} \in \mathbb{R}^3; x_3 = 0\}$. Let $B = \{\mathbf{x} \in \mathbb{R}^3; |\mathbf{x}-\mathbf{x}_0| < r\}$ denote an open ball with center $\mathbf{x}_0$ and radius $r > 0$ such that the upper and lower hemispheres satisfy

$$B^+ = B \cap \{\mathbf{x} \in \mathbb{R}^3; x_3 > 0\} \subseteq \overline{U}, \tag{20}$$

$$B^- = B \cap \{\mathbf{x} \in \mathbb{R}^3; x_3 < 0\} \subseteq \mathbb{R}^3 \setminus \overline{U} \tag{21}$$

respectively.

Suppose a vector $\mathbf{w}(\mathbf{x}) = (w^1(\mathbf{x}), w^2(\mathbf{x}), w^3(\mathbf{x}))^\top \in \mathbf{C}^\infty(B^+)$ and its corresponding extension formula is

$$\widetilde{\mathbf{w}(\mathbf{x})} = \begin{cases} \mathbf{w}(\mathbf{x}), & \text{in } B^+, \\ \begin{pmatrix} \sum_{j=1}^3 \lambda_j w^1(x_1, x_2, -\frac{1}{j}x_3) \\ \sum_{j=1}^3 \lambda_j w^2(x_1, x_2, -\frac{1}{j}x_3) \\ \sum_{j=1}^3 (-\frac{1}{j})\lambda_j w^3(x_1, x_2, -\frac{1}{j}x_3) \end{pmatrix}, & \text{in } B^-, \end{cases}$$

where parameters $(\lambda_1, \lambda_2, \lambda_3) = (6, -32, 27)$ satisfy $\sum_{j=1}^{3}(-\frac{1}{j})^k \lambda_j = 1,\ k = 0, 1, 2$ uniquely. This uniquely determines $\widetilde{\mathbf{w}(\mathbf{x})} \in \mathbf{C^1}(B)$. Moreover, since $\widetilde{\mathbf{w}(\mathbf{x})} \in \mathbf{H^2}(B^+ \cup B^-)$, there holds $\widetilde{\mathbf{w}(\mathbf{x})} \in \mathbf{H^2}(B)$.

Now we show the extension from $\mathbf{H^2}(B^+)$ to $\mathbf{H^2}(B)$ is continuous. By a similar mirror reflection technique used in (3.7) of [28] applied on very component of $D^\theta \widetilde{\mathbf{w}(\mathbf{x})}$, we have

$$\int_B \sum_{|\boldsymbol{\theta}|=2} |D^{\boldsymbol{\theta}} \widetilde{\mathbf{w}(\mathbf{x})}|^2 d\mathbf{x} \leq C \int_{B^+} \sum_{|\boldsymbol{\theta}|=2} |D^{\boldsymbol{\theta}} \mathbf{w}(\mathbf{x})|^2 d\mathbf{x} \tag{22}$$

with $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$, $|\boldsymbol{\theta}| = \sum_{i=1}^{3} \theta_i = 2$ and $D^{\boldsymbol{\theta}} \boldsymbol{\varphi}(\mathbf{x})$ denoting all second derivative terms of the vector function $\boldsymbol{\varphi}(\mathbf{x})$. The constant C is related to at most the second order in terms of $\lambda_i,\ i = 1, 2, 3$.

When $\partial U$ is not flat near $\mathbf{x}_0$, one can use the flattening technique and partition of unity localization in order to reduce this situation to the flat one. □

Since that the interface $\Gamma$ is $C^2$-smooth, we can get the following corollary. Note that the assumption of a $C^2$ smooth interface $\Gamma$ limits, in principle, the applicability of our results to "bubble-like" subdomains as illustrated on Figures 1 and 6.

**Corollary 3.6.** *There exist two bounded linear extension operators for $i = 1, 2$,*

$$\mathbf{E^i} :\ \mathbf{H^2}(\Omega_i) \to \mathbf{H^2}(\Omega) \tag{23}$$

*such that for any $\mathbf{v} \in \mathbf{H^2}(\Omega_i)$ with $\Omega_i$ a subdomain, there hold*

- $\mathbf{E^i v} = \mathbf{v}$ *a.e. in* $\Omega_i$;
- $||\mathbf{E^i v}||_{\mathbf{H^1}(\mathbf{curl};\Omega)} \leq C||\mathbf{v}||_{\mathbf{H^1}(\mathbf{curl};\Omega_i)}$;
- $||\mathbf{E^i v}||_{\mathbf{H^2}(\Omega)} \leq C||\mathbf{v}||_{\mathbf{H^2}(\Omega_i)}$.

To ensure the curl conserving of transformations between different geometric domains, we should introduce a special transformation known as the Piola's transformation (refer to [36], [7])

$$\mathbf{F_K} :\ \widehat{K} \to K \tag{24}$$

which is a continuously differentiable, invertible and surjective mapping.

Let $\mathbf{B_K} \triangleq \mathbf{dF_K}$ denote the corresponding Jacobian matrix of $\mathbf{F_K}$, the transformation $\widehat{\mathbf{u}} \in \mathbf{H}(\mathbf{curl}; \widehat{K})$ to $\mathbf{u} \in \mathbf{H}(\mathbf{curl}; K)$ is via

$$\mathbf{u} \circ \mathbf{F_K} = \mathbf{B_K^{-\top}} \widehat{\mathbf{u}}, \tag{25}$$

then the transformation from $\widehat{\mathbf{curl}}\ \widehat{\mathbf{u}}$ to $\mathbf{curl}\ \mathbf{u}$ is by

$$\mathbf{curl}\ \mathbf{u} = \frac{1}{\det(\mathbf{B_K})} \mathbf{B_K} \widehat{\mathbf{curl}}\ \widehat{\mathbf{u}}. \tag{26}$$

Let $\widehat{\tau}$ be a unit vector in the direction of an edge $\widehat{e}$ of the tetrahedron $\widehat{K}$. Then the vector "$\tau$" given by

$$\boldsymbol{\tau} = \frac{\mathbf{B_K} \widehat{\boldsymbol{\tau}}}{|\mathbf{B_K} \widehat{\boldsymbol{\tau}}|}. \tag{27}$$

is the unit tangent vector to the edge $e$ of $K$.

Refer to [36], through transformation (25), there holds the following result.

**Lemma 3.7.** *Suppose the mesh $\mathcal{T}_h$ is regular and $s \geq 0$. The vector $\widehat{\mathbf{v}} \in \mathbf{H^s}(\mathbf{curl};\widehat{K})$ is transformed to $\mathbf{v} \in \mathbf{H^s}(\mathbf{curl};K)$ by (25). Then there hold the following equivalent relations.*

$$|\widehat{\mathbf{v}}|_{\mathbf{H^s}(\widehat{K})} \backsimeq h_K^{s-1/2}|\mathbf{v}|_{\mathbf{H^s}(K)}, \tag{28}$$

$$|\widehat{\mathbf{curl}}\ \widehat{\mathbf{v}}|_{\mathbf{H^s}(\widehat{K})} \backsimeq h_K^{s+1/2}|\mathbf{curl}\ \mathbf{v}|_{\mathbf{H^s}(K)}. \tag{29}$$

## 4. Stability analysis

For any $\mathbf{v}_h \in \mathbf{V}_h$, define the parameter-dependent norm

$$\begin{aligned}||\mathbf{v}_h||_{h,\mathrm{curl}} = (&\sum_{i=1}^{2}||\alpha_i^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_{h,i}||^2_{0,\Omega_i} + \sum_{K\in\mathcal{T}_h^\Gamma}\frac{\{\alpha\}}{h}||[\mathbf{v}_h \times \mathbf{n}_\Gamma]||^2_{0,\Gamma_K}\\ &+\sum_{i=1}^{2}||\beta^{\frac{1}{2}}\mathbf{v}_{h,i}||^2_{0,\Omega_i} + \sum_{i=1}^{2}\sum_{f\in\mathcal{F}_h^{\Gamma,i}}h\alpha_i||[\mathbf{curl}\ \mathbf{v}_{h,i}]_f||^2_{0,f})^{\frac{1}{2}}.\end{aligned} \tag{30}$$

For any $\mathbf{v} \in \mathbf{V} + \mathbf{V}_h$, define the norm

$$||\mathbf{v}||_{*,\mathrm{curl}} = (||\mathbf{v}||^2_{h,\mathrm{curl}} + \sum_{K\in\mathcal{T}_h^\Gamma}\frac{h}{\{\alpha\}}||\{\mathbf{n}_\Gamma \times (\alpha\mathbf{curl}\ \mathbf{v})\}||^2_{0,\Gamma_K})^{\frac{1}{2}}. \tag{31}$$

The following lemma shows that two norms $||\cdot||_{h,\mathrm{curl}}$ and $||\cdot||_{*,\mathrm{curl}}$ are equivalent in the discrete space $\mathbf{V}_h$, and the equivalence constants are independent of the mesh size $h$ and physical parameters.

