INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 19, Number 2-3, Pages 194–219

STRUCTURE-PRESERVING NUMERICAL METHODS FOR A CLASS OF STOCHASTIC POISSON SYSTEMS

YUCHAO WANG, LIJIN WANG, AND YANZHAO CAO*

Abstract. We propose a type of numerical methods for a class of stochastic Poisson systems with invariant energy. The proposed numerical methods preserve both the energy and the Casimir functions of the systems. In addition, we provide a new approach of constructing stochastic Poisson integrators which respect the Poisson structure and the Casimir functions of stochastic Poisson systems based on coordinate transformations on the midpoint method. Numerical tests are performed to demonstrate our theoretical analysis.

Key words. stochastic Poisson systems, structure-preserving algorithms, Poisson structure, Casimir functions, Poisson integrators.

1. Introduction

Stochastic Poisson systems (SPSs) are generalizations of stochastic Hamiltonian systems ([2, 9, 19]) and have the following form ([9]):

$$
\begin{aligned}
&d\boldsymbol{y}(t) = \boldsymbol{B}(\boldsymbol{y}(t)) \left(\nabla H_0(\boldsymbol{y}(t))dt + \sum_{r=1}^{s} \nabla H_r(\boldsymbol{y}(t)) \circ dW_r(t) \right), \\
&\boldsymbol{y}(0) = \boldsymbol{y}_0,
\end{aligned} \tag{1}
$$

where $\boldsymbol{y} = (y^1, \ldots, y^m)^T \in \mathbb{R}^m$, $H_r(\boldsymbol{y})$ $(r = 0, \ldots, s)$ are smooth functions of $\boldsymbol{y}$, $\boldsymbol{\mathcal{W}}(t) := (W_1(t), \ldots, W_s(t))$ is an s-dimensional standard Wiener process defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$, '$\circ$' denotes the Stratonovich differential, and $\boldsymbol{B}(\boldsymbol{y}) = (b_{ij}(\boldsymbol{y}))$ is a smooth $m \times m$ matrix–valued function of $\boldsymbol{y}$ which is skew-symmetric $(b_{ij}(\boldsymbol{y}) = -b_{ji}(\boldsymbol{y}))$ and satisfies

$$
\sum_{l=1}^{m} \left(\frac{\partial b_{ij}(\boldsymbol{y})}{\partial y^l} b_{lk}(\boldsymbol{y}) + \frac{\partial b_{jk}(\boldsymbol{y})}{\partial y^l} b_{li}(\boldsymbol{y}) + \frac{\partial b_{ki}(\boldsymbol{y})}{\partial y^l} b_{lj}(\boldsymbol{y}) \right) = 0, \tag{2}
$$

for all $i, j, k \in \{1, \ldots, m\}$.

If the dimension $m = 2d$ is an even integer, and

$$
\boldsymbol{B}(\boldsymbol{y}) \equiv \boldsymbol{J}^{-1} = \begin{pmatrix} \mathbf{0}_d & -\boldsymbol{I}_d \\ \boldsymbol{I}_d & \mathbf{0}_d \end{pmatrix}
$$

where $\boldsymbol{I}_d$ denotes the d-dimensional identity matrix, then SPSs (1) degenerate to stochastic Hamiltonian systems ([17, 18, 19]). It was proved in [9] that almost surely the phase flow of a SPS $\boldsymbol{\varphi}_t : \boldsymbol{y} \to \boldsymbol{\varphi}_t(\boldsymbol{y})$ possesses the Poisson structure:

$$
\frac{\partial \boldsymbol{\varphi}_t(\boldsymbol{y})}{\partial \boldsymbol{y}} \boldsymbol{B}(\boldsymbol{y}) \frac{\partial \boldsymbol{\varphi}_t(\boldsymbol{y})}{\partial \boldsymbol{y}}^T = \boldsymbol{B}(\boldsymbol{\varphi}_t(\boldsymbol{y})), \quad \forall t \geq 0, \quad a.s. \tag{3}
$$

Moreover, if the rank of $\boldsymbol{B}(\boldsymbol{y})$ is not full, there exists at least one Casimir function $C(\boldsymbol{y})$ with the property $\nabla C(\boldsymbol{y})^T \boldsymbol{B}(\boldsymbol{y}) \equiv \mathbf{0}$ $(\forall \boldsymbol{y})$ ([7]). Casimir functions are invariants of the SPSs almost surely ([9]), i.e. $C(\boldsymbol{y}(t)) \equiv C(\boldsymbol{y}_0)$ along the solution $\boldsymbol{y}(t)$

Received by the editors August 30, 2021 and, in revised form, March 10, 2022.
2000 *Mathematics Subject Classification.* 60H35, 60H15, 65C30, 60H10, 65D30.
*Corresponding author: Yanzhao Cao.

of (1) $\forall t \geq 0$ almost surely, since

$$\begin{aligned} dC(\boldsymbol{y}) =& \nabla C(\boldsymbol{y})^T d\boldsymbol{y} \\ =& \nabla C(\boldsymbol{y})^T \boldsymbol{B}(\boldsymbol{y}) \left(\nabla H_0(\boldsymbol{y}) dt + \sum_{r=1}^{s} \nabla H_r(\boldsymbol{y}) \circ dW_r(t) \right) \\ =& 0. \end{aligned}$$

A numerical method $\{\boldsymbol{y}_n : n \in \mathbb{N}\}$ of (1) is said to preserve the Casimir function $C(\boldsymbol{y})$ if

$$C(\boldsymbol{y}_{n+1}) = C(\boldsymbol{y}_n), \quad \forall n \in \mathbb{N}, \text{ a.s.}.$$

It is not difficult to see that, for any $i \in \{0, \dots, s\}$, if

$$\{H_i(\boldsymbol{y}), H_j(\boldsymbol{y})\} := \nabla H_i(\boldsymbol{y})^T \boldsymbol{B}(\boldsymbol{y}) \nabla H_j(\boldsymbol{y}) = 0 \text{ for all } j = 0, \dots, s \text{ and all } \boldsymbol{y},$$

where $\{H_i(\boldsymbol{y}), H_j(\boldsymbol{y})\}$ is called the Poisson bracket of $H_i(\boldsymbol{y})$ and $H_j(\boldsymbol{y})$, then

$$dH_i(\boldsymbol{y}) = \nabla H_i(\boldsymbol{y})^T d\boldsymbol{y} = 0.$$

In this case $H_i(\boldsymbol{y})$ is an invariant Hamiltonian of (1). When $H_r \equiv 0$ for $r = 1, \dots, s$, SPSs (1) degenerate to deterministic Poisson systems ([7, 13]).

Poisson systems find applications in many scientific and engineering areas such as astronomy, robotics, quantum mechanics, electrodynamics and so on ([31]). Given the characterization of the Poisson structure (3), a numerical method $\{\boldsymbol{y}_n : n \in \mathbb{N}\}$ is said to preserve the Poisson structure of the system if it satisfies (see e.g. [7, 9])

$$\frac{\partial \boldsymbol{y}_{n+1}}{\partial \boldsymbol{y}_n} \boldsymbol{B}(\boldsymbol{y}_n) \frac{\partial \boldsymbol{y}_{n+1}}{\partial \boldsymbol{y}_n}^T = \boldsymbol{B}(\boldsymbol{y}_{n+1}) \text{ (a.s. in stochastic cases)}, \ n \in \mathbb{N}. \tag{4}$$

Numerical methods for Poisson systems that can preserve both the Poisson structure and the Casimir functions are called Poisson integrators. Even for deterministic Poisson systems, it is challenging to construct general Poisson integrators in case the structure matrix $\boldsymbol{B}(\boldsymbol{y})$ is nonconstant (p. 270 of [7], [10]). During the last decades, there have been numerous studies exploiting special structures of particular deterministic Poisson systems to construct Poisson integrators or other structure-preserving numerical methods for them. Such methods have been shown to produce much better long-time numerical behavior than other general-purpose methods (see e.g. [1, 3, 4, 5, 6, 7, 15, 16, 21, 22, 23, 24, 28, 29, 30] and references therein).

Stochastic Poisson systems were recently proposed and numerically studied (see e.g. [2, 8, 9, 12, 25, 26]), where stochastic Poisson integrators or energy (Hamiltonian)-preserving methods were investigated. For the following stochastic Poisson system

$$d\boldsymbol{y}(t) = \boldsymbol{B}(\boldsymbol{y}(t)) \nabla H(\boldsymbol{y}(t))(dt + c \circ dW(t)), \tag{5}$$

where $\boldsymbol{B}(\boldsymbol{y})$ is a skew-symmetric matrix satisfying (2) and c is a non-zero constant, $H(\boldsymbol{y})$ is obviously an invariant Hamiltonian and called the energy of the system ([2]). [2] proposed a class of numerical methods that can preserve the energy $H(\boldsymbol{y})$ and quadratic Casimir functions of the system. [12] constructed a class of explicit parametric stochastic Runge–Kutta methods which preserve the energy $H(\boldsymbol{y})$ for suitable parameters and can achieve any prescribed mean-square orders.

In (5), when $c = 0$ and

$$\begin{aligned} \boldsymbol{B}(\boldsymbol{y}) &= \left(b_{ij}^0 y^i y^j\right) = diag(y^1, \dots, y^m) \boldsymbol{B_0}\, diag(y^1, \dots, y^m), \\ H(\boldsymbol{y}) &= \sum_{i=1}^{m} \beta_i y^i - p_i \ln y^i, \end{aligned} \tag{6}$$

where $\boldsymbol{B_0} = (b^0_{ij})$ is a skew-symmetric constant matrix, and $\beta_i \neq 0$ $(i = 1, ..., m)$, the system is a deterministic Lokta-Volterra system studied in [20] where the Poisson structure of the system was revealed. Therefore the system (5) with $\boldsymbol{B}(\boldsymbol{y})$ and $H(\boldsymbol{y})$ defined by (6), denoted by (5)–(6) in the sequel, is a stochastic extension of the Lotka-Volterra system in [20], which includes the test Lotka-Volterra model in [2] as a special case.

For the system (5)–(6), motivated by the discussion in [14], we proved in [27] the almost sure existence and uniqueness of the solution under certain conditions, which we call the well-posedness conditions, and further verified that the solution is almost surely positive given positive initial values and bounded under the well-posedness conditions. It is not difficult to verify that the functions

$$C(\boldsymbol{y}) = \alpha_1 \ln y^1 + \cdots + \alpha_m \ln y^m \tag{7}$$

are Casimir functions of the system (5)–(6) where $\alpha = (\alpha_1, \ldots, \alpha_m)^T \in Ker\boldsymbol{B_0}$. Obviously, these Casimir functions are not quadratic so that they are not guaranteed to be preserved by the energy-preserving method in [2].

In this paper, we propose a new numerical method for the system (5)–(6) which preserves both the energy and the Casimir functions of the system. For brevity we call it the energy-Casimir-preserving scheme in the sequel. In addition, we prove that the scheme also inherits the almost sure positiveness (given positive initial values) and boundedness of the exact solution under the well-posedness conditions, and the root mean-square convergence order of the scheme is 1. Furthermore, we shall show that the midpoint scheme applied to a stochastic Poisson system with constant structure matrix is a Poisson integrator, and that Poisson integrators are invariant under invertible coordinate transformations. Based on these we construct Poisson integrator for the system (5)–(6) by firstly transforming it via coordinate transformation to a system with constant structure matrix to which we apply the midpoint method, and then using the inverse coordinate transformation to transform the midpoint method back to a Poisson integrator for the original system (5)–(6). Numerical tests confirm our theoretical analysis.

The rest of the paper is arranged as follows. In Section 2 we construct the energy-Casimir-preserving scheme for (5)–(6), and prove its preservation of the positiveness and boundedness of the exact solution in addition to its preservation of the energy and Casimir functions. In Section 3 we analyze the root mean-square convergence order of the method. In Section 4 we show that the midpoint method is a Poisson integrator for any SPS with constant structure matrix, and that Poisson integrators are invariant under invertible coordinate transformations. Then we construct a stochastic Poisson integrator for the considered system (5)–(6) via transforming a midpoint scheme. In Section 5 we conduct a variety of numerical experiments to demonstrate our theoretical results, followed by a brief conclusion in Section 6.

2. The energy-Casimir-preserving scheme

In the following we write a vector $\boldsymbol{a} > 0$ to mean that each of its elements is positive.

2.1. Construction of the numerical scheme. For convenience we write the system (5)–(6) more compactly as the following system (8):

$$d\boldsymbol{y}(t) = \boldsymbol{B}(\boldsymbol{y}(t))\nabla H(\boldsymbol{y}(t))(dt + c \circ dW(t)), \quad \boldsymbol{y}(0) = \boldsymbol{y}_0, \text{ where } \boldsymbol{y}_0 > 0, \text{ and}$$
$$\boldsymbol{B}(\boldsymbol{y}) = \left(b_{ij}^0 y^i y^j\right) = diag(y^1, ..., y^m)\boldsymbol{B_0}\, diag(y^1, ..., y^m),$$
$$H(\boldsymbol{y}) = \sum_{i=1}^{m} \beta_i y^i - p_i \ln y^i. \tag{8}$$

We first state its well-posedness conditions in Assumption 2.1 which also guarantee the almost sure positiveness and boundedness of the solution ([27]):

Assumption 2.1. *([27]) For the parameters $\beta = (\beta_1, \dots, \beta_m)^T, p = (p_1, \dots, p_m)^T$ of system (8), there exists a real number $s_0 \in R$ and a vector $\boldsymbol{\alpha} \in Ker\boldsymbol{B_0}$ such that*

$$\begin{cases} s_0\beta > 0, \\ -s_0 p + \alpha < 0. \end{cases}$$

In the sequel we assume that Assumption 2.1 is valid. We propose the following numerical scheme for (8):

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \boldsymbol{B}\left(\frac{\boldsymbol{y}_{n+1} - \boldsymbol{y}_n}{\ln|\boldsymbol{y}_{n+1}| - \ln|\boldsymbol{y}_n|}\right) \int_0^1 \nabla H(\boldsymbol{y}_n + \tau(\boldsymbol{y}_{n+1} - \boldsymbol{y}_n))d\tau \left(h + c\Delta\widehat{W}_n\right), \tag{9}$$

where

$$\frac{\boldsymbol{y}_{n+1} - \boldsymbol{y}_n}{\ln|\boldsymbol{y}_{n+1}| - \ln|\boldsymbol{y}_n|} := \left(\frac{y_{n+1}^1 - y_n^1}{\ln|y_{n+1}^1| - \ln|y_n^1|}, \dots, \frac{y_{n+1}^m - y_n^m}{\ln|y_{n+1}^m| - \ln|y_n^m|}\right)^T,$$

and $\Delta\widehat{W}_n$ is a truncation of the Wiener process increment $\Delta W_n := W(t_{n+1}) - W(t_n)$, which was proposed for implicit stochastic schemes in [18] and $\Delta\widehat{W}_n = \sqrt{h}\zeta_{h,n}$ with

$$\zeta_{h,n} = \begin{cases} A_h, & \xi_n > A_h, \\ \xi_n, & |\xi_n| \le A_h, \\ -A_h, & \xi_n < -A_h, \end{cases}$$

where $A_h = \sqrt{2k|\ln h|}$ for $k \ge 2$, $\xi_n \sim \mathcal{N}(0,1)$ and $\Delta W_n = \sqrt{h}\xi_n$.

