INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 19, Number 1, Pages 52–84

APPROXIMATE SOLUTION OF A NONLINEAR FRACTIONAL-ORDER HIV MODEL USING HOMOTOPY ANALYSIS METHOD

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Abstract. In the present paper, we propose and analyze a nonlinear fractional-order SEIR (susceptible-exposed-infected-recovered) epidemic model to transmit HIV. The fixed points of the model and their stability results are obtained. Using the fractional derivatives, we relied on the Caputo fractional derivative. Further, we employed the homotopy analysis method (HAM) to get an approximate solution of the dynamic fractional derivatives of the model. The purpose of using HAM as a solution technique is its reliability, easy to handle, that utilizes a simple process to adjust and control the convergence region of the obtained infinite series solution. It uses an auxiliary parameter and allows to obtain a one-parametric family of explicit series solutions. Firstly, several $\hbar$-curves are plotted to demonstrate the regions of convergence, then the residual and square residual errors are obtained for different values of these regions. In the end, numerical solutions are presented for various iterations to show the accuracy of the HAM. Besides, the convergence theorem of HAM is also proved. The obtained results show the effectiveness and strength of the applied HAM on the proposed fractional-order SEIR model. Also, from the sensitivity analysis results, it is seen that the parameters μ and σ are more sensitive than ϵ and ρ in disease transmission.

Key words. SEIR epidemic model, Caputo fractional derivative, Homotopy analysis method, Stability analysis, Basic reproduction number $\mathfrak{R}_0$.

1. Introduction

The increased logicality of modern computer capability has enhanced the prospect of mathematical modeling extraordinarily makes it possible to study very complex systems in a better way. Epidemiology in medicine deals with infectious and non-infectious diseases for their incidence, distribution, and possible control, and other factors relating to health. Initially, the branch was limited to infectious diseases, but nowadays, it finds applications to other diseases. HIV is the virus that causes destruction of the immune system by lowering down the $CD4^+$ T-cells that fight infection and makes the person host for many diseases that causes death. HIV has become the first global pandemic incurable due to the nonavailability of possible vaccines and is one of the major public health problems in the world today [1, 2, 3]. HIV will not survive outside the body, so the infection cannot be transmitted through daily activities like hugging an infected person, greeting by shaking hands, or kissing. This disease is transmitted via contaminated body fluids, including blood, semen, and vaginal secretions and through sexual intercourse (anal, vaginal, or oral) or by the use of the same needles among drug addicts [4, 5, 6, 7, 8, 9, 10]. Thus, HIV has very different characteristics. It is important to refine the basic ideas and extend the available literature models for a better understanding of this virus.

Infectious disease modeling has become an area of much attentiveness in recent years. Such accomplishments are useful to biomedical scientists for the prevention

Received by the editors November 3, 2020 and, in revised form, 8 December 2021.
2000 *Mathematics Subject Classification.* 26A33, 34D20, 37M05, 37N25, 65L20, 92B05, 93A30.
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and control of disease outbreaks. In recent years, many fractional-order epidemic models have been proposed to describe the dynamics of various infectious diseases. Kumar and Kumar [11] introduced a fractional-order SIR model with a constant vaccination rate. By their analysis, they have shown that the model has two equilibria, namely disease-free equilibrium and the endemic equilibrium. They have analyzed their model for local stability. Obtained results showed the effectiveness and reliability of their applied method through the numerical procedure. Wiah et al. [12] developed a fractional SIRC model, in which they presented a detailed analysis of the two existing equilibrium points. Firstly, they have shown the positive solution of their model in fractional order. They used the multi-step generalized differential transform method to obtain the approximate numerical solution. Finally, they compared their numerical results with a nonstandard numerical method and fourth-order Runge-Kutta method for accuracy. Kheiri and Jafari [13] formulated a multi-patch HIV/AIDS epidemic model with fractional order derivative to investigate the effect of human movement on the spread of HIV/AIDS among patches. They derived the basic reproduction number $\mathfrak{R}_0$ and proved that if $\mathfrak{R}_0 < 1$, the disease-free equilibrium (DFE) is locally and globally asymptotically stable. In the case of $\mathfrak{R}_0 > 1$, they obtained sufficient conditions under which the endemic equilibrium is unique and globally asymptotically stable. They also formulated a fractional optimal control problem, in which the state and co-state equations were given in terms of the left fractional derivatives. The necessary conditions for fractional optimal control of the disease were obtained. Their numerical results show that implementing all the control efforts decreases HIV-infected and AIDS people in both patches significantly. Silva and Torres [14], in their paper, proposed and studied the local and uniform stability of a fractional HIV/AIDS model. They also carried out numerical simulations to illustrate their theoretical results. Naik et al. [15] proposed and analyzed a nonlinear fractional-order epidemic model for HIV transmission with two infectious stages. In their study, they took the Caputo type fractional derivative and generalized Adams-Bashforth-Moulton method for the numerical solution of the model. They also determined the model equilibria and studied their stability results. They also formulated a fractional optimality condition for their proposed model. The effectiveness of the used control strategies is shown through numerical simulations, which suggested the adopted control measures efficiently increase the life cycle of the HIV patients.

Recently, Ali et al. [16], in their manuscript, proposed a SIATR compartmental model for HIV/AIDS epidemics under fractal-fractional-order derivative. They constructed the existence theory utilizing Schaefer- and Banach-type fixed point theorems to solve their considered model. Besides, Ulam-Hyers and generalized Ulam-Hyers stability conditions via nonlinear functional analysis were established. A fractional Adams-Bashforth method based on two-step Lagrange polynomial is employed for numerical simulation of the considered model. They tested their simulated results for various fractal-fractional orders on some existing real data of disease spread in South Africa and shows that the values of compartments SIAT decrease as the treatment starts. Tamilalagan et al. [17] proposed an article in which they extended a classical HIV infection model to the fractional-order case under the influence of antibody and cytotoxic T-Lymphocyte (CTL) immune responses. They studied the effectiveness of antiretroviral therapy drugs, namely, protease inhibitors and reverse transcriptase, in suppressing HIV infection. They first discussed the dynamical behavior of their model through stationary states' linear stability and then figured out the stable regions of the infectious and infection-free steady states.

They compared the results of integer order and fractional order of their model for better understanding the role played by fractional-order derivatives in HIV infection.

Leibniz was the first who introduced fractional calculus or fractional derivative in 1695, represents a calculus that extends the classical operators of differentiation and integration to non-integer and complex order. Many epidemic models with fractional derivatives have been used to deal with some epidemic behaviors [18, 19, 20, 21, 22, 23]. The main advantage of the fractional-order differential equation is the involvement of memory and hereditary properties which are absent in the integer-order models, where such effects are neglected or difficult to incorporate. Also, it helps to reduce errors arising from the neglected parameters in modeling real-life problems. The fractional calculus is not limited to any particular branch but finds applications in several fields [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37]. However, development still needs to be achieved before the ordinary derivatives could be interpreted as a subset of the fractional derivatives.

HAM as a solution technique has been used in recent years to solve various nonlinear system of differential equations in mathematics as well as in sciences [38, 39, 40, 41, 42, 43, 44] and nonlinear fractional-order equations [45, 46]. The HAM was first developed by Liao [47, 48]. It allows for fine-tuning of convergence region and rate of convergence by allowing an auxiliary parameter to vary. Compared to the homotopy perturbation method [49], the homotopy series solution will be convergent by considering two factors: the auxiliary linear operator and the initial guess. This paper aims to introduce and apply an effective method, so-called HAM, to obtain a convergent series solution of a nonlinear fractional-order HIV epidemic model. The fractional-order models possess memory, gives us a more realistic way to model HIV/AIDS epidemics. The memory property of the fractional models allows the integration of more information from the past, which translates into more accurate predictions for the model. With respect to the HIV epidemics, this memory property may be used to devise adequate therapeutics directed to each individual since distinct patients present different disease progression routes. The latter is associated with age, the status of the immune system, and genetic profile. Clinicians can, thus, use the information (in terms of behavior predictions) of fractional-order systems to fit patients' data with the most appropriate non-integer-order index. The purpose of considering such a study in fractional-order derivatives is that there is no such HIV model with fractional-order derivatives solved by HAM, as this method contains an auxiliary parameter $\hbar$ that can be used to adjust the convergence region of the obtained series solutions. Besides this, the convergence theorem of HAM is also proved in the paper that provides the HAM solutions are convergent, which differs the current paper from the available literature on HIV and increases the novelty of the paper.

The rest of the paper is decorated as follows. After the introduction in Section 1, Section 2 presents some preliminary results required to formulate the mathematical model. Section 3 gives the formulation of the model and its well-posedness. In Section 4, we discuss the mathematical analysis of the proposed fractional-order HIV epidemic model along with equilibrium points and the stability of equilibrium points. Also, in Section 4, Routh-Hurwitz stability conditions for the fractional system are discussed. In Section 5, a sensitivity analysis of the parameters involved in the threshold parameter is discussed. Furthermore, in Section 6, the HAM application is performed on the proposed fractional-order HIV epidemic model, and the numerical simulations are done to validate the analytical studies. In this

section, we also prove the convergence theorem of HAM. In Section 7, numerical results are given to illustrate the capability of HAM. In Section 8, we discuss the solution obtained by using HAM. In this Section, we also improve the solution obtained by applying the least-squares method. Finally, Section 9 concludes all the major findings of the present research study.

2. Some Preliminaries

Mathematicians have continuously modified the definitions of fractional-order derivatives and appeared with the derivatives of the type Riemann-Liouville, the Caputo, Caputo-Fabrizio, Atangana-Baleanu, the Grunwald-Letnikov, the Weyl, the Marchand, the Riesz, and the Miller and Ross, some with nonsingular kernel and without singular kernel [50, 51, 52, 53, 54, 55, 56, 57, 58, 59].

2.1. Definition. A real function $\xi(x)$, $x > 0$ is said to be in the space C_μ, $\mu \in R$, if there exists a real number $q > \mu$, such that $\xi(x) = x^q \xi_1(x)$, where $\xi_1(x) \in C[0, \infty)$ and it is said to be in the space C_μ^n, if and only if $\xi^n(x) \in C_\mu$, $n \in N$

2.2. Definition. The Riemann-Liouville form of fractional integral operator ${}_0^{RL}D_t^{-\alpha}$ of order $\alpha > 0$ for a function $\xi : \Re^+ \to \Re$ is defined as

$$ {}_0^{RL}D_t^{-\alpha}\xi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1}\xi(x)dx, \quad t > 0 $$

or

$$ {}_0^{RL}I_t^{\alpha}\xi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1}\xi(x)dx, \quad t > 0 $$

where $\alpha > 0$ and Γ is a well-known Gamma function.

2.3. Definition. The Riemann-Liouville form of fractional derivative of $\xi(x)$ order $\alpha > 0$ is defined as

$$ {}_0^{RL}D_t^{\alpha}\xi(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n \int_0^t \frac{\xi(x)}{(t-x)^{\alpha-n+1}}dx, & 0 \le n-1 < \alpha < n, \quad n = [\alpha], n \in N, \\ (\frac{d}{dt})^n \xi(t), & \alpha = n, n \in N \end{cases} $$

2.4. Definition. The Caputo fractional derivative of $\xi(x)$ order α is defined as

$$ {}_0^{C}D_t^{\alpha}\xi(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{(\frac{d}{dx})^n \xi(x)}{(t-x)^{\alpha-n+1}}dx, & 0 \le n-1 < \alpha < n, \quad n = [\alpha], n \in N, \\ (\frac{d}{dt})^n \xi(t), & \alpha = n, n \in N \end{cases} $$

where the operator ${}_0^{C}D_t^{\alpha}$ satisfies the following two basic properties:

${}_0^{C}D_t^{\alpha}\,{}_0^{RL}I_t^{\alpha}\xi(t) = \xi(t)$ and ${}_0^{RL}I_t^{\alpha}\,{}_0^{C}D_t^{\alpha}\xi(t) = \xi(t) - \sum_{k=0}^{n-1} \frac{\xi^{(k)}(a)}{k!}(t-a)^k, \ \ t > a.$

The definition 2.3 and definition 2.4 are not equivalent to each other, and their difference is expressed by

$$ {}_0^{C}D_t^{\alpha}\xi(t) = {}_0^{RL}D_t^{\alpha}\xi(t) - \sum_{y=0}^{n-1} r_y^{\alpha}(t)\xi^{(y)}(0), \quad r_y^{\alpha}(t) = \frac{t^{y-\alpha}}{\Gamma(y+1-\alpha)}. $$

The Caputo operator ${}_0^C D_t^\alpha$, has advantages for differential equations with initial values. In the case of Riemann-Liouville and Caputo derivatives, respectively, the initial values are usually given as [59]

$$ {}_0^{RL} D_t^\alpha \xi(0) = b_\nu, \qquad {}_0^C D_t^\alpha \xi(0) = b_\nu, \quad \nu = 1, 2, 3, \ldots, n. $$

A direct definition of the fractional derivative $D_t^\alpha \xi(t)$, is based on finite differences of an equidistant grid in $[0, t]$. Assume that the function $D_t^\alpha \xi(\tau)$, satisfies some smoothness conditions in every finite interval $(0, t), t \leq T$. Choosing the grid

$$ 0 = \tau_0 < \tau_1 < \ldots < t = \tau_{n+1} = (n+1)u, \;\; \tau_{n+1} - \tau_n = u $$

and using the classical notation of finite differences,

$$ \frac{1}{u^\alpha} \Delta_u^\alpha \xi(t) = \frac{1}{u^\alpha} \left(\xi(\tau_{n+1}) - \sum_{\nu=1}^{n+1} c_\nu^\alpha \xi(\tau_{n+1-\nu}) \right) $$

where

$$ c_\nu^\alpha = (-1)^{\nu-1} \left(\begin{array}{c} \alpha \\ \nu \end{array} \right). $$