**Lemma 4.1.** *For any $\mathbf{v}_h \in \mathbf{V}_h$, there holds*

$$\begin{aligned}\sum_{K\in\mathcal{T}_h^\Gamma}\frac{h}{\{\alpha\}}||\{\mathbf{n}_\Gamma \times (\alpha\mathbf{curl}\ \mathbf{v}_h)\}||^2_{0,\Gamma_K} \leq & C_1(\sum_{i=1,2}||\alpha_i^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_{h,i}||^2_{0,\Omega_i}\\ &+\sum_{i=1}^{2}\sum_{f\in\mathcal{F}_h^{\Gamma,i}}h\alpha_i||[\mathbf{curl}\ \mathbf{v}_{h,i}]_f||^2_{0,f}).\end{aligned} \tag{32}$$

*Proof.* Since $\mathbf{curl}\mathbf{v}_{h,i}$ is a piecewise constant in $\Omega_i$, $\kappa_i = \dfrac{1/\alpha_i}{1/\alpha_1 + 1/\alpha_2}$, $i = 1, 2$.

$$\begin{aligned}\sum_{K\in\mathcal{T}_h^\Gamma}\frac{h}{\{\alpha\}}||\{\mathbf{n}_\Gamma \times (\alpha\mathbf{curl}\ \mathbf{v}_h)\}||^2_{0,\Gamma_K} &\leq 2\sum_{i=1}^{2}\sum_{K\in\mathcal{T}_h^\Gamma}\frac{h}{\{\alpha\}}||\kappa_i\alpha_i\mathbf{curl}\ \mathbf{v}_{h,i}||^2_{0,\Gamma_K}\\ &= 2\sum_{i=1}^{2}\sum_{K\in\mathcal{T}_h^\Gamma}\frac{h}{\{\alpha\}}|\kappa_i\alpha_i\mathbf{curl}\ \mathbf{v}_{h,i}|^2|\Gamma_K|\\ &\leq 2\sum_{i=1}^{2}\sum_{K\in\mathcal{T}_h^\Gamma}h\frac{\kappa_i\alpha_i}{\{\alpha\}}|\alpha_i^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_{h,i}|^2|\Gamma_K|\\ &\leq 2\sum_{i=1}^{2}\sum_{K\in\mathcal{T}_h^\Gamma}\frac{h|\Gamma_K|}{|K|}|\alpha_i^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_{h,i}|^2|K|\\ &\leq C\sum_{i=1}^{2}\sum_{K\in\mathcal{T}_h^\Gamma}||\alpha_i^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_{h,i}||^2_{0,K}.\end{aligned} \tag{33}$$

In the last step, we use the fact that $|\Gamma_K| \le h_K^2$ and $|K| = O(h_K^3)$.

Further, by Lemma 3.4, we get the result (32). The proof is completed. $\square$

From the norm definitions (30), (31) and Lemma 4.1, we immediately obtain the norm equivalence corollary.

**Corollary 4.2.** *For any $\mathbf{v}_h \in \mathbf{V}_h$, there exist two positive constants $c$ and $C$ such that*

$$c||\mathbf{v}_h||_{h,\mathrm{curl}} \le ||\mathbf{v}_h||_{*,\mathrm{curl}} \le C||\mathbf{v}_h||_{h,\mathrm{curl}}. \tag{34}$$

Then we give the properties of the bilinear discretization form $a_h(\cdot,\cdot)$. First, there holds the continuity property.

**Theorem 4.3.** *For any $\mathbf{w},\ \mathbf{v} \in \mathbf{V} + \mathbf{V}_h$, there holds*

$$a_h(\mathbf{w},\mathbf{v}) \le C||\mathbf{w}||_{*,\mathrm{curl}}||\mathbf{v}||_{*,\mathrm{curl}}. \tag{35}$$

*Proof.* By Cauchy-Schwarz inequality,

$$\begin{aligned}
a_h(\mathbf{w},\mathbf{v}) &= \sum_{i=1}^{2}(\alpha_i \mathbf{curl}\ \mathbf{w}_i, \mathbf{curl}\ \mathbf{v}_i)_{\Omega_i} + (\{\mathbf{n}_\Gamma \times (\alpha \mathbf{curl}\ \mathbf{w})\}, [\mathbf{n}_\Gamma \times (\mathbf{v} \times \mathbf{n}_\Gamma)])_\Gamma \\
&+ \sum_{i=1}^{2}(\beta \mathbf{w}_i, \mathbf{v}_i)_{\Omega_i} + (\{\mathbf{n}_\Gamma \times (\alpha \mathbf{curl}\ \mathbf{v})\}, [\mathbf{n}_\Gamma \times (\mathbf{w} \times \mathbf{n}_\Gamma)])_\Gamma \\
&+ \gamma_1 \sum_{K\in\mathcal{T}_h^\Gamma} h^{-1}\{\alpha\}([\mathbf{w}\times\mathbf{n}_\Gamma], [\mathbf{v}\times\mathbf{n}_\Gamma])_{\Gamma_K} \\
&+ \gamma_2 \sum_{i=1}^{2}\sum_{f\in\mathcal{F}_h^{\Gamma,i}} h\alpha_i([\mathbf{curl}\ \mathbf{w}_i]_f, [\mathbf{curl}\ \mathbf{v}_i]_f)_f \\
&\le \sum_{i=1}^{2}||\alpha_i^{\frac{1}{2}}\mathbf{curl}\ \mathbf{w}_i||_{0,\Omega_i}||\alpha_i^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_i||_{0,\Omega_i} + \sum_{i=1}^{2}||\beta^{\frac{1}{2}}\mathbf{w}_i||_{0,\Omega_i}||\beta^{\frac{1}{2}}\mathbf{v}_i||_{0,\Omega_i} \\
&+ \sum_{K\in\mathcal{T}_h^\Gamma}(\frac{h}{\{\alpha\}})^{\frac{1}{2}}||\{\mathbf{n}_\Gamma \times (\alpha \mathbf{curl}\ \mathbf{w})\}||_{0,\Gamma_K}(\frac{\{\alpha\}}{h})^{\frac{1}{2}}||[\mathbf{n}_\Gamma \times (\mathbf{v}\times\mathbf{n}_\Gamma)]||_{0,\Gamma_K} \\
&+ \sum_{K\in\mathcal{T}_h^\Gamma}(\frac{h}{\{\alpha\}})^{\frac{1}{2}}||\{\mathbf{n}_\Gamma \times (\alpha \mathbf{curl}\ \mathbf{v})\}||_{0,\Gamma_K}(\frac{\{\alpha\}}{h})^{\frac{1}{2}}||[\mathbf{n}_\Gamma \times (\mathbf{w}\times\mathbf{n}_\Gamma)]||_{0,\Gamma_K} \\
&+ \gamma_1 \sum_{K\in\mathcal{T}_h^\Gamma}(\frac{\{\alpha\}}{h})^{\frac{1}{2}}||[\mathbf{w}\times\mathbf{n}_\Gamma]||_{0,\Gamma_K}(\frac{\{\alpha\}}{h})^{\frac{1}{2}}||[\mathbf{v}\times\mathbf{n}_\Gamma]||_{0,\Gamma_K} \\
&+ \gamma_2 \sum_{i=1}^{2}\sum_{f\in\mathcal{F}_h^{\Gamma,i}}(h\alpha_i)^{\frac{1}{2}}||[\mathbf{curl}\ \mathbf{w}_i]_f||_{0,f}(h\alpha_i)^{\frac{1}{2}}||[\mathbf{curl}\ \mathbf{v}_i]_f||_{0,f}
\end{aligned}$$

