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BOUNDARY ELEMENT METHOD WITH HIGH ORDER IMPEDANCE BOUNDARY CONDITIONS IN ELECTROMAGNETICS

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Abstract. In this paper, we study boundary element method with high order impedance boundary conditions (HOIBC) to solve Maxwell's equations. The unknowns are electric and magnetic currents **J** and **M**. We propose several formulations and study the existence and uniqueness of the solution. Then, we discretize these formulations with a finite element method based on Lagrange elements. We give numerical tests of the HOIBC solution.

Key words. Boundary element method, scattering problem, high order impedance boundary condition.

1. Introduction

Radar and antenna system designers are interested in the theoretical study of the scattering of electromagnetic waves. Interest in this topic has prompted intensive research in this area long time ago. However rigorous analysis was not performed until recently. The development of the computing technology improves modeling possibility and it increases the interest in the scattering problem of electromagnetic waves. The difficulties of numerical methods include the necessity of using a large number of unknowns in the description of high frequency electromagnetic fields. The scattering problem is being studied for conducting bodies and for a perfect conducting body covered by a complex layer. The complex layer is considered as a homogeneous surface, as a chiral surface or as a frequency selective surface. Presently, the frequency selective surface is important for design artificial coatings of antenna.

There are many methods for solving the Maxwell's equations in harmonic regime. The first method is the volume method. It locates their computations all over the volume internal and external objects. It uses a domain containing the obstacles bounded by an artificial border. It considers the physical characteristics of the media, in particular the effects of anisotropy, but it requires a large number of unknowns and the management of explicit boundary conditions. Another method is the discontinuous Galerkin method, [12] that we used to solve elasticity problem [13].

Here, we choose the method of moments. It places unknowns on the boundaries of the object and it takes into account the boundary conditions. It allows reducing the exterior problem to a system of integral equations defined on the surface of the obstacle and we calculate equivalent magnetic and electric currents **M** and **J** which produce the true scattered fields in the exterior region. However, they can only be applied to homogeneous bodies. Here, we choose this method to solve time-harmonic scattering problem for a coated body.

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In order to ensure a unique solution to this boundary value problem it is necessary to apply boundary condition. Generally, we add impedance boundary condition on the surface of the object where impedance operator is a constant. It is known as standard impedance boundary conditions or Leontovith condition. But this approximation does not depend on incident angle at all. In this paper, we deal with higher order impedance boundary conditions to take account incidence angle. Recently, the higher order impedance boundary conditions have been studied in [2, 3]. This list is not exhaustive. These conditions take into account the incident angle at each point of the surface and include derivatives of tangential components of the fields that are equivalent to transverse wave numbers. The authors give several numerical results for body of revolution.

Later, the higher order impedance boundary conditions is applied to study the scattering problem from a finite planar or curved infinitesimally thin frequency selective surface embedded in a dielectric layer [8, 4, 5, 6]. The author introduces differential operators to express higher order impedance boundary conditions.

The organization of this paper is as follows. In section 2, we present the physical model. Then, in section 3, we give the high order approximation of the impedance boundary condition. In section 4, we establish formulations and we study existence and uniqueness of the solutions. In section 5, we give a discretization of this formulation and in section 6 we give several numerical tests.

2. Mathematical model of physical problem

We consider the scattering problem of electromagnetic waves ($\mathbf{E}$, $\mathbf{H}$) by a perfect conducting body with a complex coating. We denote Ω an open domain in $\mathbb{R}^2$ with a Lipschitz-continuous boundary $\Gamma = \partial\Omega$, which can be equipped with an exterior unit normal vector field $\mathbf{n}$, (see Figure 1). Electromagnetic waves propagate in $\Omega^+ = \mathbb{R}^2 \backslash \overline{\Omega}$. We illuminate this system by incident electromagnetic waves. Scattering waves occur when incident waves bounce off an object in a variety of directions. The amount of scattering waves that take place depends on the wavelength of the incident waves and structure of the object. We determine total electromagnetic fields $(\mathbf{E}, \mathbf{H})$ in Ω^+ as:

$$\begin{cases} \mathbf{E} = \mathbf{E}^{inc} + \mathbf{E}^{sc}, \\ \mathbf{H} = \mathbf{H}^{inc} + \mathbf{H}^{sc}. \end{cases} \tag{1}$$

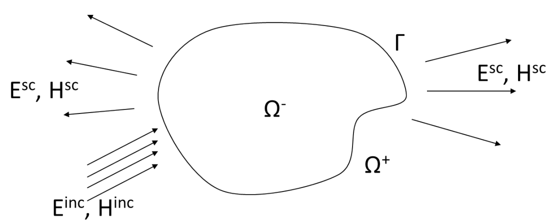

FIGURE 1. Scattering problem.

Superscripts *inc* and *sc* characterize incident and scattered fields, respectively. Waves propagation medium is described by two values ϵ (electrical permittivity) and μ (magnetic permeability), where we have $\epsilon = \epsilon_0$ and $\mu = \mu_0$ for free space. We are interested in the time-harmonic electromagnetic fields that are defined as

$$\begin{cases} \mathbf{E}(x,t) = \Re(\mathbf{E}(x)e^{i\omega t}), \\ \mathbf{H}(x,t) = \Re(\mathbf{H}(x)e^{i\omega t}), \end{cases} \tag{2}$$

where ω denotes the pulsation. The fields outside the body are governed by Maxwell's equations for a free space. The harmonic solution verifies following e-quations:

$$\begin{cases} \mathbf{rot}\mathbf{E} + i\omega\mu\mathbf{H} = 0, \\ \mathbf{rot}\mathbf{H} - i\omega\epsilon\mathbf{E} = 0. \end{cases} \tag{3}$$

The fields inside the coating are governed by a set of equations that take into account the detailed electromagnetic properties of the coating. We consider boundary condition that binds the tangent electric and magnetic fields. For two vectors $u = u_x i + u_y j$ et $v = v_x i + v_y j$ in a cartesian coordinate repere with i and j are unitaire vectors in the plane, we have $u \times v = (u_x v_y - v_x u_y)k$ with (i, j, k) a direct repere in $\mathbb{R}^3$. The medium characteristics give an impedance at each point of the surface Γ:

$$\mathbf{E}_{tg} - Z(\mathbf{n} \times \mathbf{H}) = 0, \quad on\ \Gamma, \tag{4}$$

where Z is impedance operator that depends on incident angle, medium thickness and characteristics ϵ and μ. Subscript tg denotes tangent component on the surface Γ defined as:

$$\mathbf{E}_{tg} = \mathbf{n} \times (\mathbf{E} \times \mathbf{n}).$$

The boundary condition (4) is called impedance boundary condition (IBC). The simplest form of which is known as Leontovich IBC or standard IBC (SIBC), where $Z = constant$. The IBC can be partially constant (if the object is formed by different materials) or more different. For the correct formulation of the problem, we should introduce asymptotic behavior of the fields $(\mathbf{E}, \mathbf{H})$, the Silver-Müller radiation condition:

$$\lim_{r \to \infty} r(\mathbf{E} \times \mathbf{n}_r + \mathbf{H}) = 0, \tag{5}$$

where $r = |\mathbf{x}|$ and $\mathbf{n}_r = \dfrac{\mathbf{x}}{|\mathbf{x}|}$, $\mathbf{x} \in \mathbb{R}^2$.

Then, we have the next problem:

Problem 2.1. *Find $(\boldsymbol{E}, \boldsymbol{H})$ such that*

$$\begin{cases} \mathbf{rot}\boldsymbol{E} + ik_0\mu\boldsymbol{H} = 0 \quad in\ \Omega^+ \\ \mathbf{rot}\boldsymbol{H} - ik_0\epsilon\boldsymbol{E} = 0 \quad in\ \Omega^+ \\ \boldsymbol{E}_{tg} - Z(\boldsymbol{n} \times \boldsymbol{H}) = 0 \quad on\ \Gamma \\ \lim_{r\to\infty} r(\boldsymbol{E} \times \boldsymbol{n}_r + \boldsymbol{H}) = 0. \end{cases} \tag{6}$$

In the first time, we obtain the following result.

Theorem 2.1. *The problem 2.1 admits a unique solution, if following relations are verified:*

$$\begin{cases} \Im(\mu) \le 0, \\ \Im(\epsilon) \le 0, \\ \Re(k_0 \int_\Gamma \mathbf{E}^* \cdot (\mathbf{n} \times \mathbf{H}) ds) \ge 0. \end{cases} \tag{7}$$

where $\mathbf{E}^*$ *is the adjoint of* $\mathbf{E}$.

Proof 1. *It suffies to apply Rellich's lemma to obtain the result with the conditions 7.*

In the next section, we give the approximation of the impedance operator Z with integral operators to derive a variational formulation for the scattering problem (6).

3. Approximation of impedance operator

3.1. High order impedance boundary condition. We assume that the plane-wave fields are written in the following forms:

$$\mathbf{E}(\mathbf{r},t)=\mathbf{e}_1E_0e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t},$$

$$\mathbf{H}(\mathbf{r},t)=\mathbf{e}_2H_0e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t},$$

where $\mathbf{e}_1,\ \mathbf{e}_2$ are two constant real unit vectors; $E_0,\ H_0$ are complex amplitudes which are constant in space and time.

$$\begin{aligned}\mathbf{E}(\mathbf{r}) =&\mathbf{E}_0e^{-i(k_x\hat{\mathbf{x}}+k_y\hat{\mathbf{y}}+k_z\hat{\mathbf{z}})\cdot\mathbf{r}},\\ \partial_x\mathbf{E}(\mathbf{r}) =&-ik_x\mathbf{E}(\mathbf{r}),\\ \partial_x^2\mathbf{E}(\mathbf{r}) =&-k_x^2\mathbf{E}(\mathbf{r}).\end{aligned}$$

So we can replace partial derivatives by k_x and k_y components

$$\partial_x=-ik_x\ \ \textit{and}\ \ \partial_y=-ik_y \tag{8}$$

or

$$\partial_x^2=-k_x^2,\ \partial_{xy}^2=-k_xk_y\ \ \textit{and}\ \ \partial_y^2=-k_y^2. \tag{9}$$

In [R-S] the impedance boundary conditions are written using the spectral domain approach and are approximated as a ratio of second order polynomials for a coating, invariant under rotation. Those approximation equations could be written as

$$(1+b_1\partial_x^2+b_2\partial_y^2)E_x+(b_1-b_2)\partial_{xy}^2E_y=(a_1-a_2)\partial_{xy}^2H_x-(a_0+a_1\partial_x^2+a_2\partial_y^2)H_y \tag{10}$$

and

$$(b_1-b_2)\partial_{xy}^2E_x+(1+b_2\partial_x^2+b_1\partial_y^2)E_y=(a_0+a_2\partial_x^2+a_1\partial_y^2)H_x+(a_2-a_1)\partial_{xy}^2H_y. \tag{11}$$

Note that $\mathbf{n}\times\mathbf{H}=-H_y\mathbf{x}+H_x\mathbf{y}$. And the high order impedance condition is written in matrix form

$$\begin{aligned}&\begin{bmatrix}1+b_1\partial_x^2+b_2\partial_y^2 & (b_1-b_2)\partial_{xy}^2\\ (b_1-b_2)\partial_{xy}^2 & 1+b_2\partial_x^2+b_1\partial_y^2\end{bmatrix}\begin{pmatrix}E_x\\ E_y\end{pmatrix}\\ =&\begin{bmatrix}a_0+a_1\partial_x^2+a_2\partial_y^2 & (a_1-a_2)\partial_{xy}^2\\ (a_1-a_2)\partial_{xy}^2 & a_0+a_2\partial_x^2+a_1\partial_y^2\end{bmatrix}\begin{pmatrix}-H_y\\ H_x\end{pmatrix}.\end{aligned} \tag{12}$$

In the next, we give high order impedance boundary condition.

3.2. Approximation of higher order impedance boundary condition. Here, we need to consider two different situations. In the first case, we consider case when the electric field is perpendicular to the incident plane, as shown on figure (2 a). Incident, scattered and transverse electric fields are directed toward the viewer. The direction of magnetic field was chosen such that energy current has positive direction, i.e. direction of wave propagation. We call this case, transverse-electric (TE) polarization. In the second case, electric fields are parallel to incident plane, as shown in figure (2 b). In this case we call it transverse-magnetic (TM) polarization.