2.2. Properties of the scheme.

2.2.1. Preservation of the energy and Casimir functions.

Theorem 2.1. *Applied to system (8), the numerical scheme (9) exactly preserves the energy and the Casimir functions of (8).*

Proof. From the fundamental theorem of calculus, we have

$$H(\boldsymbol{y}_{n+1}) - H(\boldsymbol{y}_n) = \int_0^1 \nabla H(\boldsymbol{y}_n + \theta(\boldsymbol{y}_{n+1} - \boldsymbol{y}_n))^T (\boldsymbol{y}_{n+1} - \boldsymbol{y}_n) d\theta.$$

By the scheme (8),

$$\int_0^1 \nabla H(\boldsymbol{y}_n + \theta(\boldsymbol{y}_{n+1} - \boldsymbol{y}_n))^T d\theta$$
$$\cdot \boldsymbol{B}\left(\frac{\boldsymbol{y}_{n+1} - \boldsymbol{y}_n}{\ln|\boldsymbol{y}_{n+1}| - \ln|\boldsymbol{y}_n|}\right) \int_0^1 \nabla H(\boldsymbol{y}_n + \tau(\boldsymbol{y}_{n+1} - \boldsymbol{y}_n))d\tau \left(h + c\Delta\widehat{W}_n\right) = 0,$$

due to skew-symmetry of the structure matrix $\boldsymbol{B}(\boldsymbol{y})$.

The gradient of the Casimir function C given by $C(\boldsymbol{y}) = \sum_{i=1}^m \alpha_i \ln y^i$ is $\left(\frac{\alpha_1}{y^1}, \ldots, \frac{\alpha_m}{y^m}\right)^T$. By straightforward calculations we obtain

$$\begin{aligned}
&\int_0^1 \nabla C(\boldsymbol{y}_n + \theta(\boldsymbol{y}_{n+1} - \boldsymbol{y}_n))^T d\theta \\
&= \left(\alpha_1 \frac{\ln|y^1_{n+1}| - \ln|y^1_n|}{y^1_{n+1} - y^1_n}, \ldots, \alpha_m \frac{\ln|y^m_{n+1}| - \ln|y^m_n|}{y^m_{n+1} - y^m_n}\right) \\
&= \nabla C\left(\frac{\boldsymbol{y}_{n+1} - \boldsymbol{y}_n}{\ln|\boldsymbol{y}_{n+1}| - \ln|\boldsymbol{y}_n|}\right)^T.
\end{aligned}$$

Therefore,

$$\begin{aligned}
C(\boldsymbol{y}_{n+1}) - C(\boldsymbol{y}_n) &= \int_0^1 \nabla C(\boldsymbol{y}_n + \theta(\boldsymbol{y}_{n+1} - \boldsymbol{y}_n))^T (\boldsymbol{y}_{n+1} - \boldsymbol{y}_n) d\theta \\
= \nabla C&\left(\frac{\boldsymbol{y}_{n+1} - \boldsymbol{y}_n}{\ln|\boldsymbol{y}_{n+1}| - \ln|\boldsymbol{y}_n|}\right)^T \boldsymbol{B}\left(\frac{\boldsymbol{y}_{n+1} - \boldsymbol{y}_n}{\ln|\boldsymbol{y}_{n+1}| - \ln|\boldsymbol{y}_n|}\right) \\
\cdot &\int_0^1 \nabla H(\boldsymbol{y}_n + \tau(\boldsymbol{y}_{n+1} - \boldsymbol{y}_n)) d\tau \left(h + c\Delta\widehat{W}_n\right) = 0,
\end{aligned}$$

owing to the definition of Casimir functions. □

2.2.2. Positiveness and boundedness of the numerical solution. For any given initial value $\boldsymbol{y}_0 \in \mathbb{R}^m_+$, the almost sure existence (global non-explosion) and uniqueness of the solution of the system (8) can be ensured, and the solution is positive and bounded almost surely([27]). Next we show that the almost sure positiveness and boundedness can be preserved by the scheme (9). In what follows, equalities and inequalities between random variables are in the 'almost sure' sense.

Write $\boldsymbol{B}(\cdot)\nabla H(\cdot)$ as $(\boldsymbol{B}\nabla H)(\cdot)$. From

$$\int_0^1 \nabla H(\boldsymbol{y}_n + \tau(\boldsymbol{y}_{n+1} - \boldsymbol{y}_n)) d\tau = \nabla H\left(\frac{\boldsymbol{y}_{n+1} - \boldsymbol{y}_n}{\ln|\boldsymbol{y}_{n+1}| - \ln|\boldsymbol{y}_n|}\right),$$

the scheme (9) can be rewritten as

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + (\boldsymbol{B}\nabla H)\left(\frac{\boldsymbol{y}_{n+1} - \boldsymbol{y}_n}{\ln|\boldsymbol{y}_{n+1}| - \ln|\boldsymbol{y}_n|}\right)\left(h + c\Delta\widehat{W}_n\right),$$

Calculating $\boldsymbol{B}(\boldsymbol{y})\nabla H(\boldsymbol{y})$ out, we have that system (8) is equivalent to

$$\begin{pmatrix} dy^1(t) \\ \vdots \\ dy^m(t) \end{pmatrix} = \begin{pmatrix} y^1\left(\sum_{j\neq 1} b^0_{1j}\left(\beta_j y^j - p_j\right)\right) \\ \vdots \\ y^m\left(\sum_{j\neq m} b^0_{mj}\left(\beta_j y^j - p_j\right)\right) \end{pmatrix} (dt + c \circ dW(t)),$$

and the component form of the scheme (9) is

$$y^i_{n+1} = y^i_n + \frac{y^i_{n+1} - y^i_n}{\ln|y^i_{n+1}| - \ln|y^i_n|}\left(\sum_{j\neq i} b^0_{ij}\left(\beta_j \frac{y^j_{n+1} - y^j_n}{\ln|y^j_{n+1}| - \ln|y^j_n|} - p_j\right)\right)\left(h + c\Delta\widehat{W}_n\right),$$

for $i = 1, \ldots, m$. Note that $b^0_{ii} = 0$ since $\boldsymbol{B_0}$ is skew-symmetric.

We denote $[l, L]^m := \underbrace{[l, L] \times \cdots \times [l, L]}_{m}$, i.e. the m-fold Cartesian product of $[l, L]$ for the following lemma, and a vector $\boldsymbol{y} = (y^1, \ldots, y^m) \in [l, L]^m$ means $y^i \in [l, L]$ for $i = 1, \ldots, m$.

Lemma 2.1. *Given Assumption 2.1 and $\boldsymbol{y}_0 > 0$, for any vector $\boldsymbol{y} = (y^1, \ldots, y^m) \in \mathbb{R}_+^m$ which satisfies $H(\boldsymbol{y}) = H(\boldsymbol{y}_0)$ and $C(\boldsymbol{y}) = C(\boldsymbol{y}_0)$, there exist positive numbers $l > 0$ and $L > 0$ such that $\boldsymbol{y} \in [l, L]^m$.*

Proof. By Assumption 2.1,

$$\begin{aligned}(a_1, a_2, \ldots, a_m) &:= s_0\beta > 0,\\ (d_1, d_2, \ldots, d_m) &:= -s_0 p + \alpha < 0.\end{aligned}$$

Now we construct the function $G(\boldsymbol{y})$ for $\boldsymbol{y} > 0$:

$$\begin{aligned}G(\boldsymbol{y}) &= s_0 H(\boldsymbol{y}) + C(\boldsymbol{y}) + \sum_{j=1}^{n} d_j - d_j \ln(-d_j/a_j)\\ &= \sum_{j=1}^{m} a_j y^j + d_j \ln y^j + d_j - d_j \ln(-d_j/a_j) =: \sum_{j=1}^{m} G_j(y^j)\end{aligned}$$

with

$$G_j(y^j) := a_j y^j + d_j \ln y^j + d_j - d_j \ln(-d_j/a_j), \quad j = 1, \ldots, m.$$

Clearly, the function $G_j(y^j)$ has the minimum value $G_j(-\frac{d_j}{a_j}) = 0$ which implies $G_j(y^j) \geq 0$ on $(0, +\infty)$ for $j = 1, \ldots, m$. Since $H(\boldsymbol{y}) = H(\boldsymbol{y}_0)$ and $C(\boldsymbol{y}) = C(\boldsymbol{y}_0)$, we have

$$G(\boldsymbol{y}) = s_0 H(\boldsymbol{y}) + C(\boldsymbol{y}) + \sum_{j=1}^{m} d_j - d_j \ln(-d_j/a_j) \equiv G(\boldsymbol{y}_0).$$

Therefore,

$$G_j(y^j) = G(\boldsymbol{y}) - \sum_{i \neq j}^{m} G_i(y^i) \leq G(\boldsymbol{y}_0), \quad j = 1, \ldots, m. \tag{10}$$

The second derivative $G_j''(y^j) = -\frac{d_j}{(y^j)^2} > 0$. It is not difficult to see that, for $j = 1, \ldots, m$,

$$\begin{aligned}&i)\quad G_j(y^j) \text{ convexes down on } (0, +\infty);\\ &ii)\quad \lim_{y^j \to 0^+} G_j(y^j) = +\infty, \qquad \lim_{y^j \to +\infty} G_j(y^j) = +\infty.\end{aligned} \tag{11}$$

(10)–(11) implies that there exist $l > 0$ and $L > 0$ such that $y^j \in [l, L]$ $(j = 1, \ldots, m)$. More explicitly for L, the equation of the tangent line of G_j at the point $(-\frac{2d_j}{a_j}, G_j(-\frac{2d_j}{a_j}))$ is

$$z = \frac{a_j}{2}\left(y^j + \frac{2d_j}{a_j}\right) + G_j\left(-\frac{2d_j}{a_j}\right).$$

Then by the convexity of $G_j(y^j)$, we obtain

$$\frac{a_j}{2}\left(y^j + \frac{2d_j}{a_j}\right) + G_j\left(-\frac{2d_j}{a_j}\right) \leq G_j(y^j) \leq G(\boldsymbol{y}_0),$$

which implies

$$y^j \leq \frac{2\,[G(\boldsymbol{y}_0) - d_j \ln 2]}{a_j} =: L_j, \quad j = 1, \ldots, m,$$

which concludes the proof of the lemma with $L := \max_{j\in\{1,\ldots,m\}} L_j$. $\square$

In the following, for vectors $\rho = (\rho^1, \ldots, \rho^m)^T$ and $r = (r^1, \ldots, r^m)^T$, $|\rho| \leq |r|$ means that $|\rho^i| \leq |r^i|$ for $i = 1, \ldots, m$, and $\|\cdot\|$ denotes the Euclidean norm.

Theorem 2.2. *Under the Assumption 2.1 and $\boldsymbol{y}_0 > 0$, there exist constants $K > 0$ and $h_0 > 0$ such that for any $h < h_0$, solution $\boldsymbol{y}_{n+1}$, $n = 0, 1, \cdots$, of the scheme* (9) *is positive and*

$$|\boldsymbol{y}_{n+1} - \boldsymbol{y}_n| \leq \frac{K(h + c|\zeta_{h,n}|\sqrt{h})}{1 - K(h + c|\zeta_{h,n}|\sqrt{h})}|\boldsymbol{y}_n| \leq \frac{1}{2}\boldsymbol{y}_n. \tag{12}$$

Proof. We prove the theorem by induction. Given $\boldsymbol{y}_n > 0$ and h, let

$$\boldsymbol{\phi}(\boldsymbol{z}) = \boldsymbol{y}_n + (\boldsymbol{B}\nabla H)\left(\frac{\boldsymbol{z} - \boldsymbol{y}_n}{\ln|\boldsymbol{z}| - \ln|\boldsymbol{y}_n|}\right)\left(h + c\Delta\widehat{W}_n\right).$$

Then the scheme (9) can be written as

$$\boldsymbol{y}_{n+1} = \boldsymbol{\phi}(\boldsymbol{y}_{n+1})$$

which we solve by the fixed-point iteration with an initial value $\boldsymbol{z}$ satisfying

$$0 < |\boldsymbol{z} - \boldsymbol{y}_n| \leq \frac{K(h + c|\zeta_{h,n}|\sqrt{h})}{1 - K(h + c|\zeta_{h,n}|\sqrt{h})}|\boldsymbol{y}_n| \tag{13}$$

for certain $K > 0$ and $h_1 > 0$ such that when $h \leq h_1$

$$K(h + c|\zeta_{h,n}|\sqrt{h}) \leq \frac{1}{3}, \quad \frac{K(h + c|\zeta_{h,n}|\sqrt{h})}{1 - K(h + c|\zeta_{h,n}|\sqrt{h})} \leq \frac{1}{2}, \tag{14}$$

which implies

$$0 < |\boldsymbol{z} - \boldsymbol{y}_n| \leq \frac{1}{2}|\boldsymbol{y}_n| \tag{15}$$

so that $\boldsymbol{z} > 0$. Since $\boldsymbol{y}_n > 0$ satisfies $H(\boldsymbol{y}_n) = H(\boldsymbol{y}_0)$ and $C(\boldsymbol{y}_n) = C(\boldsymbol{y}_0)$, it follows from Lemma 2.1 that there exist $l > 0$ and $L > 0$ dependent only on the system (8) such that $\boldsymbol{y}_n \in [l, L]^m$. (15) then implies that $\boldsymbol{z} \in [\frac{l}{2}, 2L]^m$.