$$
\begin{aligned}
\leq & \Big(\sum_{i=1}^{2}||\alpha_i^{\frac{1}{2}}\mathbf{curl}\ \mathbf{w}_i||_{0,\Omega_i}^2+\sum_{i=1}^{2}||\beta^{\frac{1}{2}}\mathbf{w}_i||_{0,\Omega_i}^2+\sum_{K\in\mathcal{T}_h^\Gamma}\frac{h}{\{\alpha\}}||\{\mathbf{n}_\Gamma\times(\alpha\mathbf{curl}\ \mathbf{w})\}||_{0,\Gamma_K}^2 \\
& +(1+\gamma_1)\sum_{K\in\mathcal{T}_h^\Gamma}\frac{\{\alpha\}}{h}||[\mathbf{w}\times\mathbf{n}_\Gamma]||_{0,\Gamma_K}^2+\gamma_2\sum_{i=1}^{2}\sum_{f\in\mathcal{F}_h^{\Gamma,i}}(h\alpha_i)^{\frac{1}{2}}||[\mathbf{curl}\ \mathbf{w}_i]_f||_{0,f}^2\Big)^{\frac{1}{2}} \\
& \cdot\Big(\sum_{i=1}^{2}||\alpha_i^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_i||_{0,\Omega_i}^2+\sum_{i=1}^{2}||\beta^{\frac{1}{2}}\mathbf{v}_i||_{0,\Omega_i}^2+\sum_{K\in\mathcal{T}_h^\Gamma}\frac{h}{\{\alpha\}}||\{\mathbf{n}_\Gamma\times(\alpha\mathbf{curl}\ \mathbf{v})\}||_{0,\Gamma_K}^2 \\
& +(1+\gamma_1)\sum_{K\in\mathcal{T}_h^\Gamma}\frac{\{\alpha\}}{h}||[\mathbf{v}\times\mathbf{n}_\Gamma]||_{0,\Gamma_K}^2+\gamma_2\sum_{i=1}^{2}\sum_{f\in\mathcal{F}_h^{\Gamma,i}}(h\alpha_i)^{\frac{1}{2}}||[\mathbf{curl}\ \mathbf{v}_i]_f||_{0,f}^2\Big)^{\frac{1}{2}} \\
\leq & \max\{1+\gamma_1,\gamma_2\}||\mathbf{w}||_{*,\mathrm{curl}}||\mathbf{v}||_{*,\mathrm{curl}}.
\end{aligned}
$$

Choose $C=\max\{1+\gamma_1,\gamma_2\}$, the proof is completed. □

Then the following result shows that the coercivity holds for a weaker norm than the one that is used in the final a priori bound.

**Theorem 4.4.** *For any $\mathbf{v}_h\in\mathbf{V}_h$, there holds*

$$a_h(\mathbf{v}_h,\mathbf{v}_h)\geq C||\mathbf{v}_h||_{h,\mathrm{curl}}^2. \tag{36}$$

*Proof.* By Cauchy-Schwarz inequality and Young inequality with $\epsilon$,

$$
\begin{aligned}
& (\{\mathbf{n}_\Gamma\times(\alpha\mathbf{curl}\ \mathbf{v}_h)\},[\mathbf{n}_\Gamma\times(\mathbf{v}_h\times\mathbf{n}_\Gamma)])_\Gamma \\
\leq & \sum_{K\in\mathcal{T}_h^\Gamma}\Big(\frac{h}{\{\alpha\}}\Big)^{\frac{1}{2}}||\{\mathbf{n}_\Gamma\times(\alpha\mathbf{curl}\ \mathbf{v}_h)\}||_{0,\Gamma_K}\Big(\frac{\{\alpha\}}{h}\Big)^{\frac{1}{2}}||[\mathbf{n}_\Gamma\times(\mathbf{v}_h\times\mathbf{n}_\Gamma)]||_{0,\Gamma_K}
\end{aligned}
$$

$$\leq\frac{\epsilon}{2}\sum_{K\in\mathcal{T}_h^\Gamma}\frac{h}{\{\alpha\}}||\{\mathbf{n}_\Gamma\times(\alpha\mathbf{curl}\ \mathbf{v}_h)\}||_{0,\Gamma_K}^2+\frac{1}{2\epsilon}\sum_{K\in\mathcal{T}_h^\Gamma}\frac{\{\alpha\}}{h}||[\mathbf{v}_h\times\mathbf{n}_\Gamma]||_{0,\Gamma_K}^2. \tag{37}$$

Then according to Lemma 4.1, there is

$$
\begin{aligned}
a_h(\mathbf{v}_h,\mathbf{v}_h)= & \sum_{i=1}^{2}||\alpha_i^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_{h,i}||_{0,\Omega_i}^2+\gamma_1\sum_{K\in\mathcal{T}_h^\Gamma}\frac{\{\alpha\}}{h}||[\mathbf{v}_h\times\mathbf{n}_\Gamma]||_{0,\Gamma_K}^2 \\
& +\sum_{i=1}^{2}||\beta^{\frac{1}{2}}\mathbf{v}_{h,i}||_{0,\Omega_i}^2+\gamma_2\sum_{i=1}^{2}\sum_{f\in\mathcal{F}_h^{\Gamma,i}}h\alpha_i||[\mathbf{curl}\ \mathbf{v}_{h,i}]_f||_{0,f}^2 \\
& +2(\{\mathbf{n}_\Gamma\times(\alpha\mathbf{curl}\ \mathbf{v}_h)\},[\mathbf{n}_\Gamma\times(\mathbf{v}_h\times\mathbf{n}_\Gamma)])_\Gamma \\
\geq & \sum_{i=1}^{2}||\alpha_i^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_{h,i}||_{0,\Omega_i}^2+\sum_{i=1}^{2}||\beta^{\frac{1}{2}}\mathbf{v}_{h,i}||_{0,\Omega_i}^2+\gamma_1\sum_{K\in\mathcal{T}_h^\Gamma}\frac{\{\alpha\}}{h}||[\mathbf{v}_h\times\mathbf{n}_\Gamma]||_{0,\Gamma_K}^2 \\
& -\epsilon\sum_{K\in\mathcal{T}_h^\Gamma}\frac{h}{\{\alpha\}}||\{\mathbf{n}_\Gamma\times(\alpha\mathbf{curl}\ \mathbf{v}_h)\}||_{0,\Gamma_K}^2-\frac{1}{\epsilon}\sum_{K\in\mathcal{T}_h^\Gamma}\frac{\{\alpha\}}{h}||[\mathbf{v}_h\times\mathbf{n}_\Gamma]||_{0,\Gamma_K}^2 \\
& +\gamma_2\sum_{i=1}^{2}\sum_{f\in\mathcal{F}_h^{\Gamma,i}}h\alpha_i||[\mathbf{curl}\ \mathbf{v}_{h,i}]_f||_{0,f}^2
\end{aligned}
$$

$$
\begin{aligned}
&\geq \frac{1}{2}\sum_{i=1}^{2}||\alpha_i^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_{h,i}||_{0,\Omega_i}^2 + (\frac{1}{2} - C_1\epsilon)\sum_{i=1}^{2}||\alpha_i^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_{h,i}||_{0,\Omega_i}^2 \\
&\quad + \sum_{i=1}^{2}||\beta^{\frac{1}{2}}\mathbf{v}_{h,i}||_{0,\Omega_i}^2 + (\gamma_1 - \frac{1}{\epsilon})\sum_{K\in\mathcal{T}_h^\Gamma}\frac{\{\alpha\}}{h}||[\mathbf{v}_h\times\mathbf{n}_\Gamma]||_{0,\Gamma_K}^2 \\
&\quad + (\gamma_2 - C_1\epsilon)\sum_{i=1}^{2}\sum_{f\in\mathcal{F}_h^{\Gamma,i}} h\alpha_i||[\mathbf{curl}\ \mathbf{v}_{h,i}]_f||_{0,f}^2 \\
&\geq C||\mathbf{v}_h||_{h,\mathrm{curl}}^2.
\end{aligned}
$$

In the last step, choose $\epsilon = \dfrac{1}{2C_1}$, $\gamma_1 \geq 2C_1$ and $\gamma_2 \geq 1$, then the proof is completed with

$$C = \min\{\frac{1}{2}, \gamma_1 - \frac{1}{\epsilon}, \gamma_2 - C_1\epsilon\}.$$

□

According to Theorems 4.3-4.4 and Lax-Milgram Theorem, we obtain that the discrete formulation (7) has a unique solution.