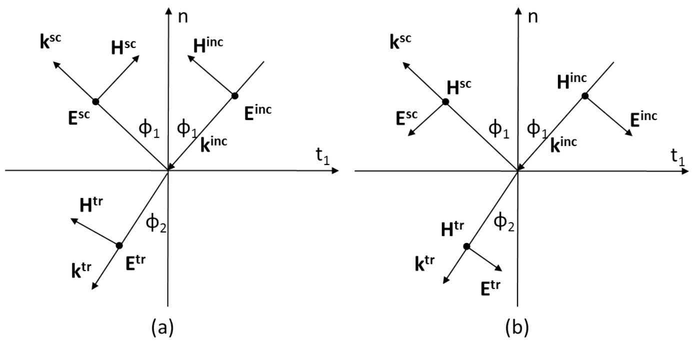

FIGURE 2. Reflection and refraction with (a) TE and (b) TM polarizations.

We assume that the incident fields propagate perpendicular to the cylinder axis, so that $\partial/\partial y = 0$. And fields are polarized either with the electric field in the y direction (E polarization or TM), or with the magnetic field in the y direction (H polarization or TE). Then, we have that $\partial_y \equiv 0$ and we obtain:

$$\begin{pmatrix} 1+b_1\partial_x^2 & 0 \\ 0 & 1+b_2\partial_x^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} a_0+a_1\partial_x^2 & 0 \\ 0 & a_0+a_2\partial_x^2 \end{pmatrix} \begin{pmatrix} -H_y \\ H_x \end{pmatrix}. \tag{13}$$

So we get that in two dimensional TE polarization, we have

$$(1+b_1\partial_x^2)E_x = -(a_0+a_1\partial_x^2)H_y$$

and in TM polarization, we have

$$(1+b_2\partial_x^2)E_y = (a_0+a_2\partial_x^2)H_x.$$

Then, for a plane wave the first order IBC (13) can be written as

$$\begin{pmatrix} 1-b_1k_x^2 & 0 \\ 0 & 1-b_2k_x^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} a_0-a_1k_x^2 & 0 \\ 0 & a_0-a_2k_x^2 \end{pmatrix} \begin{pmatrix} -H_y \\ H_x \end{pmatrix}. \tag{14}$$

According to (9), we get first order approximation of impedance in two dimensional cases for each polarization

$$\boxed{Z_{2Dj}: \quad (1+b_j\partial_x^2)\mathbf{E}_{tg} = (a_0+a_j\partial_x^2)\mathbf{n}\times\mathbf{H}} \tag{15}$$

and the impedance Z_{2Dj} is the following rational function of k_x^2

$$Z_{2Dj} = \frac{a_0-a_jk_x^2}{1-b_jk_x^2}, \quad j=1,2. \tag{16}$$

The coefficients indicated by $j = 1,2$ correspond to polarizations TE and TM respectively. These coefficients (a_0, a_j, and b_j) are determined by equating this

first order impedance Z_{2Dj} and the exact impedance. Besides, we can express exact impedance for TE and TM polarization as follows:
(17)
$$Z_{TE}^{ex} = \sqrt{\frac{\mu}{\epsilon}}\frac{k_z}{k}\tan(k_z d) = z_0\sqrt{\mu_r\epsilon_r - \left(\frac{k_x}{k_0}\right)^2}\tan\left(\sqrt{\mu_r\epsilon_r - \left(\frac{k_x}{k_0}\right)^2}k_0 d\right)/\epsilon_r,$$

$$Z_{TM}^{ex} = \sqrt{\frac{\mu}{\epsilon}}\frac{k}{k_z}\tan(k_z d) = \frac{z_0\mu_r\tan\left(\sqrt{\mu_r\epsilon_r - \left(\frac{k_x}{k_0}\right)^2}k_0 d\right)}{\sqrt{\mu_r\epsilon_r - \left(\frac{k_x}{k_0}\right)^2}}. \quad (18)$$

If $\theta = 0$, we get that $a_0 = Z(0) = Z^{ex}(0)$, for a normally incident wave, which is known as the Leontovich boundary condition and we get

$$a_0 = \sqrt{\frac{\mu_0\mu_r}{\epsilon_0\epsilon_r}}\tan(\omega\sqrt{\mu_0\mu_r\epsilon_0\epsilon_r}d).$$

We calculate other coefficients a_j and b_j, using two arbitrary angles θ_1 and θ_2 by:

$$\begin{pmatrix} a_j \\ b_j \end{pmatrix} = \begin{bmatrix} -k_x^2(\theta_1) & k_x^2 Z_j^{ex}(\theta_1) \\ -k_x^2(\theta_2) & k_x^2 Z_j^{ex}(\theta_2) \end{bmatrix}^{-1} \begin{pmatrix} Z_j^{ex}(\theta_1) - a_0 \\ Z_j^{ex}(\theta_2) - a_0 \end{pmatrix}.$$

The indices correspond to TE and TM polarizations, as in (16). The arbitrary angles θ_1 and θ_2 should be in the angle range $]0, \pi/2[$. Here we take $k_x^2 = k_0^2\sin^2(\theta)$ as [8].
The equation (15) can be extended to second order polynomials in ∂_x^2:

$$\boxed{\mathbf{E}_{tg} + b_j\partial_x^2\mathbf{E}_{tg} + b_j'\partial_x^4\mathbf{E}_{tg} = a_0(\mathbf{n}\times\mathbf{H}) + a_j\partial_x^2(\mathbf{n}\times\mathbf{H}) + a_j'\partial_x^4(\mathbf{n}\times\mathbf{H})} \quad (19)$$

or it can be reduced to constant:

$$\mathbf{E}_{tg} = a_0(\mathbf{n}\times\mathbf{H}). \quad (20)$$

We will call the equation (19) second order IBC (IBC2), the equation (20) zeroth order IBC (IBC0), which is also known as Leontovich IBC. And we will call the equation (15) as first order IBC (IBC1). Note that (19) with $a_j' = b_j' = 0$ derives to (15). As well as with $a_j = b_j = 0$, the equation (15) derives to (20).

In the following, we explain the process to calculate the coefficients of HOIBC. In this case, we determine two coefficients for IBC1 and five coefficients for IBC2 by matching the impedances exactly for normal incidence resulting in Eq; (17) and (18) as well as two or five values of θ_k resulting in linear equations for the remaining coefficients.

3.3. Calculus of coefficients of the approximation of the HOIBC. The simplest IBC is Leontovich IBC, as were already mentioned several times $Z = const$. Usually, it is taken for incident wave perpendicular to plane

$$Z_1 = Z_2 = Z_{1,2}^{ex}(\theta = 0),$$

$$a_0 = z_0\sqrt{\frac{\mu_r}{\epsilon_r}}\tan(\sqrt{\mu_r\epsilon_r}k_0 d). \quad (LIBC)$$

We take an arbitrary angle value in permitted range $[0, \pi/2]$. If the angle is not zero, then impedance are different to each other

$$Z_1 = Z_1^{ex}(\theta) \ and \ Z_2 = Z_2^{ex}(\theta),$$

in different polarizations.
For IBC1, we solve the system for different $\theta_k \in]0, \pi/2[, \ k = 1, 2$

$$\begin{pmatrix} a_j \\ b_j \end{pmatrix} = \begin{bmatrix} \xi_1 & -\xi_1 Z_j^{ex}(\xi_1) \\ \xi_2 & -\xi_2 Z_j^{ex}(\xi_2) \end{bmatrix}^{-1} \begin{pmatrix} Z_j^{ex}(\xi_1) - a_0 \\ Z_j^{ex}(\xi_2) - a_0 \end{pmatrix}$$

and for IBC2 condition, the coefficients are calculated by solving system for different $\theta_k \in]0, \pi/2[, \ k = 1, 2, 3, 4$

$$\begin{pmatrix} a_j \\ a'_j \\ b_j \\ b'_j \end{pmatrix} = \begin{bmatrix} \xi_1 & \xi_1^2 & -\xi_1 Z_j^{ex}(\xi_1) & -\xi_1^2 Z_j^{ex}(\xi_1)^2 \\ ... & ... & ... & ... \\ \xi_4 & \xi_4^2 & -\xi_4 Z_j^{ex}(\xi_4) & -\xi_4^2 Z_j^{ex}(\xi_4)^2 \end{bmatrix}^{-1} \begin{pmatrix} Z_j^{ex}(\xi_1) - a_0 \\ ... \\ ... \\ Z_j^{ex}(\xi_4) - a_0 \end{pmatrix}.$$

Then, in the next part, we give numerical tests that validate our approximation.

3.4. Numerical tests for the approximation of the HOIBC. In order to illustrate the relative accuracy of approximated boundary conditions compared to the exact IBC, we present here some examples. And we will see that HOIBC consider the incident angle parameter. We will see the difference between IBC0 and exact IBC.

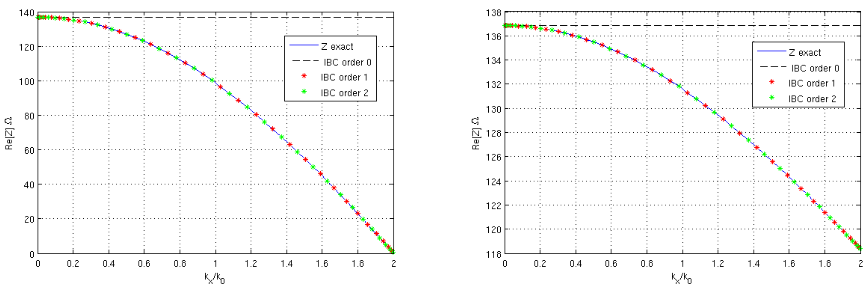

FIGURE 3. Comparison of the exact impedance, Leontovich impedance, first-order and second-order IBC in TE polarisation (left) and TM polarisation (right).

Let us consider a mono-layer dielectric coating with characteristics $\varepsilon_r = 4.0$, $\mu_r = 1.0$ and $d = 0.005\lambda_0$. Figure 3 shows values exact IBC, SIBC, first order and second order impedance boundary conditions, in TE polarization where the angle of incidence of the plane wave ϕ has angle range $]0, \pi[$. The IBC0 was taken as an impedance of a perpendicular incidence wave. To calculate first-order IBC approximation we used $\phi = 0 \ , \pi/6 \ , \pi/3$ and to calculate second-order IBC we used $\phi = 0 \ , \pi/8 \ , \pi/6 \ , \pi/4 \ , \pi/3$. On the figure 3, we can easily see that the difference between IBC0 and exact IBC increases. While the difference between exact IBC and IBC1 is very small, as the difference between exact IBC and IBC2.

But we can see the error of IBC1 and IBC2 approximations on the figure 4. As the angle of incidence increases the error of first-order IBC approximation reaches 0.39Ω.

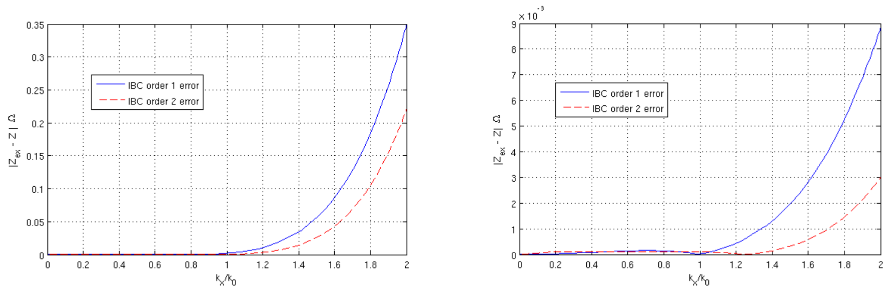

FIGURE 4. Errors of first-order (IBC1) and second-order (IBC2) IBC in TE polarisation (left) and TM polarisation (right).

With our approximation of impedance boundary condition, we derive variational formulation to solve the scattering problem with boundary element method. The higher order boundary conditions of the scattering problem begins by defining two equivalent problems, one for the exterior region, and another for interior region. For the exterior region the material are replaced by equivalent magnetic and electric currents J and M. We use stratton-Chu formulae to obtain variational formulations.

4. Variational Formulations of Problem 2.1

The higher order impedance boundary conditions solution of the scattering problem begins by defining two equivalent problems. For the exterior region, we introduce equivalent magnetic and electric on Γ defined by:

$$\mathbf{M} = [\mathbf{E} \times \mathbf{n}]_{-}^{+} \quad \mathbf{J} = [\mathbf{n} \times \mathbf{H}]_{-}^{+},$$

where $[\ ]_{-}^{+}$ denotes difference between upper (+) and lower (-) values of interface, $\mathbf{n}$ is the exterior normal vector to the surface. We use the following integral operators.