By the mean value theorem, $\frac{\boldsymbol{z}-\boldsymbol{y}_n}{\ln\boldsymbol{z}-\ln\boldsymbol{y}_n} := \left(\frac{z^1-y_n^1}{\ln z^1-\ln y_n^1}, \ldots, \frac{z^m-y_n^m}{\ln z^m-\ln y_n^m}\right)^T$ is a vector between $\boldsymbol{y}_n$ and $\boldsymbol{z}$, thus

$$\left|\frac{\boldsymbol{z} - \boldsymbol{y}_n}{\ln\boldsymbol{z} - \ln\boldsymbol{y}_n} - \boldsymbol{y}_n\right| \leq |\boldsymbol{z} - \boldsymbol{y}_n|, \quad \frac{\boldsymbol{z} - \boldsymbol{y}_n}{\ln\boldsymbol{z} - \ln\boldsymbol{y}_n} \in \left[\frac{l}{2}, 2L\right]^m. \tag{16}$$

Next, we show that the vector $\boldsymbol{\phi}(\boldsymbol{z})$ form the iteration also satisfies the inequality (13), i.e.,

$$0 < |\boldsymbol{\phi}(\boldsymbol{z}) - \boldsymbol{y}_n| \leq \frac{K(h + c|\zeta_{h,n}|\sqrt{h})}{1 - K(h + c|\zeta_{h,n}|\sqrt{h})}|\boldsymbol{y}_n|.$$

We consider the i-th component of $\boldsymbol{\phi}(\boldsymbol{z})$, $i \in \{1, \dots, m\}$:

$$\begin{aligned}\phi^i(\boldsymbol{z}) &= y_n^i + \frac{z^i - y_n^i}{\ln|z^i| - \ln|y_n^i|}\left(\sum_{j\neq i} b_{ij}^0\left(\beta_j \frac{z^j - y_n^j}{\ln|z^j| - \ln|y_n^j|} - p_j\right)\right)\left(h + c\Delta\widehat{W}_n\right) \\ &= y_n^i + \frac{z^i - y_n^i}{\ln z^i - \ln y_n^i}\left(\sum_{j\neq i} b_{ij}^0\left(\beta_j \frac{z^j - y_n^j}{\ln z^j - \ln y_n^j} - p_j\right)\right)\left(h + c\Delta\widehat{W}_n\right),\end{aligned}$$

Due to (16), there exists $K_1 > 0$ such that for $i = 1, \dots, m$

$$\left|\sum_{j\neq i} b_{ij}^0\left(\beta_j \frac{z^j - y_n^j}{\ln z^j - \ln y_n^j} - p_j\right)\right| \leq K_1. \tag{17}$$

Now we update the K by letting $K := \max\{K, K_1\}$, and adjust the h_1 for the validity of (14) accordingly. Then by (16), (13) and (14) we have

$$\begin{aligned}\left|\phi^i(\boldsymbol{z}) - y_n^i\right| &\leq \left|\frac{z^i - y_n^i}{\ln z^i - \ln y_n^i} - y_n^i\right|\left|\sum_{j\neq i} b_{ij}^0\left(\beta_j \frac{z^j - y_n^j}{\ln z^j - \ln y_n^j} - p_j\right)\right|(h + c|\zeta_{h,n}|\sqrt{h}) \\ &\quad + \left|y_n^i\right|\left|\sum_{j\neq i} b_{ij}^0\left(\beta_j \frac{z^j - y_n^j}{\ln z^j - \ln y_n^j} - p_j\right)\right|(h + c|\zeta_{h,n}|\sqrt{h}) \\ &\leq K\left|z^i - y_n^i\right|\left(h + c|\zeta_{h,n}|\sqrt{h}\right) + K\left|y_n^i\right|(h + c|\zeta_{h,n}|\sqrt{h}) \\ &\leq K(h + c|\zeta_{h,n}|\sqrt{h})\left(\frac{K(h + c|\zeta_{h,n}|\sqrt{h})}{1 - K(h + c|\zeta_{h,n}|\sqrt{h})} + 1\right)|y_n^i| \\ &= \frac{K(h + c|\zeta_{h,n}|\sqrt{h})}{1 - K(h + c|\zeta_h|\sqrt{h})}|y_n^i| \leq \frac{1}{2}|y_n^i|,\end{aligned}$$

for $i = 1, \dots, m$. Since $h + c\zeta_{h,n}\sqrt{h} \neq 0$ a.s., $\boldsymbol{\phi}(\boldsymbol{z}) \neq \boldsymbol{y}_n$ a.s.. Therefore, almost surely the mapping $\boldsymbol{\phi}$ maps $\left[\frac{l}{2}, 2L\right]^m \setminus \{\boldsymbol{y}_n\}$ to $\left[\frac{l}{2}, 2L\right]^m \setminus \{\boldsymbol{y}_n\}$ itself. Next we show that $\boldsymbol{\phi}$ is also a contraction mapping on $\left[\frac{l}{2}, 2L\right]^m \setminus \{\boldsymbol{y}_n\}$ for sufficiently small h. For $\boldsymbol{z} \in \left[\frac{l}{2}, 2L\right]^m \setminus \{\boldsymbol{y}_n\}$, denoting

$$\boldsymbol{f}(\boldsymbol{z}) = (\boldsymbol{B}\nabla H)\left(\frac{\boldsymbol{z} - \boldsymbol{y}_n}{\ln \boldsymbol{z} - \ln \boldsymbol{y}_n}\right) =: (f^1(\boldsymbol{z}), \dots, f^m(\boldsymbol{z}))^T,$$

we have

$$\boldsymbol{\phi}(\boldsymbol{z}) = \boldsymbol{y}_n + \boldsymbol{f}(\boldsymbol{z})(h + c\zeta_{h,n}\sqrt{h}),$$

and for $i = 1, \dots, m$,

$$f^i(\boldsymbol{z}) = \left(\frac{z^i - y_n^i}{\ln z^i - \ln y_n^i}\right)\sum_{j\neq i} b_{ij}^0\left(\beta_j \frac{z^j - y_n^j}{\ln z^j - \ln y_n^j} - p_j\right).$$

For $k = 1, \dots, m$, the partial derivative of $f^i(\boldsymbol{z})$ with respect to z^k is

$$f_k^i(\boldsymbol{z}) := \frac{\partial f^i}{\partial z^k}(\boldsymbol{z}) = \begin{cases} \frac{\ln z^i - \ln y_n^i - (z^i - y_n^i)/z^i}{(\ln z^i - \ln y_n^i)^2}\sum_{j\neq i} b_{ij}^0\left(\beta_j \frac{z^j - y_n^j}{\ln z^j - \ln y_n^j} - p_j\right), & k = i, \\ \frac{z^i - y_n^i}{\ln z^i - \ln y_n^i} b_{ik}^0 \beta_k \frac{\ln z^k - \ln y_n^k - (z^k - y_n^k)/z^k}{(\ln z^k - \ln y_n^k)^2}, & k \neq i. \end{cases}$$

A simple calculation gives $\lim_{z^k \to y_n^k} \frac{\ln z^k - \ln y_n^k - \frac{z^k - y_n^k}{z^k}}{(\ln z^k - \ln y_n^k)^2} = \frac{1}{2}$ (see also (19)), wherefore defining $\frac{\ln z^k - \ln y_n^k - \frac{z^k - y_n^k}{z^k}}{(\ln z^k - \ln y_n^k)^2} = \frac{1}{2}$ when $z^k = y_n^k$, the function $\frac{\ln z^k - \ln y_n^k - \frac{z^k - y_n^k}{z^k}}{(\ln z^k - \ln y_n^k)^2}$ is continuous on $z^k \in [\frac{l}{2}, 2L]$ and thus bounded by a certain $m_k \geq \frac{1}{2}$ for $k = 1, \dots, m$. Owing to (16) and (17), there exists $M_k^i > 0$ such that $|f_k^i(\boldsymbol{z})| \leq M_k^i$ for $i, k = 1, \dots, m$, which implies that $\|\boldsymbol{f}'(\boldsymbol{z})\| \leq M$ on $[\frac{l}{2}, 2L]^m \backslash \{\boldsymbol{y}_n\}$ for certain $M > 0$.

Thus for $z_1, z_2 \in [\frac{l}{2}, 2L]^m \backslash \{\boldsymbol{y}_n\}$,

$$\begin{aligned} \|\boldsymbol{\phi}(\boldsymbol{z}_1) - \boldsymbol{\phi}(\boldsymbol{z}_2)\| &= \|\boldsymbol{f}(\boldsymbol{z}_1) - \boldsymbol{f}(\boldsymbol{z}_2)\|(h + c|\zeta_{h,n}|\sqrt{h}) \\ &\leq M(h + c|\zeta_{h,n}|\sqrt{h})\|\boldsymbol{z}_1 - \boldsymbol{z}_2\|. \end{aligned} \tag{18}$$

Note that there exists $h_2 > 0$ and $C_0 > 0$ such that for $h \leq h_2$,

$$M(h + c|\zeta_{h,n}|\sqrt{h}) \leq C_0 < 1,$$

which means that $\boldsymbol{\phi}$ is a contraction map on $[\frac{l}{2}, 2L]^m \backslash \{\boldsymbol{y}_n\}$. Then by the contraction mapping theorem, when $h \leq h_0 := \min\{h_1, h_2\}$, the iteration based on $\boldsymbol{\phi}$ converges and $\boldsymbol{y}_{n+1} \in [\frac{l}{2}, 2L]^m \backslash \{\boldsymbol{y}_n\}$ almost surely. Since $H(\boldsymbol{y}_{n+1}) = H(\boldsymbol{y}_0)$ and $C(\boldsymbol{y}_{n+1}) = C(\boldsymbol{y}_0)$, Lemma 2.1 indicates that $\boldsymbol{y}_{n+1} \in [l, L]^m$ almost surely. Therefore, given $\boldsymbol{y}_0 \in \mathbb{R}_+^m$, the numerical solutions $\{\boldsymbol{y}_n,\ n = 0, 1, \dots\}$ form the scheme (9) are positive and bounded almost surely. $\square$

Remark 2.1. Note that the bound for the numerical solution derived in the proof of Lemma 2.1 is the same with that of the exact solution given in [27].

From the above analysis, since the scheme (9) preserves the positiveness of the solution, we can rewrite it without the absolute value notations as

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \boldsymbol{B}\left(\frac{\boldsymbol{y}_{n+1} - \boldsymbol{y}_n}{\ln \boldsymbol{y}_{n+1} - \ln \boldsymbol{y}_n}\right) \int_0^1 \nabla H(\boldsymbol{y}_n + \tau(\boldsymbol{y}_{n+1} - \boldsymbol{y}_n)) d\tau \left(h + c\Delta\widehat{W}_n\right).$$

3. Root mean-square convergence order of the method

In this section we derive the root mean-square error estimate of our numerical method proposed in Section 2. To this end, we need the following three lemmas.

Lemma 3.1. *([2]) For all $\gamma \in [1, +\infty)$, there exists a positive constant $C_\gamma > 0$ such that, for all $n \in \mathbb{N}_+$ and all $h \in (0, 1)$, we have*

$$\left(E\left(|\Delta W_n|^\gamma\right)\right)^{\frac{1}{\gamma}} \leq C_\gamma h^{\frac{1}{2}} \quad \textit{and} \quad \left(E\left(\left|\Delta\widehat{W}_n\right|^\gamma\right)\right)^{\frac{1}{\gamma}} \leq C_\gamma h^{\frac{1}{2}}.$$

Define a function $\mathscr{H}$ as

$$\mathscr{H}(x) = \begin{cases} \frac{x - x_0}{\ln x - \ln x_0}, & x \neq x_0, \\ x_0, & x = x_0, \end{cases}$$

with $x \in \mathbb{R}$ and $x_0 > 0$.

Lemma 3.2. *The function $\mathscr{H}$ is at least three times continuously differentiable at x_0, and has the following expansion*

$$\mathscr{H}(x) = x_0 + \frac{1}{2}(x - x_0) - \frac{1}{12x_0}(x - x_0)^2 + \frac{1}{6}h^{(3)}(\xi)(x - x_0)^3,$$

where ξ is between x_0 and x.

Proof. Clearly $\mathscr{H}(x)$ is continuous on $(0,+\infty)$, and when $x \neq x_0$,

$$\mathscr{H}'(x) = \frac{(\ln x - \ln x_0) - \frac{1}{x}(x - x_0)}{(\ln x - \ln x_0)^2}.$$

By the derivative limit theorem,

$$\begin{aligned}
\mathscr{H}'(x_0) &= \lim_{x\to x_0} \frac{(\ln x - \ln x_0) - \frac{1}{x}(x - x_0)}{(\ln x - \ln x_0)^2} \\
&= \lim_{x\to x_0} \frac{\frac{1}{x_0}(x - x_0) - \frac{1}{2x_0^2}(x - x_0)^2 + O((x - x_0)^3) - \frac{1}{x}(x - x_0)}{\left(\frac{1}{x_0}(x - x_0) + O\left((x - x_0)^2\right)\right)^2} \\
&= \lim_{x\to x_0} \frac{\frac{1}{xx_0}(x - x_0)^2 - \frac{1}{2x_0^2}(x - x_0)^2 + O\left((x - x_0)^3\right)}{\frac{1}{x_0^2}(x - x_0)^2 + O\left((x - x_0)^3\right)} \\
&= \lim_{x\to x_0} \frac{\frac{1}{xx_0} - \frac{1}{2x_0^2} + O((x - x_0))}{\frac{1}{x_0^2} + O((x - x_0))} = \frac{1}{2}. \qquad (19)
\end{aligned}$$

Using the derivative limit theorem again, we obtain the second derivative function

$$\mathscr{H}''(x) = \begin{cases} \frac{2(x-x_0)-(x+x_0)(\ln x-\ln x_0)}{x^2(\ln x-\ln x_0)^3}, & x \neq x_0, \\ -\frac{1}{6x_0}, & x = x_0, \end{cases}$$

and similarly,

$$\mathscr{H}^{(3)}(x) = \begin{cases} \frac{(x+2x_0)(\ln x-\ln x_0)^2+6x_0(\ln x-\ln x_0)-6(x-x_0)}{x^3(\ln x-\ln x_0)^4}, & x \neq x_0, \\ \frac{1}{4x_0^2}, & x = x_0. \end{cases}$$

Then by the Taylor's formula, Lamma 3.2 holds. $\square$

Lemma 3.3. *If the numerical solutions* $\{\boldsymbol{y}_n, n = 0, 1, ..., N\}$ *of system* (8) *based on the scheme* (9) *are positive, then* $\boldsymbol{y}_n$ *has the asymptotic expansion:*

$$\boldsymbol{y}_{n+1} - \boldsymbol{y}_n = \boldsymbol{a}(\boldsymbol{y}_n)(h + c\Delta\widehat{W}_n) + \boldsymbol{b}(\boldsymbol{y}_n)(h + c\Delta\widehat{W}_n)^2 + \boldsymbol{c}(\boldsymbol{y}_n)(h + c\Delta\widehat{W}_n)^3 + \boldsymbol{R}_n, \qquad (20)$$

where $\boldsymbol{a}(\boldsymbol{y}_n) = \boldsymbol{B}\nabla H(\boldsymbol{y}_n)$, $\boldsymbol{b}(\boldsymbol{y}_n) = (\boldsymbol{B}\nabla H)'(\boldsymbol{y}_n)\,(\boldsymbol{B}\nabla H)\,(\boldsymbol{y}_n)/2$, $\boldsymbol{c}(\boldsymbol{y}_n)$ *is a continuous function of* $\boldsymbol{y}_n \in [l, L]^m$ *and independent of* $\boldsymbol{y}_{n+1}$ *and* $\Delta\widehat{W}_n$, *and* $\boldsymbol{R}_n$ *depends on* $\boldsymbol{y}_{n+1}$ *and satisfies:*

$$E[\|\boldsymbol{R}_n\|] \le O(h^2), \quad (E[\|\boldsymbol{R}_n\|^2])^{\frac{1}{2}} \le O(h^2). \qquad (21)$$

Proof. For simplicity, we write $\boldsymbol{g} := \nabla H$ in the following discussion, and $\boldsymbol{R}$ denotes random vectors or matrices whose norms possess finite moments bounded by a constant independent of h. Since $\{\boldsymbol{y}_n\}$ are positive, the scheme (9) can be written as

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \boldsymbol{B}\left(\frac{\boldsymbol{y}_{n+1} - \boldsymbol{y}_n}{\ln \boldsymbol{y}_{n+1} - \ln \boldsymbol{y}_n}\right)\int_0^1 \boldsymbol{g}(\boldsymbol{y}_n + \tau(\boldsymbol{y}_{n+1} - \boldsymbol{y}_n))d\tau\left(h + c\Delta\widehat{W}_n\right). \qquad (22)$$