3. Model formulation

To describe the transmission dynamics of HIV/AIDS, we formulate a deterministic compartmental mathematical model consists of a system of four first-order nonlinear ordinary differential equations for the four independent functions (susceptible population $\xi_S(t)$, exposed population $\xi_E(t)$, infected population $\xi_I(t)$ and recovered population $\xi_R(t)$) that takes the following form

$$ \left\{ \begin{array}{l} \frac{d\xi_S(t)}{dt} = \Lambda - \mu \xi_S(t) \xi_I(t) - \lambda \xi_S(t), \\ \frac{d\xi_E(t)}{dt} = \mu \xi_S(t) \xi_I(t) - (\lambda + \epsilon + \sigma) \xi_E(t), \\ \frac{d\xi_I(t)}{dt} = \sigma \xi_E(t) - (\rho + \lambda) \xi_I(t), \\ \frac{d\xi_R(t)}{dt} = \rho \xi_I(t) + \epsilon \xi_E(t) - \lambda \xi_R(t). \end{array} \right. \tag{1} $$

The total population $N(t)$ is divided into four sub-population compartments namely susceptible, exposed, infected and recovered such that $N(t) = \xi_S(t) + \xi_E(t) + \xi_I(t) + \xi_R(t)$ for all t. The proposed model is considered as the generalization of the original Kermack-Mckendrick model [2], where only three compartments were considered, but here the exposed compartment is included contains those susceptibles in the population who have sexual intercourse with the sex workers (male/female) without knowing their disease status who themselves don't know but are infectious as a result by having sexual contact they are exposed to the infection. The following description is associated with the above classical model. The susceptibles $\xi_S(t)$ in the population becomes infected at a rate μ on contact with infected individuals, recruited at a rate Λ and decreased by a natural death at a rate λ. The exposed individuals $\xi_E(t)$ generated through contact with infected individuals at rate μ, which is breakthrough into infected class at a rate σ, decreased by testing or HIV therapy at a rate ϵ and removed at a rate λ. The class of infected individuals $\xi_I(t)$ is generated from exposed individuals at a rate ϵ, which is decreased by recovery at a rate ρ from infection and diminished at a rate λ. This generates a completely protected class $\xi_R(t)$ against the disease of individuals. The natural death at a rate λ is diminished by $\xi_I(t)$ recovered individual. It may further be noted that $N(t) = \xi_S'(t) + \xi_E'(t) + \xi_I'(t) + \xi_R'(t) = 0$ reveals that the total population is bounded.

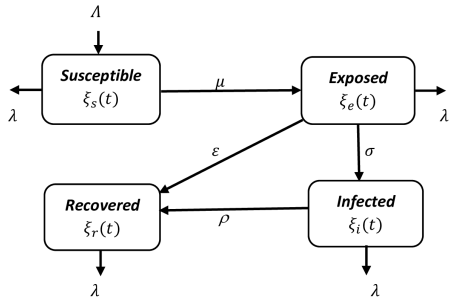

FIGURE 1. Schematic diagram of the proposed fractional-order SEIR epidemic model.

Following the derivation techniques applied in Podlubany [60], Kilbas et al. [61], Atangana and Owolabi [62], we derive the proposed model in fractional-order case by replacing each standard order derivative in time Caputo fractional operator. The advantage of this replacement is the involvement of the memory effect in the present work. Also, considering a fractional-order system helps reduce the errors arising from the neglected parameters in real-life modeling phenomena. Thus, the proposed fractional-order SEIR model for HIV transmission has the following form [18, 20, 23, 60, 61, 62]

$$\begin{cases} {}_0^C D_t^\alpha \xi_S(t) = \Lambda - \mu\xi_S(t)\xi_I(t) - \lambda\xi_S(t), \\ {}_0^C D_t^\alpha \xi_E(t) = \mu\xi_S(t)\xi_I(t) - (\lambda + \epsilon + \sigma)\xi_E(t), \\ {}_0^C D_t^\alpha \xi_I(t) = \sigma\xi_E(t) - (\rho + \lambda)\xi_I(t), \\ {}_0^C D_t^\alpha \xi_R(t) = \rho\xi_I(t) + \epsilon\xi_E(t) - \lambda\xi_R(t). \end{cases} \tag{2}$$

Usually, the second set of dependent variables, representing the fraction of the total population in each of the four categories, $\xi_s(t) = \frac{\xi_S(t)}{N}$, $\xi_e(t) = \frac{\xi_E(t)}{N}$, $\xi_i(t) = \frac{\xi_I(t)}{N}$, and $\xi_r(t) = \frac{\xi_R(t)}{N}$ is considered. In this case, model (2) becomes

$$\begin{cases} {}_0^C D_t^\alpha \xi_s(t) = \Lambda - \mu\xi_s(t)\xi_i(t) - \lambda\xi_s(t), \\ {}_0^C D_t^\alpha \xi_e(t) = \mu\xi_s(t)\xi_i(t) - (\lambda + \epsilon + \sigma)\xi_e(t), \\ {}_0^C D_t^\alpha \xi_i(t) = \sigma\xi_e(t) - (\rho + \lambda)\xi_i(t), \\ {}_0^C D_t^\alpha \xi_r(t) = \rho\xi_i(t) + \epsilon\xi_e(t) - \lambda\xi_r(t). \end{cases} \tag{3}$$

subject to the initial conditions

$$\xi_s(0) = \xi_{s,0}, \quad \xi_e(0) = \xi_{e,0}, \quad \xi_i(0) = \xi_{i,0}, \quad \xi_r(0) = \xi_{r,0}, \tag{4}$$

where $0 < \alpha \leq 1$, $N(t) = \xi_s(t) + \xi_e(t) + \xi_i(t) + \xi_r(t)$, $(\xi_s, \xi_e, \xi_i, \xi_r) \in R_+^4$. We assume that the functions $\xi_s(t), \xi_e(t), \xi_i(t), \xi_r(t)$ and their Caputo fractional derivatives are continuous at $t \geq 0$. To start, the existence, uniqueness, and non-negativity of the solution of system (3) are analyzed.

TABLE 1. Parameter values for the simulation of the proposed fractional-order SEIR epidemic model.

Parameter	Meaning	Values	Source
Λ	Recruitment rate	0.32	[15, 63]
μ	Infection rate	0.05	[20]
λ	Death rate	0.2	[15, 63]
σ	Infected class rate	0.01	[15, 63]
ϵ	Testing rate	0.25	[15, 63]
ρ	Recovery rate	0.03	[18, 22]
$\xi_{s,0}$	Initially susceptible individuals	20	[22, 64]
$\xi_{e,0}$	Initially exposed individuals	0.01	[20]
$\xi_{i,0}$	Initially infected individuals	0.02	[20]
$\xi_{r,0}$	Initially recovered individuals	0.00	[20, 22]

4. Mathematical analysis of the model

4.1. Positivity and boundedness. Positivity implies that the population survives, and boundedness may be interpreted as a natural restriction to growth due to limited resources. Let us denote $R^4_+ = \{\xi(t) \in R^4 : \xi(t) \geq 0\}$ and let $\xi(t) = [\xi_s(t), \xi_e(t), \xi_i(t), \xi_r(t)]^T$. For the proof of the main theorem about the non-negativity of the solutions, we recall the following lemma [2, 20, 65, 66].

Lemma 4.1.1. (Generalized Mean Value Theorem [2, 20, 65]). Let $\xi(x) \in C[a, b]$ and Caputo fractional derivative ${}^C_0D^\alpha_t\xi(x) \in C(a, b]$ for $0 < \alpha \leq 1$, then we have

$$\xi(x) = \xi(a) + \frac{1}{\Gamma(\alpha)}{}^C_0D^\alpha_t\xi(\gamma)(x-a)^\alpha,$$

with $0 \leq \gamma \leq x$, $\forall x \in (a, b]$.

Remark 4.1.1. If $\xi(x) \in C[0, b]$ and Caputo fractional derivative ${}^C_0D^\alpha_t\xi(x) \in (0, b]$ for $0 < \alpha \leq 1$. It is clear from the lemma 4.1.1 that if ${}^C_0D^\alpha_t\xi(x) \geq 0\ \forall x \in (0, b]$, then the function $\xi(x)$ is non-decreasing and if ${}^C_0D^\alpha_t\xi(x) \leq 0\ \forall x \in (0, b]$, then the function $\xi(x)$ is non-increasing for all $x \in [0, b]$.

Theorem 4.1.2. There is a unique solution $\xi(t) = [\xi_s(t), \xi_e(t), \xi_i(t), \xi_r(t)]^T$ for the initial value problem given by (3) and initial condition (4) on $t \geq 0$ in $(0, \alpha)$ and the solution will remain in R^4_+. Furthermore, the solutions are all bounded.

Proof: According to Lin [66] from Theorem 3.2 [66] and Remark 3.2 [66], we can determine the solution on $(0, \infty)$ by solving the model (3) along with initial conditions (4), which is not only existent but also unique. Subsequently, we have to explain the non-negative domain R^4_+, is a positively invariant region. From model (3), we find

$${}^C_0D^\alpha_t\xi_s(t)\big|_{\xi_s=0} = \Lambda > 0,\ \ {}^C_0D^\alpha_t\xi_e(t)\big|_{\xi_e=0} = \mu\xi_s(t)\xi_i(t) \geq 0,$$

$${}^C_0D^\alpha_t\xi_i(t)\big|_{\xi_i=0} = \sigma\xi_e(t) \geq 0,\ \ {}^C_0D^\alpha_t\xi_r(t)\big|_{\xi_r=0} = \rho\xi_i(t) + \epsilon\xi_e(t) \geq 0.$$

On each hyperplane bounding the non-negative orthant, the vector field points into R^4_+. Furthermore, from the system (3)

$${}^C_0D^\alpha_t N(t) + \lambda N(t) \leq \Lambda.$$

Thus, by Lemma 4.1.1, in the case of HIV infection, the total population $N(t)$, i.e., the subpopulations $\xi_s(t), \xi_e(t), \xi_i(t)$ and $\xi_r(t)$ are bounded.□

Therefore, the biologically feasible region for the system (3) is

$$\Upsilon = \big\{(\xi_s(t), \xi_e(t), \xi_i(t), \xi_r(t)) \in R_+^4 \big| 0 < \xi_s(t) + \xi_e(t) + \xi_i(t) + \xi_r(t) \leq \frac{\Lambda}{\lambda}\big\}.$$

4.2. Routh-Hurwitz stability conditions for fractional system. Consider the following general form of a fractional nonlinear dynamical system [67]

$$D^\alpha \eta_j(t) = g_j(\eta_1, \eta_2, \eta_3, \eta_4), \quad j = 1, 2, 3, 4, \quad 0 < \alpha \leq 1, \tag{5}$$

with the initial conditions $\eta_1(0) = \eta_{1,0}, \eta_2(0) = \eta_{2,0}, \eta_3(0) = \eta_{3,0}, \eta_4(0) = \eta_{4,0}$.

To evaluate equilibrium points of Eq. (5), consider $D^\alpha \eta_j(t) = 0$, this implies that $g_j(\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*) = 0$. Let $\chi^*(\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*)$ be an equilibrium point of system (5) and perturb the equilibrium point by adding a positive term $\epsilon_j(t)$ that is $\eta_j(t) = \eta_j^*(t) + \epsilon_j(t)$, then

$$D^\alpha \left(\eta_j^*(t) + \epsilon_j(t)\right) = g_j(\eta_1^* + \epsilon_1, \eta_2^* + \epsilon_2, \eta_3^* + \epsilon_3, \eta_4^* + \epsilon_4).$$

This implies

$$D^\alpha \left(\epsilon_j(t)\right) = g_j(\eta_1^* + \epsilon_1, \eta_2^* + \epsilon_2, \eta_3^* + \epsilon_3, \eta_4^* + \epsilon_4).$$

Using the Taylor series expansion, we get

$$\begin{aligned} D^\alpha \left(\epsilon_j(t)\right) =& g_j(\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*) + \frac{\partial g_j}{\partial \eta_1}\Big|_{eq\epsilon_1} + \frac{\partial g_j}{\partial \eta_2}\Big|_{eq\epsilon_2} \\ &+ \frac{\partial g_j}{\partial \eta_3}\Big|_{eq\epsilon_3} + \frac{\partial g_j}{\partial \eta_4}\Big|_{eq\epsilon_4} + higher\ \ order\ \ terms \end{aligned}$$

Since $g_j(\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*) = 0$, then

$$D^\alpha \left(\epsilon_j(t)\right) \cong \frac{\partial g_j}{\partial \eta_1}\Big|_{eq\epsilon_1} + \frac{\partial g_j}{\partial \eta_2}\Big|_{eq\epsilon_2} + \frac{\partial g_j}{\partial \eta_3}\Big|_{eq\epsilon_3} + \frac{\partial g_j}{\partial \eta_4}\Big|_{eq\epsilon_4}. \tag{6}$$