Definition 4.1. *We introduce the integral operators $(B-S)$, $(P+Q)$ and I defined by:*

$$\langle (B-S)\boldsymbol{A}, \boldsymbol{\psi}\rangle = i \iint_{\Gamma} kG\boldsymbol{A} \cdot \boldsymbol{\psi} - \frac{1}{k} G\nabla_y \cdot \boldsymbol{A}\nabla_x \cdot \boldsymbol{\psi} dydx, \tag{21}$$

$$\langle (P+Q)\boldsymbol{A}, \boldsymbol{\psi}\rangle = \frac{1}{2}\int_{\Gamma} \boldsymbol{\psi} \cdot (\boldsymbol{n} \times \boldsymbol{A})dx + \iint_{\Gamma} (\boldsymbol{\psi} \times \boldsymbol{A}) \cdot \nabla_x G dydx, \tag{22}$$

$$\langle I\boldsymbol{A}, \boldsymbol{\psi}\rangle = \int_{\Gamma} \boldsymbol{A} \cdot \boldsymbol{\psi} dx, \tag{23}$$

and $G(x,y)$ is the Green kernel giving the outgoing solutions to the scalar Helmholtz equation:

$$G(x,y) := \frac{\pi}{i} H_0^{(2)}(k|x-y|), \quad \nabla_x G(x,y) := -\frac{\pi k}{i|x-y|} H_1^{(2)}(k_j|x-y|)(\mathbf{x}-\mathbf{y}). \tag{24}$$

We give the following results about these operators.

Theorem 4.1. *The operator Q is continuous from $H^{-1/2}(\mathrm{div},\Gamma)$ to $H^{-1/2}(\mathrm{rot},\Gamma)$ and we have that:*

$$|(\mathbf{n} \times Q + \frac{I}{2})\mathbf{M}|_{-1/2,\mathrm{div}_\Gamma} \leq C|\mathbf{M}|_{-1/2,\mathrm{div}_\Gamma} \quad \forall \mathbf{M} \in H^{-1/2}(\mathrm{div},\Gamma). \tag{25}$$

And we have, in [11](Chapter II, p.61) :

Theorem 4.2. *The operator* $(B-S)$ *is an isomorphisme from* $H^{-1/2}(\text{div},\Gamma)$ *to* $H^{-1/2}(\text{rot},\Gamma)$ *and it verifies the inequality:*

$$\|(B-S)\boldsymbol{\phi}\|_{-1/2,\text{rot}_\Gamma} \le C\|\boldsymbol{\phi}\|_{-1/2,\text{div}_\Gamma} \tag{26}$$

and the coercivity relation $\forall\boldsymbol{\phi}\in H^{-1/2}(\text{div},\Gamma)$*:*

$$\Re(<\boldsymbol{\phi},(B-S)\boldsymbol{\phi}>) \ge C\|\boldsymbol{\phi}\|^2_{-1/2,\text{div}_\Gamma}. \tag{27}$$

4.1. Integral method-EFIE-MFIE and HOIBC. Here we apply the first and second order HOIBC for two dimensional problem that were defined in the last section. The problems for TE and TM polarizations will be presented separately.

Two dimensional case system is invariant in one direction, so object surface Γ becomes a curved contour, that we will call C. We have the curvilinear abscissa l along C and normal to the contour unit vector $\mathbf{n}$. We set the local frame ($\boldsymbol{\tau},\ \boldsymbol{\nu},\ \mathbf{n}$), where $\boldsymbol{\tau}$ is a unit vector tangent to the contour C in l direction, and $\boldsymbol{\nu}$ can be defined as $\boldsymbol{\nu}=\mathbf{n}\times\boldsymbol{\tau}$. We suppose that our two dimensional system does not depend on ν parameter, however variable ν component is depend on l.

In the first time, we can write that if E and H are solutions of problem (2.1) then $\mathbf{J}$ and $\mathbf{M}$ verify the EFIE and the MFIE:

$$< Z_0(B-S)\mathbf{J},\boldsymbol{\Psi}_J> + <(P+Q)\mathbf{M},\boldsymbol{\Psi}_J> = < IE^{inc},\boldsymbol{\Psi}_J>, \tag{28}$$

$$-<(P+Q)\mathbf{J},\boldsymbol{\Psi}_M> + <\frac{1}{Z_0}(B-S)\mathbf{M},\boldsymbol{\Psi}_M> = < IH^{inc},\boldsymbol{\Psi}_M>. \tag{29}$$

Now, we give a variational formulation of the impedance boundary condition and we insert it in the below equations.

We said that impedance boundary conditions are described by the following

$$\mathbf{E}_{tg} = Z(\mathbf{n}\times\mathbf{H}).$$

According to the definition of electromagnetic current densities, we have

$$\mathbf{E}_{tg} = \mathbf{n}\times(\mathbf{E}\times\mathbf{n}) = \mathbf{n}\times\mathbf{M} \quad on\ \Gamma;$$

$$\mathbf{n}\times\mathbf{H} = \mathbf{J} \quad on\ \Gamma.$$

So we rewrite impedance boundary condition as follows

$$\mathbf{n}\times\mathbf{M} = Z\mathbf{J}. \tag{30}$$

And we approximate the operator Z, as a ratio of polynomials of differential operators. So, we recall first order IBC

$$\boxed{(1+b_jd_l^2)(\mathbf{n}\times\mathbf{M}) = (a_0+a_jd_l^2)\mathbf{J}} \tag{31}$$

and the second order IBC

$$\boxed{(1+b_jd_l^2+b'_jd_l^4)(\mathbf{n}\times\mathbf{M}) = (a_0+a_jd_l^2+a'_jd_l^4)\mathbf{J}} \tag{32}$$

where $j=1,2$ correspond to TE and TM polarizations, respectively. The invariance in one direction for two dimensional model allow us to simplify the Hodge operator as the second partial derivative on the contour where the electromagnetic current densities $\mathbf{n}\times\mathbf{M}$ and $\mathbf{J}$ have $\boldsymbol{\tau}$ direction for TE polarization, and $\boldsymbol{\nu}$ direction for TM polarization such as:

- TE: $\partial_x^2\mathbf{J} = \boldsymbol{\tau}\partial_x^2 J_\tau = \boldsymbol{\tau}d_l^2 J_\tau$ and $\partial_x^2(\mathbf{n}\times\mathbf{M}) = -\boldsymbol{\tau}\partial_x^2 M_\nu = -\boldsymbol{\tau}d_l^2 M_\nu$;

- TM: $\partial_x^2 \mathbf{J} = \boldsymbol{\nu}\partial_x^2 J_\nu = \boldsymbol{\nu} d_l^2 J_\nu$ and $\partial_x^2(\mathbf{n}\times\mathbf{M}) = \boldsymbol{\nu}\partial_x^2 M_\tau = \boldsymbol{\nu} d_l^2 M_\tau$.

Then, we establish two integral formulations in TE and two in TM polarization with ibc1 and ibc2. In these formulations, the principal variables are electric and magnetic densities J end M. Here, we only present the formulations in TE polarization since we obtain the formulations with the same way in TM polarization.

In the next, we establish formulations on a contour C since we suppose that Ω has an invariance in one direction.

4.1.1. Variational formulations with IBC1. Employing the standard method of moments technique to solve the boundary condition equation and using a function Ψ_j for testing the equation (31) along the contour C the following equation is obtained:

$$\int_C (1 + b_j d_l^2)(\mathbf{n}\times\mathbf{M})\cdot\mathbf{\Psi}_J dl = \int_C (a_0 + a_j d_l^2)\mathbf{J}\cdot\mathbf{\Psi}_J dl.$$

Therefore, we have:

$$\int_C (\mathbf{n}\times\mathbf{M})\cdot\mathbf{\Psi}_J dl = \int_C (a_0 + a_j d_l^2)\mathbf{J}\cdot\mathbf{\Psi}_J dl - \int_C b_j d_l^2(\mathbf{n}\times\mathbf{M})\cdot\mathbf{\Psi}_J dl.$$

We put it in the operator P and obtain:

$$\begin{aligned} < P\mathbf{M}, \mathbf{\Psi}_J > &= \frac{1}{2}\int_C (\mathbf{n}\times\mathbf{M})\cdot\mathbf{\Psi}_J dl \\ &= \frac{a_0}{2}\int_C \mathbf{J}\cdot\mathbf{\Psi}_J dl + \frac{a_j}{2}\int_C d_l^2\mathbf{J}\cdot\mathbf{\Psi}_J dl - \frac{b_j}{2}\int_C d_l^2(\mathbf{n}\times\mathbf{M})\cdot\mathbf{\Psi}_J dl. \end{aligned} \tag{33}$$

Now, we take $(\mathbf{n}\times\mathbf{\Psi}_M)$ a function for testing the equation (31) in another form, and we have:

$$\int_C (1 + b_j d_l^2)(\mathbf{n}\times\mathbf{M})\cdot(\mathbf{n}\times\mathbf{\Psi}_M) dl = \int_C (a_0 + a_j d_l^2)\mathbf{J}\cdot(\mathbf{n}\times\mathbf{\Psi}_M) dl.$$

We take the first part of right side

$$\begin{aligned} \int_C \mathbf{J}\cdot(\mathbf{n}\times\mathbf{\Psi}_M) dl = &\frac{1}{a_0}\int_C (1 + b_j d_l^2)(\mathbf{n}\times\mathbf{M})\cdot(\mathbf{n}\times\mathbf{\Psi}_M) dl \\ &- \frac{1}{a_0}\int_C a_j d_l^2\mathbf{J}\cdot(\mathbf{n}\times\mathbf{\Psi}_M) dl. \end{aligned}$$

And using the formula of vector analysis

$$\mathbf{\Psi}_M\cdot(\mathbf{n}\times\mathbf{J}) = -\mathbf{J}\cdot(\mathbf{n}\times\mathbf{\Psi}_M),$$

we put it in P operator with weakly form of IBC1

$$\begin{aligned} < P\mathbf{J}, \mathbf{\Psi}_M > &= \frac{1}{2}\int_C (\mathbf{n}\times\mathbf{J})\cdot\mathbf{\Psi}_M dl = -\frac{1}{2}\int_C \mathbf{J}\cdot(\mathbf{n}\times\mathbf{\Psi}_M) dl \\ &= -\frac{1}{2a_0}\int_C (\mathbf{n}\times\mathbf{M})\cdot(\mathbf{n}\times\mathbf{\Psi}_M) dl \\ &\quad - \frac{b_j}{2a_0}\int_C d_l^2(\mathbf{n}\times\mathbf{M})\cdot(\mathbf{n}\times\mathbf{\Psi}_M) dl + \frac{a_j}{2a_0}\int_C d_l^2\mathbf{J}\cdot(\mathbf{n}\times\mathbf{\Psi}_M) dl. \end{aligned} \tag{34}$$

First, we observe TE polarization, where P operator becomes:

$$\int_C PM_\nu\ \Psi_{J\tau} dl = \frac{a_0}{2}\int_C J_\tau\ \Psi_{J\tau} dl + \frac{a_1}{2}\int_C d_l^2 J_\tau\ \Psi_{J\tau} dl + \frac{b_1}{2}\int_C d_l^2 M_\nu\ \Psi_{J\tau} dl$$

and

$$\int_C PJ_\tau\ \Psi_{M\nu}dl = -\frac{1}{2a_0}\int_C M_\nu\ \Psi_{M\nu}dl - \frac{b_1}{2a_0}\int_C d_l^2 M_\nu\ \Psi_{M\nu}dl - \frac{a_1}{2a_0}\int_C d_l^2 J_\tau\ \Psi_{M\nu}dl$$

for EFIE and MFIE, respectively.

We put them in the variational equations (28) and (29) and get:

$$\begin{aligned} &iZ_0 \iint_C kG(l,l')\ J_\tau(l')\ \Psi_{J\tau}\ [\boldsymbol{\tau}(l')\cdot\boldsymbol{\tau}(l)] - \frac{1}{k}G(l,l')\ d_l' J_\tau(l')\ d_l\Psi_{J\tau}(l)\ dl'dl \\ &+ \iint_C \Psi_{J\tau}(l)\ M_\nu(l')\ [\boldsymbol{\tau}(l)\times\boldsymbol{\nu}(l')]\cdot\nabla_l G(l,l')\ dl'dl \\ &+ \frac{a_0}{2}\int_C J_\tau\ \Psi_{J\tau}dl + \frac{a_1}{2}\int_C d_l^2 J_\tau\ \Psi_{J\tau}dl + \frac{b_1}{2}\int_C d_l^2 M_\nu\ \Psi_{J\tau}dl \\ &= \int_C E_\tau^{inc}\ \Psi_{J\tau}dl \end{aligned} \tag{35}$$

and

$$\begin{aligned} &-\iint_C \Psi_{M\nu}(l)\ J_\tau(l')\ [\boldsymbol{\nu(l)}\times\boldsymbol{\tau(l')}]\cdot\nabla_l G(l,l')dl'dl \\ &+ \frac{i}{Z_0}\iint_C kG(l,l')\ M_\nu(l')\ \Psi_{M\nu}(l)[\boldsymbol{\nu(l')}\cdot\boldsymbol{\nu(l)}]dl'dl \\ &+ \frac{1}{2a_0}\int_C M_\nu\ \Psi_{M\nu}dl + \frac{b_j}{2a_0}\int_C d_l^2 M_\nu\ \Psi_{M\nu}dl + \frac{a_j}{2a_0}\int_C d_l^2 J_\tau\ \Psi_{M\nu}dl \\ &= \int_C H_\nu^{inc}\ \Psi_{M\nu}dl, \end{aligned} \tag{36}$$

for EFIE and MFIE, respectively.