## 5. Error estimation

The main result of this section is formulated in the following theorem.

**Theorem 5.1.** *Let $\mathbf{u} \in \mathbf{H_0}(\mathbf{curl};\Omega)\cap\mathbf{H^2}(\Omega_1\cup\Omega_2)$ be the solution of the problem* (1) *and $\mathbf{u}_h \in \mathbf{V}_h$ be the solution of the discrete problem* (7)*. There exists a positive constant $C$ such that*

$$||\mathbf{u}-\mathbf{u}_h||_{*,\mathrm{curl}} \leq Ch\sum_{i=1}^{2}(\alpha_i^{\frac{1}{2}}||\mathbf{u}||_{\mathbf{H^2}(\Omega_i)} + \beta^{\frac{1}{2}}||\mathbf{u}||_{\mathbf{H^1}(\Omega_i)}). \tag{38}$$

In order to get the above finite element error estimate, we need to analyse the consistency and interpolation error estimates. With the definition of bilinear form, we have the following consistency relation firstly.

**Lemma 5.2.** *Let $\mathbf{u} \in H_0(\mathbf{curl};\Omega)\cap\mathbf{H}^2(\Omega_1\cup\Omega_2)$ be the solution of the problem* (1) *and $\mathbf{u}_h \in \mathbf{V}_h$ be the solution of the discrete problem* (7)*. Then for any $\mathbf{v}_h \in \mathbf{V}_h$, there holds*

$$a_h(\mathbf{u}-\mathbf{u}_h, \mathbf{v}_h) = 0. \tag{39}$$

*Proof.* According to the definition of the bilinear form (8),

$$
\begin{aligned}
a_h(\mathbf{u},\mathbf{v}_h) =& \sum_{i=1}^{2}(\alpha_i\mathbf{curl}\ \mathbf{u}_i, \mathbf{curl}\ \mathbf{v}_{h,i})_{\Omega_i} + \sum_{i=1}^{2}(\beta\mathbf{u}_i, \mathbf{v}_{h,i})_{\Omega_i} \\
&+ (\{\mathbf{n}_\Gamma\times(\alpha\mathbf{curl}\ \mathbf{u})\}, [\mathbf{n}_\Gamma\times(\mathbf{v}_h\times\mathbf{n}_\Gamma)])_\Gamma.
\end{aligned} \tag{40}
$$

Using Green's formula,

$$
\begin{aligned}
&\sum_{i=1}^{2}(\alpha_i\mathbf{curl}\ \mathbf{u}_i, \mathbf{curl}\ \mathbf{v}_{h,i})_{\Omega_i} \\
=&\sum_{i=1}^{2}(\mathbf{curl}(\alpha_i\mathbf{curl}\ \mathbf{u}_i), \mathbf{v}_{h,i})_{\Omega_i} + \sum_{i=1}^{2}(\alpha_i\mathbf{curl}\ \mathbf{u}_i, \mathbf{n}\times\mathbf{v}_{h,i})_{\partial\Omega_i}
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{2} (\mathbf{curl}(\alpha_i \mathbf{curl}\ \mathbf{u}_i), \mathbf{v}_{h,i})_{\Omega_i} - \sum_{i=1}^{2} (\mathbf{n} \times (\alpha_i \mathbf{curl}\ \mathbf{u}_i), \mathbf{v}_{h,i})_{\partial\Omega_i} \\
&= \sum_{i=1}^{2} (\mathbf{curl}(\alpha_i \mathbf{curl}\ \mathbf{u}_i), \mathbf{v}_{h,i})_{\Omega_i} - \sum_{i=1}^{2} (\mathbf{n} \times (\alpha_i \mathbf{curl}\ \mathbf{u}_i), \mathbf{n} \times (\mathbf{v}_{h,i} \times \mathbf{n}))_{\partial\Omega_i} \\
&= \sum_{i=1}^{2} (\mathbf{curl}(\alpha_i \mathbf{curl}\ \mathbf{u}_i), \mathbf{v}_{h,i})_{\Omega_i} - (\{\mathbf{n}_\Gamma \times (\alpha \mathbf{curl}\ \mathbf{u})\}, [\mathbf{n}_\Gamma \times (\mathbf{v}_h \times \mathbf{n}_\Gamma)])_\Gamma \\
&\quad - ([\mathbf{n}_\Gamma \times (\alpha \mathbf{curl}\ \mathbf{u})], \{\mathbf{n}_\Gamma \times (\mathbf{v}_h \times \mathbf{n}_\Gamma)\})_\Gamma - (\mathbf{n}_\Gamma \times (\alpha \mathbf{curl}\ \mathbf{u}), \mathbf{n}_\Gamma \times (\mathbf{v}_h \times \mathbf{n}_\Gamma))_{\partial\Omega} \\
&= \sum_{i=1}^{2} (\mathbf{curl}(\alpha_i \mathbf{curl}\ \mathbf{u}_i), \mathbf{v}_{h,i})_{\Omega_i} - (\{\mathbf{n}_\Gamma \times (\alpha \mathbf{curl}\ \mathbf{u})\}, [\mathbf{n}_\Gamma \times (\mathbf{v}_h \times \mathbf{n}_\Gamma)])_\Gamma.
\end{aligned}
$$

Combining (40) with (41),

$$
a_h(\mathbf{u}, \mathbf{v}_h) = \sum_{i=1}^{2} (\mathbf{curl}(\alpha_i \mathbf{curl}\ \mathbf{u}_i), \mathbf{v}_{h,i})_{\Omega_i} + \sum_{i=1}^{2} (\beta \mathbf{u}_i, \mathbf{v}_{h,i})_{\Omega_i} = (\mathbf{f}, \mathbf{v}_h)_\Omega, \tag{42}
$$

which implies $a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) = 0$. The proof is completed. □

Then we have the following *Céa*-type lemma.

**Lemma 5.3.** *Let* $\mathbf{u} \in \mathbf{H_0}(\mathbf{curl}; \Omega) \cap \mathbf{H}^2(\Omega_1 \cup \Omega_2)$ *be the solution of the problem* (1) *and* $\mathbf{u}_h \in \mathbf{V}_h$ *be the solution of the discrete problem* (7)*. There holds*

$$
||\mathbf{u} - \mathbf{u}_h||_{*,\mathrm{curl}} \le C \inf_{\forall \mathbf{v}_h \in \mathbf{V}_h} ||\mathbf{u} - \mathbf{v}_h||_{*,\mathrm{curl}}. \tag{43}
$$

*Proof.* Using the triangular inequality and Corollary 4.2, for any $\mathbf{v}_h \in \mathbf{V}_h$,

$$
\begin{aligned}
||\mathbf{u} - \mathbf{u}_h||_{*,\mathrm{curl}} &\le ||\mathbf{u} - \mathbf{v}_h||_{*,\mathrm{curl}} + ||\mathbf{v}_h - \mathbf{u}_h||_{*,\mathrm{curl}} \\
&\le ||\mathbf{u} - \mathbf{v}_h||_{*,\mathrm{curl}} + C||\mathbf{v}_h - \mathbf{u}_h||_{h,\mathrm{curl}}. \qquad (44)
\end{aligned}
$$

For the second term in (44), by Theorem 4.4, Lemma 5.2 and Theorem 4.3,

$$
\begin{aligned}
||\mathbf{v}_h - \mathbf{u}_h||_{h,\mathrm{curl}} &\le C\ \frac{a_h(\mathbf{v}_h - \mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h)}{||\mathbf{v}_h - \mathbf{u}_h||_{h,\mathrm{curl}}} \\
&\le C\ \frac{a_h(\mathbf{v}_h - \mathbf{u}, \mathbf{v}_h - \mathbf{u}_h)}{||\mathbf{v}_h - \mathbf{u}_h||_{h,\mathrm{curl}}} \\
&\le C\ ||\mathbf{u} - \mathbf{v}_h||_{*,\mathrm{curl}}. \qquad (45)
\end{aligned}
$$

Combining (44) and (45), the result (43) is derived. □

From Lemma 5.3, we know that the finite element error can be controlled by one proper interpolation error. We introduce such an interpolation operator in the following.