We can write the system of Eq. (6) in the following matrix form

$$D^\alpha \left(\epsilon(t)\right) = J_\epsilon(t), \ \ \epsilon(t) = \left(\epsilon_1(t), \epsilon_2(t), \epsilon_3(t), \epsilon_4(t)\right)^T, \tag{7}$$

where

$$J(\chi^*) = \begin{pmatrix} \frac{\partial g_1}{\partial \eta_1} & \frac{\partial g_1}{\partial \eta_2} & \frac{\partial g_1}{\partial \eta_3} & \frac{\partial g_1}{\partial \eta_4} \\ \frac{\partial g_2}{\partial \eta_1} & \frac{\partial g_2}{\partial \eta_2} & \frac{\partial g_2}{\partial \eta_3} & \frac{\partial g_2}{\partial \eta_4} \\ \frac{\partial g_3}{\partial \eta_1} & \frac{\partial g_3}{\partial \eta_2} & \frac{\partial g_3}{\partial \eta_3} & \frac{\partial g_3}{\partial \eta_4} \\ \frac{\partial g_4}{\partial \eta_1} & \frac{\partial g_4}{\partial \eta_2} & \frac{\partial g_4}{\partial \eta_3} & \frac{\partial g_4}{\partial \eta_4} \end{pmatrix},$$

is the Jacobian matrix evaluated at the equilibrium point χ^* and satisfies the following relation

$$\Phi^{-1} J \Phi = \Pi, \quad \Pi = Diagonal\ \ matrix(\lambda_1, \lambda_2, \lambda_3, \lambda_4), \tag{8}$$

where $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are the eigenvalues of J and Φ is the eigenvectors of J. System (5) has the initial conditions:

$$\epsilon_1(0) = \eta_1(0) - \eta_1^*, \ \epsilon_2(0) = \eta_2(0) - \eta_2^*, \ \epsilon_3(0) = \eta_3(0) - \eta_3^*, \ \epsilon_4(0) = \eta_4(0) - \eta_4^*.$$

Using (7) and (8), we obtain

$$D^\alpha \epsilon(t) = (\Phi \Pi \Phi^{-1}) \epsilon(t).$$

This implies

$$D^\alpha (\Phi^{-1} \epsilon(t)) = \Pi (\Phi^{-1} \epsilon(t)),$$

and hence

$$D^\alpha \psi(t) = \Pi \psi(t), \ \psi(t) = \Phi^{-1} \epsilon(t), \ \psi(t) = \big(\psi_1(t), \psi_2(t), \psi_3(t), \psi_4(t)\big)^T. \tag{9}$$

Therefore

(10)
$$D^{\alpha}\psi_1(t) = \lambda_1\psi_1(t),\ \ D^{\alpha}\psi_2(t) = \lambda_2\psi_2(t),\ \ D^{\alpha}\psi_3(t) = \lambda_3\psi_3(t),\ \ D^{\alpha}\psi_4(t) = \lambda_4\psi_4(t).$$

The solution of Eq. (10) is given by

$$\psi_j(t) = E_{\alpha}\big(\lambda_j t^{\alpha}\big)\psi_j(0), \quad j = 1, 2, 3, 4,$$

where $E_{\alpha}\big(\lambda t^{\alpha}\big) = \Sigma_{n=0}^{\infty}\frac{\lambda^n t^{n\alpha}}{\Gamma(n\alpha+1)}$ is the Mittag-Leffler function. Then, $\psi_1(t), \psi_2(t)$, $\psi_3(t)$ and $\psi_4(t)$ are decreasing, and thus, $\epsilon_1, \epsilon_2, \epsilon_3$, and ϵ_4,are decreasing. Therefore, the equilibrium point χ^* is locally asymptotically stable if the Matignon's conditions [68] given by $\big|arg(\lambda_j)\big| > \alpha\frac{\pi}{2},\ \ j = 1, 2, 3, 4$ are satisfied.

4.3. Equilibirium points and their stability. To evaluate the equilibrium points, setting the right-hand side of the system (3) equal to zero, we obtain equilibrium points as

(11)
$$\begin{cases} {}_0^C D_t^{\alpha}\xi_s(t) = 0, \\ {}_0^C D_t^{\alpha}\xi_e(t) = 0, \\ {}_0^C D_t^{\alpha}\xi_i(t) = 0, \\ {}_0^C D_t^{\alpha}\xi_r(t) = 0. \end{cases}$$

Then the equilibrium points are $E^0 = (1, 0, 0, 0)$ and $E^* = (\xi_s^*, \xi_e^*, \xi_i^*, \xi_r^*)$ where

$$\xi_s^* = \frac{(\lambda+\epsilon+\sigma)(\rho+\lambda)}{\mu\sigma}, \quad \xi_e^* = \frac{(\rho+\lambda)}{\sigma}\xi_i^*,$$
$$\xi_i^* = \frac{(\Lambda-\lambda\xi_s^*)}{\mu\xi_s^*}, \quad \xi_r^* = \frac{(\sigma\rho+\epsilon(\rho+\lambda))}{\lambda\sigma}\xi_i^*.$$

If $\xi_s^*, \xi_e^*, \xi_i^*, \xi_r^*$ are between 0 and 1.

Thus, the proposed nonlinear fractional-order HIV epidemic model has two equilibria, namely disease-free equilibrium and endemic equilibrium.

Disease-free equilibrium (E^0): The equilibrium state with the absence of infection is known as disease-free equilibrium or zero equilibrium. The disease-free equilibrium has always been feasible, as in this equilibrium, the infection dies out from the population. The disease-free equilibrium is given by $E^0 = (1, 0, 0, 0)$.

Endemic equilibrium (E^*): The positive endemic equilibrium is that state of the system where the infection spreads throughout the population and causes the disease persistence. For system (3), the endemic equilibrium is considered as $E^* = (\xi_s^*, \xi_e^*, \xi_i^*, \xi_r^*)$. i.e., the state in which infection spreads in the susceptible population.

To study the disease-free equilibrium's local stability, we compute the basic reproduction number by using the next-generation matrix method [69, 70, 71].

Basic reproduction number $\Re_0$: The basic reproduction number $\Re_0$ is defined as the number of secondary cases produced by a typical infected individual during its entire period of infectiousness in a completely susceptible population. We find $\Re_0$ using the next-generation matrix [69, 70, 71]. Therefore

$$\Re_0 = \frac{\mu\sigma}{(\lambda+\epsilon+\sigma)(\rho+\lambda)}.$$

It shows that if $\Re_0 < 1$, then the disease/infection does not spread, and the infection dies. On the other hand, if $\Re_0 > 1$, then the disease persists in the whole population. From the definition of $\Re_0$, we conclude that for the endemic equilibrium point $E^* = (\xi_s^*, \xi_e^*, \xi_i^*, \xi_r^*)$, we have the following relations

$$E^* = \left(\frac{1}{\Re_0}, \frac{\lambda(\rho+\lambda)}{\sigma\mu}(\Re_0 - 1), \frac{\lambda}{\mu}(\Re_0 - 1), \frac{(\sigma\rho + \epsilon(\rho+\lambda))}{\sigma\mu}(\Re_0 - 1) \right),$$

and so, it exists only if $\Re_0 > 1$.

Local stability of equilibria

Now, we will discuss the local stability of the equilibrium points. For this, we state the results in the form of theorems and prove them.

Theorem 4.2.1: The disease-free equilibrium E^0 of proposed fractional-order HIV epidemic model is asymptotically stable if $\Re_0 < 1$ i.e., if $\frac{\mu\sigma}{(\lambda+\epsilon+\sigma)(\rho+\lambda)} < 1$ and is unstable if $\Re_0 > 1$.

Proof: In an epidemiological sense, the above result depicts that small inflow of infected individuals will not be able to spread infection if $\Re_0 < 1$. In this case, the spread of infection is dependent on the initial sizes of the sub-population. To prove the above theorem 4.2.1, the general Jacobian matrix and the matrices corresponding to each equilibrium point will be obtained. Therefore, the Jacobian matrix is given by

$$\Omega = \begin{bmatrix} -\mu\xi_i - \lambda & 0 & -\mu\xi_s & 0 \\ \mu\xi_i & -\lambda - \epsilon - \sigma & \mu\xi_s & 0 \\ 0 & \sigma & -\rho - \lambda & 0 \\ 0 & \epsilon & \rho & -\lambda \end{bmatrix}.$$

Now at the disease-free equilibrium E^0,

$$\Omega = \begin{bmatrix} -\lambda & 0 & -\mu & 0 \\ 0 & -\lambda - \epsilon - \sigma & \mu & 0 \\ 0 & \sigma & -\rho - \lambda & 0 \\ 0 & \epsilon & \rho & -\lambda \end{bmatrix}.$$

Therefore, the Routh-Hurwitz stability conditions for fractional-order systems discussed in subsection 4.2 describes that the necessary and sufficient condition

$$|arg(eig(\Omega))| = |arg(\delta_i)| > \alpha\frac{\pi}{2}, \tag{12}$$

for various fractional-order models. Therefore, the disease-free equilibrium of system (3) is asymptotically stable if all of the eigenvalues, $\delta_j, j = 1, 2, 3, 4$ of $\Omega(E^0)$ satisfy the condition (12). Hence, a sufficient condition for the local asymptotic stability of the equilibrium points is that the eigenvalues $\delta_j, j = 1, 2, 3, 4$ of the Jacobian matrix $\Omega(E^0)$ satisfy the condition $|arg(\delta_j)| > \alpha\frac{\pi}{2}$. This confirms that fractional-order differential equations are, at least, as stable as their integer order counterpart. By solving the characteristic equation, the eigenvalues can be obtained as

$$|\Omega(E^0) - \delta I| = 0.$$

The simplification allows us to get the following algebraic equation

$$(\lambda + \delta)^2(\delta^2 + (\theta_1 + \theta_2)\delta + \theta_1\theta_2 - \theta_3) = 0,$$

where $\theta_1 = \lambda + \epsilon + \sigma$, $\theta_2 = \rho + \lambda$ and $\theta_3 = \mu\sigma$.
Therefore, the roots of the characteristic equation are

$$\delta_{1,2} = -\lambda,$$

$$\delta_{3,4} = \frac{-(\theta_1 + \theta_2) \pm \sqrt{(\theta_1 + \theta_2)^2 - 4(\theta_1\theta_2 - \theta_3)}}{2},$$

and because $\theta_1 + \theta_2 > 0$, then if $\theta_1\theta_2 > \theta_3$, all of the eigenvalues $\delta_j, j = 1, 2, 3, 4$, satisfy the condition given by (12). Therefore, all the eigenvalues have negative real parts if $\frac{\mu\sigma}{(\lambda+\epsilon+\sigma)(\rho+\lambda)} < 1$ i.e., if $\Re_0 < 1$.□

In the next theorem 4.2.2, we discuss the asymptotic stability of the endemic equilibrium of the system given by (3).

Theorem 4.2.2.: The endemic equilibrium E^* is asymptotically stable whenever $\Re_0 > 1$ and unstable otherwise.

Proof: The characteristic equation of the endemic equilibrium point is expressed as the following polynomial:

$$P(\delta) = -(\lambda + \delta)\Phi(\delta),$$

where

$$\Phi(\delta) = \delta^3 + \gamma_1\delta^2 + \gamma_2\delta + \gamma_3 = 0. \tag{13}$$

Now, expressing the discriminant of $\Phi(\delta)$ as

$$D(\delta) = 18\gamma_1\gamma_2\gamma_2 + (\gamma_1\gamma_2)^2 - 4\gamma_3\gamma_1^2 - 4\gamma_2^2 - 27\gamma_3^2, \tag{14}$$

and using the construction of results by Ahmed et al. [72], the following fractional Routh-Hurwitz conditions associated with are observed:

(1) If $D(\delta) > 0$, then the necessary and sufficient condition for the equilibrium point to be locally asymptotically stable is $\gamma_1 > 0$, $\gamma_3 > 0$, $\gamma_1\gamma_2 > \gamma_3$.
(2) If $D(\delta) < 0$, $\gamma_1 \geq 0$, $\gamma_2 \geq 0$, $\gamma_3 > 0$, then the equilibrium point is locally asymptotically stable if $\alpha < \frac{2}{3}$.
(3) If $D(\delta) < 0$, $\gamma_1 < 0$, $\gamma_2 < 0$, and $\alpha > \frac{2}{3}$, then all roots of the Eq. (13) satisfy the condition then $|arg(\delta_j)| < \alpha\frac{\pi}{2}, j = 1, 2, 3, 4$.

To discuss the local stability of the endemic equilibrium point, we have from the model constraints $\xi_s(t) + \xi_e(t) + \xi_i(t) + \xi_r(t) = 1$, let us consider the Jacobian matrix of the system (3) evaluated at endemic equilibrium E^* as

$$\Omega(E^*) = \begin{bmatrix} -\mu\xi_i - \lambda & 0 & -\mu\xi_s & 0 \\ \mu\xi_i & -\lambda - \epsilon - \sigma & \mu\xi_s & 0 \\ 0 & \sigma & -\rho - \lambda & 0 \\ 0 & \epsilon & \rho & -\lambda \end{bmatrix}.$$

This further implies that

$$\Omega(E^*) = \begin{bmatrix} -\mu\Re_0 & 0 & -\frac{\mu}{\Re_0} & 0 \\ \mu(\Re_0 - 1) & -\lambda - \epsilon - \sigma & \frac{\mu}{\Re_0} & 0 \\ 0 & \sigma & -\rho - \lambda & 0 \\ 0 & \epsilon & \rho & -\lambda \end{bmatrix},$$

and so the eigenvalues are

$$\delta_j = \frac{-\Re_0(\theta_1 + \theta_2) \pm \sqrt{(\theta_2 - \theta_1\Re_0)^2\Re_0^2 + 4\theta_3\Re_0}}{2}, j = 1, 2, 3, 4$$

It is apparent from the above result that the eigenvalues of the system (3) at endemic equilibrium are all real and negative if $\Re_0 > 1$, i.e., the basic reproduction number $\Re_0$ is greater than one.□

Global stability of equilibria

The global existence of the solution of the fractional differential equation always becomes a most important concern, which is carried out in the following section.