In the equations (35) and (36) we have scalar products $[\boldsymbol{\tau}(l')\cdot\boldsymbol{\tau}(l)] = 1$ and $[\boldsymbol{\nu}(l)\cdot\boldsymbol{\nu}(l')] = 1$, and vector products $[\boldsymbol{\tau}(l)\times\boldsymbol{\nu}(l')] = \mathbf{n}(l)$ and $[\boldsymbol{\nu}(l)\times\boldsymbol{\tau}(l')] = -\mathbf{n}(l')$. The operator S contains surface divergence operator that becomes differential operator

$$\begin{aligned} \mathrm{div}_\Gamma\mathbf{J} =& \mathrm{div}_\Gamma(\boldsymbol{\tau}J_\tau) = d_l J_\tau; \\ \mathrm{div}_\Gamma\mathbf{M} =& \mathrm{div}_\Gamma(\boldsymbol{\nu}M_\nu) = d_\nu M_\nu \equiv 0, \end{aligned}$$

because the model is invariance in ν parameter.

By doing integration by parts, we have

$$\frac{b_1}{2}\int_C d_l^2 M_\nu(l)\Psi_{J\tau}(l)dl = -\frac{b_1}{2}\int_C d_l M_\nu(l)\ d_l\Psi_{J\tau}(l)dl. \tag{37}$$

Finally we combine two equations (35)-(36) to present next variational problem:

Problem 4.1. *Find $U = (J_\tau, M_\nu) \in [H^1(C)]^2$ such that:*

$$\boxed{A(U,\Psi) = \int_C E_\tau^{inc}\Psi_{J\tau}dl + \int_C H_\nu^{inc}\Psi_{M\nu}dl}, \tag{38}$$

for all $\Psi = (\Psi_{J\tau}, \Psi_{M\nu}) \in [H^1(C)]^2$, where the bilinear form A is defined as:

$$
\begin{aligned}
A(U,\Psi) =& iZ_0 \iint_C kG(l,l')J_\tau(l')\Psi_{J\tau}\ [\tau(l)\cdot\tau(l')] - \frac{1}{k}G(l,l')d_l' J_\tau(l')d_l\Psi_{J\tau}(l)dl'dl \\
&+ \iint_C \Psi_{J\tau}(l)M_\nu(l')\ \mathbf{n}(l)\cdot\nabla_l G(l,l')dl'dl \\
&+ \iint_C \Psi_{M\nu}J_\tau\ \mathbf{n}(l')\cdot\nabla_l G(l,l')dl'dl \\
&+ \frac{i}{Z_0}\iint_C kG(l,l')M_\nu(l')\Psi_{M\nu}(l)dl'dl \\
&+ \frac{a_0}{2}\int_C J_\tau\Psi_{J\tau}dl + \frac{1}{2a_0}\int_C M_\nu\Psi_{M\nu}dl \\
&- \frac{a_1}{2}\int_C d_lJ\ d_l\Psi_{J\tau}dl - \frac{b_1}{2}\int_C d_lM\ d_l\Psi_{J\tau}dl \\
&- \frac{b_1}{2a_0}\int_C d_lM\ d_l\Psi_{M\nu}dl - \frac{a_1}{2a_0}\int_C d_lJ\ d_l\Psi_{M\nu}dl.
\end{aligned}
\tag{39}
$$

We present similar variational problem for TM polarization:

Problem 4.2. *Find $U=(J_\nu, M_\tau)\in[H^1(C)]^2$ such that:*

$$
\boxed{A(U,\Psi) = \int_C E_\nu^{inc}\Psi_{J\nu}dl + \int_C H_\tau^{inc}\Psi_{M\tau}dl}
\tag{40}
$$

for all $\Psi=(\Psi_{J\nu},\Psi_{M\tau})\in[H^1(C)]^2$, where the bilinear form A is defined as:

$$
\begin{aligned}
A(U,\Psi) =& iZ_0 \iint_C kG(l,l')J_\nu(l')\Psi_{J\nu}dl'dl \\
&- \iint_C \Psi_{J\nu}(l)M_\tau(l')\ \mathbf{n}(l')\cdot\nabla_l G(l,l')dl'dl \\
&- \iint_C \Psi_{M\tau}J_\nu\ \mathbf{n}(l)\cdot\nabla_l G(l,l')dl'dl \\
&+ \frac{i}{Z_0}\iint_C kG(l,l')M_\tau(l')\Psi_{M\tau}(l)[\tau(l)\cdot\tau(l')] \\
&- \frac{1}{k}G(l,l')d_l' M_\tau(l')d_l\Psi_{M\tau}(l)dl'dl \\
&+ \frac{a_0}{2}\int_C J_\nu\Psi_{J\nu}dl - \frac{1}{2a_0}\int_C M_\tau\Psi_{M\tau}dl \\
&- \frac{a_2}{2}\int_C d_lJ\ d_l\Psi_{J\nu}dl + \frac{b_2}{2}\int_C d_lM\ d_l\Psi_{J\nu}dl \\
&+ \frac{b_2}{2a_0}\int_C d_lM\ d_l\Psi_{M\tau}dl - \frac{a_2}{2a_0}\int_C d_lJ\ d_l\Psi_{M\tau}dl.
\end{aligned}
\tag{41}
$$

4.1.2. Variational formulations with IBC2. The equation (32) passes the same way as IBC1 to become weak. The weak formulations replace operator P in EFIE and MFIE equations. Finally, we assemble them to define the bilinear form:

$$
\begin{aligned}
A(U,\Psi) =& iZ_0 \iint_C kG(l,l')J_\tau(l')\Psi_{J\tau}\ [\boldsymbol{\tau}(l)\cdot\boldsymbol{\tau}(l')] \\
&- \frac{1}{k}G(l,l')d'_l J_\tau(l')d_l\Psi_{J\tau}(l)dl'dl \\
&+ \iint_C \Psi_{J\tau}(l)M_\nu(l')\ \mathbf{n}(l)\cdot\nabla_l G(l,l')dl'dl \\
&+ \iint_C \Psi_{M\nu}J_\tau\ \mathbf{n}(l')\cdot\nabla_l G(l,l')dl'dl \\
&+ \frac{i}{Z_0}\iint_C kG(l,l')M_\nu(l')\Psi_{M\nu}(l)dl'dl \\
&+ \frac{a_0}{2}\int_C J_\tau\Psi_{J\tau}dl + \frac{1}{2a_0}\int_C M_\nu\Psi_{M\nu}dl \\
&+ \frac{a_1}{2}\int_C d_l^2 J_\tau\Psi_{J\tau}dl + \frac{b_1}{2}\int_C d_l^2 M_\nu\Psi_{J\tau}dl \\
&+ \frac{b_1}{2a_0}\int_C d_l^2 M_\nu\Psi_{M\nu}dl + \frac{a_1}{2a_0}\int_C d_l^2 J_\tau\Psi_{M\nu}dl \\
&+ \frac{a'_1}{2}\int_C d_l^4 M_\nu\ \Psi_{J\tau}dl + \frac{b'_1}{2}\int_C d_l^4 M_\nu\ \Psi_{J\tau} \\
&+ \frac{b'_1}{2a_0}\int_C d_l^4 M_\nu\ \Psi_{M\nu}dl + \frac{a'_1}{2a_0}\int_C d_l^4 J_\tau\ \Psi_{M\nu}dl,
\end{aligned}
$$

for TE polarization. And with integration by parts, we get for TM polarization

$$
\begin{aligned}
A(U,\Psi) =& iZ_0 \iint_C kG(l,l')J_\tau(l')\Psi_{J\tau}\ [\boldsymbol{\tau}(l)\cdot\boldsymbol{\tau}(l')] - \frac{1}{k}G(l,l')d'_l J_\tau(l')d_l\Psi_{J\tau}(l)dl'dl \\
&+ \iint_C \Psi_{J\tau}(l)M_\nu(l')\ \mathbf{n}(l)\cdot\nabla_l G(l,l')dl'dl \\
&+ \iint_C \Psi_{M\nu}J_\tau\ \mathbf{n}(l')\cdot\nabla_l G(l,l')dl'dl \\
&+ \frac{i}{Z_0}\iint_C kG(l,l')M_\nu(l')\Psi_{M\nu}(l)dl'dl \\
&+ \frac{a_0}{2}\int_C J_\tau\Psi_{J\tau}dl + \frac{1}{2a_0}\int_C M_\nu\Psi_{M\nu}dl \\
&- \frac{a_1}{2}\int_C d_l J_\tau\ d_l\Psi_{J\tau}dl - \frac{b_1}{2}\int_C d_l M_\nu\ d_l\Psi_{J\tau}dl \\
&- \frac{b_1}{2a_0}\int_C d_l M_\nu\ d_l\Psi_{M\nu}dl - \frac{a_1}{2a_0}\int_C d_l J_\tau\ d_l\Psi_{M\nu}dl \\
&+ \frac{a'_1}{2}\int_C d_l^2 M_\nu\ d_l^2\Psi_{J\tau}dl + \frac{b'_1}{2}\int_C d_l^2 M_\nu\ d_l^2\Psi_{J\tau} \\
&+ \frac{b'_1}{2a_0}\int_C d_l^2 M_\nu\ d_l^2\Psi_{M\nu}dl + \frac{a'_1}{2a_0}\int_C d_l^2 J_\tau\ d_l^2\Psi_{M\nu}dl.
\end{aligned}
$$

We can write variational formulation for TE and TM polarization as:

Problem 4.3. *Find* $U = (J_\tau, M_\nu) \in [H^1(C)]^2$ *such that*

$$A(U,\Psi) = \int_C E_\tau^{inc}\Psi_{J\tau}dl + \int_C H_\nu^{inc}\Psi_{M\nu}dl,$$

for all $\Psi = (\Psi_{J\tau}, \Psi_{M\nu}) \in [H^1(C)]^2$.

4.2. Existence and uniqueness theorem for problem 4.1. In the next, we are going to show that our variational problem in TE has a unique solution using the Fredholm alternative. Here we do not study the existence and uniqueness For TM problem 4.2 because the procedure is the same.

In the first time, it is necessary to determine the continuity and the coercivity of the bilinear form $A(U, \Psi)$. Then we consider the operator $A(U, \Psi)$ as a sum of three bilinear operators:

$$\begin{aligned} A_1(U,\Psi) = &\iint_C Z_0(B-S)J_\tau \Psi_{J\tau} dl' dl + \iint_C \frac{1}{Z_0}(B-S)M_\nu \Psi_{M\nu} dl' dl \\ &+ \iint_C QM_\nu \Psi_{J\tau} dl' dl + \iint_C QJ_\tau \Psi_{M\nu} dl' dl \\ &+ \frac{a_0}{2}\int_C J_\tau \Psi_{J\tau} dl + \frac{1}{2a_0}\int_C M_\nu \Psi_{M\nu} dl \\ A_2(U,\Psi) = &- \frac{a_1}{2}\int_C d_l J \; d_l \Psi_{J\tau} dl - \frac{b_1}{2a_0}\int_C d_l M \; d_l \Psi_{M\nu} dl \end{aligned}$$

and

$$A_3(U,\Psi) = -\frac{b_1}{2}\int_C d_l M \; d_l \Psi_{J\tau} dl - \frac{a_1}{2a_0}\int_C d_l J \; d_l \Psi_{M\nu} dl,$$

where

$$A = A_1 + A_2 + A_3.$$

4.2.1. Continuity of the bilinear form A.

Lemma 4.1. *The bilinear form $A(U, \Psi)$ (39) is continuous on $[H^1(C)]^2$.*

Proof. : We are going to show that exists $\beta > 0$ such that

$$|A(U,\Psi)| \le \beta \|U\|_{H^1(C)} \|\Psi\|_{H^1(C)}, \tag{42}$$

for all $U, \Psi \in H^1(C)$. In the first time, we have from [11] a constant $\beta_1 > 0$ such that:

$$|A_1(U,\Psi)| \le \beta_1 \|U\|_{H^1(C)} \|\Psi\|_{H^1(C)}.$$

Besides, using Cauchy-Schwarz inequality, we get:

$$\begin{aligned} |A_2(U,\Psi) + A_3(U,\Psi)| \le &\left|\frac{a_1}{2}\int_C d_l J \; d_l \Psi_{J\tau} dl\right| + \left|\frac{b_1}{2}\int_C d_l M \; d_l \Psi_{J\tau} dl\right| \\ &+ \left|\frac{b_1}{2a_0}\int_C d_l M \; d_l \Psi_{M\nu} dl\right| + \left|\frac{a_1}{2a_0}\int_C d_l J \; d_l \Psi_{M\nu} dl\right| \\ \le &\left|\frac{a_1}{2}\right| \|d_l J\|_{L^2} \|\Psi_{J\tau}\|_{L^2} + \left|\frac{b_1}{2}\right| \|d_l M\|_{L^2} \|\Psi_{J\tau}\|_{L^2} \\ &+ \left|\frac{b_1}{2a_0}\right| \|d_l M\|_{L^2} \|\Psi_{M\nu}\|_{L^2} + \left|\frac{a_1}{2a_0}\right| \|d_l J\|_{L^2} \|\Psi_{M\nu}\|_{L^2} \\ \le &\beta_2 \|U\|_{H^1(C)} \|\Psi\|_{H^1(C)} \quad , \; \textit{where } \beta_2 \ge 0. \end{aligned}$$