Let $I_h$ be the interpolation operator introduced in [38] and define a new interpolation operator $\mathbf{I}_h^*$ on the extended finite element space $\mathbf{V}_h$ by

$$
\mathbf{I}_h^* \mathbf{u} = (\mathbf{I}_{h,1}^* \mathbf{u}_1, \mathbf{I}_{h,2}^* \mathbf{u}_2) = ((\mathbf{I}_h \mathbf{E}^1 \mathbf{u}_1)|_{\Omega_1}, (\mathbf{I}_h \mathbf{E}^2 \mathbf{u}_2)|_{\Omega_2}), \tag{46}
$$

where $\mathbf{u} \in \mathbf{H_0}(\mathbf{curl}; \Omega) \cap \mathbf{H}^2(\Omega_1 \cup \Omega_2)$, $\mathbf{u}_1 = \mathbf{u}|_{\Omega_1}$, $\mathbf{u}_2 = \mathbf{u}|_{\Omega_2}$ and $\mathbf{E}^i$ is the extension operator in Corollary 3.6.

Refer to the Proposition 3 in [38], based on the Piola's transformation (25) and Bramble-Hilbert Lemma, the standard approximation properties for the lowest order of second family Nédélec edge elements are derived as follows.

**Lemma 5.4.** *For any* $\mathbf{u} \in \mathbf{H^2}(\Omega)$, *there holds*

$$||\mathbf{u} - \mathbf{I}_h \mathbf{u}||_{\mathbf{L^2}(\Omega)} \leq Ch^2 |\mathbf{u}|_{\mathbf{H^2}(\Omega)}, \tag{47}$$

$$||\mathbf{u} - \mathbf{I}_h \mathbf{u}||_{\mathbf{L^2}(\Omega)} \leq Ch |\mathbf{u}|_{\mathbf{H^1}(\Omega)}, \tag{48}$$

$$||\nabla(\mathbf{u} - \mathbf{I}_h \mathbf{u})||_{\mathbf{L^2}(\Omega)} \leq Ch |\mathbf{u}|_{\mathbf{H^2}(\Omega)}, \tag{49}$$

$$||\mathrm{curl}(\mathbf{u} - \mathbf{I}_\mathrm{h} \mathbf{u})||_{\mathbf{L^2}(\Omega)} \leq \mathrm{Ch} |\mathrm{curl}\ \mathbf{u}|_{\mathbf{H^1}(\Omega)}. \tag{50}$$

Then the following theorem is valid.

**Theorem 5.5.** *Let* $\mathbf{u} \in \mathbf{H_0}(\mathbf{curl};\Omega) \cap \mathbf{H^2}(\Omega_1 \cup \Omega_2)$ *be the solution of the problem* (1). *The interpolation operator* $\mathbf{I}_h^*$ *is defined as* (46). *There exists a positive constant* $C$ *such that*

$$||\mathbf{u} - \mathbf{I}_h^* \mathbf{u}||_{*,\mathrm{curl}} \leq Ch \sum_{i=1}^{2} (\alpha_i^{\frac{1}{2}} ||\mathbf{u}||_{\mathbf{H^2}(\Omega_i)} + \beta^{\frac{1}{2}} ||\mathbf{u}||_{\mathbf{H^1}(\Omega_i)}). \tag{51}$$

*Proof.* By the definition (31),

$$\begin{aligned}
||\mathbf{u} - \mathbf{I}_h^* \mathbf{u}||_{*,\mathrm{curl}}^2 = & \sum_{i=1}^{2} ||\alpha_i^{\frac{1}{2}} \mathbf{curl}(\mathbf{u}_i - \mathbf{I}_{h,i}^* \mathbf{u}_i)||_{0,\Omega_i}^2 + \sum_{i=1}^{2} ||\beta^{\frac{1}{2}} (\mathbf{u}_i - \mathbf{I}_{h,i}^* \mathbf{u}_i)||_{0,\Omega_i}^2 \\
& + \sum_{K \in \mathcal{T}_h^\Gamma} \frac{\{\alpha\}}{h} ||[(\mathbf{u} - \mathbf{I}_h^* \mathbf{u}) \times \mathbf{n}_\Gamma]||_{0,\Gamma_K}^2 \\
& + \sum_{K \in \mathcal{T}_h^\Gamma} \frac{h}{\{\alpha\}} ||\{\mathbf{n}_\Gamma \times (\alpha \mathbf{curl}(\mathbf{u} - \mathbf{I}_h^* \mathbf{u}))\}||_{0,\Gamma_K}^2 \\
& + \sum_{i=1}^{2} \sum_{f \in \mathcal{F}_h^{\Gamma,i}} h\alpha_i ||[\mathbf{curl}(\mathbf{u}_i - \mathbf{I}_{h,i}^* \mathbf{u}_i)]_f||_{0,f}^2.
\end{aligned} \tag{52}$$

First, using Corollary 3.6 and Lemma 5.4, we have

$$\begin{aligned}
\sum_{i=1}^{2} ||\alpha_i^{\frac{1}{2}} \mathbf{curl}(\mathbf{u}_i - \mathbf{I}_{h,i}^* \mathbf{u}_i)||_{0,\Omega_i}^2 &= \sum_{i=1}^{2} ||\alpha_i^{\frac{1}{2}} \mathbf{curl}(\mathbf{E^i} \mathbf{u}_i - \mathbf{I}_h \mathbf{E^i} \mathbf{u}_i)||_{0,\Omega_i}^2 \\
&\leq \sum_{i=1}^{2} ||\alpha_i^{\frac{1}{2}} \mathbf{curl}(\mathbf{E^i} \mathbf{u}_i - \mathbf{I}_h \mathbf{E^i} \mathbf{u}_i)||_{0,\Omega}^2 \\
&\leq C \sum_{i=1}^{2} h^2 |\alpha_i^{\frac{1}{2}} \mathbf{curl}(\mathbf{E^i} \mathbf{u}_i)|_{1,\Omega}^2 \\
&\leq C \sum_{i=1}^{2} h^2 ||\alpha_i^{\frac{1}{2}} \mathbf{curl}\ \mathbf{u}_i||_{1,\Omega_i}^2,
\end{aligned} \tag{53}$$

and

$$
\begin{aligned}
\sum_{i=1}^{2}||\beta^{\frac{1}{2}}(\mathbf{u}_i-\mathbf{I}^*_{h,i}\mathbf{u}_i)||^2_{0,\Omega_i} &= \sum_{i=1}^{2}||\beta^{\frac{1}{2}}(\mathbf{E^i}\mathbf{u}_i-\mathbf{I}_h\mathbf{E^i}\mathbf{u}_i)||^2_{0,\Omega_i} \\
&\leq \sum_{i=1}^{2}||\beta^{\frac{1}{2}}(\mathbf{E^i}\mathbf{u}_i-\mathbf{I}_h\mathbf{E^i}\mathbf{u}_i)||^2_{0,\Omega} \\
&\leq C\sum_{i=1}^{2}h^2|\beta^{\frac{1}{2}}\mathbf{E^i}\mathbf{u}_i|^2_{1,\Omega} \\
&\leq C\sum_{i=1}^{2}h^2||\beta^{\frac{1}{2}}\mathbf{u}_i||^2_{1,\Omega_i}. \qquad (54)
\end{aligned}
$$