Theorem 4.2.3: Assume that the function $\Psi : \mathfrak{R}_+ \times \mathfrak{R}^4 \to \mathfrak{R}^4$ satisfies the following conditions in the global space[19, 66]

(1) The function $\Psi(t, \xi(t))$ is Lebesgue measurable for t on $\mathfrak{R}$.
(2) The function $\Psi(t, \xi(t))$ is continuous for $\xi(t)$ on $\mathfrak{R}^4$.
(3) The function $\frac{\partial \Psi(t,\xi(t))}{\partial \xi}$ is continuous for $\xi(t)$ on $\mathfrak{R}^4$.
(4) $\|\Psi(t, \xi(t))\| \leq \beta_1 + \beta_2 \|\xi(t)\|$, for all most every $t \in \mathfrak{R}$ and all $\xi(t) \in \mathfrak{R}^4$.

Here β_1, β_2 are two positive constants and $\xi(t) = [\xi_s(t), \xi_e(t), \xi_i(t), \xi_r(t)]^T$. Then, the initial value problem

$$\begin{cases} {}_0^C D_t^\alpha \xi(t) = \Psi(t, \xi(t)), \alpha \in (0, 1], \\ \xi(t_0) = \xi_0, \end{cases} \tag{15}$$

has a unique solution.

Theorem 4.2.4: The system (3) has a unique solution, and the solution remains in $\mathfrak{R}_+^4$.

Proof: From Theorem 4.2.3, we obtain the unique solution on $(0, \infty)$ by solving the system (3). Firstly, Lin [66] discussed the proof of the theorem and shows that the solution exists and unique. In Theorem 4.1.2, we already proved that the solution of model (3) would remain in $\mathfrak{R}_+^4$.□

The stability region of the fractional-order HIV epidemic model (3) with order α is illustrated in Fig.2. *Re* refers to the real and *Im* refers to the imaginary parts of the eigenvalues, respectively. From Fig.2, it is easy to show that the stability region of the fractional-order case is greater than the stability region of the integer-order case.

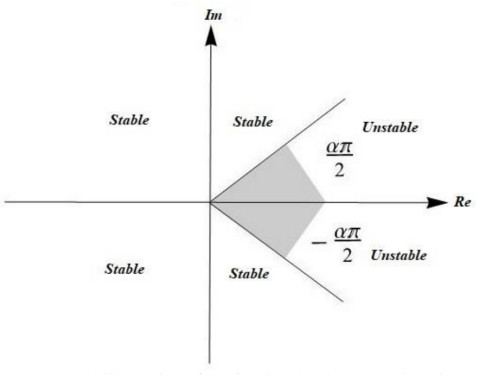

FIGURE 2. Stability region of the fractional-order HIV epidemic system.

5. Sensitivity analysis

This section is devoted to the sensitivity analysis of the basic reproduction number $\mathfrak{R}_0$. Because of the uncertainties associated with certain parameter values, it

is necessary to determine the model robustness as the parameter values changes. Here it should be noted that the positive sign of the index means the parameter contributes to the increasing value of $\mathfrak{R}_0$, while the negative sign indicates the decrease in $\mathfrak{R}_0$. Using the approach in Chitnis et al. [73] and Makinde et al. [74], we analyze the reproduction number to determine which parameter is more sensitive towards disease dynamics.

Definition 5.1
The normalized forward sensitivity index of a variable, V that depends differentially on a parameter p is denoted by Π_p^V and defined as:

$$\Pi_p^V = \frac{p}{V} \times \frac{\partial V}{\partial p}.$$

5.1. Sensitivity indices of $\mathfrak{R}_0$. We derive the sensitivity of $\mathfrak{R}_0$ to each of the different parameters μ, σ, ϵ and ρ . The sensitivity index of $\mathfrak{R}_0$ for μ is presented by

$$\Pi_\mu^{\mathfrak{R}_0} = \frac{\mu}{\mathfrak{R}_0} \times \frac{\partial \mathfrak{R}_0}{\partial \mu} = \frac{\mu(\lambda+\epsilon+\sigma)(\rho+\lambda)}{\mu\sigma} \times \frac{\sigma}{(\lambda+\epsilon+\sigma)(\rho+\lambda)} > 0.$$

Again, the sensitivity indices of $\mathfrak{R}_0$ resulting from the evaluation of the other parameters of the model are shown below.

$$\Pi_\sigma^{\mathfrak{R}_0} = \frac{\sigma}{\mathfrak{R}_0} \times \frac{\partial \mathfrak{R}_0}{\partial \sigma} = \frac{\sigma(\lambda+\epsilon+\sigma)(\rho+\lambda)}{\mu\sigma} \times \frac{\mu}{(\lambda+\epsilon+\sigma)(\rho+\lambda)} > 0,$$

$$= \frac{\lambda+\epsilon}{\lambda+\epsilon+\sigma} > 0.$$

Similarly,

$$\Pi_\epsilon^{\mathfrak{R}_0} = \frac{\epsilon}{\mathfrak{R}_0} \times \frac{\partial \mathfrak{R}_0}{\partial \epsilon} = \frac{\epsilon(\lambda+\epsilon+\sigma)(\rho+\lambda)}{\mu\sigma} \times \frac{\mu}{(\lambda+\epsilon+\sigma)(\rho+\lambda)} < 0,$$

$$= -\frac{\epsilon}{\lambda+\epsilon+\sigma} < 0.$$

Again

$$\Pi_\rho^{\mathfrak{R}_0} = \frac{\rho}{\mathfrak{R}_0} \times \frac{\partial \mathfrak{R}_0}{\partial \rho} = \frac{\rho(\lambda+\epsilon+\sigma)(\rho+\lambda)}{\mu\sigma} \times \frac{\mu}{(\lambda+\epsilon+\sigma)(\rho+\lambda)} > 0,$$

$$= -\frac{\rho}{\lambda+\rho} < 0.$$

From the above-obtained sensitivity indices, it is seen that $\mathfrak{R}_0$ is increasing with μ and σ while decreasing with ϵ and ρ that means μ, σ are more sensitive in disease transmission than ϵ and ρ.

6. Methodology for the solution

In this section, an approximate solution for the proposed fractional-order HIV epidemic model is presented. Because there are no general methods to obtain an analytical solution of the nonlinear fractional system (3), further, the fractional case is more difficult to handle even numerically [65]. So, we use the so-called technique HAM to obtain the series solution of the system (3), as it is a flexible method that contains the auxiliary parameters and functions that make to adjust the convergence region of the obtained series solution.

6.1. Application of HAM to the fractional-order SEIR epidemic model. Here we apply the algorithm of HAM to find an approximate solution in terms of convergent series for the equations of the nonlinear fractional-order system (3), which gives an accurate solution over a longer time frame.

To construction the homotopy series solution for the proposed nonlinear fractional-order HIV epidemic model (3), let $q \in [0,1]$ be the so-called embedding parameter. The HAM is based on a kind of continuous mappings

$$\xi_s(t) \to \psi_S(t;q), \;\; \xi_e(t) \to \psi_E(t;q), \;\; \xi_i(t) \to \psi_I(t;q), \;\; \xi_r(t) \to \psi_R(t;q).$$

Due to governing equations, we choose the auxiliary linear operators

$$\begin{cases} L_1[\psi_S(t;q)] = {}_0^C D_t^\alpha, \\ L_2[\psi_E(t;q)] = {}_0^C D_t^\alpha, \\ L_3[\psi_I(t;q)] = {}_0^C D_t^\alpha, \\ L_4[\psi_R(t;q)] = {}_0^C D_t^\alpha. \end{cases} \tag{16}$$

We define the homotopy maps as [27, 38, 39, 41]

$$\begin{aligned}
&\Psi_S\left(\psi_S(t;q), \psi_E(t;q), \psi_I(t;q), \psi_R(t;q)\right) \\
=&(1-q)L_1[\psi_S(t;q) - \xi_{s,0}] - q\hbar H_1(t) N_S[\psi_S(t;q), \psi_E(t;q), \psi_I(t;q), \psi_R(t;q)], \\
&\Psi_E\left(\psi_S(t;q), \psi_E(t;q), \psi_I(t;q), \psi_R(t;q)\right) \\
=&(1-q)L_2[\psi_E(t;q) - \xi_{e,0}] - q\hbar H_2(t) N_E[\psi_S(t;q), \psi_E(t;q), \psi_I(t;q), \psi_R(t;q)], \\
&\Psi_I\left(\psi_S(t;q), \psi_E(t;q), \psi_I(t;q), \psi_R(t;q)\right) \\
=&(1-q)L_3[\psi_I(t;q) - \xi_{i,0}] - q\hbar H_3(t) N_I[\psi_S(t;q), \psi_E(t;q), \psi_I(t;q), \psi_R(t;q)], \\
&\Psi_R\left(\psi_S(t;q), \psi_E(t;q), \psi_I(t;q), \psi_R(t;q)\right) \\
=&(1-q)L_4[\psi_R(t;q) - \xi_{r,0}] - q\hbar H_4(t) N_R[\psi_S(t;q), \psi_E(t;q), \psi_I(t;q), \psi_R(t;q)],
\end{aligned}$$

where $\hbar \neq 0$ and $H_j(t) \neq 0, j = 1,2,3,4$ denote the so-called auxiliary parameter and auxiliary function, respectively, and N_S, N_E, N_I, N_R are nonlinear operators that define as

$$\begin{aligned}
&N_S[\psi_S(t;q), \psi_E(t;q), \psi_I(t;q), \psi_R(t;q)] \\
=&{}_0^C D_t^\alpha[\psi_S(t;q)] - \Lambda + \mu\psi_S(t;q)\psi_I(t;q) + \lambda\psi_S(t;q),
\end{aligned}$$

$$\begin{aligned}
&N_E[\psi_S(t;q), \psi_E(t;q), \psi_I(t;q), \psi_R(t;q)] \\
=&{}_0^C D_t^\alpha[\psi_E(t;q)] - \mu\psi_S(t;q)\psi_I(t;q) + (\lambda + \epsilon + \sigma)\psi_E(t;q), \\
&N_I[\psi_S(t;q), \psi_E(t;q), \psi_I(t;q), \psi_R(t;q)] \\
=&{}_0^C D_t^\alpha[\psi_I(t;q)] - \sigma\psi_E(t;q) + (\rho + \lambda)\psi_I(t;q), \\
&N_R[\psi_S(t;q), \psi_E(t;q), \psi_I(t;q), \psi_R(t;q)] \\
=&{}_0^C D_t^\alpha[\psi_R(t;q)] - \rho\psi_I(t;q) - \epsilon\psi_E(t;q) + \lambda\psi_R(t;q).
\end{aligned}$$

When $q = 0$, we have the homotopy maps as

$$\begin{aligned}
\Psi_S[\psi_S(t;0), \psi_E(t;0), \psi_I(t;0), \psi_R(t;0)] &= L_1[\psi_S(t;0) - \xi_{s,0}], \\
\Psi_E[\psi_S(t;0), \psi_E(t;0), \psi_I(t;0), \psi_R(t;0)] &= L_2[\psi_E(t;0) - \xi_{e,0}], \\
\Psi_I[\psi_S(t;0), \psi_E(t;0), \psi_I(t;0), \psi_R(t;0)] &= L_3[\psi_I(t;0) - \xi_{i,0}], \\
\Psi_R[\psi_S(t;0), \psi_E(t;0), \psi_I(t;0), \psi_R(t;0)] &= L_4[\psi_R(t;0) - \xi_{r,0}],
\end{aligned}$$

and when $q = 1$, will be

$$\begin{aligned}
&\Psi_S\left(\psi_S(t;1),\psi_E(t;1),\psi_I(t;1),\psi_R(t;1)\right)\\
=&-\hbar H_1(t)N_S[\psi_S(t;1),\psi_E(t;1),\psi_I(t;1),\psi_R(t;1)],\\
&\Psi_E\left(\psi_S(t;1),\psi_E(t;1),\psi_I(t;1),\psi_R(t;1)\right)\\
=&-\hbar H_2(t)N_E[\psi_S(t;1),\psi_E(t;1),\psi_I(t;1),\psi_R(t;1)],\\
&\Psi_I\left(\psi_S(t;1),\psi_E(t;1),\psi_I(t;1),\psi_R(t;1)\right)\\
=&-\hbar H_3(t)N_I[\psi_S(t;1),\psi_E(t;1),\psi_I(t;1),\psi_R(t;1)],\\
&\Psi_R\left(\psi_S(t;1),\psi_E(t;1),\psi_I(t;1),\psi_R(t;1)\right)\\
=&-\hbar H_4(t)N_R[\psi_S(t;1),\psi_E(t;1),\psi_I(t;1),\psi_R(t;1)].
\end{aligned}$$

Thus, by requiring

$$\begin{aligned}
&\Psi_S\left(\psi_S(t;q),\psi_E(t;q),\psi_I(t;q),\psi_R(t;q)\right)\\
=&\Psi_E\left(\psi_S(t;q),\psi_E(t;q),\psi_I(t;q),\psi_R(t;q)\right)=0,\\
&\Psi_I\left(\psi_S(t;q),\psi_E(t;q),\psi_I(t;q),\psi_R(t;q)\right)\\
=&\Psi_R\left(\psi_S(t;q),\psi_E(t;q),\psi_I(t;q),\psi_R(t;q)\right)=0,
\end{aligned}$$

and using the embedding parameter q, we construct a family of equations the zeroth-order deformation equations in the following form