Finally, we take $\beta = \beta_1 + \beta_2 \ge 0$. ∎

4.2.2. Coercivity of the bilinear form A**.** We give a coercivity result for A to apply Fredholm alternative.

Lemma 4.2. *The bilinear form $A(U,\Psi)$ is coercive on $H^1(C)$; i.e., there exists $\gamma > 0$ and γ' such that*

$$\Re[A(U,U^*)] \geq \gamma \|U\|^2_{H^1(C)} - \gamma' \|U\|^2_{L^2(C)}, \quad \forall U \in [H^1(C)]^4,$$

if coefficients satisfy

$$\Re(a_1) + \frac{|a_0||b_1 + a_1^*/a_0^*|}{2} = 0 \text{ and } \Re(a_1) = \Re(b_1 a_0^*). \tag{43}$$

Proof. We take firstly $\Psi = U^*$ and get

$$\begin{aligned} A(U,U^*) =& iZ_0 \iint_C kGJ_\tau J_\tau^* [\boldsymbol{\tau}(l') \cdot \boldsymbol{\tau}(l)] - \frac{1}{k} d_l' J_\tau(l') d_l J_\tau^*(l) dl' dl \\ &+ \iint_C J_\tau^* M_\nu \mathbf{n}(l) \cdot \nabla_l G dl' dl + \iint_C M_\nu^* J_\tau \mathbf{n}(l') \cdot \nabla_l G dl' dl \\ &+ \frac{i}{Z_0} \iint_C kGM_\nu M_\nu^* [\boldsymbol{\nu}(l') \cdot \boldsymbol{\nu}(l)] dl' dl \\ &+ \frac{a_0}{2} \int_C J_\tau J_\tau^* dl + \frac{1}{2a_0} \int_C M_\nu M_\nu^* dl \\ &- \frac{a_1}{2} \int_C d_l J_\tau \ d_l J_\tau^* dl - \frac{b_1}{2} \int_C d_l M_\nu \ d_l J_\tau^* dl \\ &- \frac{b_1}{2a_0} \int_C d_l M_\nu \ d_l M_\nu^* dl - \frac{a_1}{2a_0} \int_C d_l J_\tau \ d_l M_\nu^* dl. \end{aligned} \tag{44}$$

From [11], we have $\gamma_1 > 0$ such as:

$$\begin{aligned} \Re[A_1(U,U^*)] \geq & \frac{\Re(a_0)}{2} \|J_\tau\|^2_{L^2(C)} + \frac{\Re(a_0)}{2|a_0|^2} \|M_\nu\|^2_{L^2(C)} \\ &+ \gamma_1 \left(\|J_\tau\|^2_{H^1(C)} + \|M_\nu\|^2_{H^1(C)} \right). \end{aligned}$$

Next for operator A_2, we have

$$A_2 = -\frac{a_1}{2} \int_C d_l J_\tau \ d_l J_\tau^* dl - \frac{b_1}{2a_0} \int_C d_l M_\nu \ d_l M_\nu^* dl,$$

where real part is

$$\Re(A_2) = -\frac{\Re(a_1)}{2} \|d_l J_\tau\|^2_{L^2(C)} - \Re(\frac{b_1}{2a_0}) \|d_l M_\nu\|^2_{L^2(C)}.$$

And it gets

$$A_3 = -\frac{b_1}{2} \int_C d_l M_\nu \ d_l J_\tau^* dl - \frac{a_1}{2a_0} \int_C d_l J_\tau \ d_l M_\nu^* dl,$$

where real part is

$$\begin{aligned} \Re(A_3) =& \Re \left(-\frac{b_1}{2} \int_C d_l M_\nu \ d_l J_\tau^* dl - \frac{a_1}{2a_0} \int_C d_l J_l \ d_l M_\nu^* dl \right) \\ =& -\Re \left[\left(\frac{b_1}{2} + \frac{a_1^* a_0}{2|a_0|^2} \right) \int_C d_l M_\nu \ d_l J_\tau^* dl \right] \\ =& -\Re \left[\int_C \frac{1}{|a_0|^{\frac{1}{2}}} \left(\frac{b_1}{2} + \frac{a_1^* a_0}{2|a_0|^2} \right)^{\frac{1}{2}} d_l M_\nu \cdot |a_0|^{\frac{1}{2}} \left(\frac{b_1}{2} + \frac{a_1^* a_0}{2|a_0|^2} \right)^{\frac{1}{2}} d_l J_l^* dl \right]. \end{aligned}$$

We denote $q = b_1|a_0| + a_1^* a_0/|a_0|$, then we obtain

$$\Re(A_3) \geq -\frac{|q|}{4}\|d_l J_\tau\|^2_{L^2(C)} - \frac{|q|}{4|a_0|^2}\|d_l M_\nu\|^2_{L^2(C)}.$$

If we have $\Re(a_1 - b_1^* a_0) = 0$ or $\Re(a_1) = \Re(b_1 a_0^*)$. Finally, the sum of operators A_2 and A_3 verifies

$$\begin{aligned}\Re(A_2) + \Re(A_3) \geq & -\frac{1}{2}\left(\Re(a_1) + \frac{|q|}{2}\right)\|d_l J_\tau\|^2_{L^2(C)} \\ & -\frac{1}{2|a_0|^2}\left(\Re(a_1) + \frac{|q|}{2}\right)\|d_l M_\nu\|^2_{L^2(C)}.\end{aligned}$$

Then, if $\Re(a_1) + \frac{|q|}{2} = 0$, we get that

$$\Re(A) = \Re(A_1) + \Re(A_2) + \Re(A_3) \geq \gamma_1\|U\|^2_{H^1(C)} - c\|U\|^2_{L^2(C)}.$$

That gives us coercivity of $A(U, \Psi)$. □

We give the main result.

Theorem 4.3. *The problem (38) admits a unique solution $U \in [H^1(C)]^2$ for any $\Psi \in [H^1(C)]^2$, if coefficients satisfy*

$$\Re(a_1) + \frac{|a_0||b_1 + a_1^*/a_0^*|}{2} = 0 \text{ and } \Re(a_1) = \Re(b_1 a_0^*). \tag{45}$$

Proof. With lemmas 4.1-4.2 we can apply the Fredholm alternative to show that problem (38) admits a unique solution $U \in [H^1(C)]^2$. □

4.3. Second variational formulation for problem 4.1. We use auxiliary variables X, Y as in [8] to avoid integration by parts. Then, we obtain variational formulation such as:

Problem 4.4. *Find $U = (J_\tau, M_\nu, X, Y) \in [H^1(C)^2 \times L^2(C)^2]$ such that:*

$$\boxed{A(U, \Psi) = \int_C E_\tau^{inc}\Psi_{J_\tau} dl + \int_C H_\nu^{inc}\Psi_{M_\nu} dl}, \tag{46}$$

for all $(\Psi_{J\tau}, \Psi_{M\nu}, X', Y') \in [H^1(C)^2 \times L^2(C)^2]$, where the bilinear form A is defined as:

$$
\begin{aligned}
A(U,\Psi) =& iZ_0 \iint_C kG(l,l')\ J_\tau(l')\ \Psi_{J\tau}\ [\boldsymbol{\tau}(l')\cdot\boldsymbol{\tau}(l)] \\
& - \frac{1}{k} G(l,l')\ d'_l J_\tau(l')\ d_l \Psi_{J\tau}(l) dl' dl \\
& + \iint_C \Psi_{J\tau}(l)\ M_\nu(l')\ \mathbf{n}(l)\cdot\nabla_l G(l,l') dl' dl \\
& + \iint_C \Psi_{M\nu}\ J_\tau\ \mathbf{n}(l')\cdot\nabla_l G(l,l') dl' dl \\
& + \frac{i}{Z_0} \iint_C kG(l,l')\ M_\nu(l')\ \Psi_{M\nu}(l) dl' dl \\
& + \frac{a_0}{2}\int_C J_\tau\ \Psi_{J\tau} dl + \frac{1}{2a_0}\int_C M_\nu\ \Psi_{M\nu} dl \\
& + \frac{a_1}{2}\int_C d_l X\ \Psi_{J\tau} dl + \frac{b_1}{2}\int_C d_l Y\ \Psi_{J\tau} dl \\
& + \frac{b_1}{2a_0}\int_C d_l Y\ \Psi_{M\nu} dl + \frac{a_1}{2a_0}\int_C d_l X\ \Psi_{M\nu} dl \\
& + c_1 \int_C X\ X' dl - c_1 \int_C d_l J_\tau\ X' dl + d_1 \int_C Y\ Y' dl - d_1 \int_C d_l M_\nu\ Y' dl
\end{aligned}
\tag{47}
$$

for all $\Psi = (\Psi_{J\tau}, \Psi_{M\nu}, X', Y') \in [H^1(C)^2 \times L^2(C)^2]$.

In the next section, we explain the discretization of this formulation by a finite element method with Lagrange elements.

5. Discretization of the formulation (46)

We approximate the unknowns J and M by a finite element method based on Lagrange elements. We approximate the curve C by means of N straight *line segments* C_i. We denote nodes from 1 to N and we consider V_h a finite dimensional subspace defined by

$$V_h = \left\{ v_h : C^h \to \mathbb{R},\ v_h \in H^1(C^h),\ v_h|_{C_i} \in P_1,\ \forall i \in 1,...,N \right\} \subset H^1(C^h),$$

where P_1 is the space of first degree polynomials, and

$$W_h = \left\{ w_h : C^h \to \mathbb{R},\ w_h \in H^1(C^h),\ w_h|_{C_i} \in P_0,\ \forall i \in 1,...,N \right\} \subset L^2(C^h),$$

where P_0 is the space of constant functions.

We discretize the unknowns with basis functions defined by:

$$J_\tau \approx J_\tau^h(l) = \sum_{i=1}^N J_{\tau i}\phi_i(l)\ \in V_h, \tag{48}$$

$$M_\nu \approx M_\nu^h(l) = \sum_{i=1}^N M_{\nu i}\psi_i(l)\ \in W_h, \tag{49}$$

$$X \approx X^h(l) = \sum_{i=1}^N X_i\psi_i(l)\ \in W_h, \tag{50}$$

and

$$Y \approx Y^h(l) = \sum_{i=1}^{N} Y_i \psi_i(l) \ \in W_h. \tag{51}$$

where $\phi_i \in V_h$ and $\psi_i \in W_h$

Then, the discretization of the bilinear form $A(U, \Psi)$ in (46) is :

$$\begin{aligned}
A(U^h, \Psi^h) =& iZ_0 \sum_{i,j=1}^{N} \left(\iint_{C^h} kG \ \phi_j \ \phi_i \ [\vec{\tau}_j(l') \cdot \vec{\tau}_i(l)] - \frac{1}{k} G \ d_l' \phi_j \ d_l \phi_i \ dl' dl \right) J_{\tau j}^h \\
&+ \sum_{i,j=1}^{N} \left(\iint_{C^h} \psi_j \ \phi_i \ \mathbf{n}_i \cdot \nabla_l G dl' dl \right) M_{\nu j}^h \\
&+ \sum_{i,j=1}^{N} \left(\iint_{C^h} \phi_j \ \psi_i \ \mathbf{n}_j \cdot \nabla_l G dl' dl \right) J_{\tau j}^h \\
&+ \frac{i}{Z_0} \sum_{i,j=1}^{N} \left(\iint_{C^h} kG \ \psi_j \ \psi_i \ dl' dl \right) M_{\nu j}^h \\
&+ \frac{a_0}{2} \sum_{i,j=1}^{N} \left(\int_{C^h} \phi_j \ \phi_i dl \right) J_{\tau j}^h + \frac{1}{2a_0} \sum_{i,j=1}^{N} \left(\int_{C^h} \psi_j \ \psi_i dl \right) M_{\nu j}^h \\
&+ \frac{a_1}{2} \sum_{i,j=1}^{N} \left(\int_{C^h} d_l \psi_j \ \phi_i dl \right) X_j^h + \frac{b_1}{2} \sum_{i,j=1}^{N} \left(\int_{C^h} d_l \psi_j \ \phi_i dl \right) Y_j^h \\
&+ \frac{b_1}{2a_0} \sum_{i,j=1}^{N} \left(\int_{C^h} d_l \psi_j \ \psi_i dl \right) Y_j^h + \frac{a_1}{2a_0} \sum_{i,j=1}^{N} \left(\int_{C^h} d_l \psi_j \ \psi_i dl \right) X_j^h \\
&+ \sum_{i,j=1}^{N} \left(\int_{C^h} \psi_j \ \psi_i dl \right) X_j^h - \sum_{i,j=1}^{N} \left(\int_{C^h} d_l \psi_j \ \psi_i dl \right) J_{\tau j}^h \\
&+ \sum_{i,j=1}^{N} \left(\int_{C^h} \psi_j \ \psi_i dl \right) Y_j^h - \sum_{i,j=1}^{N} \left(\int_{C^h} d_l \psi_j \ \psi_i dl \right) M_{\nu j}^h.
\end{aligned}$$