For the interface terms, using the trace inequality (10),

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h^\Gamma}\frac{\{\alpha\}}{h}||[(\mathbf{u}-\mathbf{I}^*_h\mathbf{u})\times\mathbf{n}_\Gamma]||^2_{0,\Gamma_K} \\
\leq& 2\sum_{i=1}^{2}\sum_{K\in\mathcal{T}_h^\Gamma}\frac{\{\alpha\}}{h}||\mathbf{E^i}\mathbf{u}_i-\mathbf{I}_h\mathbf{E^i}\mathbf{u}_i||^2_{0,\Gamma_K} \\
\leq& C\sum_{i=1}^{2}\sum_{K\in\mathcal{T}_h^\Gamma}\frac{\{\alpha\}}{h}(h_K^{-1}||\mathbf{E^i}\mathbf{u}_i-\mathbf{I}_h\mathbf{E^i}\mathbf{u}_i||^2_{0,K}+h_K||\nabla(\mathbf{E^i}\mathbf{u}_i-\mathbf{I}_h\mathbf{E^i}\mathbf{u}_i)||^2_{0,K}) \\
\leq& C\sum_{i=1}^{2}\sum_{K\in\mathcal{T}_h^\Gamma}\{\alpha\}(h_K^2|\mathbf{E^i}\mathbf{u}_i|^2_{2,K}+h_K^2|\mathbf{E^i}\mathbf{u}_i|^2_{2,K}) \\
\leq& Ch^2\sum_{i=1}^{2}||\alpha_i^{\frac{1}{2}}\mathbf{u}_i||^2_{2,\Omega_i}, \qquad (55)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h^\Gamma}\frac{h}{\{\alpha\}}||\{\mathbf{n}_\Gamma\times(\alpha\mathbf{curl}(\mathbf{u}-\mathbf{I}^*_h\mathbf{u}))\}||^2_{0,\Gamma_K} \\
\leq& 2\sum_{i=1}^{2}\sum_{K\in\mathcal{T}_h^\Gamma}\frac{h}{\{\alpha\}}||\kappa_i\alpha_i\mathbf{curl}(\mathbf{u}_i-\mathbf{I}^*_{h,i}\mathbf{u}_i)||^2_{0,\Gamma_K} \\
\leq& C\sum_{i=1}^{2}\sum_{K\in\mathcal{T}_h^\Gamma}\frac{\kappa_i\alpha_i}{\{\alpha\}}h||\alpha_i^{\frac{1}{2}}\mathbf{curl}(\mathbf{E^i}\mathbf{u}_i-\mathbf{I}_h\mathbf{E^i}\mathbf{u}_i)||^2_{0,\Gamma_K} \\
\leq& C\sum_{i=1}^{2}\sum_{K\in\mathcal{T}_h^\Gamma}h(h_K^{-1}||\alpha_i^{\frac{1}{2}}\mathbf{curl}(\mathbf{E^i}\mathbf{u}_i-\mathbf{I}_h\mathbf{E^i}\mathbf{u}_i)||^2_{0,K} \\
&+h_K|\alpha_i^{\frac{1}{2}}\mathbf{curl}(\mathbf{E^i}\mathbf{u}_i-\mathbf{I}_h\mathbf{E^i}\mathbf{u}_i)|^2_{1,K}) \qquad (56)
\end{aligned}
$$

$$\leq C \sum_{i=1}^{2} \sum_{K\in\mathcal{T}_h^\Gamma} h_K^2 |\alpha_i^{\frac{1}{2}} \mathbf{curl}(\mathbf{E^i u}_i)|_{1,K}^2$$

$$\leq Ch^2 \sum_{i=1}^{2} ||\alpha_i^{\frac{1}{2}} \mathbf{curl}\ \mathbf{u}_i||_{1,\Omega_i}^2.$$

In the last step of (55), we use the following property

$$\{\alpha\} = \kappa_1\alpha_1 + \kappa_2\alpha_2 = \frac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2} \leq 2\alpha_i,\ i=1,2.$$

For the transmission terms, assuming the elements located on the different sides of the transmission face $f$ are denoted by $K_l$ and $K_r$ respectively, then there holds

$$\sum_{i=1}^{2} \sum_{f\in\mathcal{F}_h^{\Gamma,i}} h\alpha_i ||[\mathbf{curl}(\mathbf{u}_i - \mathbf{I}_{h,i}^* \mathbf{u}_i)]_f||_{0,f}^2$$

$$\leq 2\sum_{i=1}^{2} \sum_{f\in\mathcal{F}_h^{\Gamma,i}} \sum_{j=l}^{r} h||\alpha_i^{\frac{1}{2}} \mathbf{curl}(\mathbf{E^i u}_i - \mathbf{I}_h \mathbf{E^i u}_i)|_{K_j}||_{0,f}^2$$

$$\leq C\sum_{i=1}^{2} \sum_{f\in\mathcal{F}_h^{\Gamma,i}} \sum_{j=l}^{r} h(h_{K_j}^{-1}||\alpha_i^{\frac{1}{2}} \mathbf{curl}(\mathbf{E^i u}_i - \mathbf{I}_h \mathbf{E^i u}_i)||_{0,K_j}^2$$

$$+ h_{K_j}|\alpha_i^{\frac{1}{2}} \mathbf{curl}(\mathbf{E^i u}_i - \mathbf{I}_h \mathbf{E^i u}_i)|_{1,K_j}^2)$$

$$\leq C\sum_{i=1}^{2} \sum_{f\in\mathcal{F}_h^{\Gamma,i}} \sum_{j=l}^{r} h_{K_j}^2 |\alpha_i^{\frac{1}{2}} \mathbf{curl}(\mathbf{E^i u}_i)|_{1,K_j}^2$$

$$\leq C\sum_{i=1}^{2} \sum_{K\in\mathcal{T}_h} h^2 |\alpha_i^{\frac{1}{2}} \mathbf{curl}(\mathbf{E^i u}_i)|_{1,K}^2$$

$$\leq Ch^2 \sum_{i=1}^{2} ||\alpha_i^{\frac{1}{2}} \mathbf{curl}\ \mathbf{u}_i||_{1,\Omega_i}^2. \tag{57}$$

Combining the inequalities (53)-(57), the result (51) is derived, and the proof is completed. □

**Remark 5.6.** *Note that the major problem originates from the term* (55)*. When dealing with* (55)*, in order to obtain an optimal convergence order, we have to draw support from the $H^1$-based trace inequality* (10)*, which leads to the high regularity assumption based on $\mathbf{H^2}(\Omega_1\cup\Omega_2)$ not usual $H^1(\mathbf{curl};\Omega_1\cup\Omega_2)$.*

According to Lemma 5.3 and Theorem 5.5, we can obtain our main result Theorem 5.1.

## 6. Numerical examples

In this section, refer to the iFEM package [12], we present three three-dimensional numerical examples for verification by using the extended lowest order of second family Nédélec edge elements. In the following, we will check the optimality of convergence orders and the robustness of results with respect to the mesh size, physical parameters and the interface position both under the norm (31). Define a relative error $r_e$ as follows

$$r_e = \frac{||\mathbf{u} - \mathbf{u}_h||_{*,\mathrm{curl}}}{||\mathbf{u}||_*} \tag{58}$$

with $||\mathbf{u}||_* = \sum_{i=1}^{2} (\alpha_i^{\frac{1}{2}} ||\mathbf{u}||_{\mathbf{H^2}(\Omega_i)} + \beta^{\frac{1}{2}} ||\mathbf{u}||_{\mathbf{H^1}(\Omega_i)})$.

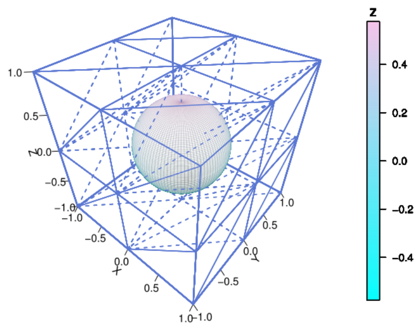
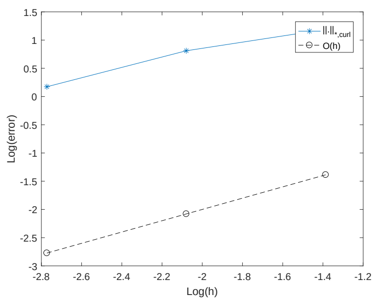
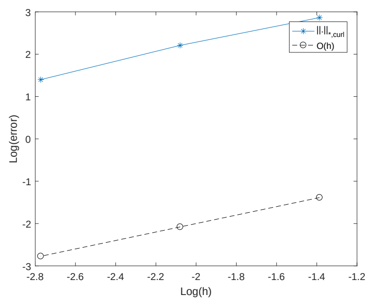


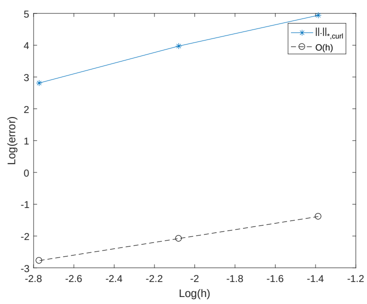

FIGURE 5. Illustration of the coarsest tetrahedral mesh.