By the mean value theorem,

$$\frac{\boldsymbol{y}_{n+1} - \boldsymbol{y}_n}{\ln \boldsymbol{y}_{n+1} - \ln \boldsymbol{y}_n} = \boldsymbol{y}_n + \theta_n(\boldsymbol{y}_{n+1} - \boldsymbol{y}_n) \qquad (23)$$

with $\theta_n \in (0,1)$. Thus

$$\begin{aligned}\boldsymbol{B}\left(\frac{\boldsymbol{y}_{n+1}-\boldsymbol{y}_n}{\ln \boldsymbol{y}_{n+1}-\ln \boldsymbol{y}_n}\right) &= \boldsymbol{B}(\boldsymbol{y}_n) + \left(\frac{\boldsymbol{y}_{n+1}-\boldsymbol{y}_n}{\ln \boldsymbol{y}_{n+1}-\ln \boldsymbol{y}_n} - \boldsymbol{y}_n\right) \\ &\quad \cdot \int_0^1 \boldsymbol{B}'\left(\boldsymbol{y}_n + \tau\left(\frac{\boldsymbol{y}_{n+1}-\boldsymbol{y}_n}{\ln \boldsymbol{y}_{n+1}-\ln \boldsymbol{y}_n} - \boldsymbol{y}_n\right)\right) d\tau \\ &= \boldsymbol{B}(\boldsymbol{y}_n) + \theta_n \int_0^1 \boldsymbol{B}'\left(\boldsymbol{y}_n + \tau\left(\frac{\boldsymbol{y}_{n+1}-\boldsymbol{y}_n}{\ln \boldsymbol{y}_{n+1}-\ln \boldsymbol{y}_n} - \boldsymbol{y}_n\right)\right) d\tau \\ &\quad \cdot (\boldsymbol{y}_{n+1}-\boldsymbol{y}_n). \end{aligned} \tag{24}$$

By Lemma 2.1 and the proof of Theorem 2.2 we know that given $\boldsymbol{y}_0 > 0$, $\boldsymbol{y}_n \in [l, L]^m$ almost surely for $n \in \mathbb{N}$, wherefore almost surely $\boldsymbol{z}_{n,\theta} := \boldsymbol{y}_n + \theta(\boldsymbol{y}_{n+1} - \boldsymbol{y}_n) \in [l, L]^m$ for $\theta \in [0,1]$. Since $\boldsymbol{B}(\boldsymbol{y})$, $\boldsymbol{B}'(\boldsymbol{y})$ and $\boldsymbol{g}(\boldsymbol{y}) = \nabla H(\boldsymbol{y})$ are continuous functions on $[l, L]^m$, they are bounded almost surely at $\boldsymbol{y} = \boldsymbol{z}_{n,\theta}$ by a constant independent of h for all $n \in \mathbb{N}$ and $\theta \in [0,1]$, implying that their moments at $\boldsymbol{y} = \boldsymbol{z}_{n,\theta}$ are bounded by constants independent of h, n and θ. Substituting $\boldsymbol{y}_{n+1} - \boldsymbol{y}_n$ from (22) into the right-hand side of (24), we can write

$$\boldsymbol{B}\left(\frac{\boldsymbol{y}_{n+1}-\boldsymbol{y}_n}{\ln \boldsymbol{y}_{n+1}-\ln \boldsymbol{y}_n}\right) = \boldsymbol{B}(\boldsymbol{y}_n) + (h + c\Delta\widehat{W}_n)\boldsymbol{R}, \tag{25}$$

where $\boldsymbol{R}$ represents a random matrix whose norm has finite moments bounded by constants independent of h.

Similarly,

$$\begin{aligned}&\int_0^1 \boldsymbol{g}(\boldsymbol{y}_n + \tau(\boldsymbol{y}_{n+1}-\boldsymbol{y}_n))d\tau \\ =&\int_0^1 \left[\boldsymbol{g}(\boldsymbol{y}_n) + \int_0^\tau \boldsymbol{g}'(\boldsymbol{y}_n + s(\boldsymbol{y}_{n+1}-\boldsymbol{y}_n))ds(\boldsymbol{y}_{n+1}-\boldsymbol{y}_n)\right] d\tau \\ =&\boldsymbol{g}(\boldsymbol{y}_n) + \int_0^1\int_0^\tau \boldsymbol{g}'(\boldsymbol{y}_n + s(\boldsymbol{y}_{n+1}-\boldsymbol{y}_n))dsd\tau(\boldsymbol{y}_{n+1}-\boldsymbol{y}_n) \\ =&\boldsymbol{g}(\boldsymbol{y}_n) + (h + c\Delta\widehat{W}_n)\boldsymbol{R}, \end{aligned} \tag{26}$$

where $\boldsymbol{R}$ is a random vector whose norm has finite moments bounded by constants independent of h. Taking the product of (25) and (26), we obtain the coefficient of the first power of $(h + c\Delta\widehat{W}_n)$ in the expansion (20)

$$\boldsymbol{y}_{n+1} - \boldsymbol{y}_n = \boldsymbol{B}(\boldsymbol{y}_n)\boldsymbol{g}(\boldsymbol{y}_n)(h + c\Delta\widehat{W}_n) + (h + c\Delta\widehat{W}_n)^2\boldsymbol{R}. \tag{27}$$

Next, we deduce the explicit form of the coefficient of $(h + c\Delta\widehat{W}_n)^2$ in the expansion (20). For the function $\mathscr{H}(x)$ in Lemma 3.2, if $\boldsymbol{x} \in \mathbb{R}^m$ we let $\mathscr{H}(\boldsymbol{x}) = (\mathscr{H}(x^1), \ldots, \mathscr{H}(x^m))^T$. Then by Lemma 3.2 and the equality (27) we have:

$$\begin{aligned}\frac{\boldsymbol{y}_{n+1}-\boldsymbol{y}_n}{\ln \boldsymbol{y}_{n+1}-\ln \boldsymbol{y}_n} - \boldsymbol{y}_n &= \frac{1}{2}\boldsymbol{I}_m(\boldsymbol{y}_{n+1}-\boldsymbol{y}_n) + \frac{1}{2}\mathscr{H}''(\boldsymbol{\xi}_n)(\boldsymbol{y}_{n+1}-\boldsymbol{y}_n)^2 \\ &= \frac{1}{2}\boldsymbol{B}(\boldsymbol{y}_n)\boldsymbol{g}(\boldsymbol{y}_n)(h + c\Delta\widehat{W}_n) + (h + c\Delta\widehat{W}_n)^2\boldsymbol{R}, \end{aligned} \tag{28}$$

where $\boldsymbol{\xi}_n$ is a vector between $\boldsymbol{y}_n$ and $\boldsymbol{y}_{n+1}$ and $\boldsymbol{I}_m$ is the m-dimensional identity matrix. Note that, for a vector $\boldsymbol{y} \in \mathbb{R}^m$ and a tensor $\boldsymbol{\mathcal{T}}$ that can function on $\boldsymbol{y}$, we write $\boldsymbol{\mathcal{T}}(\boldsymbol{y})^2$ to represent $\boldsymbol{\mathcal{T}}(\boldsymbol{y}, \boldsymbol{y})$ for brevity. Thanks to the smoothness of $\boldsymbol{B}$ and

$\boldsymbol{g}$, and using (28) we have

$$
\begin{aligned}
&\boldsymbol{B}\left(\frac{\boldsymbol{y}_{n+1}-\boldsymbol{y}_n}{\ln \boldsymbol{y}_{n+1}-\ln \boldsymbol{y}_n}\right)=\boldsymbol{B}(\boldsymbol{y}_n)+\boldsymbol{B}'(\boldsymbol{y}_n)\left(\frac{\boldsymbol{y}_{n+1}-\boldsymbol{y}_n}{\ln \boldsymbol{y}_{n+1}-\ln \boldsymbol{y}_n}-\boldsymbol{y}_n\right)\\
&+\int_0^1\int_0^\tau \boldsymbol{B}''\left(\boldsymbol{y}_n+s\left(\frac{\boldsymbol{y}_{n+1}-\boldsymbol{y}_n}{\ln \boldsymbol{y}_{n+1}-\ln \boldsymbol{y}_n}-\boldsymbol{y}_n\right)\right)dsd\tau\left(\frac{\boldsymbol{y}_{n+1}-\boldsymbol{y}_n}{\ln \boldsymbol{y}_{n+1}-\ln \boldsymbol{y}_n}-\boldsymbol{y}_n\right)^2\\
&=\boldsymbol{B}(\boldsymbol{y}_n)+\frac{1}{2}\boldsymbol{B}'(\boldsymbol{y}_n)\left(\boldsymbol{B}(\boldsymbol{y}_n)\boldsymbol{g}(\boldsymbol{y}_n)\right)(h+c\Delta\widehat{W}_n)+(h+c\Delta\widehat{W}_n)^2\boldsymbol{R},
\end{aligned}
\tag{29}
$$

and

$$
\begin{aligned}
&\int_0^1 \boldsymbol{g}(\boldsymbol{y}_n+\tau(\boldsymbol{y}_{n+1}-\boldsymbol{y}_n))d\tau=\boldsymbol{g}(\boldsymbol{y}_n)+\frac{1}{2}\boldsymbol{g}'(\boldsymbol{y}_n)(\boldsymbol{y}_{n+1}-\boldsymbol{y}_n)\\
&+\int_0^1\int_0^\tau\int_0^s \boldsymbol{g}''(\boldsymbol{y}_n+s_1(\boldsymbol{y}_{n+1}-\boldsymbol{y}_n))ds_1dsd\tau(\boldsymbol{y}_{n+1}-\boldsymbol{y}_n)^2\\
&=\boldsymbol{g}(\boldsymbol{y}_n)+\frac{1}{2}\boldsymbol{g}'(\boldsymbol{y}_n)\left(\boldsymbol{B}(\boldsymbol{y}_n)\boldsymbol{g}(\boldsymbol{y}_n)\right)(h+c\Delta\widehat{W}_n)+(h+c\Delta\widehat{W}_n)^2\boldsymbol{R}.
\end{aligned}
\tag{30}
$$

Taking the product of the two equalities (29) and (30), we obtain

$$
\begin{aligned}
&\boldsymbol{y}_{n+1}-\boldsymbol{y}_n=\boldsymbol{B}(\boldsymbol{y}_n)\boldsymbol{g}(\boldsymbol{y}_n)(h+c\Delta\widehat{W}_n)\\
&+\frac{1}{2}\left(\boldsymbol{B}'(\boldsymbol{y}_n)(\boldsymbol{B}(\boldsymbol{y}_n)\boldsymbol{g}(\boldsymbol{y}_n),\boldsymbol{g}(\boldsymbol{y}_n))+\boldsymbol{B}(\boldsymbol{y}_n)\boldsymbol{g}'(\boldsymbol{y}_n)(\boldsymbol{B}(\boldsymbol{y}_n)\boldsymbol{g}(\boldsymbol{y}_n))\right)(h+c\Delta\widehat{W}_n)^2\\
&+(h+c\Delta\widehat{W}_n)^3\boldsymbol{R}.
\end{aligned}
\tag{31}
$$

Following this procedure, we can futher obtain the coefficient $\boldsymbol{c}(\boldsymbol{y}_n)$ of $(h+c\Delta\widehat{W}_n)^3$ in the expansion (20), which is a continuous function of $\boldsymbol{y}_n\in[l,L]^m$ and independent of $\boldsymbol{y}_{n+1}$ and $\Delta\widehat{W}_n$, and $\boldsymbol{R}_n=(h+c\Delta\widehat{W}_n)^4\boldsymbol{R}$. By Lemma 3.1 and the moment boundedness of $\boldsymbol{R}$ we have

$$
\begin{aligned}
E\left(\|\boldsymbol{R}_n\|\right)&=E\left(\left\|(h+c\Delta\widehat{W}_n)^4\boldsymbol{R}\right\|\right)=E\left((h+c\Delta\widehat{W}_n)^4\left\|\boldsymbol{R}\right\|\right)\\
&\le\left(E(h+c\Delta\widehat{W}_n)^8\right)^{\frac{1}{2}}\left(E\left\|\boldsymbol{R}\right\|^2\right)^{\frac{1}{2}}\le O(h^2),
\end{aligned}
$$

and

$$
\begin{aligned}
\left(E(\|\boldsymbol{R}_n\|^2)\right)^{\frac{1}{2}}&=\left(E\left((h+c\Delta\widehat{W}_n)^8\left\|\boldsymbol{R}\right\|^2\right)\right)^{\frac{1}{2}}\le\left(E(h+c\Delta\widehat{W}_n)^{16}\right)^{\frac{1}{4}}\left(E\left\|\boldsymbol{R}\right\|^4\right)^{\frac{1}{4}}\\
&\le O(h^2).
\end{aligned}
$$

□

We are now ready to prove the following result on the root mean-square convergence order of the method (9).

Theorem 3.1. *Under Assumption 2.1 and $\boldsymbol{y}_0>0$, the numerical scheme* (9) *applied to the system* (8) *is of root mean-square convergence order 1.*

Proof. The energy-preserving scheme proposed in [2] applied to the system (8) has the following form

$$
\tilde{\boldsymbol{y}}_{n+1}=\tilde{\boldsymbol{y}}_n+\boldsymbol{B}\left(\frac{\tilde{\boldsymbol{y}}_n+\tilde{\boldsymbol{y}}_{n+1}}{2}\right)\int_0^1\nabla H(\tilde{\boldsymbol{y}}_n+\tau(\tilde{\boldsymbol{y}}_{n+1}-\tilde{\boldsymbol{y}}_n))d\tau(h+c\Delta\widehat{W}_n).
\tag{32}
$$

According to Lemma 3.2 of [2], the scheme (32) has the expansion:

(33)
$$\tilde{\boldsymbol{y}}_{n+1} - \tilde{\boldsymbol{y}}_n = \boldsymbol{a}(\tilde{\boldsymbol{y}}_n)(h + c\Delta\widehat{W}_n) + \boldsymbol{b}(\tilde{\boldsymbol{y}}_n)(h + c\Delta\widehat{W}_n)^2 + \tilde{\boldsymbol{c}}(\boldsymbol{y}_n)(h + c\Delta\widehat{W}_n)^3 + \tilde{\boldsymbol{R}}_n,$$

where the functions $\boldsymbol{a}$ and $\boldsymbol{b}$ are the same with those in the expansion (20) for the scheme (9), $\tilde{\boldsymbol{c}}(\tilde{\boldsymbol{y}}_n)$ is a continuous function of $\tilde{\boldsymbol{y}}_n > 0$ independent of $\tilde{\boldsymbol{y}}_{n+1}$ and $\Delta\widehat{W}_n$, and $\tilde{\boldsymbol{R}}_n$ satisfies the following estimates:

$$E\left(\|\tilde{\boldsymbol{R}}_n\|\right) = O(h^2), \quad \left(E\left(\|\tilde{\boldsymbol{R}}_n\|^2\right)\right)^{\frac{1}{2}} = O(h^2). \tag{34}$$