We then obtain the following matrices:

$$\begin{aligned}
(B - S)_{ij} =& i \iint_{C^h} kG(l, l') \ \phi_j(l') \ \phi_i \ [\boldsymbol{\tau}_j \cdot \boldsymbol{\tau}_i] - \frac{1}{k} G(l, l') \ d_l' \phi_j(l') \ d_l \phi_i(l) \ dl' dl, \\
Q_{ij} =& \iint_{C^h} \phi_i(l) \ \psi_j(l') \ \mathbf{n}_i \cdot \nabla_l G(l, l') \ dl' dl, \\
B_{ij} =& i \iint_{C^h} kG(l, l') \ \psi_j(l') \ \psi_i(l) \ dl' dl, \\
I1_{ij} =& \int_{C^h} \phi_i(l) \ \phi_j(l) \ dl, \\
I2_{ij} =& \int_{C^h} \psi_i(l) \ \psi_j(l) \ dl,
\end{aligned}$$

$$D1_{ij} = \int_{C^h} \phi_i(l)\ d_l\psi_j(l)\ dl,$$

$$D3_{ij} = \int_{C^h} \psi_i(l)\ d_l\psi_j(l)\ dl,$$

$$D5_{ij} = \int_{C^h} \psi_i(l)\ d_l\phi_j(l)\ dl.$$

We can write the linear system in IBC1 case:

$$\left[\begin{array}{cccc} Z_0[B-S]+\frac{a_0}{2}[I1] & [Q] & \frac{a_1}{2}[D1] & \frac{b_1}{2}[D1] \\ [Q]^T & \frac{1}{Z_0}[B]+\frac{1}{2a_0}[I2] & \frac{a_1}{2a_0}[D3] & \frac{b_1}{2a_0}[D3] \\ -[D5] & 0 & [I2] & 0 \\ 0 & -[D3] & 0 & [I2] \end{array}\right] \begin{pmatrix} \overline{J}^h \\ \overline{M}^h \\ \overline{X}^h \\ \overline{Y}^h \end{pmatrix} = \begin{pmatrix} \overline{E}^h \\ \overline{H}^h \\ 0 \\ 0 \end{pmatrix}, \tag{52}$$

where right-side vectors $\overline{E}^h$, $\overline{H}^h$ are defined as follows:

$$E_i^h = \int_{C^h} \mathbf{E}^{inc} \cdot \boldsymbol{\phi}_i dl;$$

$$H_i^h = \int_{C^h} \mathbf{H}^{inc} \cdot \boldsymbol{\psi}_i dl.$$

Then, we are going to eliminate the vectors $\overline{J}^h$ and $\overline{M}^h$. From the last two lines in (60), we get

$$-[D5]\ \overline{J}^h + [I2]\ \overline{X}^h = 0\ \rightarrow\ \overline{X}^h = [I2]^{-1}\ [D5]\ \overline{J}^h;$$

$$-[D3]\ \overline{M}^h + [I2]\ \overline{Y}^h = 0\ \rightarrow\ \overline{Y}^h = [I2]^{-1}\ [D3]\ \overline{M}^h.$$

We obtain final system:

$$\left[\begin{array}{cc} [A1] & [A2] \\ [A3] & [A4] \end{array}\right] \begin{pmatrix} \overline{J}^h \\ \overline{M}^h \end{pmatrix} = \begin{pmatrix} \overline{E}^h \\ \overline{H}^h \end{pmatrix}, \tag{53}$$

where matrices are defined as

$$[A1] = Z_0[B-S] + \frac{a_0}{2}[I1] + \frac{a_1}{2}[D1]\ [I2]^{-1}\ [D5],$$

$$[A2] = [Q] + \frac{b_1}{2}[D1]\ [I2]^{-1}\ [D3],$$

$$[A3] = [Q]^T + \frac{a_1}{2a_0}[D3]\ [I2]^{-1}\ [D5],$$

$$[A4] = \frac{1}{Z_0}[B] + \frac{1}{2a_0}[I2] + \frac{b_1}{2a_0}[D3]\ [I2]^{-1}\ [D3].$$

In the next part, we brievely explain the calculation of matrices for (B-S) and Q operators and the matrices for integral operators IBC1 and IBC2.

5.1. Calculation of matrices for the approximation of the impedance in IBC1. Here, we use basis functions ϕ_i and ψ_j defined by:

$$\phi_i(x) = \begin{cases} \frac{x-x_{i-1}}{x_i-x_{i-1}} & x \in [x_{i-1}, x_i] \\ \frac{x_{i+1}-x}{x_{i+1}-x_i} & x \in [x_i, x_{i+1}] \\ 0 & x \notin [x_{i-1}, x_{i+1}] \end{cases}. \tag{54}$$

$$\psi_j(x) = \begin{cases} \frac{1}{x_{j+1}-x_j} & x \in [x_j, x_{j+1}] \\ 0 & x \notin [x_j, x_{j+1}] \end{cases}. \tag{55}$$

We observe that the derivative of a P_1 function is a function of class P_0. Thus we can express $d_l\phi_i$ with the basis functions ψ_j. Whereas the derivative of functions ψ_j, we express as difference of Dirac functions in breaking points:

$$d_l\phi_i(l) = \psi_{i-1}(l) - \psi_i(l); \tag{56}$$

$$d_l\psi_j(l) = \delta_j - \delta_{j+1}. \tag{57}$$

The element of the matrix $(B-S)$ are calculated on segments associated to functions ϕ_i and ϕ_j':

$$(B-S)_{ij} = i\iint_C G(l,l')(k\phi_j'\phi_i[\vec{\tau}_j'\cdot\vec{\tau}_i] - \frac{1}{k}d_l'\phi_j'd_l\phi_i)dl'dl$$

$$= i\int_{C_i+C_{i-1}} dl \int_{C_j'+C_{j-1}'} G(l,l')(k\phi_j'\phi_i[\vec{\tau}_j'\cdot\vec{\tau}_i] - \frac{1}{k}(\psi_{j-1}'-\psi_j')(\psi_{i-1}-\psi_i))dl'.$$

For the sake of simplification, we want to show calculation of the simple part

$$Int_{ij} = i\iint_{C_iC_j'} G(l,l')(k\phi_j'\phi_i[\vec{\tau}_j'\cdot\vec{\tau}_i] - \frac{1}{k}\psi_j'\psi_i)dl'dl,$$

where $\psi_i = \frac{1}{x_{i+1}-x_i}$ is a constant on the element C_i and $\psi_j' = \frac{1}{x_{j+1}-x_j}$ on C_j'.

According to features of Green's function $G(l,l')$, we separate calculation into two cases. First case when arguments l and l' are apart from each other.

- Apart elements: if the elements have enough big distance from each other, we can be sure in convergence of integral and we use Gaussian quadrature to calculate the integral:

$$Int_{ij} \approx i\sum_{g=1}^{n_g}\sum_{g'=1}^{n_g'} p_g p_g' \frac{\pi}{i}H_0^{(2)}(k\rho_{gg'})\left[k\phi_{jg'}'\phi_{ig}[\vec{\tau}_j'\cdot\vec{\tau}_i] - \frac{1}{kh_j'h_i}\right].$$

- Closed elements: if the elements are close to each other, we should expand Green function:

$$G = \frac{\pi}{i}H_0^{(2)}(k\rho) = \underbrace{\frac{\pi}{i}H_0^{(2)}(k\rho) + 2\ln(\rho)}_{\to G|_1} - \underbrace{2\ln(\rho)}_{\to G|_2}. \tag{58}$$

When $\rho \to 0$, we have

$$G|_1 = \frac{\pi}{i}H_0^{(2)}(k\rho) - 2\ln(\rho) \to \frac{\pi}{i} - 2(\gamma + \ln(\frac{k}{2})). \tag{59}$$

So for the calculation of double integral $Int_{ij}|_1$ we can use Gauss points approach:

$$Int_{ij}|_1 \approx i\left\{\frac{\pi}{i} - 2(\gamma+\ln(\frac{k}{2}))\right\}\sum_{g=1}^{n_g}\sum_{g'=1}^{n_g'} p_g p_g'\left[k\phi_{jg'}'\phi_{ig}[\vec{\tau}_j'\cdot\vec{\tau}_i] - \frac{1}{kh_j'h_i}\right]$$

and to calculate the remaining part, we integrate over Γ with help of Gauss points and we obtain:

$$Int_{ij}|_2 \approx -2i\sum_{g=1}^{n_g} p_g \int_{C_j} ln(\rho(l_g,l'))\left[k\phi_j'\phi_{ig}[\vec{\tau}_j'\cdot\vec{\tau}_i] - \frac{1}{kh_j'h_i}\right]dl'.$$

5.2. Calculation of the matrix Q. Here, we explain the calculation of the matrix for the operator Q. The elements of matrix Q are calculated on segments associated to functions ϕ_i and ψ'_j.

$$Q_{ij} = -i \iint_C \phi_i(l)\psi_j(l')\mathbf{n}_i \cdot \nabla_l G(l,l')dl'dl,$$

where function ψ'_j is defined only on a segment C_j and gradient of Green function ∇G is expressed

$$\nabla G(l,l') = -\frac{\pi k}{i\rho} H_1^{(2)}(k\rho)\vec{\rho}.$$

So we can write

$$Q_{ij} = i \int_{C_i+C_{i-1}} \int_{C_j} \phi_i(l)\psi_j(l')\frac{\pi k}{\rho} H_1^{(2)}(k\rho)\mathbf{n}_i \cdot \vec{\rho}.$$

As in $(B-S)$ matrix, for apart elements we use Gauss points approach.

On the another hand if elements are closed, according to the property of $H_1^{(2)}(k\rho)$ for $\rho \to 0$

$$\frac{k}{\rho}\left[H_1^{(2)}(k\rho) - \frac{2i}{\pi k\rho} + \frac{i}{\pi}k\rho\ln(\rho)\right] \to -\frac{i}{\pi}k^2\ln(k/2) + k^2\left(\frac{1}{2} + \frac{i}{2\pi}(1-2\gamma)\right),$$

$$\nabla G(l,l') = \underbrace{-\frac{\pi k_j}{i\rho}\left[H_1^{(2)}(k\rho) - \frac{2i}{\pi k\rho} + \frac{i}{\pi}k\rho\ln(\rho)\right]\vec{\rho}}_{\to GG|_1} - \underbrace{\left[\frac{2}{\rho^2} - k^2\ln(\rho)\right]\vec{\rho}}_{\to GG|_2}.$$

For ρ small enough, we can write

$$GG|_1 \approx -i\left[k\ln(k/2) - \frac{\pi}{i}k^2\left(\frac{1}{2} + \frac{i}{2\pi}(1-2\gamma)\right)\right]\sum_{g=1}^{n_g}\sum_{g'=1}^{n_{g'}} p_g p'_g \phi_{ig}\psi'_{jg'}\mathbf{n}_i \cdot \vec{\rho}_{gg'}$$

and

$$GG|_2 \approx i\sum_{g=1}^{n_g}\sum_{g'=1}^{n_{g'}} p_g p'_g \phi_{ig}\psi'_{jg'}\left[\frac{2}{\rho_{gg'}} - k^2\ln(\rho_{gg'})\right]\mathbf{n}_i \cdot \vec{\rho}_{gg'}.$$

In the next, we give linear system in IBC2.