(a) $\alpha_1 = 1,\ \alpha_2 = 0.1$.




(b) $\alpha_1 = 1,\ \alpha_2 = 0.01$.




(c) $\alpha_1 = 1,\ \alpha_2 = 0.001$.

FIGURE 6. Illustrations of convergence order under different coefficient pairs.

**6.1. Numerical example with curved interface.** The computational domain $\Omega$ is $(-1,1)^3$ and the interface $\Gamma$ is a sphere surface $\{(x,y,z): x^2+y^2+z^2=\frac{1}{3}\}$. Set $\Omega_1 = \{(x,y,z)\in\Omega : x^2+y^2+z^2<\frac{1}{3}\}$ and $\Omega_2 = \Omega\setminus\overline{\Omega}_1$. Corresponding coarsest subdivision area with the mesh size $h=0.5$ is as shown in Figure 5. Refine the mesh in a regular way which divides a coarse element into eight smaller ones.

Give the exact solution as follows

$$\mathbf{u} = \begin{cases} \boldsymbol{\psi(x,y,z)}/\alpha_1, & \text{in } \Omega_1, \\ \boldsymbol{\psi(x,y,z)}/\alpha_2, & \text{in } \Omega_2, \end{cases} \tag{59}$$

where $\boldsymbol{\psi(x,y,z)} = [(x^2-1)(y^2-1)(z^2-1)(x^2+y^2+z^2-\frac{1}{3})^2, 0, 0]^\top$.

We choose $\beta=1, \gamma_1=100000, \gamma_2=10$ and derive the source function $\mathbf{f}$ through (1). It is easy to verify that the construction of $\mathbf{u}$ satisfies the homogeneous boundary condition and the jump conditions in (1). By the refinement process, we can see the optimal first order of the convergence rate under different physical parameter pairs, see Figures 6.

In addition, refer to numerical experiments in [28], we check relation between the relative error $r_e$ and different physical coefficient pairs in the following table. Table 1 shows that the optimal error estimate is independent of the physical coefficients. If we choose $\gamma_1=100, \gamma_2=10$, the result of the optimal convergence order can

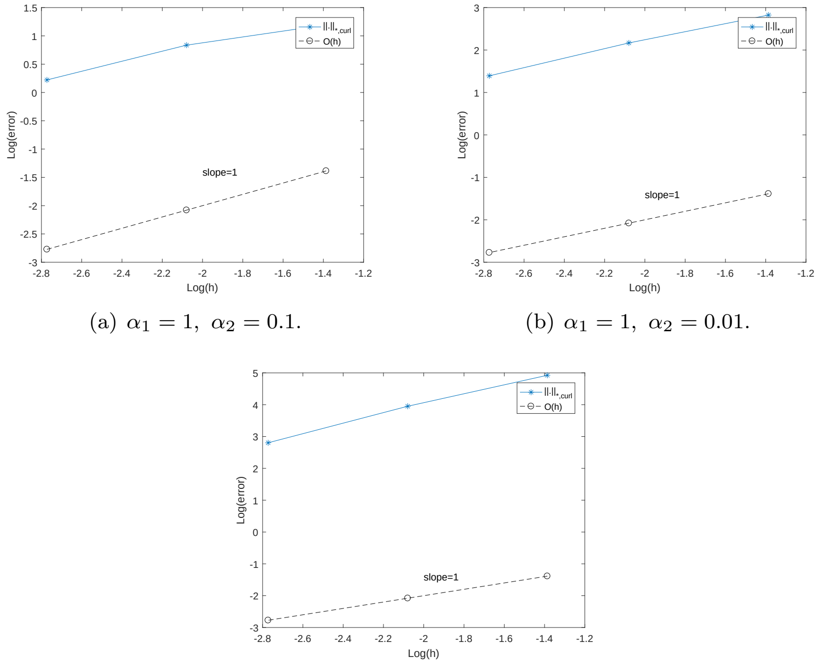
TABLE 1. Relative errors $r_e$ under different physical coefficients and refinement levels.

| $(\alpha_1,\alpha_2)$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $(1,10)$ | 0.1009 | 0.0832 | 0.0619 | 0.0303 |
| $(1,100)$ | 0.2086 | 0.1644 | 0.1234 | 0.0549 |
| $(1,1000)$ | 0.3051 | 0.2335 | 0.1734 | 0.0752 |
| $(1,10000)$ | 0.3254 | 0.2473 | 0.1831 | 0.0791 |
| $(1,100000)$ | 0.3277 | 0.2489 | 0.1842 | 0.0795 |
| $(1,1000000)$ | 0.3279 | 0.2490 | 0.1843 | 0.0795 |

be seen in Figure 7. And relative errors $r_e$ under different physical coefficients and refinement levels can be seen in Table 2. From Figure 7, we know that the choice of $\gamma_1$ and $\gamma_2$ do not affect the optimal convergence order, Table 2 shows that the relative errors produced under $\gamma_1=100$ and $\gamma_2=10$ are slightly larger than ones in the case with $\gamma_1=100000$ and $\gamma_2=10$.

TABLE 2. Relative errors $r_e$ under different physical coefficients and refinement levels.

| $(\alpha_1,\alpha_2)$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $(1,10)$ | 0.1041 | 0.0846 | 0.0932 | 0.0613 |
| $(1,100)$ | 0.2188 | 0.1786 | 0.2110 | 0.1414 |
| $(1,1000)$ | 0.3211 | 0.2543 | 0.3029 | 0.2018 |
| $(1,10000)$ | 0.3426 | 0.2644 | 0.3208 | 0.2133 |
| $(1,100000)$ | 0.3450 | 0.2711 | 0.3228 | 0.2146 |
| $(1,1000000)$ | 0.3452 | 0.2712 | 0.3230 | 0.2147 |




(a) $\alpha_1 = 1,\ \alpha_2 = 0.1.$

(b) $\alpha_1 = 1,\ \alpha_2 = 0.01.$

(c) $\alpha_1 = 1,\ \alpha_2 = 0.001.$

FIGURE 7. Illustrations of convergence order under different coefficient pairs.

In addition, to study the relationship between the error estimate and the interface location, we fix the mesh size $h = 0.0625$, move the centre of circle interface $\Gamma$ from $(x_0, y_0, z_0) = (0, 0, 0)$ to (0.0625,0,0) along the direction of X-axis, and keep the radius of circle unchanged. Then the relative errors $r_e$ under different interface positions can be seen in Table 6, which shows that the error estimate is independent of the interface position relative to the mesh.

TABLE 3. Relative error values $r_e$ under different interface positions.

| $x_0$ | $y_0$ | $z_0$ | $r_e$ |
|---|---|---|---|
| 0 | 0 | 0 | 0.0241 |
| 0.00625 | 0 | 0 | 0.0239 |
| 0.00625*2 | 0 | 0 | 0.0237 |
| 0.00625*3 | 0 | 0 | 0.0240 |
| 0.00625*4 | 0 | 0 | 0.0237 |
| 0.00625*5 | 0 | 0 | 0.0237 |
| 0.00625*6 | 0 | 0 | 0.0237 |
| 0.00625*7 | 0 | 0 | 0.0238 |
| 0.00625*8 | 0 | 0 | 0.0235 |
| 0.00625*9 | 0 | 0 | 0.0235 |
| 0.00625*10 | 0 | 0 | 0.0238 |

**6.2. Numerical example with moving direct interface.** Choose the computational domain $\Omega = (-1,1)^3$ and the interface face $y = \frac{\pi}{5}$, let $\Omega_1 = \{(x,y,z) \in \Omega : y < \frac{\pi}{5}\}$ and $\Omega_2 = \Omega \setminus \overline{\Omega}_1$, see Figure 8. The exact solution $\mathbf{u}$ is given by

$$\mathbf{u} = \begin{cases} \boldsymbol{\psi(x,y,z)}/\alpha_1, & \text{in } \Omega_1, \\ \boldsymbol{\psi(x,y,z)}/\alpha_2, & \text{in } \Omega_2, \end{cases} \tag{60}$$

where $\boldsymbol{\psi(x,y,z)} = [\sin(\pi x)\sin(\pi y)\sin(\pi z)(y - \frac{\pi}{5})^2, 0, 0]^\top$.