Therefore, starting from the same point $\boldsymbol{y}_n = \tilde{\boldsymbol{y}}_n$, the local difference between $\tilde{\boldsymbol{y}}_{n+1}$ and $\boldsymbol{y}_{n+1}$ produced by the scheme (9) is

$$\boldsymbol{y}_{n+1} - \tilde{\boldsymbol{y}}_{n+1} = \left(\boldsymbol{c}(\boldsymbol{y}_n) - \tilde{\boldsymbol{c}}(\boldsymbol{y}_n)\right)(h + c\Delta\widehat{W}_n)^3 + (\boldsymbol{R}_n - \tilde{\boldsymbol{R}}_n). \tag{35}$$

By (21) and (34), we have

$$\|E(\boldsymbol{R}_n - \tilde{\boldsymbol{R}}_n)\| \le E\left(\|\boldsymbol{R}_n - \tilde{\boldsymbol{R}}_n\|\right) \le O(h^2),$$
$$E\|\boldsymbol{R}_n - \tilde{\boldsymbol{R}}_n\|^2 \le E\|\boldsymbol{R}_n\|^2 + E\|\tilde{\boldsymbol{R}}_n\|^2 = O(h^4).$$

Since $\boldsymbol{c}(\boldsymbol{y}_n)$ and $\tilde{\boldsymbol{c}}(\boldsymbol{y}_n)$ depend only on $\boldsymbol{y}_n (\in [l, L]^m a.s.)$ and are continuous on $[l, L]^m$, they are almost surely bounded such that $E(\|\boldsymbol{c}(\boldsymbol{y}_n) - \tilde{\boldsymbol{c}}(\boldsymbol{y}_n)\|) \le \tilde{M}_1$ with $\tilde{M}_1$ independent of h and n. Consequently,

$$\begin{aligned}\|E[(\boldsymbol{c}(\boldsymbol{y}_n) - \tilde{\boldsymbol{c}}(\boldsymbol{y}_n))(h + c\Delta\widehat{W}_n)^3]\| &= \|E[\boldsymbol{c}(\boldsymbol{y}_n) - \tilde{\boldsymbol{c}}(\boldsymbol{y}_n)]\|\, E(h + c\Delta\widehat{W}_n)^3 \\ &\le (h^3 + 3c^2h^2)E\|\boldsymbol{c}(\boldsymbol{y}_n) - \tilde{\boldsymbol{c}}(\boldsymbol{y}_n)\| \\ &= O(h^2).\end{aligned}$$

Therefore,

$$\begin{aligned}\|E(\boldsymbol{y}_{n+1} - \tilde{\boldsymbol{y}}_{n+1})\| \le& \|E[(\boldsymbol{c}(\boldsymbol{y}_n) - \tilde{\boldsymbol{c}}(\boldsymbol{y}_n))(h + c\Delta\widehat{W}_n)^3]\| + \|E(\boldsymbol{R}_n - \tilde{\boldsymbol{R}}_n)\| \\ \le& O(h^2). \end{aligned}\tag{36}$$

Moreover, $E\|\boldsymbol{c}(\boldsymbol{y}_n) - \boldsymbol{c}(\tilde{\boldsymbol{y}}_n)\|^2 \le \tilde{M}_2$ with a certain constant $\tilde{M}_2$ independent of h and n, due to the almost sure boundedness of $\boldsymbol{c}(\boldsymbol{y}_n)$ and $\boldsymbol{c}(\tilde{\boldsymbol{y}}_n)$ as discussed above. Thus we have

$$\begin{aligned}E\|\boldsymbol{y}_{n+1} - \tilde{\boldsymbol{y}}_{n+1}\|^2 &\le E\|(\boldsymbol{c}(\boldsymbol{y}_n) - \tilde{\boldsymbol{c}}(\boldsymbol{y}_n))(h + c\Delta\widehat{W}_n)^3\|^2 + E\|\boldsymbol{R}_n - \tilde{\boldsymbol{R}}_n\|^2 \\ &= E\|\boldsymbol{c}(\boldsymbol{y}_n) - \tilde{\boldsymbol{c}}(\boldsymbol{y}_n)\|^2(h + c\Delta\widehat{W}_n)^6 + E\|\boldsymbol{R}_n - \tilde{\boldsymbol{R}}_n\|^2 \\ &\le O(h^3),\end{aligned}\tag{37}$$

that is,

$$\left(E\|\boldsymbol{y}_{n+1} - \tilde{\boldsymbol{y}}_{n+1}\|^2\right)^{\frac{1}{2}} \le O(h^{\frac{3}{2}}). \tag{38}$$

It has been shown that the energy-preserving scheme (32) is of root mean-square order 1 ([2]). With the local difference between $\tilde{\boldsymbol{y}}_{n+1}$ and $\boldsymbol{y}_{n+1}$ given in (36) and (38), the Lemma 2.1 in [17] with $p_1 = 2$ and $p_2 = \frac{3}{2}$ then implies that the numerical scheme (9) also has root mean-square convergence order 1. □

Remark 3.1. Similar to the discussion in Remark 3.5 of [2], the validity of our result of Theorem 3.1 is based on the fact that, the coefficient $\boldsymbol{B}(\boldsymbol{y})\nabla H(\boldsymbol{y}) =: \boldsymbol{a}(\boldsymbol{y})$ of the system (8) is smooth such that all its derivatives are bounded on $[l, L]^m$ where the numerical solution $\{\boldsymbol{y}_n\}$ and the exact solution $\boldsymbol{y}(t_n)$ almost surely locate for all $n \in \mathbb{N}$, given $\boldsymbol{y}_0 > 0$ and the Assumption 2.1. Moreover, denoting the exact flow of the system (8) by $\tilde{\boldsymbol{\varphi}}_t : \boldsymbol{y} \to \tilde{\boldsymbol{\varphi}}_t(\boldsymbol{y})$, it is not difficult to see that also $\tilde{\boldsymbol{\varphi}}_t(\boldsymbol{y}_n)$ and $\tilde{\boldsymbol{\varphi}}_t(\boldsymbol{y}(t_n))$ belong to $[l, L]^m$ almost surely for all $t \geq 0$ and $n \in \mathbb{N}$, under the given $\boldsymbol{y}_0 > 0$ and the Assumption 2.1. Owing to these, the Lipschitz continuity of $\boldsymbol{a}(\boldsymbol{y})$ on $[l, L]^m$ can guarantee the applicability of Lemma 2.1 in [17] to our problem, similar to the case for Theorem 3.3 in [2].

4. The midpoint-related Poisson integrators

In this section we give an approach of constructing Poisson integrators for stochastic Poisson systems based on the midpoint scheme, and apply it to the stochastic Poisson system (8).

4.1. The midpoint scheme for stochastic Poisson systems with constant structure matrices.

For stochastic Poisson systems with constant skew-symmetric structure matrix $\bar{\boldsymbol{B}}$ defined by

$$d\boldsymbol{y}(t) = \bar{\boldsymbol{B}} \left(\nabla H_0(\boldsymbol{y}(t))dt + \sum_{r=1}^{s} \nabla H_r(\boldsymbol{y}(t)) \circ dW_r(t) \right), \tag{39}$$

we consider the following midpoint scheme:

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \bar{\boldsymbol{B}} \left(\nabla H_0 \left(\frac{\boldsymbol{y}_n + \boldsymbol{y}_{n+1}}{2} \right) h + \sum_{r=1}^{s} \nabla H_r \left(\frac{\boldsymbol{y}_n + \boldsymbol{y}_{n+1}}{2} \right) \Delta \widehat{W}_{rn} \right). \tag{40}$$

First we prove the following theorem.

Theorem 4.1. *The midpoint scheme* (40) *is a Poisson integrator for the stochastic Poisson system* (39).

Proof. Rewrite the midpoint scheme (40) to the following form:

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{y}_n, \boldsymbol{y}_{n+1}) :=& \boldsymbol{y}_{n+1} - \boldsymbol{y}_n - \bar{\boldsymbol{B}} \left(\nabla H_0 \left(\frac{\boldsymbol{y}_n + \boldsymbol{y}_{n+1}}{2} \right) h \right. \\ & \left. + \sum_{r=1}^{s} \nabla H_r \left(\frac{\boldsymbol{y}_n + \boldsymbol{y}_{n+1}}{2} \right) \Delta \widehat{W}_{rn} \right) \\ =& \mathbf{0}. \end{aligned} \tag{41}$$

By the implicit function theorem, we have

$$\frac{\partial \boldsymbol{y}_{n+1}}{\partial \boldsymbol{y}_n} = - \left(\frac{\partial \boldsymbol{u}(\boldsymbol{y}_n, \boldsymbol{y}_{n+1})}{\partial \boldsymbol{y}_{n+1}} \right)^{-1} \frac{\partial \boldsymbol{u}(\boldsymbol{y}_n, \boldsymbol{y}_{n+1})}{\partial \boldsymbol{y}_n}. \tag{42}$$

Thus, to prove $\frac{\partial \boldsymbol{y}_{n+1}}{\partial \boldsymbol{y}_n} \bar{\boldsymbol{B}} \frac{\partial \boldsymbol{y}_{n+1}}{\partial \boldsymbol{y}_n}^T = \bar{\boldsymbol{B}}$ is equivalent to show

$$\frac{\partial \boldsymbol{u}(\boldsymbol{y}_n, \boldsymbol{y}_{n+1})}{\partial \boldsymbol{y}_{n+1}} \bar{\boldsymbol{B}} \frac{\partial \boldsymbol{u}(\boldsymbol{y}_n, \boldsymbol{y}_{n+1})}{\partial \boldsymbol{y}_{n+1}}^T = \frac{\partial \boldsymbol{u}(\boldsymbol{y}_n, \boldsymbol{y}_{n+1})}{\partial \boldsymbol{y}_n} \bar{\boldsymbol{B}} \frac{\partial \boldsymbol{u}(\boldsymbol{y}_n, \boldsymbol{y}_{n+1})}{\partial \boldsymbol{y}_n}^T. \tag{43}$$

(41) implies

$$\begin{aligned} \frac{\partial \boldsymbol{u}(\boldsymbol{y}_n, \boldsymbol{y}_{n+1})}{\partial \boldsymbol{y}_n} &= -\boldsymbol{I} - \bar{\boldsymbol{B}} \left[\frac{h}{2} H_0^{''} \left(\frac{\boldsymbol{y}_n + \boldsymbol{y}_{n+1}}{2} \right) + \sum_{r=1}^{s} \frac{\Delta \widehat{W}_{rn}}{2} H_r^{''} \left(\frac{\boldsymbol{y}_n + \boldsymbol{y}_{n+1}}{2} \right) \right] \\ &= -\boldsymbol{I} - \frac{\bar{\boldsymbol{B}}}{2} \left(h\boldsymbol{Q}_0 + \sum_{r=1}^{s} \Delta \widehat{W}_{rn} \boldsymbol{Q}_r \right), \end{aligned} \tag{44}$$

where $\boldsymbol{Q}_0 := H_0''\left(\frac{\boldsymbol{y}_n+\boldsymbol{y}_{n+1}}{2}\right)$, $\boldsymbol{Q}_r := H_r''\left(\frac{\boldsymbol{y}_n+\boldsymbol{y}_{n+1}}{2}\right)$ are both symmetric matrices. Similarly,

$$\begin{aligned}\frac{\partial \boldsymbol{u}(\boldsymbol{y}_n,\boldsymbol{y}_{n+1})}{\partial \boldsymbol{y}_{n+1}} &= \boldsymbol{I} - \bar{\boldsymbol{B}}\left[\frac{h}{2}H_0''\left(\frac{\boldsymbol{y}_n+\boldsymbol{y}_{n+1}}{2}\right) + \sum_{r=1}^{s}\frac{\Delta\widehat{W}_{rn}}{2}H_r''\left(\frac{\boldsymbol{y}_n+\boldsymbol{y}_{n+1}}{2}\right)\right]\\ &= \boldsymbol{I} - \frac{\bar{\boldsymbol{B}}}{2}\left(h\boldsymbol{Q}_0 + \sum_{r=1}^{s}\Delta\widehat{W}_{rn}\boldsymbol{Q}_r\right).\end{aligned}\tag{45}$$

Then the transposes of (45) and (44) are

$$\frac{\partial \boldsymbol{u}(\boldsymbol{y}_n,\boldsymbol{y}_{n+1})}{\partial \boldsymbol{y}_n}^T = -\boldsymbol{I} + \frac{1}{2}\left(h\boldsymbol{Q}_0 + \sum_{r=1}^{s}\Delta\widehat{W}_{rn}\boldsymbol{Q}_r\right)\bar{\boldsymbol{B}}\tag{46}$$

and

$$\frac{\partial \boldsymbol{u}(\boldsymbol{y}_n,\boldsymbol{y}_{n+1})}{\partial \boldsymbol{y}_{n+1}}^T = \boldsymbol{I} + \frac{1}{2}\left(h\boldsymbol{Q}_0 + \sum_{r=1}^{s}\Delta\widehat{W}_{rn}\boldsymbol{Q}_r\right)\bar{\boldsymbol{B}},\tag{47}$$

respectively. By (44) and (46) we have

$$\begin{aligned}&\frac{\partial \boldsymbol{u}(\boldsymbol{y}_n,\boldsymbol{y}_{n+1})}{\partial \boldsymbol{y}_n}\bar{\boldsymbol{B}}\frac{\partial \boldsymbol{u}(\boldsymbol{y}_n,\boldsymbol{y}_{n+1})}{\partial \boldsymbol{y}_n}^T\\ &= \left[-\boldsymbol{I} - \frac{\bar{\boldsymbol{B}}}{2}\left(h\boldsymbol{Q}_0 + \sum_{r=1}^{s}\Delta\widehat{W}_{rn}\boldsymbol{Q}_r\right)\right]\bar{\boldsymbol{B}}\left[-\boldsymbol{I} + \frac{1}{2}\left(h\boldsymbol{Q}_0 + \sum_{r=1}^{s}\Delta\widehat{W}_{rn}\boldsymbol{Q}_r\right)\bar{\boldsymbol{B}}\right]\\ &= \bar{\boldsymbol{B}} - \frac{\bar{\boldsymbol{B}}}{4}\left(h\boldsymbol{Q}_0 + \sum_{r=1}^{s}\Delta\widehat{W}_{rn}\boldsymbol{Q}_r\right)\bar{\boldsymbol{B}}\left(h\boldsymbol{Q}_0 + \sum_{r=1}^{s}\Delta\widehat{W}_{rn}\boldsymbol{Q}_r\right)\bar{\boldsymbol{B}}.\end{aligned}$$

Similarly (45) and (47) give

$$\begin{aligned}&\frac{\partial \boldsymbol{u}(\boldsymbol{y}_n,\boldsymbol{y}_{n+1})}{\partial \boldsymbol{y}_{n+1}}\bar{\boldsymbol{B}}\frac{\partial \boldsymbol{u}(\boldsymbol{y}_n,\boldsymbol{y}_{n+1})}{\partial \boldsymbol{y}_{n+1}}^T\\ &= \bar{\boldsymbol{B}} - \frac{\bar{\boldsymbol{B}}}{4}\left(h\boldsymbol{Q}_0 + \sum_{r=1}^{s}\Delta\widehat{W}_{rn}\boldsymbol{Q}_r\right)\bar{\boldsymbol{B}}\left(h\boldsymbol{Q}_0 + \sum_{r=1}^{s}\Delta\widehat{W}_{rn}\boldsymbol{Q}_r\right)\bar{\boldsymbol{B}}.\end{aligned}$$

Thus (43) holds, which implies the midpoint scheme (40) preserves the Poisson structure of the stochastic Poisson system (39).