(17)

$$\begin{cases}
(1-q)L_1[\psi_S(t;q)-\xi_{s,0}] = q\hbar H_1(t)N_S[\psi_S(t;q),\psi_E(t;q),\psi_I(t;q),\psi_R(t;q)],\\
(1-q)L_2[\psi_E(t;q)-\xi_{e,0}] = q\hbar H_2(t)N_E[\psi_S(t;q),\psi_E(t;q),\psi_I(t;q),\psi_R(t;q)],\\
(1-q)L_3[\psi_I(t;q)-\xi_{i,0}] = q\hbar H_3(t)N_I[\psi_S(t;q),\psi_E(t;q),\psi_I(t;q),\psi_R(t;q)],\\
(1-q)L_4[\psi_R(t;q)-\xi_{r,0}] = q\hbar H_4(t)N_R[\psi_S(t;q),\psi_E(t;q),\psi_I(t;q),\psi_R(t;q)],
\end{cases}$$

subject to the initial conditions

$$\psi_S(0;q)=\xi_{s,0},\ \ \psi_E(0;q)=\xi_{e,0},\ \ \psi_I(0;q)=\xi_{i,0},\ \ \psi_R(0;q)=\xi_{r,0}. \tag{18}$$

For $q = 0$ and $q = 1$, the above zeroth-order Eq. (17) has the solutions

$$\psi_S(t;0)=\xi_{s,0}(t),\ \ \psi_E(t;0)=\xi_{e,0}(t),\ \ \psi_I(t;0)=\xi_{i,0}(t),\ \ \psi_R(t;0)=\xi_{r,0}(t), \tag{19}$$

and

$$\psi_S(t;1)=\xi_s(t),\ \ \psi_E(t;1)=\xi_e(t),\ \ \psi_I(t)=\xi_i(t),\ \ \psi_R(t;1)=\xi_r(t), \tag{20}$$

Therefore, as the embedding parameter q increases from 0 to 1 , the functions $\psi_S(t;q),\psi_E(t;q),\psi_I(t;q)$ and $\psi_R(t;q)$ vary from the initial values $\xi_{s,0},\xi_{e,0},\xi_{i,0}$ and $\xi_{r,0}$ to the exact solution $\xi_s(t),\xi_e(t),\xi_i(t)$, and $\xi_r(t)$, respectively. This is the basic idea of homotopy, and this kind of variation is called deformations in topology. Expanding $\psi_S(t;q),\psi_E(t;q),\psi_I(t;q)$ and $\psi_R(t;q)$ in Taylor series for q, we have the homotopy-Maclaurin series as follows

$$\begin{cases}
\psi_S(t;q) = \xi_{s,0}(t) + \sum_{m=1}^{\infty}\xi_{s,m}(t)q^m,\\
\psi_E(t;q) = \xi_{e,0}(t) + \sum_{m=1}^{\infty}\xi_{e,m}(t)q^m,\\
\psi_I(t;q) = \xi_{i,0}(t) + \sum_{m=1}^{\infty}\xi_{i,m}(t)q^m,\\
\psi_R(t;q) = \xi_{r,0}(t) + \sum_{m=1}^{\infty}\xi_{r,m}(t)q^m,
\end{cases} \tag{21}$$

where

$$\xi_{s,m}(t)=\frac{1}{m!}\left.\frac{\partial^m\psi_S(t;q)}{\partial q^m}\right|_{q=0},\quad \xi_{e,m}(t)=\frac{1}{m!}\left.\frac{\partial^m\psi_E(t;q)}{\partial q^m}\right|_{q=0}$$

$$\xi_{i,m}(t) = \frac{1}{m!}\left.\frac{\partial^m \psi_I(t;q)}{\partial q^m}\right|_{q=0}, \quad \xi_{r,m}(t) = \frac{1}{m!}\left.\frac{\partial^m \psi_R(t;q)}{\partial q^m}\right|_{q=0}$$

where $\hbar$ is chosen in such a way that these series are convergent at $q = 1$. Thus, at $q = 1$, the homotopy series solutions become

$$\begin{cases} \psi_S(t;1) = \xi_s(t) = \xi_{s,0}(t) + \sum_{m=1}^{\infty} \xi_{s,m}(t), \\ \psi_E(t;1) = \xi_e(t) = \xi_{e,0}(t) + \sum_{m=1}^{\infty} \xi_{e,m}(t), \\ \psi_I(t;1) = \xi_i(t) = \xi_{i,0}(t) + \sum_{m=1}^{\infty} \xi_{i,m}(t), \\ \psi_R(t;1) = \xi_r(t) = \xi_{r,0}(t) + \sum_{m=1}^{\infty} \xi_{r,m}(t). \end{cases} \tag{22}$$

Now, the equation so-called m^{th}-order deformation equation is obtained as

$$\begin{cases} L_1[\xi_{s,m}(t) - \chi_m \xi_{s,m-1}(t)] = \hbar H_1(t) \mathfrak{R}_{\xi_s,m}\left(\vec{\xi}_{s,m-1}(t)\right), & m = 1, 2, ..., n \\ L_2[\xi_{e,m}(t) - \chi_m \xi_{e,m-1}(t)] = \hbar H_2(t) \mathfrak{R}_{\xi_e,m}\left(\vec{\xi}_{e,m-1}(t)\right), & m = 1, 2, ..., n \\ L_3[\xi_{i,m}(t) - \chi_m \xi_{i,m-1}(t)] = \hbar H_3(t) \mathfrak{R}_{\xi_i,m}\left(\vec{\xi}_{i,m-1}(t)\right), & m = 1, 2, ..., n \\ L_4[\xi_{r,m}(t) - \chi_m \xi_{r,m-1}(t)] = \hbar H_4(t) \mathfrak{R}_{\xi_r,m}\left(\vec{\xi}_{r,m-1}(t)\right), & m = 1, 2, ..., n \end{cases} \tag{23}$$

where

$$\chi_m = \begin{cases} 0 & m \leq 1 \\ 1 & m > 1 \end{cases}$$

with initial conditions

$$\xi_{s,m}(0) = 0, \quad \xi_{e,m}(0) = 0, \quad \xi_{i,m}(0) = 0, \quad \xi_{r,m}(0) = 0. \tag{24}$$

Defining the vector

$$\vec{\xi}_{m-1} = \{\xi_{s,m-1}(t), \xi_{e,m-1}(t), \xi_{i,m-1}(t), \xi_{r,m-1}(t)\},$$

we derive

$$\begin{aligned} \mathfrak{R}_{\xi_s,m}\left(\vec{\xi}_{s,m-1}(t)\right) =& {}_0^C D_t^\alpha \xi_{s,m-1}(t) - (1 - \chi_m)\Lambda \\ &+ \mu \sum_{k=1}^{m-1} \xi_{s,k}(t)\xi_{i,m-1-k}(t) + \lambda \xi_{s,m-1}(t), \end{aligned} \tag{25}$$

$$\begin{aligned} \mathfrak{R}_{\xi_e,m}\left(\vec{\xi}_{e,m-1}(t)\right) =& {}_0^C D_t^\alpha \xi_{e,m-1}(t) - \mu \sum_{k=1}^{m-1} \xi_{s,k}(t)\xi_{i,m-1-k}(t) \\ &+ (\lambda + \epsilon + \sigma)\xi_{e,m-1}(t), \end{aligned} \tag{26}$$

$$\mathfrak{R}_{\xi_i,m}\left(\vec{\xi}_{i,m-1}(t)\right) = {}_0^C D_t^\alpha \xi_{i,m-1}(t) - \sigma \xi_{e,m-1}(t) + (\lambda + \rho)\xi_{i,m-1}(t), \tag{27}$$

$$\mathfrak{R}_{\xi_r,m}\left(\vec{\xi}_{r,m-1}(t)\right) = {}_0^C D_t^\alpha \xi_{r,m-1}(t) - \rho \xi_{i,m-1}(t) - \epsilon \xi_{e,m-1}(t) + \lambda \xi_{r,m-1}(t). \tag{28}$$

For simplicity, we can choose the auxiliary functions $H_j(t) = 1, j = 1, 2, 3, 4$ and take $L_j = {}_0^C D_t^\alpha, j = 1, 2, 3, 4$, then the right inverse of ${}_0^C D_t^\alpha$ will be I_t^α, the Riemann-Liouville fractional integral operator. Hence, the m^{th}-order deformation Eq. (23) for $m \geq 1$ becomes

$$\begin{cases} \xi_{s,m}(t) = \chi_m \xi_{s,m-1}(t) + \hbar I_t^\alpha \left(\mathfrak{R}_{\xi_s,m}\left(\vec{\xi}_{s,m-1}(t)\right)\right), \\ \xi_{e,m}(t) = \chi_m \xi_{e,m-1}(t) + \hbar I_t^\alpha \left(\mathfrak{R}_{\xi_e,m}\left(\vec{\xi}_{e,m-1}(t)\right)\right), \\ \xi_{i,m}(t) = \chi_m \xi_{i,m-1}(t) + \hbar I_t^\alpha \left(\mathfrak{R}_{\xi_i,m}\left(\vec{\xi}_{i,m-1}(t)\right)\right), \\ \xi_{r,m}(t) = \chi_m \xi_{r,m-1}(t) + \hbar I_t^\alpha \left(\mathfrak{R}_{\xi_r,m}\left(\vec{\xi}_{r,m-1}(t)\right)\right). \end{cases} \tag{29}$$

If we choose $\xi_s(t) = \xi_s(0) = \xi_{s,0}, \xi_e(t) = \xi_e(0) = \xi_{e,0}, \xi_i(t) = \xi_i(0) = \xi_{i,0}$ and $\xi_r(t) = \xi_r(0) = \xi_{r,0}$, as initial guess approximations of $\xi_s(t), \xi_e(t), \xi_i(t)$ and $\xi_r(t)$ respectively, then two terms approximations for $\xi_s(t), \xi_e(t), \xi_i(t)$ and $\xi_r(t)$ are calculated and presented below

$$\xi_{s,1}(t) = \frac{3.7}{\alpha\Gamma(\alpha)}\hbar t^{\alpha}, \tag{30}$$

$$\xi_{e,1}(t) = \frac{0.0154}{\alpha\Gamma(\alpha)}\hbar t^{\alpha}, \tag{31}$$

$$\xi_{i,1}(t) = \frac{0.0045}{\alpha\Gamma(\alpha)}\hbar t^{\alpha}, \tag{32}$$

$$\xi_{r,1}(t) = \frac{0.0031}{\alpha\Gamma(\alpha)}\hbar t^{\alpha}, \tag{33}$$

and

$$\xi_{s,2}(t) = \frac{3.7}{\alpha\Gamma(\alpha)}\hbar t^{\alpha} + \frac{3.7}{\alpha\Gamma(\alpha)}\hbar^2 t^{\alpha} + \frac{1.326149971247507}{\alpha\Gamma(\alpha)\Gamma(\frac{1}{2}+\alpha)}4^{-\alpha}\hbar^2 t^{2\alpha}, \tag{34}$$

$$\xi_{e,2}(t) = \frac{0.0154}{\alpha\Gamma(\alpha)}\hbar t^{\alpha} - \frac{0.0154}{\alpha\Gamma(\alpha)}\hbar^2 t^{\alpha} - \frac{0.027090184657239905}{\alpha\Gamma(\alpha)\Gamma(\frac{1}{2}+\alpha)}4^{-\alpha}\hbar^2 t^{2\alpha}, \tag{35}$$

$$\xi_{i,2}(t) = \frac{0.0045}{\alpha\Gamma(\alpha)}\hbar t^{\alpha} + \frac{0.0045}{\alpha\Gamma(\alpha)}\hbar^2 t^{\alpha} + \frac{0.002107447628726658}{\alpha\Gamma(\alpha)\Gamma(\frac{1}{2}+\alpha)}4^{-\alpha}\hbar^2 t^{2\alpha}, \tag{36}$$

$$\xi_{r,2}(t) = \frac{0.0031}{\alpha\Gamma(\alpha)}\hbar t^{\alpha} + \frac{0.0031}{\alpha\Gamma(\alpha)}\hbar^2 t^{\alpha} + \frac{0.005485744668552572}{\alpha\Gamma(\alpha)\Gamma(\frac{1}{2}+\alpha)}4^{-\alpha}\hbar^2 t^{2\alpha}. \tag{37}$$

Finally, we approximate the solution $\xi_s(t), \xi_e(t), \xi_i(t)$ and $\xi_r(t)$ of the model (3) by the j^{th} truncated series

$$\begin{cases} \phi_{\xi_s,j}(t) = \sum_{m=0}^{j-1} \xi_{s,m}(t), \\ \phi_{\xi_e,j}(t) = \sum_{m=0}^{j-1} \xi_{e,m}(t), \\ \phi_{\xi_i,j}(t) = \sum_{m=0}^{j-1} \xi_{i,m}(t), \\ \phi_{\xi_r,j}(t) = \sum_{m=0}^{j-1} \xi_{r,m}(t). \end{cases} \tag{38}$$

We mention here that if we set the auxiliary parameter $\hbar = -1$ and $\alpha = 1$, then the HAM solution is the same as the Adomian decomposition solution obtained in [75] and the homotopy perturbation solution obtained in [40, 47, 48].