5.3. Matrix form in IBC2. We introduce basis matrices,

$$E_{ij} = \left\{\begin{array}{cc} 1 & i=j-1 \\ -1 & i=j \end{array}\right\},$$

$$M_{ij} =< \psi_j, \psi_i >= \{\frac{1}{h_i} \quad i=j\},$$

where $[M]$ is an invertible diagonal matrix.

$$S_{ij} =< d_l\psi_j, \phi_i >= \left\{\begin{array}{cc} \frac{1}{h_j} & i=j \\ -\frac{1}{h_j} & i=j+1 \end{array}\right\},$$

$$P_{ij} =< d_l\psi_j, \psi_i >= \frac{1}{2}\left\{\begin{array}{cc} \frac{1}{h_ih_j} & i=j-1 \\ -\frac{1}{h_ih_j} & i=j+1 \end{array}\right\},$$

where basis functions ϕ and ψ are defined earlier in (54)-(57). Here matrices $[M]$, $[S]$ and $[P]$ correspond to matrices $[I2]$, $[D1]$ and $[D3]$ respectively.

Now we define matrix $[T]$ that corresponds to matrix $[D5]$:

$$T_{ij} =< d_l\phi_j, \psi_i >=< \psi_{j-1} - \psi_j, \psi_i >= (ME)_{ij}.$$

And matrices:

$$M^{-1}_{ij} = \{h_i \quad i = j\},$$

$$(M^{-1}P)_{ij} = \frac{1}{2}\left\{ \begin{array}{cc} \frac{1}{h_j} & i = j-1 \\ -\frac{1}{h_j} & i = j+1 \end{array} \right\}.$$

We need to find next matrices from (53):

$$[D1][I2]^{-1}[D5] \equiv [S][M]^{-1}[M][E] \equiv [S][E],$$

$$\boxed{(SE)_{ij} = \left\{ \begin{array}{cc} \frac{1}{h_i} & i = j-1 \\ -(\frac{1}{h_i} + \frac{1}{h_{i-1}}) & i = j \\ \frac{1}{h_{i-1}} & i = j+1 \end{array} \right\}},$$

$$[D1][I2]^{-1}[D3] \equiv [S][M]^{-1}[P],$$

$$\boxed{(SM^{-1}P)_{ij} = \frac{1}{2}\left\{ \begin{array}{cc} \frac{1}{h_ih_j} & i = j-1 \\ -\frac{1}{h_{i-1}h_j} & i = j \\ -\frac{1}{h_ih_j} & i = j+1 \\ \frac{1}{h_{i-1}h_j} & i = j+2 \end{array} \right\}},$$

$$[D3][I2]^{-1}[D5] \equiv [P][M]^{-1}[M][E] \equiv [P][E],$$

$$\boxed{(PE)_{ij} = \frac{1}{2}\left\{ \begin{array}{cc} \frac{1}{h_ih_{j-1}} & i = j-2 \\ -\frac{1}{h_ih_j} & i = j-1 \\ -\frac{1}{h_ih_{j-1}} & i = j \\ \frac{1}{h_ih_j} & i = j+1 \end{array} \right\}},$$

$$[D3][I2]^{-1}[D3] \equiv [P][M]^{-1}[P],$$

$$\boxed{(PM^{-1}P)_{ij} = \frac{1}{4}\left\{ \begin{array}{cc} \frac{1}{h_ih_jh_{j+1}} & i = j-2 \\ -\frac{1}{h_ih_j}(\frac{1}{h_{j-1}} + \frac{1}{h_{j+1}}) & i = j \\ \frac{1}{h_ih_{i-1}h_j} & i = j+2 \end{array} \right\}}.$$

For IBC2 we have next matrix of a problem:

$$\left[\begin{array}{cccccccc} Z_0[B-S] + \frac{a_0}{2}[I1] & [Q] & \frac{a_1}{2}[D1] & \frac{b_1}{2}[D1] & 0 & 0 & \frac{a'_1}{2}[D1] & \frac{b'_1}{2}[D1] \\ [Q]^T & \frac{1}{Z_0}[B] + \frac{1}{2a_0}[I2] & \frac{a_1}{2a_0}[D3] & \frac{b_1}{2a_0}[D3] & 0 & 0 & \frac{a'_1}{2a_0}[D3] & \frac{b'_1}{2a_0}[D3] \\ -[D5] & 0 & [I2] & 0 & 0 & 0 & 0 & 0 \\ 0 & -[D3] & 0 & [I2] & 0 & 0 & 0 & 0 \\ 0 & 0 & -[D3] & 0 & [I2] & 0 & 0 & 0 \\ 0 & 0 & 0 & -[D3] & 0 & [I2] & 0 & 0 \\ 0 & 0 & 0 & 0 & -[D3] & 0 & [I2] & 0 \\ 0 & 0 & & 0 & 0 & 0-[D3] & 0 & [I2] \end{array} \right] \tag{60}$$

with auxiliary unkowns X_2, Y_2, X_3, Y_3, such that

$$< d_l^4 J, \psi_j >=< d_l^3 X_1, \psi_j >=< d_l^2 X_2, \psi_j >=< d_l X_3, \psi_j > .$$

We have next equations form

$$< X_3, \psi_j >=< d_l X_2, \psi_j > \Rightarrow \; [I2]\overline{X_3} = [D3]\overline{X_2} \; \Rightarrow \; \overline{X_3} = [I2]^{-1}[D3]\overline{X_2},$$

$$< X_2, \psi_j > = < d_l X_1, \psi_j > \Rightarrow \ [I2]\overline{X_2} = [D3]\overline{X_1} \ \Rightarrow \ \overline{X_2} = [I2]^{-1}[D3]\overline{X_1},$$

$$< X_1, \psi_j > = < d_l J, \psi_j > \Rightarrow \ [I2]\overline{X_1} = [D3]\overline{J} \ \Rightarrow \ \overline{X_1} = [I2]^{-1}[D3]\overline{J}.$$

The same equations for Y_3, Y_2, Y_1 and M. Finally, we need to find next matrices

$$[D1] \left([I2]^{-1}[D3]\right)^2 [I2]^{-1}[D5] \equiv [S] \left([M]^{-1}[P]\right)^2 [M]^{-1}[M][E] \equiv [S] \left([M]^{-1}[P]\right)^2 [E],$$

$$(S(M^{-1}P)^2E)_{ij} = \frac{1}{4} \left\{ \begin{array}{cc} \frac{1}{h_i h_{i+1} h_{j-1}} & i = j-3 \\ -\frac{1}{h_i h_{i+1}}\left(\frac{1}{h_j} + \frac{1}{h_{i-1}}\right) & i = j-2 \\ -\frac{1}{h_i}\left(\frac{1}{h_i h_{i-1}} + \frac{1}{h_i h_{i+1}} - \frac{1}{h_{i-1} h_{i+1}}\right) & i = j-1 \\ \frac{1}{h_{i-1} h_{i-1}}\left(\frac{1}{h_{i-2}} + \frac{1}{h_i}\right) + \frac{1}{h_i h_i}\left(\frac{1}{h_{i-1}} + \frac{1}{h_{i+1}}\right) & i = j \\ \frac{1}{h_{i-1}}\left(\frac{1}{h_{i-1} h_{i-2}} + \frac{1}{h_i h_{i-1}} - \frac{1}{h_i h_{i-2}}\right) & i = j+1 \\ -\frac{1}{h_{i-1} h_{i-2}}\left(\frac{1}{h_i} + \frac{1}{h_{j-1}}\right) & i = j+2 \\ \frac{1}{h_{i-1} h_j h_{j+1}} & i = j+3 \end{array} \right\},$$

$$[D1] \ \left([I2]^{-1}[D3]\right)^2 \ [I2]^{-1}[D3] \equiv [S] \ \left([M]^{-1}[P]\right)^2 \ [M]^{-1}[P] \equiv [S] \ \left([M]^{-1}[P]\right)^3,$$

$$(S(M^{-1}P)^3)_{ij} = \frac{1}{8} \left\{ \begin{array}{cc} \frac{1}{h_i h_{i+1} h_j h_{j-1}} & i = j-3 \\ -\frac{1}{h_i h_{i-1} h_j h_{j-1}} & i = j-2 \\ -\frac{1}{h_i h_j}\left(\frac{1}{h_j h_{j-1}} + \frac{1}{h_j h_{j+1}} + \frac{1}{h_{j-1} h_{j-2}}\right) & i = j-1 \\ \frac{1}{h_{i-1} h_j}\left(\frac{1}{h_j h_{j-1}} + \frac{1}{h_j h_{j+1}} + \frac{1}{h_{j-1} h_{j-2}}\right) & i = j \\ \frac{1}{h_i h_j}\left(\frac{1}{h_j h_{j-1}} + \frac{1}{h_j h_{j+1}} + \frac{1}{h_{j+1} h_{j+2}}\right) & i = j+1 \\ -\frac{1}{h_{i-1} h_j}\left(\frac{1}{h_j h_{j-1}} + \frac{1}{h_j h_{j+1}} + \frac{1}{h_{j+1} h_{j+2}}\right) & i = j+2 \\ -\frac{1}{h_i h_j h_{j+1} h_{j+2}} & i = j+3 \\ \frac{1}{h_{i-1} h_j h_{j+1} h_{j+2}} & i = j+4 \end{array} \right\},$$

$$[D3] \left([I2]^{-1}[D3]\right)^2 [I2]^{-1}[D5] \equiv [P] \left([M]^{-1}[P]\right)^2 [M]^{-1}[M][E] \equiv [P] \left([M]^{-1}[P]\right)^2 [E],$$

$$(P(M^{-1}P)^2E)_{ij} = \frac{1}{8} \left\{ \begin{array}{cc} \frac{1}{h_i h_{i+1} h_{i+2} h_{j-1}} & i = j-4 \\ -\frac{1}{h_i h_{i+1} h_j h_{j-1}} & i = j-3 \\ -\frac{1}{h_i h_{i+1}}\left(\frac{1}{h_i h_{i+1}} + \frac{1}{h_{i+1} h_j} + \frac{1}{h_i h_{i-1}}\right) & i = j-2 \\ \frac{1}{h_i h_{i+1}}\left(\frac{1}{h_i h_{i+1}} + \frac{1}{h_{i+1} h_{j+1}} + \frac{1}{h_i h_{i-1}}\right) & i = j-1 \\ \frac{1}{h_i h_{i-1}}\left(\frac{1}{h_i h_{i+1}} + \frac{1}{h_{i-1} h_{i-2}} + \frac{1}{h_i h_{i-1}}\right) & i = j \\ -\frac{1}{h_i h_{i-1}}\left(\frac{1}{h_i h_{i+1}} + \frac{1}{h_{i-1} h_{j-1}} + \frac{1}{h_i h_{i-1}}\right) & i = j+1 \\ -\frac{1}{h_i h_{i-1} h_j h_{j-1}} & i = j+2 \\ \frac{1}{h_i h_{i-1} h_j h_{j+1}} & i = j+3 \end{array} \right\},$$

$$[D3] \ \left([I2]^{-1}[D3]\right)^2 \ [I2]^{-1}[D3] \equiv [P] \ \left([M]^{-1}[P]\right)^2 \ [M]^{-1}[P] \equiv [P] \ \left([M]^{-1}[P]\right)^3,$$

$$(P(M^{-1}P)^3)_{ij} = \frac{1}{16} \left\{ \begin{array}{cc} \frac{1}{h_i h_{i+1} h_j h_{j-1} h_{j-2}} & i = j-4 \\ -\frac{1}{h_i h_{i+1} h_j h_j}\left(\frac{1}{h_{j-1}} + \frac{1}{h_{j+1}}\right) - \frac{1}{h_i h_j h_{j-1} h_{j-2}}\left(\frac{1}{h_{i-1}} + \frac{1}{h_{i+1}}\right) & i = j-2 \\ \frac{1}{h_i h_{i-1} h_j}\left(\frac{1}{h_j h_{j-1}} + \frac{1}{h_j h_{j+1}} + \frac{1}{h_{j-1} h_{j-2}}\right) & \\ + \frac{1}{h_i h_{i+1} h_j}\left(\frac{1}{h_j h_{j-1}} + \frac{1}{h_j h_{j+1}} + \frac{1}{h_{j+1} h_{j+2}}\right) & i = j \\ -\frac{1}{h_i h_{i-1} h_j h_j}\left(\frac{1}{h_{j-1}} + \frac{1}{h_{j+1}}\right) - \frac{1}{h_i h_j h_{j+1} h_{j+2}}\left(\frac{1}{h_{i-1}} + \frac{1}{h_{i+1}}\right) & i = j+2 \\ \frac{1}{h_i h_{i-1} h_j h_{j+1} h_{j+2}} & i = j+4 \end{array} \right\}.$$

6. Numerical results

6.1. Radar cross section. Radiation theory teaches us that the energy is intercepted by an object can be reflected, absorbed or transmitted through the target. We can assume that most of the energy is reflected. The spatial distribution of this energy depends on the size, shape and composition of the target, and on the frequency and nature of the incident wave. This distribution of energy is called scattering, and the target itself is often referred to as a scatterer. The radar cross section (RCS) of the body is a measure of the energy scattered in a particular direction for a given illumination [2].