Meanwhile, if $C(\boldsymbol{y})$ is a Casimir function of the system (39), we have

$$\begin{aligned}C(\boldsymbol{y}_{n+1}) - C(\boldsymbol{y}_n) &= \nabla C(\boldsymbol{y}_n^*)^T\left(\boldsymbol{y}_{n+1} - \boldsymbol{y}_n\right)\\ &= \nabla C(\boldsymbol{y}_n^*)^T\bar{\boldsymbol{B}}\left(\nabla H_0(\frac{\boldsymbol{y}_n+\boldsymbol{y}_{n+1}}{2})h + \sum_{r=1}^{s}\nabla H_r(\frac{\boldsymbol{y}_n+\boldsymbol{y}_{n+1}}{2})\Delta\widehat{W}_n\right)\\ &= 0,\end{aligned}$$

since $\nabla C(\boldsymbol{y})^T\bar{\boldsymbol{B}} \equiv \boldsymbol{0}$ $(\forall \boldsymbol{y})$ according to the definition of Casimir functions, and $\boldsymbol{y}_n^* = \boldsymbol{y}_n + \theta(\boldsymbol{y}_{n+1} - \boldsymbol{y}_n)$ $(\theta \in (0,1))$. Then we can conclude that the midpoint scheme (40) is a Poisson integrator of the stochastic Poisson system (39). $\square$

4.2. Invariance of Poisson integrators under coordinate transformations. For the general stochastic Poisson system (1):

$$d\boldsymbol{y}(t)=\boldsymbol{B}(\boldsymbol{y}(t))\left(\nabla H_0(\boldsymbol{y}(t))dt+\sum_{r=1}^{s}\nabla H_r(\boldsymbol{y}(t))\circ dW_r(t)\right),$$

we consider an invertible coordinate transformation $\boldsymbol{x}=\boldsymbol{\phi}(\boldsymbol{y})$ which transforms (1) into the following system

$$d\boldsymbol{x}(t)=\tilde{\boldsymbol{B}}(\boldsymbol{x}(t))\left(\nabla K_0(\boldsymbol{x}(t))dt+\sum_{r=1}^{s}\nabla K_r(\boldsymbol{x}(t))\circ dW_r(t)\right), \tag{48}$$

where

$$\tilde{\boldsymbol{B}}(\boldsymbol{x})=\boldsymbol{\phi}^{'}\left(\boldsymbol{\phi}^{-1}(\boldsymbol{x})\right)\boldsymbol{B}\left(\boldsymbol{\phi}^{-1}(\boldsymbol{x})\right)\boldsymbol{\phi}^{'}\left(\boldsymbol{\phi}^{-1}(\boldsymbol{x})\right)^{T}, \tag{49}$$

and $K_r(\boldsymbol{x})=H_r(\boldsymbol{y})\,(r=0,\dots,s)$. Obviously $\tilde{\boldsymbol{B}}(x)$ is skew-symmetric. If in addition its elements $\tilde{b}_{ij}(\boldsymbol{x})$ satisfy the cyclic permutation property of summation analog to (2), then the system (48) is still a stochastic Poisson system.

Here we consider whether a numerical method $\{\boldsymbol{y}_n\}$ for (1) resulted from the inverse coordinate transformation $\boldsymbol{y}_n=\boldsymbol{\phi}^{-1}(\boldsymbol{x}_n)$ from a Poisson integrator $\{\boldsymbol{x}_n\}$ for (48) is still Poisson. In general we denote a stochastic numerical scheme for the system (48) by

$$\boldsymbol{0}=\boldsymbol{f}_{\omega}(\boldsymbol{x}_n,\boldsymbol{x}_{n+1},h):=\boldsymbol{f}(\boldsymbol{x}_n,\boldsymbol{x}_{n+1},h,\boldsymbol{\mathcal{W}}(\triangle_n,\omega)),$$

where $\boldsymbol{\mathcal{W}}=(W_1,\dots,W_s)$, $\triangle_n:=[t_n,t_{n+1}]$, $\omega\in\Omega$ and $\boldsymbol{\mathcal{W}}(\triangle_n,\omega):=\{\boldsymbol{\mathcal{W}}(t,\omega),t\in[t_n,t_{n+1}],\omega\in\Omega\}$. We have the following theorem.

Theorem 4.2. *Let $\boldsymbol{x}=\boldsymbol{\phi}(\boldsymbol{y})$ be an invertible coordinate transformation that transforms the system* (1) *to* (48)*. If the numerical scheme $\boldsymbol{f}_{\omega}(\boldsymbol{x}_n,\boldsymbol{x}_{n+1},h)=\boldsymbol{0}$ is a stochastic Poisson integrator for the system* (48)*, and $\boldsymbol{y}_n=\boldsymbol{\phi}^{-1}(\boldsymbol{x}_n)$, then the numerical scheme $\boldsymbol{f}_{\omega}(\boldsymbol{\phi}(\boldsymbol{y}_n),\boldsymbol{\phi}(\boldsymbol{y}_{n+1}),h)=\boldsymbol{0}$ is a stochastic Poisson integrator for the system* (1)*.*

Proof. By the Stratonovich chain rule and (49), we have

$$\begin{aligned}
&\frac{\partial \boldsymbol{f}_{\omega}\left(\boldsymbol{\phi}(\boldsymbol{y}_n),\boldsymbol{\phi}(\boldsymbol{y}_{n+1},h)\right)}{\partial \boldsymbol{y}_n}\boldsymbol{B}(\boldsymbol{y}_n)\frac{\partial \boldsymbol{f}_{\omega}\left(\boldsymbol{\phi}(\boldsymbol{y}_n),\boldsymbol{\phi}(\boldsymbol{y}_{n+1}),h\right)}{\partial \boldsymbol{y}_n}^{T}\\
=&\frac{\partial \boldsymbol{f}_{\omega}\left(\boldsymbol{\phi}(\boldsymbol{y}_n),\boldsymbol{\phi}(\boldsymbol{y}_{n+1}),h\right)}{\partial \boldsymbol{x}_n}\boldsymbol{\phi}^{'}(\boldsymbol{y}_n)\boldsymbol{B}(\boldsymbol{y}_n)\boldsymbol{\phi}^{'}(\boldsymbol{y}_n)^{T}\frac{\partial \boldsymbol{f}_{\omega}\left(\boldsymbol{\phi}(\boldsymbol{y}_n),\boldsymbol{\phi}(\boldsymbol{y}_{n+1}),h\right)}{\partial \boldsymbol{x}_n}^{T}\\
=&\frac{\partial \boldsymbol{f}_{\omega}(\boldsymbol{x}_n,\boldsymbol{x}_{n+1},h)}{\partial \boldsymbol{x}_n}\tilde{\boldsymbol{B}}(\boldsymbol{x}_n)\frac{\partial \boldsymbol{f}_{\omega}(\boldsymbol{x}_n,\boldsymbol{x}_{n+1},h)}{\partial \boldsymbol{x}_n}^{T},
\end{aligned} \tag{50}$$

and similarly

$$\begin{aligned}
&\frac{\partial \boldsymbol{f}_{\omega}\left(\boldsymbol{\phi}(\boldsymbol{y}_n),\boldsymbol{\phi}(\boldsymbol{y}_{n+1}),h\right)}{\partial \boldsymbol{y}_{n+1}}\boldsymbol{B}(\boldsymbol{y}_{n+1})\frac{\partial \boldsymbol{f}_{\omega}\left(\boldsymbol{\phi}(\boldsymbol{y}_n),\boldsymbol{\phi}(\boldsymbol{y}_{n+1}),h\right)}{\partial \boldsymbol{y}_{n+1}}^{T}\\
=&\frac{\partial \boldsymbol{f}_{\omega}(\boldsymbol{x}_n,\boldsymbol{x}_{n+1},h)}{\partial \boldsymbol{x}_{n+1}}\tilde{\boldsymbol{B}}(\boldsymbol{x}_{n+1})\frac{\partial \boldsymbol{f}_{\omega}(\boldsymbol{x}_n,\boldsymbol{x}_{n+1},h)}{\partial \boldsymbol{x}_{n+1}}^{T}.
\end{aligned} \tag{51}$$

Similar to the discussion by (42)–(43), the right-hand sides of (50) and (51) are equal since $\boldsymbol{f}_{\omega}(\boldsymbol{x}_n,\boldsymbol{x}_{n+1},h)=\boldsymbol{0}$ is a Poisson integrator. Thus the left-hand sides of (50) and (51) are equal which implies that the scheme $\boldsymbol{f}_{\omega}(\boldsymbol{\phi}(\boldsymbol{y}_n),\boldsymbol{\phi}(\boldsymbol{y}_{n+1}),h)=\boldsymbol{0}$ preserves the Poisson structure.

In addition, under the coordinate transformation $\boldsymbol{\phi}$ the Casimir function $C(\boldsymbol{y})$ of the system (1) can be written as $C(\boldsymbol{y}) = C(\boldsymbol{\phi}^{-1}(\boldsymbol{x})) =: \tilde{C}(\boldsymbol{x}) = \tilde{C}(\boldsymbol{\phi}(\boldsymbol{y}))$, and then $\nabla_{\boldsymbol{x}}\tilde{C}(\boldsymbol{\phi}(\boldsymbol{y})) = \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}^T \nabla_{\boldsymbol{y}} C(\boldsymbol{y})$. Due to (49), we have

$$\nabla_{\boldsymbol{x}}\tilde{C}(\boldsymbol{x})^T \tilde{\boldsymbol{B}}(\boldsymbol{x}) = \nabla_{\boldsymbol{y}} C(\boldsymbol{y})^T \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{y}} \boldsymbol{B}(\boldsymbol{y}) \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{y}}^T = \nabla_{\boldsymbol{y}} C(\boldsymbol{y})^T \boldsymbol{B}(\boldsymbol{y}) \boldsymbol{\phi}'(\boldsymbol{y}) = \mathbf{0}, \tag{52}$$

where the last step is because $C(\boldsymbol{y})$ is a Casimir function of (1). (52) indicates that $\tilde{C}(\boldsymbol{x})$ is a Casimir function of (48). Therefore, $\forall n \geq 0$,

$$C(\boldsymbol{y}_{n+1}) = \tilde{C}(\boldsymbol{x}_{n+1}) = \tilde{C}(\boldsymbol{x}_n) = C(\boldsymbol{y}_n), \tag{53}$$

which means that the scheme $\boldsymbol{f}_\omega(\boldsymbol{\phi}(\boldsymbol{y}_n), \boldsymbol{\phi}(\boldsymbol{y}_{n+1}), h) = \mathbf{0}$ preserves the Casimir functions of (1).

Preserving both the Poisson structure and the Casimir functions of (1), the scheme $\boldsymbol{f}_\omega(\boldsymbol{\phi}(\boldsymbol{y}_n), \boldsymbol{\phi}(\boldsymbol{y}_{n+1}), h) = \mathbf{0}$ is therefore a stochastic Poisson integrator of (1). □

4.3. Stochastic Poisson integrator for the system (8)**.** It has been proved ([27]) that under Assumption 2.1 and given $\boldsymbol{y}_0 > 0$, the solution $\boldsymbol{y}(t)$ of the system (8) is almost surely positive. By the invertible coordinate transformation

$$x^i = \ln y^i, \quad i = 1, \ldots, m, \tag{54}$$

the system (8) can be transformed to the following stochastic Poisson system with constant structure matrix:

$$d\boldsymbol{x}(t) = \boldsymbol{B_0} \nabla K(\boldsymbol{x}(t))(dt + c \circ dW(t)), \tag{55}$$

where $K(\boldsymbol{x}) = \sum_{i=1}^m \beta_i e^{x^i} - p_i x^i$, and $\boldsymbol{B_0}$ is just the skew-symmetric constant matrix $\boldsymbol{B_0}$ in (8).

Based on Theorem 4.1 and Theorem 4.2, we can apply the midpoint method to the system (55) to obtain the stochastic Poisson integrator $\{\boldsymbol{x}_n\}$ for it, and then let $\boldsymbol{y}_n = \mathbf{exp}(\boldsymbol{x}_n) := (\exp x_n^1, \ldots, \exp x_n^m)^T$ to get a stochastic Poisson integrator $\{\boldsymbol{y}_n\}$ for the system (8), which we call the transformed midpoint method (the TM method) for the system (8) in the following. We will illustrate this by numerical tests in the next section.

5. Numerical experiments

In this section, we demonstrate the numerical behavior of the energy-Casimir-preserving scheme (9) and the transformed midpoint method via numerical experiments on several models of the form (8).

5.1. The energy-Casimir-preserving method. We test the method on a three-dimensional model and a four-dimensional model.

5.1.1. A three-dimensional Lotka-Volterra model. Consider the stochastic Lotka-Volterra system ([2])

$$d\boldsymbol{y}(t) = \begin{pmatrix} 0 & vy^1y^2 & bvy^1y^3 \\ -vy^1y^2 & 0 & -y^2y^3 \\ -bvy^1y^3 & y^2y^3 & 0 \end{pmatrix} \nabla H(\boldsymbol{y}(t))(dt + c \circ dW(t)), \tag{56}$$

where $H(\boldsymbol{y}) = aby^1 + y^2 + \gamma \ln y^2 - ay^3 - \mu \ln y^3$, $\boldsymbol{B_0} = \begin{pmatrix} 0 & v & bv \\ -v & 0 & -1 \\ -bv & 1 & 0 \end{pmatrix}$. By simple calculation one can verify that $Ker\boldsymbol{B_0} = \{k\boldsymbol{\alpha} : k \in \mathbb{R}\}$, where $\boldsymbol{\alpha} = (-1/v, -b, 1)^T$.

Then a Casimir function of the system is $C(\boldsymbol{y}) = -1/v \ln y^1 - b \ln y^2 + \ln y^3$ for $\boldsymbol{y} \in \mathbb{R}^3_+$.

In the experiments, we take $a = -0.6$, $b = -1$, $c = 0.5$, $v = -0.5$, $\gamma = 1$, $\mu = 2$, $\boldsymbol{y}_0 = (2, 0.9, 1.5)^T$ and $h = 10^{-3}$. We use the energy-preserving method (32) ([2]) as one of the comparing schemes, which we call for brevity the EP method in the sequel.