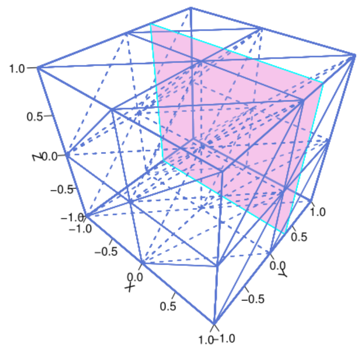



FIGURE 8. Illustration of the coarsest tetrahedral mesh.

In this case, we choose the mesh size $h = 0.0625$ and move the interface from $y = \frac{\pi}{5}$ to $y = \frac{\pi}{5} + 0.0625$. Then the relative error $r_e$ under different interface locations are calculated as follows, see Table 4. From Table 4, we see that the error estimate is independent of the interface location with respect to the mesh.

TABLE 4. Relative error values $r_e$ under different interface positions.

| $y$ | $r_e$ |
|---|---|
| $\frac{\pi}{5}$ | 0.0371 |
| $\frac{\pi}{5} + 0.00625$ | 0.0373 |
| $\frac{\pi}{5} + 0.00625 * 2$ | 0.0375 |
| $\frac{\pi}{5} + 0.00625 * 3$ | 0.0377 |
| $\frac{\pi}{5} + 0.00625 * 4$ | 0.0379 |
| $\frac{\pi}{5} + 0.00625 * 5$ | 0.0381 |
| $\frac{\pi}{5} + 0.00625 * 6$ | 0.0383 |
| $\frac{\pi}{5} + 0.00625 * 7$ | 0.0384 |
| $\frac{\pi}{5} + 0.00625 * 8$ | 0.0385 |
| $\frac{\pi}{5} + 0.00625 * 9$ | 0.0385 |
| $\frac{\pi}{5} + 0.00625 * 10$ | 0.0385 |

**6.3. Numerical example with curved interface.** We provide an authentic $\mathbf{H}(\mathbf{curl})$-interface problem in this section, whose solution satisfies merely the tangential continuity but violating the normal continuity.

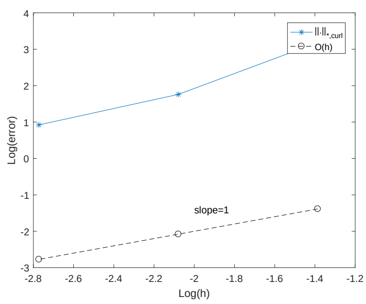
Let the domain be $\Omega = (-1, 1)^3$ and the interface be $\Gamma = \{(x, y, z) : x^2+y^2+z^2 = 1/3\}$. Set $\Omega_1 = \{(x, y, z) : x^2 + y^2 + z^2 < 1/3\}$ and $\Omega_2 = \Omega\backslash\overline{\Omega}_1$. The exact solution $\mathbf{u}$ is given by

$$\mathbf{u} = \begin{cases} \boldsymbol{\psi(x,y,z)}/\alpha_1, & \text{in } \Omega_1, \\ \boldsymbol{\psi(x,y,z)}/\alpha_2, & \text{in } \Omega_2, \end{cases} \tag{61}$$

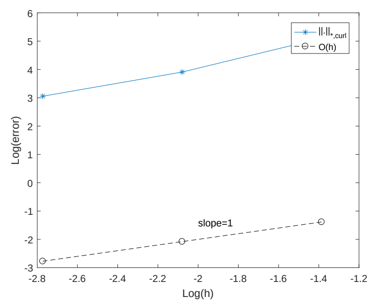
where $\boldsymbol{\psi(x,y,z)} =$

$$\begin{pmatrix} \pi\cos(\pi x)\sin(\pi y)\sin(\pi z)(x^2+y^2+z^2-1/3) + 2x\sin(\pi x)\sin(\pi y)\sin(\pi z) \\ \pi\cos(\pi y)\sin(\pi x)\sin(\pi z)(x^2+y^2+z^2-1/3) + 2y\sin(\pi x)\sin(\pi y)\sin(\pi z) \\ \pi\cos(\pi z)\sin(\pi x)\sin(\pi y)(x^2+y^2+z^2-1/3) + 2z\sin(\pi x)\sin(\pi y)\sin(\pi z) \end{pmatrix}.$$

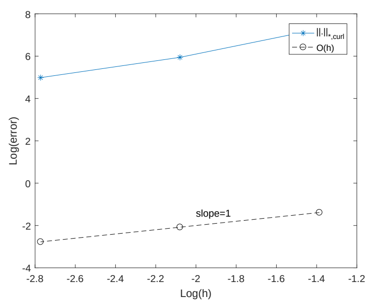
Choose $\beta = 1$ and derive the source function $f$ through the equation (1) using the exact solution (61). Set $\gamma_1 = 1000, \gamma_2 = 1$, by the refinement, the optimal convergence order is obtained, see Figure 9. Increase the relative jump of coefficients from $10^{-3}$ to $10^3$, record the corresponding relative errors $r_e$ under different physical coefficients and refinement levels in Table 5. Table 5 shows that the optimal convergence order is independent of the physical parameters. Then, fixing $\alpha_1 = 1, \alpha_2 = 0.1$ and $h = 0.0625$, moving the centre of $\Gamma$ from $(x_0, y_0, z_0) = (0, 0, 0)$ to $(x_0, y_0, z_0) = (0.0625, 0, 0)$ along the direction of X-axis, and keeping the radius of $\Gamma$ unchanged, we get the relative errors as Table 6. And Table 6 tells us that the optimal convergence order is uniform with the interface location.




(a) $\alpha_1 = 1,\ \alpha_2 = 0.1$.




(b) $\alpha_1 = 1,\ \alpha_2 = 0.01$.




(c) $\alpha_1 = 1,\ \alpha_2 = 0.001$.

FIGURE 9. Illustrations of convergence order under different coefficient pairs.

TABLE 5. Relative errors $r_e$ under different physical coefficients and refinement levels.

| $(\alpha_1, \alpha_2)$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $(1, 10^{-3})$ | 0.6017 | 0.3316 | 0.0867 | 0.0332 |
| $(1, 10^{-2})$ | 0.2091 | 0.1180 | 0.0358 | 0.0152 |
| $(1, 10^{-1})$ | 0.0948 | 0.0547 | 0.0132 | 0.0057 |
| $(1, 1)$ | 0.0716 | 0.0388 | 0.0042 | 0.0018 |
| $(1, 10^{1})$ | 0.0651 | 0.0339 | 0.0022 | 0.0016 |
| $(1, 10^{2})$ | 0.0483 | 0.0224 | 0.0041 | 0.0032 |
| $(1, 10^{3})$ | 0.0223 | 0.0105 | 0.0050 | 0.0039 |

TABLE 6. Relative error values $r_e$ under different interface positions.

| $x_0$ | $y_0$ | $z_0$ | $r_e$ |
|---|---|---|---|
| 0 | 0 | 0 | 0.0057 |
| 0.00625 | 0 | 0 | 0.0056 |
| 0.00625*2 | 0 | 0 | 0.0055 |
| 0.00625*3 | 0 | 0 | 0.0060 |
| 0.00625*4 | 0 | 0 | 0.0060 |
| 0.00625*5 | 0 | 0 | 0.0061 |
| 0.00625*6 | 0 | 0 | 0.0061 |
| 0.00625*7 | 0 | 0 | 0.0062 |
| 0.00625*8 | 0 | 0 | 0.0061 |
| 0.00625*9 | 0 | 0 | 0.0062 |
| 0.00625*10 | 0 | 0 | 0.0063 |

## 7. Conclusions

For H(curl)-elliptic interface problems in three dimensions, we propose an H(curl)-conforming Nitsche extended finite element method, based on the lowest order of second family Nédélec edge elements. A stabilization term defined on the transmission faces and harmonic weights are introduced in the approximation scheme. Stable results and the optimal convergent order are derived, which are both independent of not only the mesh size and the interface position with respect to the mesh but also the physical parameters.

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