6.2. Convergence theorem of HAM.

Theorem: As long as the series $\xi_s(t) = \xi_{s,0}(t) + \sum_{m=1}^{\infty} \xi_{s,m}$, $\xi_e(t) = \xi_{e,0}(t) + \sum_{m=1}^{\infty} \xi_{e,m}$, $\xi_i(t) = \xi_{i,0}(t) + \sum_{m=1}^{\infty} \xi_{i,m}$, $\xi_r(t) = \xi_{r,0}(t) + \sum_{m=1}^{\infty} \xi_{r,m}$, or governed by (23) under definitions (25)-(28) are convergent, then $\xi_s(t), \xi_e(t), \xi_i(t)$ and $\xi_r(t)$) must be the solutions of the system (3).

Proof: If the series $\sum_{m=0}^{\infty} \xi_{s,m}$, $\sum_{m=0}^{\infty} \xi_{e,m}$, $\sum_{m=0}^{\infty} \xi_{i,m}$ and $\sum_{m=0}^{\infty} \xi_{r,m}$ are convergent, we can write

$$\hat{\xi}_S = \sum_{m=0}^{\infty} \xi_{s,m}, \ \hat{\xi}_E = \sum_{m=0}^{\infty} \xi_{e,m}, \ \hat{\xi}_I = \sum_{m=0}^{\infty} \xi_{i,m}, \ \hat{\xi}_R = \sum_{m=1}^{\infty} \xi_{r,m},$$

and it holds

$$\lim_{m\to\infty} \xi_{s,m}(t) = \lim_{m\to\infty} \xi_{e,m}(t) = \lim_{m\to\infty} \xi_{i,m} = \lim_{m\to\infty} \xi_{r,m} = 0.$$

From (23) and using definitions of L_1, L_2, L_3 and L_4, we have [41]

$$\begin{aligned}\hbar \sum_{m=1}^{\infty} \Re_{\xi_s,m}(t) &= \sum_{m=1}^{\infty} {}_0^C D_t^{\alpha} \left(\xi_{s,m}(t) - \chi_m \xi_{s,m-1}(t)\right), \\ &= \lim_{n\to\infty} \sum_{m=1}^{n} {}_0^C D_t^{\alpha} \left(\xi_{s,m}(t) - \chi_m \xi_{s,m-1}(t)\right), \\ &= {}_0^C D_t^{\alpha} \left(\lim_{n\to\infty} \sum_{m=1}^{n} \xi_{s,m}(t) - \chi_m \xi_{s,m-1}(t) \right), \\ &= {}_0^C D_t^{\alpha} \left(\lim_{n\to\infty} \xi_{s,n}(t) \right) = 0.\end{aligned}$$

By repeating this procedure, it can be easily shown

$$\hbar \sum_{m=1}^{\infty} \Re_{\xi_e,m}(t) = \hbar \sum_{m=1}^{\infty} \Re_{\xi_i,m}(t) = \hbar \sum_{m=1}^{\infty} \Re_{\xi_r,m}(t) = 0.$$

Since $\hbar \neq 0$, then

$$\sum_{m=1}^{\infty} \Re_{\xi_s,m}(t) = 0, \tag{39}$$

$$\sum_{m=1}^{\infty} \Re_{\xi_e,m}(t) = 0, \tag{40}$$

$$\sum_{m=1}^{\infty} \Re_{\xi_i,m}(t) = 0, \tag{41}$$

$$\sum_{m=1}^{\infty} \Re_{\xi_r,m}(t) = 0. \tag{42}$$

Substituting (25) into (39) and simplifying it, we obtain [43]

$$\begin{aligned}\Re_{\xi_s,m}(t) = \sum_{m=1}^{\infty} \Big(& {}_0^C D_t^{\alpha} \xi_{s,m-1}(t) - (1-\chi_m)\Lambda \\ & + \mu \sum_{k=1}^{m-1} \xi_{s,k}(t)\xi_{i,m-1-k}(t) + \lambda \xi_{s,m-1}(t) \Big) \\ = & {}_0^C D_t^{\alpha} \sum_{m=1}^{\infty} \xi_{s,m-1}(t) - (1 - \sum_{m=1}^{\infty} \chi_m)\Lambda \\ & + \mu \sum_{m=1}^{\infty} \sum_{k=0}^{m-1} \xi_{s,k}(t)\xi_{i,m-1-k}(t) + \lambda \sum_{m=1}^{\infty} \xi_{s,m-1}(t) \\ = & {}_0^C D_t^{\alpha} \sum_{m=1}^{\infty} \xi_{s,m-1}(t) - \Lambda \\ & + \mu \sum_{k=0}^{\infty} \sum_{m=k+1}^{\infty} \xi_{s,k}(t)\xi_{i,m-1-k}(t) + \lambda \sum_{m=1}^{\infty} \xi_{s,m-1}(t)\end{aligned} \tag{43}$$

$$
\begin{aligned}
&= {}_0^C D_t^\alpha \left(\sum_{m=0}^{\infty} \xi_{s,m}(t) \right) - \Lambda \\
&\quad + \mu \left(\sum_{k=0}^{\infty} \xi_{s,k}(t) \right) \left(\sum_{j=0}^{\infty} \xi_{i,j}(t) \right) + \lambda \sum_{m=0}^{\infty} \xi_{s,m}(t) \\
&= {}_0^C D_t^\alpha \hat{\xi}_S(t) - \Lambda + \mu \hat{\xi}_S(t) \hat{\xi}_I(t) + \lambda \hat{\xi}_S(t).
\end{aligned}
$$

By repeating the above procedure and substituting (26)-(28) into (40)-(42), respectively, and simplifying those, we obtain

$$
\begin{aligned}
\mathfrak{R}_{\xi_e,m}(t) &= \sum_{m=1}^{\infty} \left({}_0^C D_t^\alpha \xi_{e,m-1}(t) - \mu \sum_{k=0}^{m-1} \xi_{s,k}(t) \xi_{i,m-1-k}(t) + (\lambda + \epsilon + \sigma) \xi_{e,m-1}(t) \right) \\
&= {}_0^C D_t^\alpha \sum_{m=1}^{\infty} \xi_{e,m-1}(t) - \mu \sum_{m=1}^{\infty} \sum_{k=0}^{m-1} \xi_{s,k}(t) \xi_{i,m-1-k}(t) \\
&\quad + (\lambda + \epsilon + \sigma) \sum_{m=1}^{\infty} \xi_{e,m-1}(t) \\
&= {}_0^C D_t^\alpha \left(\sum_{m=0}^{\infty} \xi_{e,m}(t) \right) - \mu \left(\sum_{k=0}^{\infty} \xi_{s,k}(t) \right) \left(\sum_{j=0}^{\infty} \xi_{i,j}(t) \right) \\
&\quad + (\lambda + \epsilon + \sigma) \sum_{m=0}^{\infty} \xi_{e,m}(t) \qquad (44) \\
&= {}_0^C D_t^\alpha \hat{\xi}_E(t) - \mu \hat{\xi}_S(t) \hat{\xi}_I(t) + (\lambda + \epsilon + \sigma) \hat{\xi}_E(t).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathfrak{R}_{\xi_i,m}(t) &= \sum_{m=1}^{\infty} \left({}_0^C D_t^\alpha \xi_{i,m-1}(t) - \sigma \xi_{e,m-1}(t) + (\lambda + \rho) \xi_{i,m-1}(t) \right) \\
&= {}_0^C D_t^\alpha \sum_{m=1}^{\infty} \xi_{i,m-1}(t) - \sigma \sum_{m=1}^{\infty} \xi_{e,m-1}(t) + (\lambda + \rho) \sum_{m=1}^{\infty} \xi_{i,m-1}(t) \\
&= {}_0^C D_t^\alpha \sum_{m=0}^{\infty} \xi_{i,m}(t) - \sigma \sum_{m=0}^{\infty} \xi_{e,m}(t) + (\lambda + \rho) \sum_{m=0}^{\infty} \xi_{i,m}(t) \qquad (45) \\
&= {}_0^C D_t^\alpha \hat{\xi}_I(t) - \sigma \hat{\xi}_E(t) + (\rho + \lambda) \hat{\xi}_I(t),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathfrak{R}_{\xi_r,m}(t) &= \sum_{m=1}^{\infty} \left({}_0^C D_t^\alpha \xi_{r,m-1}(t) - \rho \xi_{i,m-1}(t) - \epsilon \xi_{e,m-1}(t) + \lambda \xi_{r,m-1}(t) \right) \\
&= {}_0^C D_t^\alpha \sum_{m=1}^{\infty} \xi_{r,m-1}(t) - \rho \sum_{m=1}^{\infty} \xi_{i,m-1}(t) \\
&\quad - \epsilon \sum_{m=1}^{\infty} \xi_{e,m-1}(t) + \lambda \sum_{m=1}^{\infty} \xi_{r,m-1}(t) \qquad (46)
\end{aligned}
$$

$$
\begin{aligned}
&= {}_0^C D_t^\alpha \sum_{m=0}^{\infty} \xi_{r,m}(t) - \rho \sum_{m=0}^{\infty} \xi_{i,m}(t) - \epsilon \sum_{m=0}^{\infty} \xi_{e,m}(t) + \lambda \sum_{m=0}^{\infty} \xi_{r,m}(t) \\
&= {}_0^C D_t^\alpha \hat{\xi}_R(t) - \rho \hat{\xi}_I(t) - \epsilon \hat{\xi}_E(t) + \lambda \hat{\xi}_R(t).
\end{aligned}
$$

Thus,

$$\sum_{m=1}^{\infty} \mathfrak{R}_{\xi_s,m}(t) = {}_0^C D_t^\alpha \hat{\xi}_S(t) - \Lambda + \mu \hat{\xi}_S(t)\hat{\xi}_I(t) + \lambda \hat{\xi}_S(t) = 0, \tag{47}$$

$$\sum_{m=1}^{\infty} \mathfrak{R}_{\xi_e,m}(t) = {}_0^C D_t^\alpha \hat{\xi}_E(t) - \mu \hat{\xi}_S(t)\hat{\xi}_I(t) + (\lambda + \epsilon + \sigma)\hat{\xi}_E(t) = 0, \tag{48}$$

$$\sum_{m=1}^{\infty} \mathfrak{R}_{\xi_i,m}(t) = {}_0^C D_t^\alpha \hat{\xi}_I(t) - \sigma \hat{\xi}_E(t) + (\rho + \lambda)\hat{\xi}_I(t) = 0, \tag{49}$$

$$\sum_{m=1}^{\infty} \mathfrak{R}_{\xi_r,m}(t) = {}_0^C D_t^\alpha \hat{\xi}_R(t) - \rho \hat{\xi}_I(t) - \epsilon \hat{\xi}_E(t) + \lambda \hat{\xi}_R(t) = 0. \tag{50}$$

From (4) and (24), it holds that

$$\hat{\xi}_{S,0} = \sum_{m=0}^{\infty} \xi_{s,m}(0) = \xi_{s,0} + \sum_{m=1}^{\infty} \xi_{s,m}(0) = 20, \tag{51}$$

$$\hat{\xi}_{E,0} = \sum_{m=0}^{\infty} \xi_{e,m}(0) = \xi_{e,0} + \sum_{m=1}^{\infty} \xi_{e,m}(0) = 0.01, \tag{52}$$

$$\hat{\xi}_{I,0} = \sum_{m=0}^{\infty} \xi_{i,m}(0) = \xi_{i,0} + \sum_{m=1}^{\infty} \xi_{i,m}(0) = 0.02, \tag{53}$$

$$\hat{\xi}_{R,0} = \sum_{m=0}^{\infty} \xi_{r,m}(0) = \xi_{r,0} + \sum_{m=1}^{\infty} \xi_{r,m}(0) = 0. \tag{54}$$

Thus $\hat{\xi}_S(t), \hat{\xi}_E(t), \hat{\xi}_I(t)$ and $\hat{\xi}_R(t)$ satisfy system (3), and it must be the exact solution for system (3) with initial conditions (4).

7. Numerical results

As stated, the HAM provides an approximate analytical solution in terms of convergent power series. There is a practical need to evaluate this solution and obtain numerical values from the convergent power series. The consequent series truncation and the practical procedure are conducted to accomplish this task. The values for parameters as discussed in table 1 are considered.

To consider the behavior of solution for susceptible $\xi_s(t)$, exposed $\xi_e(t)$, infected $\xi_i(t)$ and recovered $\xi_r(t)$ for different values of $\alpha, 0 < \alpha \leq 1$, the numerical results are presented for the model (38) with Mathematica software for the homotopy analysis method. In the appendix, the 7^{th}-order approximations for susceptible $\xi_s(t)$, exposed $\xi_e(t)$, infected $\xi_i(t)$ and recovered $\xi_r(t)$, respectively were calculated for $\alpha = 1$ and $\alpha = 0.9$.

8. Discussion

HAM provides an approximate solution in the form of convergent series by using a few iterations. The solution terms depend on time t and the auxiliary parameter $\hbar$ that can be freely chosen to adjust and control the interval of convergence. This

concept plays a key role in the HAM and is generally used to gain sufficiently accurate approximations with the smallest number of homotopy terms in the homotopy series (22). According to the convergence theorem in section 6.2, the homotopy series solution contains the auxiliary parameter $\hbar$, which provides a simple way to adjust and control the convergence of the series (22). It is imperative to ensure that the series (22) is convergent. To this end, we have plotted $\hbar$-curves of $\xi_s^{'}(0), \xi_e^{'}(0), \xi_i^{'}(0)$ and $\xi_r^{'}(0)$ under 7^{th}-order approximation of the HAM in Fig.3, respectively, for $\alpha = 0.7, \alpha = 0.8, \alpha = 0.9$ and $\alpha = 1$.