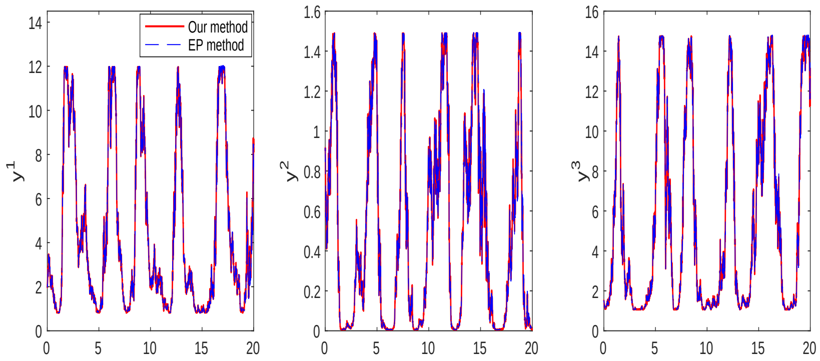
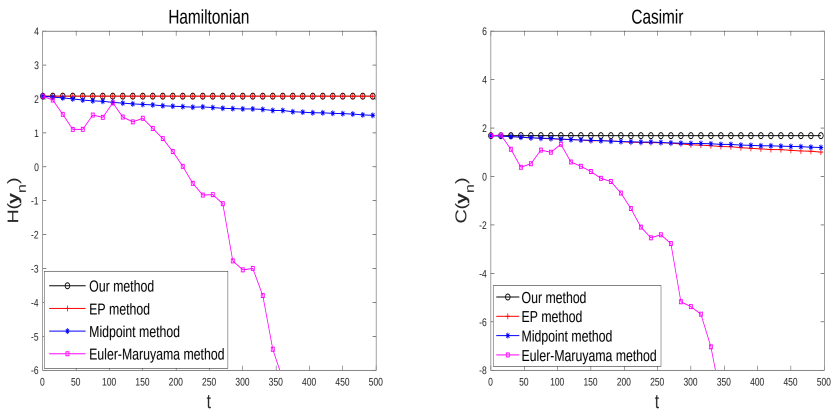


FIGURE 1. Sample trajectories of $y^i(t)$ $(i = 1, 2, 3)$ produced by our method and the EP method.

Figure 1 illustrates one sample path of y^1, y^2 and y^3 produced by our method (9) and the EP method (32), respectively. Clearly, numerical solutions from both methods are positive and bounded on the observation time interval.



(a) Energy (b) Casimir

FIGURE 2. Preservation of the energy and the Casimir function by our method, the EP method, the midpoint method, and the Euler-Maruyama method.

For the energy ($H(\boldsymbol{y})$) and Casimir ($C(\boldsymbol{y})$)-preservation, we compare our method (9) with the EP method (32), the midpoint method ([18]):

$$\overline{\boldsymbol{y}}_{n+1} = \overline{\boldsymbol{y}}_n + \boldsymbol{B}\left(\frac{\overline{\boldsymbol{y}}_n + \overline{\boldsymbol{y}}_{n+1}}{2}\right) \nabla H \left(\frac{\overline{\boldsymbol{y}}_n + \overline{\boldsymbol{y}}_{n+1}}{2}\right)(h + c\Delta \widehat{W}_n), \tag{57}$$

and the Euler-Maruyama method ([11]):

(58)

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\left((\boldsymbol{B}\nabla H) + \frac{c^2}{2}\frac{\partial(\boldsymbol{B}\nabla H)}{\partial \boldsymbol{y}}(\boldsymbol{B}\nabla H)\right)(\boldsymbol{y}_n) + c(\boldsymbol{B}\nabla H)(\boldsymbol{y}_n)\Delta W_n.$$

From Figure 2, it is clear that our method and the EP method can preserve the energy, while the other two methods can not. For the Casimir function, only our method is Casimir-preserving, and the other three methods all fail to preserve the Casimir function.

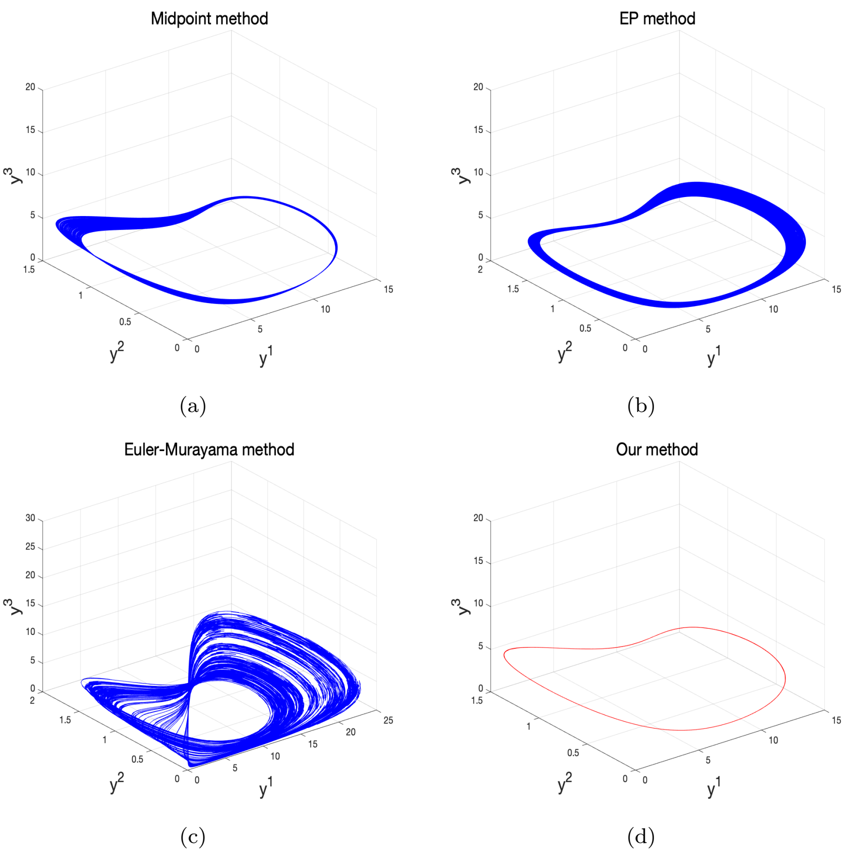
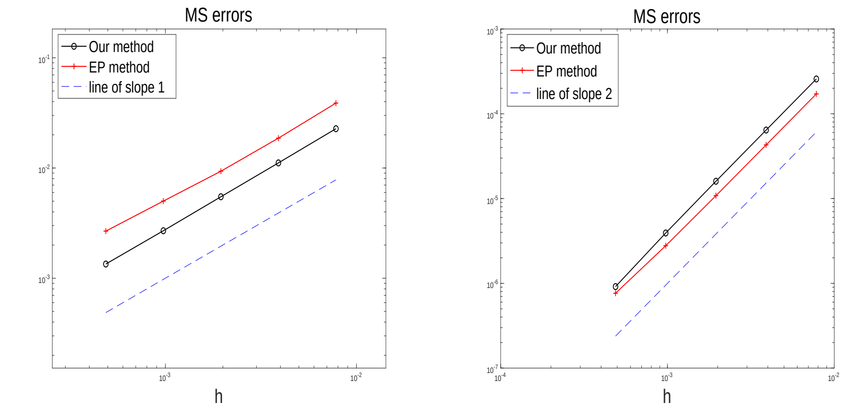


FIGURE 3. The phase orbits produced by the four methods.

Given $\boldsymbol{y}_0 \in \mathbb{R}^3_+$, the manifold $\{\boldsymbol{y} \in \mathbb{R}^3 : H(\boldsymbol{y}) = H(\boldsymbol{y}_0), C(\boldsymbol{y}) = C(\boldsymbol{y}_0)\}$ should be a curve in $\mathbb{R}^3$. Figure 3 illustrates one sample phase orbit created by our method, the EP method, the midpoint method and the Euler-Maruyama method, respectively, on the time interval $t \in [0, 500]$. One can see that only the panel (d) arising from our method is a curve, which demonstrates that our method preserves both $H(\boldsymbol{y})$ and $C(\boldsymbol{y})$.

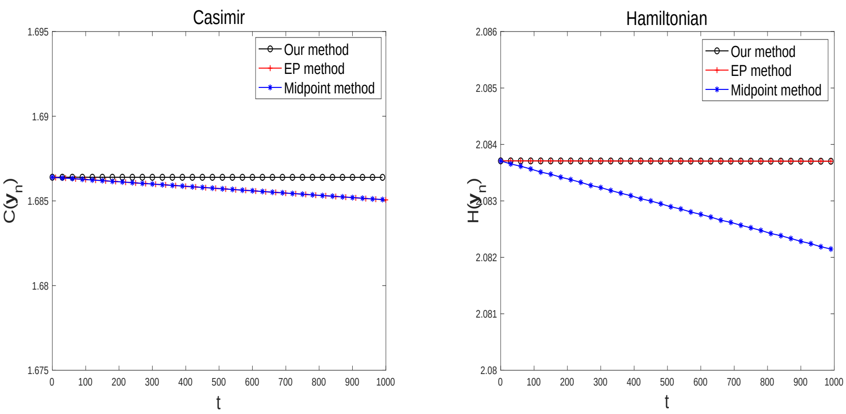
Figure 4 panel (a) is the "loglog"-plotting of the root mean-square error against h for our method and the EP method when $c = 0.5$, which indicates that both methods are of root mean-square convergence order 1. Here $h = [2^{-11}, 2^{-10}, 2^{-9}, 2^{-8}, 2^{-7}]$, and 1000 samples are taken to approximate the expectation. The exact solution is simulated by the midpoint method with the tiny time step $h = 2^{-12}$.



(a) c=0.5 (b) c=0

FIGURE 4. Root mean-square orders of our method and the EP method.

When $c = 0$, the system (56) degenerates to a three-dimensional deterministic Lotka-Volterra system studied in [20], where the Hamiltonian $H(\boldsymbol{y})$ and the Casimir function $C(\boldsymbol{y})$ are still invariants of the system. Our method (9), the EP method (32) as well as the midpoint method (57) become deterministic solvers for the system. Figure 5 illustrates the evolution of the Casimir function $C(\boldsymbol{y})$ and the



(a) Casimir (b) Energy

FIGURE 5. Casimir and energy evolution by our method, the EP method and the midpoint method when $c = 0$.

energy $H(\boldsymbol{y})$ produced by our method, the EP method and the midpoint method. It can be seen that our method preserves both the Casimir function and the energy, while the EP method fails to preserve the Casimir function, and the midpoint method preserves neither the Casimir nor the energy. Parameters here take the same values with those for Figure 2.

The "loglog" graph of the errors against the time steps h in panel (b) of Figure 4 shows that both our method and the EP method are of convergence order 2 when

$c = 0$. Other parameters and settings for producing the figure here are the same with those for panel (a) of Figure 4.

5.1.2. A four-dimensional system. Let us consider the following four-dimensional stochastic Poisson system:

$$d\boldsymbol{y}(t) = \begin{pmatrix} 0 & -y^1y^2 & -2y^1y^3 & -2y^1y^4 \\ y^1y^2 & 0 & 2y^2y^3 & 4y^2y^4 \\ 2y^1y^3 & -2y^2y^3 & 0 & 4y^3y^4 \\ 2y^1y^4 & -4y^2y^4 & -4y^3y^4 & 0 \end{pmatrix} \nabla H(\boldsymbol{y}(t))(dt + c \circ dW(t)), \tag{59}$$

where $\boldsymbol{y} = (y^1, y^2, y^3, y^4)^T$, $H(\boldsymbol{y}) = \beta_1 y^1 - p_1 \ln y^1 + \beta_2 y^2 - p_2 \ln y^2 + \beta_3 y^3 - p_3 \ln y^3 + \beta_4 y^4 - p_4 \ln y^4$, and $\boldsymbol{B_0} = \begin{pmatrix} 0 & -1 & -2 & -2 \\ 1 & 0 & 2 & 4 \\ 2 & -2 & 0 & 4 \\ 2 & -4 & -4 & 0 \end{pmatrix}$. One can verify that $C_1(\boldsymbol{y}) = -4\ln y^1 - 2\ln y^2 + \ln y^4$ and $C_2(\boldsymbol{y}) = -2\ln y^1 - 2\ln y^2 + \ln y^3$ are two different Casimir functions of the system (59). In the experiments, we set the parameters $\beta_1 = 0.4$, $p_1 = 0$, $\beta_2 = 1$, $p_2 = 1$, $\beta_3 = 0.6$, $p_3 = 0.5$, $\beta_4 = 1$, $p_4 = 2$ which satisfy the Assumption 2.1, $\boldsymbol{y}_0 = (1, 1.5, 0.5, 0.8)^T$ and $c = 0.3$.

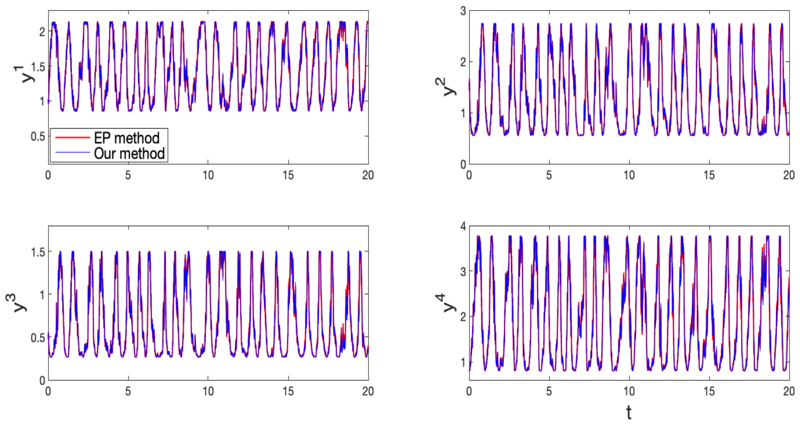
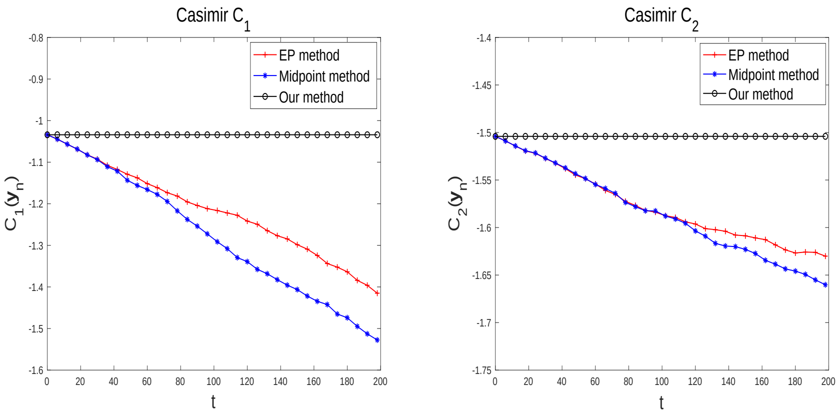


FIGURE 6. One sample trajectory of y^i $(i = 1, 2, 3, 4)$ produced by our method and the EP method.

From Figure 6 one can see that both our method and the EP method produce positive and bounded sample trajectories.

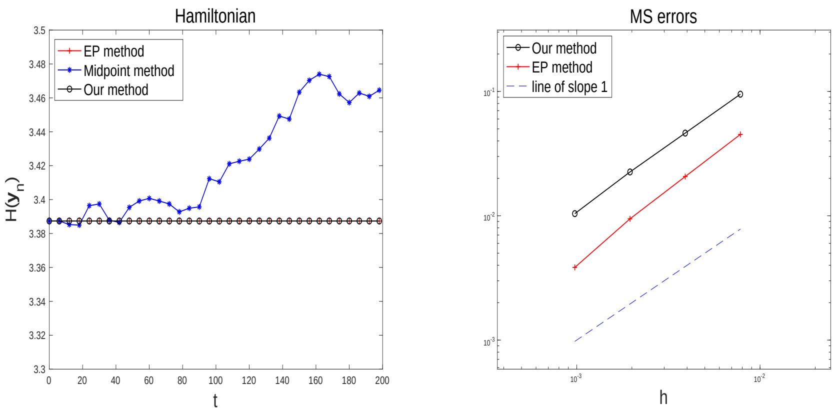
Figure 7 shows the evolution of the two Casimir functions arising from our method, the EP method and the midpoint method, respectively, from which we can see that our method preserves both Casimir functions while the other two methods can not preserve the Casimir functions.