Bistatic scattering is the name given to the situation when the scattering direction is not back toward the source of the radiation. If $\mathbf{E}$ and $\mathbf{H}$ represent fields scattered by an object illuminated by incident plane wave $\mathbf{E}^{inc}$ traveling in the direction of the unit vector $\mathbf{k}$, the *bistatic radar cross section* in the observation direction $\mathbf{r}$ is

$$\sigma(\mathbf{r},\mathbf{k}) = \lim_{r\to\infty} 4\pi r^2 \frac{|\mathbf{E}|^2}{|\mathbf{E}^{inc}|^2}.$$

This cross section is defined as the area through which an incident plane wave carries sufficient power to produce, by omnidirectional radiation, the same scattered power density as that observed in a given far field direction. The *monostatic radar cross section* is defined as the radar cross section observed in the back scattering direction, $\sigma(-\mathbf{k},\mathbf{k})$.

In two dimensions, the bistatic radar cross section for scattering by a cylindrical object illuminated by an incident plane wave $\mathbf{E}^{inc}$ traveling in the direction of the unit vector $\mathbf{k}$ normal to the cylinder axis is

$$\sigma(\rho,\mathbf{k}) = \lim_{\rho\to\infty} 2\pi\rho \frac{|\mathbf{E}|^2}{|\mathbf{E}^{inc}|^2}.$$

This cross section is the equivalent width across which an incident plane wave carries sufficient power to produce, by omnidirectional radiation, the same scattered power density as that observed in a given far field direction. The monostatic radar cross section is $\sigma(-\mathbf{k},\mathbf{k})$. That is defined for cylinders as the ratio of the total scattered power per unit length to the power density of the incident wave.

The units for RCS are square meters. As RCS can span a wide range of values, a logarithmic decibel scale is also used with a typical reference value σ_{ref} equal to $1m^2$:

$$\sigma_{dBm^2} = 10\ \log_{10}(\frac{\sigma}{\sigma_{ref}}). \tag{61}$$

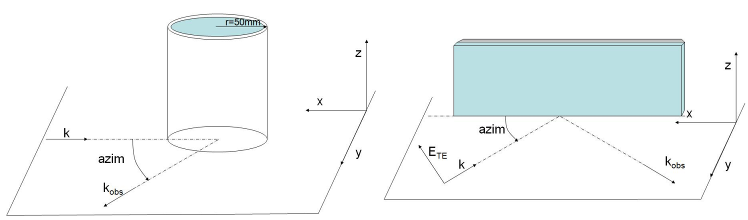

FIGURE 5. Cylinder (left) and plate with thin layer (right).

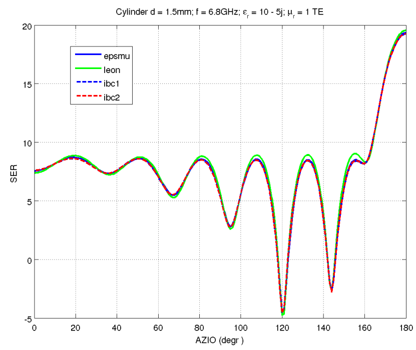

FIGURE 6. Bistatic RCS for a coated circular cylinder, when $d = 1.5mm$, $\epsilon_r = 10 - 5j$, $\mu_r = 1.0$, and $f = 6.8GHz$ with TE polarization.

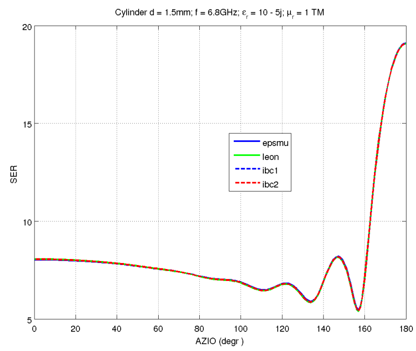

FIGURE 7. Bistatic RCS for a coated circular cylinder, when $d = 1.5mm$, $\epsilon_r = 10 - 5j$, $\mu_r = 1.0$, and $f = 6.8GHz$ with TM polarization.

6.2. Numerical tests. Let us consider conducting circular cylinder depicted in figure 5 coated with thin dielectric layer. The radius of the inner conductor is $r = 50mm$ and the thickness of the coating is d. It is assumed that the incident field is propagating normal to the axis of the cylinder. And we consider both TE and TM polarizations. In order to illustrate several key points the case of a simple dielectric coating will be considered.

An exact solution of the scattering problem depicted in figure 5 is obtained by expanding the incident field, the scattered field outside the cylinder, and the total field inside the cylinder coating in terms of a series of cylindrical wave functions and applying the appropriate boundary conditions at each interface.

Since the coefficients appearing in the HOIBC were derived by considering the planar canonical problem it is expected that the solution should be most accurate for cylinders with large radius of curvature and thin coating, where the geometrical approximation is a good one.

In order to illustrate these points scattering by three typical coated cylinders will be considered next. Figures 8-9 show the monostatic RCS for a coated conducting cylinder with inner radius λ_0, coating thickness $d = 0.1\lambda_0$, and coating parameters

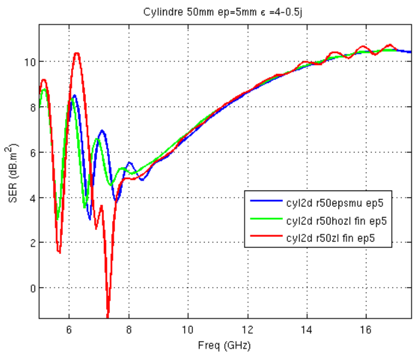

FIGURE 8. Monostatic RCS for a coated circular cylinder, when $d = 0.1\lambda_0$, $\epsilon_r = 4.0 - j0.5$ and $\mu_r = 1.0$, with TE polarization.

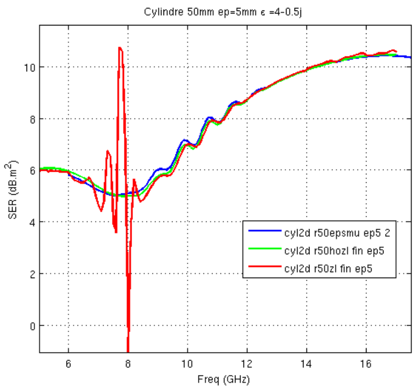

FIGURE 9. Monostatic RCS for a coated circular cylinder, when $d = 0.1\lambda_0$, $\epsilon_r = 4.0 - j0.5$ and $\mu_r = 1.0$, with TM polarization.

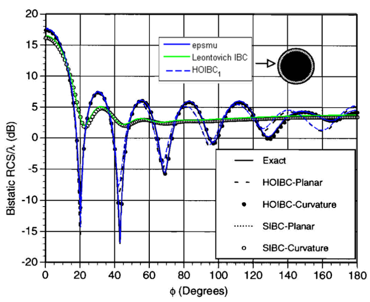

FIGURE 10. Bistatic RCS for a coated circular cylinder, when $d = 0.1\lambda_0$, $\epsilon_r = 4 - 0.5j$, $\mu_r = 1$ with TE polarization.

$\epsilon_r = 4.0 - 0.5i$ and $\mu_r = 1.0$. The exact series solution is presented along with the HOIBC and SIBC solutions. We computed monostatic RCS for different frequencies to see how do results depend on frequency. In TE-polarization we can see that results of SIBC jumps in range between $6GHz$ and $8GHz$ (see fig. 8). Much bigger difference, we can see in TM-polarization between $7GHz$ and $9GHz$ (see fig. 9).

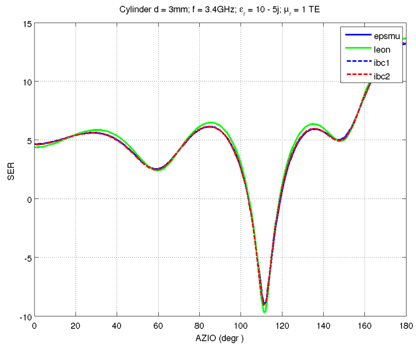

FIGURE 11. Bistatic RCS for a coated circular cylinder, when $d = 3mm$, $\epsilon_r = 10 - 5j$, $\mu_r = 1.0$, and $f = 3.4GHz$ with TE polarization.

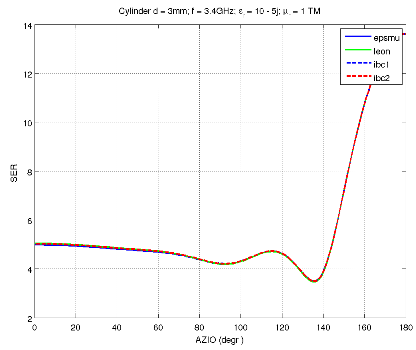

FIGURE 12. Bistatic RCS for a coated circular cylinder, when $d = 3mm$, $\epsilon_r = 10 - 5j$, $\mu_r = 1.0$, and $f = 3.4GHz$ with TM polarization.

Next we consider bistatic RCS for different scattering angles. Figures 6-7 show the bistatic radar cross section for a coated conducting cylinder with inner radius λ_0, coating thickness $d = 1.5mm$, and coating parameters $\epsilon_r = 10 - 5i$ and $\mu_r = 1$, for fixed frequency $f = 6.8GHz$ in TE and TM polarizations. The exact series solution is presented along with the SIBC and HOIBC order 1 and order 2 solutions.

After we increase thickness of a boundary and decrease frequency, so we considered bistatic RCS for different scattering angles. Figures 11-12 shows the bistatic radar cross section for a coated conducting cylinder with inner radius λ_0, coating thickness $d = 3mm$ and frequency $f = 3.4GHz$, coating parameters $\epsilon_r = 10 - 5i$ and $\mu_r = 1.0$, in TE and TM polarizations. The exact series solution is presented along with the SIBC and HOIBC order 1 and order 2 solutions.

Here we comput bistatic RCS for coated circular cylinder with parameters, $d = 0.1\lambda_0, \epsilon_r = 4 - 0.5i$ and $\mu_r = 1$. And we compare to Rahmat-Samii results for same test. The backscatter direction is $\phi = 180°$. Results for exact formulation, SIBC or

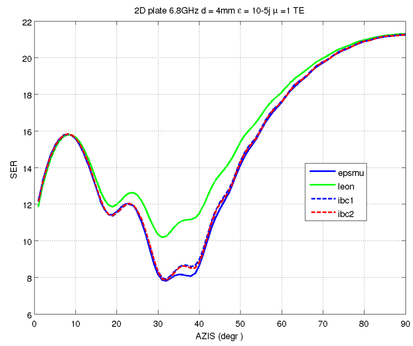

FIGURE 13. Bistatic RCS for a coated 2D plate, when $d = 4mm$, $\epsilon_r = 10 - 5j$, $\mu_r = 1.0$, and $f = 6.8GHz$ with TE polarization.

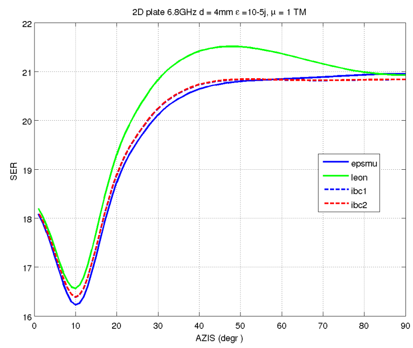

FIGURE 14. Bistatic RCS for a coated 2D plate, when $d = 4mm$, $\epsilon_r = 10 - 5j$, $\mu_r = 1.0$, and $f = 6.8GHz$ with TM polarization.

Leontovich IBC formulation and the formulation based on the planar higher order IBC are presented in the figure 10. As can be seen in the figure, the results using the planar HOIBC are in excellent agreement with the exact solution over most of the angular range, while SIBC solutions give only the average behavior of the scattered field.

Next we consider conducting plate with open boundary thin dielectric layer (see fig. 5). Figures 13-14 show the bistatic RCS for layer thickness $d = 4mm$ and frequency $f = 6.8GHz$. This example is interesting because it shows that method works even for open boundaries. And we can see that it solves problem much better than with Leontovich IBC. But it is difficult to see difference between first order and second order IBCs.

7. Conclusion

In this paper, we give integral formulations with high order impedance boundary condition to solve Maxwell's equations. We study existence and uniqueness of the solution for the formulations. Then, we give several numerical tests of the solution HOIBC over SIBC using a method of moments. The figures clearly show the increased accuracy of the HOIBC solution relative to the SIBC solution.

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