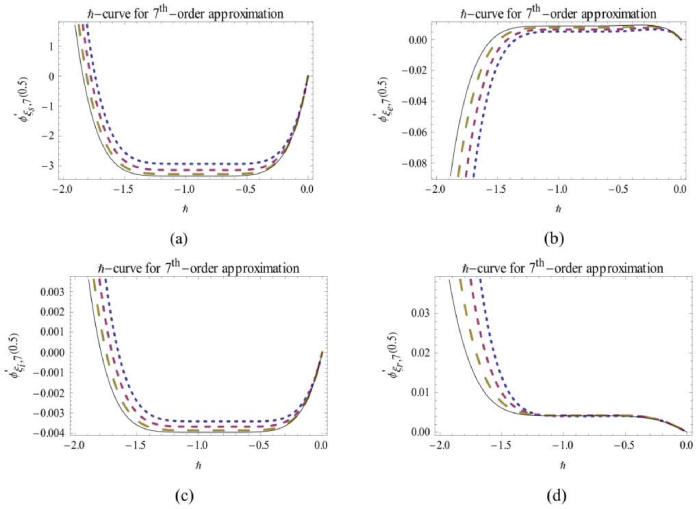

FIGURE 3. Samples of $\hbar$-curves for $\xi_s(t), \xi_e(t), \xi_i(t), and \xi_r(t)$ under 7^{th}-order approximations for various α. In these figures, dotted small line corresponds to $\alpha = 0.7$, dotted medium line corresponds to $\alpha = 0.8$, dotted large line corresponds to $\alpha = 0.9$ and solid black line corresponds to $\alpha = 1$.

According to these $\hbar$-curves, it is easy to gain the valid region the interval of convergence and optimum value for parameter $\hbar$ which corresponds to the line segment nearly parallel to the horizontal axis. For better presentation, these valid regions have been listed in table 2. We exhibit the interval of convergence of $\hbar$ and the respective optimum value $\hbar^*$ corresponding to the dynamical regime presented in Fig. 3. It is to be noted that these valid regions ensure the convergence of the obtained series.

TABLE 2. The admissible values of $\hbar$ derived from Fig. 3 under 7^{th}-order approximation.

$\alpha/\hbar$	$\hbar_1^*$	$\hbar_2^*$	$\hbar_3^*$	$\hbar_4^*$
0.7	(-1.3,-0.4)	(-1.3,-0.5)	(-1.2,-0.65)	(-1.15,-0.4)
0.8	(-1.3,-0.5)	(-1.3,-0.6)	(-1.25,-0.65)	(-1.2,-0.4)
0.9	(-1.4,-0.5)	(-1.35,-0.6)	(-1.3,-0.7)	(-1.3,-0.4)
1	(-1.4,-0.5)	(-1.35,-0.6)	(-1.3,-0.7)	(-1.3,-0.4)

TABLE 3. The minimum values of $\Delta\xi_s(\hbar_1^*)$, $\Delta\xi_e(\hbar_2^*)$, $\Delta\xi_i(\hbar_3^*)$, $\Delta\xi_r(\hbar_4^*)$ for $\alpha = 0.7$.

m	6
$\hbar_1^*$	-0.93074
$\Delta\xi_s(\hbar_1^*)$	1.91847×10^{-12}
$\hbar_2^*$	-0.90235
$\Delta\xi_e(\hbar_2^*)$	6.66134×10^{-16}
$\hbar_3^*$	-0.90043
$\Delta\xi_i(\hbar_3^*)$	3.46945×10^{-18}
$\hbar_4^*$	-0.91830
$\Delta\xi_r(\hbar_4^*)$	1.17961×10^{-18}

TABLE 4. The minimum values of $\Delta\xi_s(\hbar_1^*)$, $\Delta\xi_e(\hbar_2^*)$, $\Delta\xi_i(\hbar_3^*)$, $\Delta\xi_r(\hbar_4^*)$ for $\alpha = 0.8$.

m	6
$\hbar_1^*$	-0.94381
$\Delta\xi_s(\hbar_1^*)$	8.181907×10^{-12}
$\hbar_2^*$	-0.90262
$\Delta\xi_e(\hbar_2^*)$	5.55112×10^{-16}
$\hbar_3^*$	-0.89652
$\Delta\xi_i(\hbar_3^*)$	8.67362×10^{-18}
$\hbar_4^*$	-0.93694
$\Delta\xi_r(\hbar_4^*)$	3.46985×10^{-17}

To determine the optimal values of $\hbar$ in an interval, $[t_0, t_1]$ an error analysis is performed. A procedure to check the convergence of a homotopy-series solution is to substitute this series into the original governing equations and initial conditions, and then to evaluate the corresponding squared residual errors-the more quickly the residual error decays to zero, the faster the homotopy-series converges. In this context, an error analysis is performed in the following lines. We substitute the approximations $\phi_{\xi_s,7}, \phi_{\xi_e,7}, \phi_{\xi_i,7}$ and $\phi_{\xi_r,7}$ into the model (3) and obtain the residual functions $ER_{\xi_s}(\xi_s, \xi_e, \xi_i, \xi_r; \hbar_1), ER_{\xi_e}(\xi_s, \xi_e, \xi_i, \xi_r; \hbar_2), ER_{\xi_i}(\xi_s, \xi_e, \xi_i, \xi_r; \hbar_3)$ and $ER_{\xi_r}(\xi_s, \xi_e, \xi_i, \xi_r; \hbar_4)$ as follows

$$ER_{\xi_s,m}(\xi_s, \xi_e, \xi_i, \xi_r; \hbar_1) = {}_0^C D_t^\alpha \phi_{\xi_s,m} - \Lambda + \mu\phi_{\xi_s,m}\phi_{\xi_i,m} + \lambda\phi_{\xi_s,m}, \tag{55}$$

$$ER_{\xi_e,m}(\xi_s, \xi_e, \xi_i, \xi_r; \hbar_1) = {}_0^C D_t^\alpha \phi_{\xi_e,m} - \mu\phi_{\xi_s,m}\phi_{\xi_i,m} + (\lambda + \epsilon + \sigma)\phi_{\xi_e,m}, \tag{56}$$

$$ER_{\xi_i,m}(\xi_s, \xi_e, \xi_i, \xi_r; \hbar_1) = {}_0^C D_t^\alpha \phi_{\xi_i,m} - \sigma\phi_{\xi_e,m} + (\lambda + \rho)\phi_{\xi_i,m}, \tag{57}$$

$$ER_{\xi_r,m}(\xi_s, \xi_e, \xi_i, \xi_r; \hbar_1) = {}_0^C D_t^\alpha \phi_{\xi_r,m} - \rho\phi_{\xi_i,m} - \epsilon\phi_{\xi_e,m} + \lambda\phi_{\xi_r,m}. \tag{58}$$

Yabushita et al. [76] in 2007 and Niu and Wang [77] in 2010 suggested an optimization method for convergence control parameters. Their work is based on the squared residual error. Inspired by their approach and following the studies carried out in [40, 41], we define the square residual error for the m^{th}-order approximation

TABLE 5. The minimum values of $\Delta\xi_s(\hbar_1^*)$, $\Delta\xi_e(\hbar_2^*)$, $\Delta\xi_i(\hbar_3^*)$, $\Delta\xi_r(\hbar_4^*)$ for $\alpha = 0.9$.

m	6
$\hbar_1^*$	-0.90068
$\Delta\xi_s(\hbar_1^*)$	1.12443×10^{-12}
$\hbar_2^*$	-0.94457
$\Delta\xi_e(\hbar_2^*)$	1.11022×10^{-16}
$\hbar_3^*$	-0.89362
$\Delta\xi_i(\hbar_3^*)$	1.73472×10^{-18}
$\hbar_4^*$	-0.95145
$\Delta\xi_r(\hbar_4^*)$	3.81639×10^{-17}

TABLE 6. The minimum values of $\Delta\xi_s(\hbar_1^*)$, $\Delta\xi_e(\hbar_2^*)$, $\Delta\xi_i(\hbar_3^*)$, $\Delta\xi_r(\hbar_4^*)$ for $\alpha = 1.0$.

m	6
$\hbar_1^*$	-0.91350
$\Delta\xi_s(\hbar_1^*)$	2.51532×10^{-12}
$\hbar_2^*$	-0.86684
$\Delta\xi_e(\hbar_2^*)$	5.68434×10^{-14}
$\hbar_3^*$	-0.91145
$\Delta\xi_i(\hbar_3^*)$	6.93889×10^{-18}
$\hbar_4^*$	-0.95873
$\Delta\xi_r(\hbar_4^*)$	8.67362×10^{-18}

to be

$$\Delta\xi_{s,m}(\hbar_1) = \int_{t_0}^{t_1} \left(ER_{\xi_s,m}(\xi_s,\xi_e,\xi_i,\xi_r)\right)^2 dt, \tag{59}$$

$$\Delta\xi_{e,m}(\hbar_2) = \int_{t_0}^{t_1} (ER_{\xi_e,m}(\xi_s,\xi_e,\xi_i,\xi_r))^2 dt, \tag{60}$$

$$\Delta\xi_{i,m}(\hbar_3) = \int_{t_0}^{t_1} (ER_{\xi_i,m}(\xi_s,\xi_e,\xi_i,\xi_r))^2 dt, \tag{61}$$

$$\Delta\xi_{r,m}(\hbar_4) = \int_{t_0}^{t_1} (ER_{\xi_r,m}(\xi_s,\xi_e,\xi_i,\xi_r))^2 dt. \tag{62}$$

Values of $\hbar_1, \hbar_2, \hbar_3$ and $\hbar_4$ can be obtained for which $\Delta\xi_{s,m}(\hbar_1), \Delta\xi_{e,m}(\hbar_2)$, $\Delta\xi_{i,m}(\hbar_3)$ and $\Delta\xi_{r,m}(\hbar_4)$ are minimum. We can quickly determine the optimal values of $\hbar_1, \hbar_2, \hbar_3$ and $\hbar_4$ by using the first derivative test, i.e.,

$$\frac{d\Delta\xi_{s,m}(\hbar_1^*)}{dt} = 0,\ \frac{d\Delta\xi_{e,m}(\hbar_2^*)}{dt} = 0,\ \frac{d\Delta\xi_{i,m}(\hbar_3^*)}{dt} = 0,\ \frac{d\Delta\xi_{r,m}(\hbar_4^*)}{dt} = 0,$$

respectively. The optimal values for all of these considered cases are $\hbar_1^*, \hbar_2^*, \hbar_3^*$ and $\hbar_4^*$. The residual errors $ER_{\xi_s,7}, ER_{\xi_e,7}, ER_{\xi_i,7}$ and $ER_{\xi_r,7}$ verses $t \in (0,1)$ are demonstrated in Fig. 4. The curves of square residual errors for $\Delta\xi_{s,m}(\hbar_1)$, $\Delta\xi_{e,m}(\hbar_2)$, $\Delta\xi_{i,m}(\hbar_3)$ and $\Delta\xi_{r,m}(\hbar_4)$ under 7^{th}-order of approximation are shown in Fig. 5. For the central information regarding the order of approximation, in tables 3-6, the minimum values of $\Delta\xi_{s,m}(\hbar_1), \Delta\xi_{e,m}(\hbar_2), \Delta\xi_{i,m}(\hbar_3)$ and $\Delta\xi_{r,m}(\hbar_4)$

have been given with the optimal values of $\hbar_1^*, \hbar_2^*, \hbar_3^*$ and $\hbar_4^*$ for 7^{th}-order of approximation at different α.

In tables 7-10, the absolute error functions $ER_{\xi_s}(\xi_s, \xi_e, \xi_i, \xi_r; \hbar_1^*)$, $ER_{\xi_e}(\xi_s, \xi_e, \xi_i, \xi_r; \hbar_2^*)$, $ER_{\xi_i}(\xi_s, \xi_e, \xi_i, \xi_r; \hbar_3^*)$ and $ER_{\xi_r}(\xi_s, \xi_e, \xi_i, \xi_r; \hbar_4)$ have been calculated for various $t \in (0, 1)$ under 7^{th}-order approximation of homotopy series solution are considered. From the tables, it can be seen that the HAM provides us the accurate approximate solution for the nonlinear fractional-order HIV epidemic model (3).