Panel (a) of Figure 8 is the evolution of the energy $H(\boldsymbol{y})$ resulted from our method, the EP method and the midpoint method. Clearly our method and the EP method preserve the energy, and the midpoint method does not preserve the energy. Panel (b) is the "loglog" graph of the root mean-square errors against the time step times h for our method and the EP method. It can be seen that both methods are of root mean-square convergence order 1. Here for panel (b), $h = [2^{-10}, 2^{-9}, 2^{-8}, 2^{-7}]$, 1000 samples are taken approximating the expectation,



(a) C_1 (b) C_2

FIGURE 7. Casimirs evolution by our method, the EP method and the midpoint method.



(a) Energy (b) Root mean-square orders

FIGURE 8. (a) Evolution of the energy by our method, the EP method and the midpoint method; (b) Root mean-square orders of our method and the EP method.

and the exact solution is simulated by the midpoint method with the tiny time step $h = 2^{-12}$.

5.2. The transformed midpoint method. We still consider the stochastic Lotka-Volterra system (56), which can be written more explicitly as

$$\begin{pmatrix} dy^1 \\ dy^2 \\ dy^3 \end{pmatrix} = \begin{pmatrix} 0 & vy^1y^2 & bvy^1y^3 \\ -vy^1y^2 & 0 & -y^2y^3 \\ -bvy^1y^3 & y^2y^3 & 0 \end{pmatrix} \begin{pmatrix} ab \\ 1 + \frac{\gamma}{y^2} \\ -a - \frac{\mu}{y^3} \end{pmatrix} (dt + c \circ dW(t)) . \tag{60}$$

By the inverse coordinate transformation $x^i = \ln y^i$ $(i = 1, 2, 3)$, it can be transformed to the following stochastic Poisson system with constant structure matrix:

$$\begin{pmatrix} dx^1 \\ dx^2 \\ dx^3 \end{pmatrix} = \begin{pmatrix} 0 & v & bv \\ -v & 0 & -1 \\ -bv & 1 & 0 \end{pmatrix} \begin{pmatrix} abe^{x^1} \\ \gamma + e^{x^2} \\ -\mu - ae^{x^3} \end{pmatrix} (dt + c \circ dW(t)), \tag{61}$$

where the constant structure matrix $\boldsymbol{B_0} = \begin{pmatrix} 0 & v & bv \\ -v & 0 & -1 \\ -bv & 1 & 0 \end{pmatrix}$, and the Hamiltonian $K(x) = abe^{x^1} + \gamma x^2 + e^{x^2} - \mu x^3 - ae^{x^3}$.

Applying the midpoint method to the system (61) we have

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \boldsymbol{B_0} \nabla K \left(\frac{\boldsymbol{x}_n + \boldsymbol{x}_{n+1}}{2} \right) \left(h + c\Delta \widehat{W}_n \right). \tag{62}$$

According to Theorem 4.1, (62) is a stochastic Poisson integrator for the system (61), and by Theorem 4.2 the integrator transformed from (62) by the inverse coordinate transformation

$$y_n^i = \exp(x_n^i) \quad i = 1, 2, 3, \ \ n \in \mathbb{N} \tag{63}$$

is a stochastic Poisson integrator for the system (60). In the following we call the integrator (62)–(63) the transformed midpoint (TM) method for the system (60).

Next we perform numerical tests to demonstrate the behavior of the TM method and compare it with the midpoint method:

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + (\boldsymbol{B} \nabla H) \left(\frac{\boldsymbol{y}_n + \boldsymbol{y}_{n+1}}{2} \right) \left(h + c\Delta \widehat{W}_n \right). \tag{64}$$

We set the parameters $a = -0.6$, $b = -1$, $c = 0.3$, $v = -0.5$, $\gamma = 1$, $\mu = 2$, and $\boldsymbol{y}_0 = (2, 0.9, 1.5)^T$, $h = 10^{-3}$.

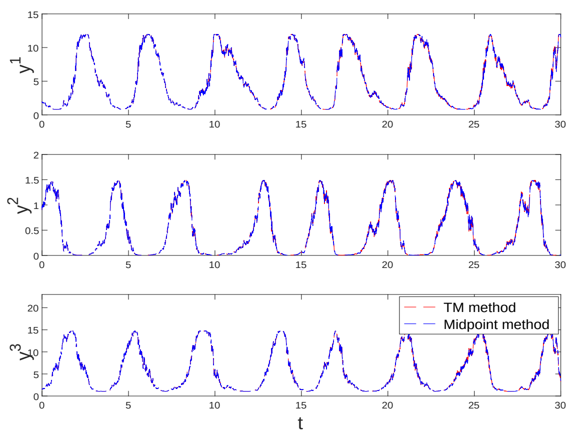
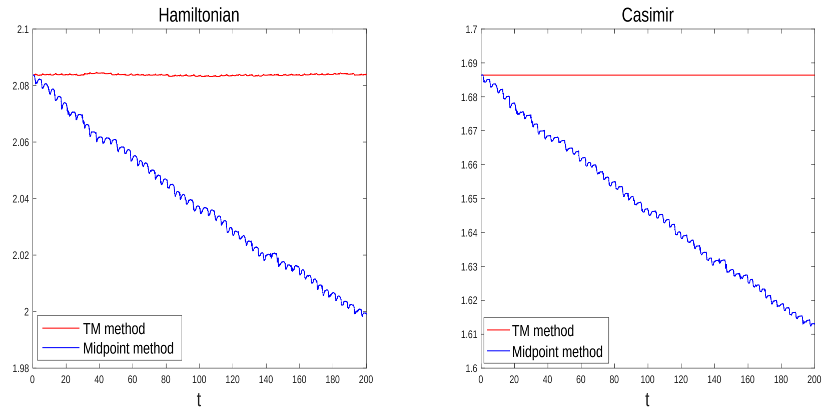


FIGURE 9. A sample trajectory of y^i $(i = 1, 2, 3)$ produced by the TM method and the midpoint method.

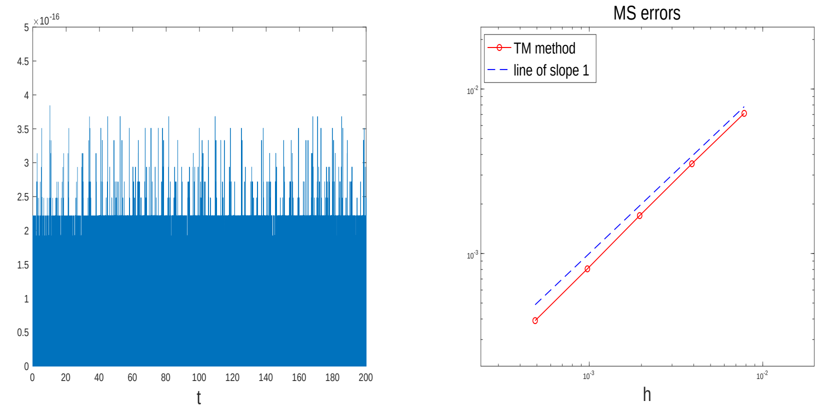
Figure 9 shows that the TM method and the midpoint method produce positive and bounded samples trajectories.



(a) Energy (b) Casimir

FIGURE 10. Hamiltonian and Casimir evolution by the TM method and the midpoint method.

Panel (a) of Figure 10 is the evolution of the energy (Hamiltonian) created by the TM method and the midpoint method, and Panel (b) is that of the Casimir function produced by the two methods. One can see that the TM method nearly preserves the energy and exactly inherits the Casimir function, while the midpoint method preserves neither the energy nor the Casimir function.



(a) Evolution of the structure error (b) Root mean-square order of the TM method

FIGURE 11. Poisson structure preservation and the root mean-square order of the TM method.

Panel (a) of Figure 11 shows the evolution of the Poisson structure error $\|\frac{\partial \boldsymbol{y}_{n+1}}{\partial \boldsymbol{y}_n}\boldsymbol{B}(\boldsymbol{y}_n)\frac{\partial \boldsymbol{y}_{n+1}}{\partial \boldsymbol{y}_n}^T - \boldsymbol{B}(\boldsymbol{y}_{n+1})\|_F$ arising from the TM method, where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix. It can be seen that the error is within the machine accuracy, indicating that the TM method preserves the Poisson structure. Panel (b) illustrates that the root mean-square convergence order of the TM method is 1, where time steps take the same values with those for panel (a) of Figure 4.

The exact solution is simulated by the midpoint method with time step 2^{-12}, and 1000 samples are taken to approximate the expectation.

6. Conclusion

The stochastic Poisson systems (SPSs) under consideration are stochastic extensions of Lotka-Volterra systems. For these systems, we proposed a class of energy-Casimir-preserving methods and analyzed the structure-preserving properties of the methods. We also proved that the convergence order of the methods is one in the root mean square sense. In addition, we proposed stochastic Poisson integrators based on midpoint method for systems which can be transformed to SPSs with constant structure matrices by invertible coordinate transformations. Numerical tests illustrate the numerical behavior of the proposed methods, and show validity of the theoretical results.

Acknowledgements

The first and second authors are supported by the National Natural Science Foundation of China (No. 11971458, No. 11471310). The third author is supported by the U.S. Department of Energy under the grant number DE-SC0022253.

References

[1] M. Beck and M.J. Gander. On the positivity of Poisson integrators for the Lotka-Volterra equations. BIT, 55:319–340, 2014.
[2] D. Cohen and G. Dujardin. Energy-preserving integrators for stochastic Poisson systems. Commun. Math. Sci., 12:1523–1539, 2014.
[3] D. Cohen and E. Hairer. Linear energy-preserving integrators for Poisson systems. BIT, 51:91–101, 2011.
[4] F. Diele and C. Marangi. Positive symplectic integrators for Predator-Prey dynamics. Discrete Cont. Dyn.-A, 23:2661–2678, 2018.
[5] K. Feng and D. L. Wang. Symplectic difference schemes for Hamiltonian systems in general symplectic structure. J. Comput. Math., 9:86–96, 1991.
[6] O. Gonzalez. Time integration and discrete Hamiltonian systems. J. Nonlinear Sci., 6:449–467, 1996.
[7] E. Hairer, C. Lubich, and G. Wanner. Geometric numerical integration: Structure-preserving algorithms for ordinary differential equations. Springer Science & Business Media, Berlin, 2nd edition, 2006.
[8] J.L. Hong, L.H. Ji, and X. Wang. Stochastic K-symplectic integrators for stochastic non-canonical Hamiltonian systems and applications to the Lotka-Volterra model. arXiv:1711.03258v1 [math.NA] 9 Nov 2017.
[9] J.L. Hong, J.L. Ruan, L.Y. Sun, and L.J. Wang. Structure-preserving numerical methods for stochastic Poisson systems. Commun. Comput. Phys., 29:802–830, 2021.
[10] B. Karasözen. Poisson integrators. Math. Comput. Model., 40:1225–1244, 2004.
[11] P.E. Klöden and E. Platen. Numerical solution of stochastic differential equations. Stochastic Modelling and Applied Probability. Springer, 1995.
[12] X.Y. Li, Q. Ma, and X.H. Ding. High-order energy-preserving methods for stochastic Poisson systems. East. Asia. J. Appl. Math., 9:465–484, 2019.
[13] S. Lie. Zur Theorie der Transformationsgruppen. Christ. Forh. Aar. 1888, Nr. 13, Christiania 1888; Gesammelte Abh., 5:553–557, 1888.
[14] X.R. Mao, G. Marion, and E. Renshaw. Environmental Brownian noise suppresses explosions in population dynamics. Stoch. Proc. Appl., 97:95–110, 2002.
[15] R.I. McLachlan. Explicit Lie-Poisson integration and the Euler equations. Phys. Rev. Lett., 71:3043–3046, 1993.
[16] R.I. McLachlan and G.R.W. Quispel. Splitting methods. Acta Numer., 11:341–434, 2002.
[17] G.N. Milstein, Y.M. Repin, and M.V. Tretyakov. Mean-square symplectic methods for Hamiltonian systems with multiplicative noise. WIAS preprint No. 670, 2001.
[18] G.N. Milstein, Y.M. Repin, and M.V. Tretyakov. Numerical methods for stochastic systems preserving symplectic structure. SIAM J. Numer. Anal., 40:1583–1604, 2002.

[19] G.N. Milstein, Y.M. Repin, and M.V. Tretyakov. Symplectic integration of Hamiltonian systems with additive noise. SIAM J. Numer. Anal., 39:2066–2088, 2002.
[20] M. Plank. Hamiltonian structures for the n-dimensional Lotka-Volterra equations. J. Math. Phys., 36:3520–3534, 1995.
[21] G.R.W. Quispel and D.I. Mclaren. A new class of energy-preserving numerical integration methods. J. Phys. A: Math. Theor., 41:045206, 2008.
[22] S. Reich. Numerical integration of the generalized Euler equations. Techn. Report 93-20, Dept. Comput. Sci., Univ. of British Columbia, 1993.
[23] J.M. Sanz-Serna. An unconventional symplectic integrator of W. Kahan. Appl. Numer. Math., 16:245–250, 1994.
[24] Y.F. Tang, V.M. Pérez-García, and L. Vázquez. Symplectic methods for the Ablowitz–Ladik model. Appl. Math. Comput., 82:17–38, 1997.
[25] L.J. Wang, P.J. Wang, and Y.Z. Cao. Numerical methods preserving multiple Hamiltonians for stochastic Poisson systems. Discrete Cont. Dyn.-S, doi: 10.3934/dcdss.2021095, 2021.
[26] P.J. Wang and L.J. Wang. Stochastic Poisson integrators based on Padé approximations for linear stochastic Poisson systems. J. U. Chinese Acad. Sci., 38(2):160–170, 2021.
[27] Y.C. Wang and L.J. Wang. Analysis of the solutions of a class of stochastic Poisson systems. J. U. Chinese Acad. Sci., accepted, 2020.
[28] R.L. Zhang, J. Liu, Y.F. Tang, H. Qin, and B.B. Zhu. Canonicalization and symplectic simulation of the gyrocenter dynamics in time-independent magnetic fields. Phys. Plasmas, 21:032504, 2014.
[29] G. Zhong and J.E. Marsden. Lie-Poisson Hamilton-Jacobi theory and Lie-Poisson integrators. Phys. Lett. A, 133(3):134–139, 1988.
[30] B.B. Zhu, R.L. Zhang, Y.F. Tang, X.B. Tu, and Y. Zhao. Splitting K-symplectic methods for non-canonical separable Hamiltonian problems. J. Comput. Phys., 322:387–399, 2016.
[31] W.J. Zhu and M.Z. Qin. Poisson schemes for Hamiltonian systems on Poisson manifolds. Comput. Math. Appl., 27(12):7–16, 1994.

School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, 100049, China
E-mail: wangyuchao18@mails.ucas.ac.cn and ljwang@ucas.ac.cn

Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849, USA
E-mail: yzc0009@auburn.edu