TABLE 7. The absolute errors $ER_{\xi_s}, ER_{\xi_e}, ER_{\xi_i}, ER_{\xi_r}$ under 7^{th}-order approximation for various $t \in (0, 1)$ at $\alpha = 0.7$.

t/ER	ER_{ξ_s}	ER_{ξ_e}	ER_{ξ_i}	ER_{ξ_r}
0.1	-9.21903×10^{-8}	-1.33562×10^{-8}	-5.36981×10^{-10}	2.74151×10^{-9}
0.2	-2.83365×10^{-9}	-9.59958×10^{-9}	-9.55992×10^{-10}	-5.94027×10^{-9}
0.3	6.40088×10^{-8}	-6.61458×10^{-8}	-9.75765×10^{-10}	-7.73039×10^{-9}
0.4	7.00496×10^{-8}	1.10682×10^{-8}	-4.05091×10^{-10}	1.25109×10^{-9}
0.5	3.21200×10^{-8}	1.86496×10^{-8}	6.97648×10^{-10}	1.36856×10^{-8}
0.6	-2.17360×10^{-8}	1.04504×10^{-8}	1.96632×10^{-9}	1.97737×10^{-8}
0.7	-6.18548×10^{-8}	-6.47373×10^{-9}	2.89522×10^{-9}	1.38468×10^{-8}
0.8	-5.855658×10^{-8}	-2.03042×10^{-8}	3.02225×10^{-9}	-4.99411×10^{-10}
0.9	1.86520×10^{-8}	-7.87222×10^{-9}	2.09473×10^{-9}	-6.27057×10^{-9}
1.0	2.02632×10^{-7}	9.17981×10^{-9}	2.14961×10^{-10}	3.03610×10^{-8}

TABLE 8. The absolute errors $ER_{\xi_s}, ER_{\xi_e}, ER_{\xi_i}, ER_{\xi_r}$ under 7^{th}-order approximation for various $t \in (0, 1)$ at $\alpha = 0.8$.

t/ER	ER_{ξ_s}	ER_{ξ_e}	ER_{ξ_i}	ER_{ξ_r}
0.1	-3.71178×10^{-7}	-1.76670×10^{-9}	-5.02209×10^{-10}	8.80147×10^{-10}
0.2	-3.57502×10^{-9}	-1.74768×10^{-8}	-1.19780×10^{-9}	-1.69860×10^{-9}
0.3	2.98221×10^{-8}	-6.29334×10^{-9}	-1.46631×10^{-9}	-2.62406×10^{-9}
0.4	3.56682×10^{-8}	1.63788×10^{-8}	-1.58270×10^{-9}	9.45837×10^{-10}
0.5	1.64586×10^{-8}	3.25153×10^{-8}	-1.26821×10^{-9}	6.52463×10^{-9}
0.6	-1.51608×10^{-8}	3.18092×10^{-8}	-2.97747×10^{-10}	9.02485×10^{-9}
0.7	-4.24430×10^{-8}	1.31467×10^{-8}	1.33463×10^{-9}	4.57266×10^{-9}
0.8	-4.61313×10^{-8}	-1.70081×10^{-8}	3.39769×10^{-9}	-5.32229×10^{-9}
0.9	-4.14500×10^{-8}	-4.75924×10^{-8}	5.46980×10^{-9}	-9.47353×10^{-9}
1.0	1.09675×10^{-7}	-6.66007×10^{-8}	7.01876×10^{-9}	1.73069×10^{-8}

We have also shown a comparison between the solution obtained by using HAM for classical and proposed fractional-order SEIR epidemic model in Fig. 6. The solid black line corresponds to the classical SEIR model (1), i.e., for $\alpha = 1.0$, and the small dotted line corresponds to the proposed fractional-order SEIR epidemic model (3) with order $\alpha = 0.85$. From Fig. 6, it is clearly visible that the fractional-order model gives better results as compared to the integer-order model.

9. Conclusion

In this paper, a nonlinear fractional-order mathematical model for HIV epidemics is formulated. The stability analysis was examined, i.e., the local and global dynamics of the system, by analyzing the basic reproduction number $\mathfrak{R}_0$. We found

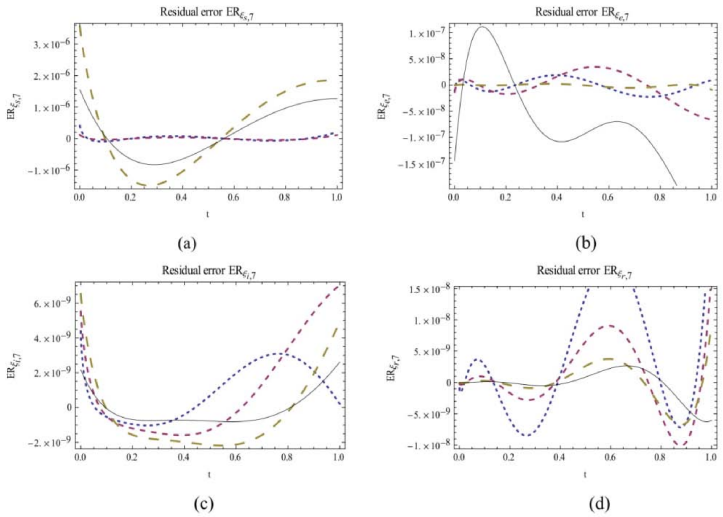

FIGURE 4. Residual errors $ER_{\xi_s,7}, ER_{\xi_e,7}, ER_{\xi_i,7}$ and $ER_{\xi_r,7}$ under 7^{th}-order approximations for various α at $t \in (0,1)$. In these figures, dotted small line corresponds to $\alpha = 0.7$, dotted medium line corresponds to $\alpha = 0.8$, dotted large line corresponds to $\alpha = 0.9$ and solid black line corresponds to $\alpha = 1$.

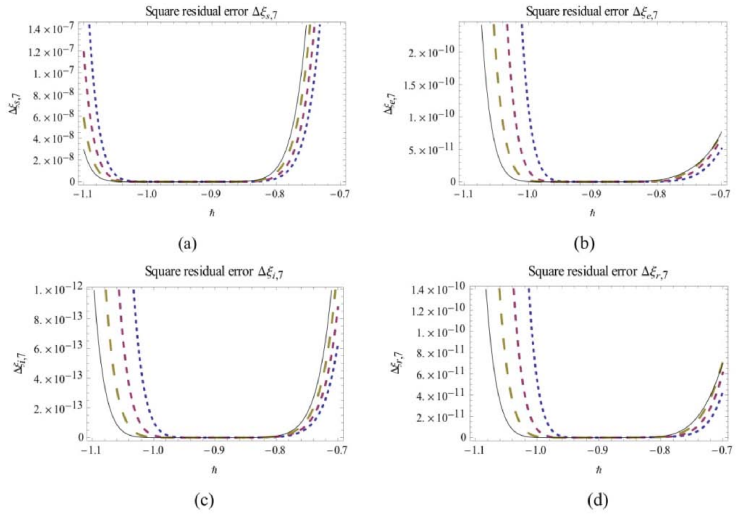

FIGURE 5. Square residual error functions for $\xi_s(t), \xi_e(t), \xi_i(t)$, and $\xi_r(t)$ under 7^{th}-order approximations versus auxiliary parameter $\hbar$ for various α. In these figures, dotted small line corresponds to $\alpha = 0.7$, dotted medium line corresponds to $\alpha = 0.8$, dotted large line corresponds to $\alpha = 0.9$ and solid black line corresponds to $\alpha = 1$.

TABLE 9. The absolute errors $ER_{\xi_s}, ER_{\xi_e}, ER_{\xi_i}, ER_{\xi_r}$ under 7^{th}-order approximation for various $t \in (0,1)$ at $\alpha = 0.9$.

t/ER	ER_{ξ_s}	ER_{ξ_e}	ER_{ξ_i}	ER_{ξ_r}
0.1	-5.71914×10^{-8}	-5.15891×10^{-10}	-1.20423×10^{-10}	2.53209×10^{-10}
0.2	-1.30749×10^{-6}	-3.55203×10^{-10}	-1.57129×10^{-9}	-4.28203×10^{-10}
0.3	-1.46270×10^{-6}	1.51138×10^{-9}	-1.83535×10^{-9}	-8.70701×10^{-10}
0.4	-1.04325×10^{-6}	1.84559×10^{-9}	-1.97975×10^{-9}	3.34607×10^{-10}
0.5	-3.66098×10^{-7}	-6.30759×10^{-10}	-2.15118×10^{-9}	2.61440×10^{-9}
0.6	3.63659×10^{-7}	-4.20502×10^{-9}	-2.12308×10^{-9}	3.74704×10^{-9}
0.7	1.01348×10^{-6}	-5.56791×10^{-9}	-1.57511×10^{-9}	1.48867×10^{-9}
0.8	1.50155×10^{-6}	-2.76920×10^{-9}	-2.52957×10^{-10}	-3.90001×10^{-9}
0.9	1.78300×10^{-6}	6.73209×10^{-10}	1.94204×10^{-9}	-6.42382×10^{-9}
1.0	1.84047×10^{-6}	9.25178×10^{-9}	4.91113×10^{-9}	9.46007×10^{-9}

TABLE 10. The absolute errors $ER_{\xi_s}, ER_{\xi_e}, ER_{\xi_i}, ER_{\xi_r}$ under 7^{th}-order approximation for various $t \in (0,1)$ at $\alpha = 1.0$.

t/ER	ER_{ξ_s}	ER_{ξ_e}	ER_{ξ_i}	ER_{ξ_r}
0.1	-5.01256×10^{-8}	1.10702×10^{-7}	-5.92502×10^{-11}	1.12652×10^{-10}
0.2	-6.63270×10^{-7}	3.90990×10^{-8}	-6.86895×10^{-10}	-7.43561×10^{-11}
0.3	-8.31066×10^{-7}	-6.58486×10^{-8}	-7.42272×10^{-10}	-4.71510×10^{-11}
0.4	-6.48510×10^{-7}	-1.07999×10^{-7}	-7.26137×10^{-10}	-2.65197×10^{-11}
0.5	-2.71119×10^{-7}	-9.48755×10^{-8}	-7.76508×10^{-10}	8.69662×10^{-11}
0.6	1.80148×10^{-7}	-7.19609×10^{-8}	-8.10260×10^{-10}	2.27310×10^{-9}
0.7	6.14076×10^{-7}	-8.09412×10^{-8}	-6.42003×10^{-10}	2.50437×10^{-9}
0.8	9.65024×10^{-7}	-1.37563×10^{-7}	-8.02285×10^{-11}	2.47102×10^{-9}
0.9	1.18921×10^{-6}	-2.24988×10^{-7}	9.99257×10^{-10}	-4.08486×10^{-9}
1.0	1.26148×10^{-6}	-2.98728×10^{-7}	2.60263×10^{-9}	-6.04070×10^{-9}

that system (3) exhibits two equilibria, namely disease-free equilibrium (E^0) and endemic equilibrium (E^*). The stability analysis, i.e., local and global stability of disease-free equilibrium (E^0) and endemic equilibrium (E^*) were studied and found that persistence or eradication of infection is independent of the initial status of the subpopulation. Sufficient conditions for the local stability of the disease-free equilibrium (E^0) point were given in terms of the basic reproduction number $\mathfrak{R}_0$ of the model. The disease-free equilibrium (E^0) has been shown to be stable for $\mathfrak{R}_0 < 1$, i.e., the disease dies out for $\mathfrak{R}_0 < 1$ and for $\mathfrak{R}_0 > 1$, it becomes unstable, and the endemic equilibrium exists. The sufficient conditions that guarantee the asymptotic stability of the endemic equilibrium(E^*) point were given. Besides this sensitivity analysis of the parameters involved in threshold parameter $\mathfrak{R}_0$ was discussed. The main goal of analyzing such techniques for the fractional-order HIV epidemic model is to help the researchers, policymakers in targeting, prevention, and treatment resources for maximum effectiveness.

Furthermore, HAM was applied to obtain an approximate analytical solution of the proposed fractional-order HIV epidemic model. It is important to note that in this method, we have some auxiliary parameters and functions. Thus, by plotting several $\hbar$-curves and finding the regions of convergence, we showed the advantages and abilities of the method. The residual and absolute errors were applied to show the efficiency and accuracy of the method. The results obtained show that the

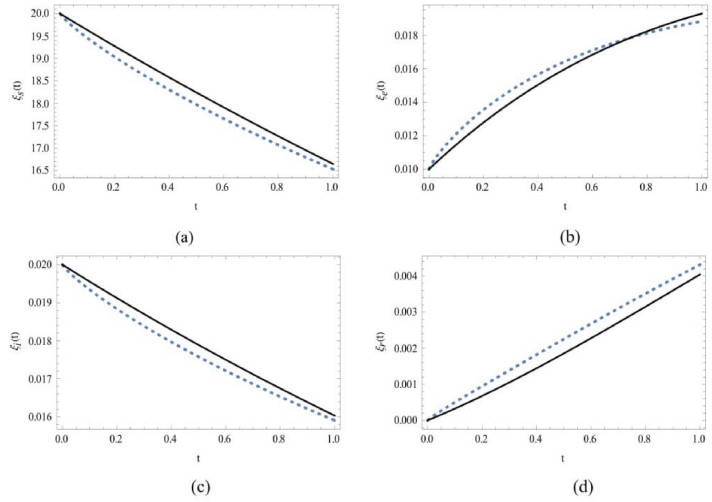

FIGURE 6. HAM solutions comparing the classical model (1) and the fractional-order model (3) with order $\alpha = 0.85$. The solid black line corresponds to the classical model (1), and the small dotted line corresponds to the fractional-order model (3).

HAM is an accurate and useful technique for getting the approximate solution of the proposed nonlinear fractional-order SEIR epidemic model. The results presented in this article are likely to inspire applications of the HAM analytical procedure for solving highly nonlinear fractional-order problems describing biological phenomena. Also, the convergence theorem of HAM to the HIV epidemic model was proved in the present paper to demonstrate the efficiency of the method.

Moreover, from the theory of fractional calculus, by considering the Caputo fractional derivatives, we realize that we have stabilized a more competent realistic model. The use of fractional calculus opens new paradigms in the area of mathematical modeling. Although the applied Caputo fractional derivative gives better results for the proposed fractional-order HIV epidemic model in the current paper, there is room for fractional-order derivatives with the operators known as Atangana-Gomez, Atangana-Baleanu, Caputo-Fabrizio, fractal-fractional, and others discovering more causative factors that are not covered in this paper, such is left for future research direction

Acknowledgments

The authors appreciate the constructive remarks and suggestions of the editor and anonymous referees that helped to improve the presentation of the paper significantly. This study was supported by the Research Fund for International Scientists (RFIS), National Natural Science Foundation of China (Grant No. 12150410306), the National Natural Science Foundation of China (Grant No. 11971375), and the China Postdoctoral Science Foundation (Grant No. 2019M663653). The funding body did not play any role in the design of the study and in writing the manuscript.

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Appendix A

The 7^{th}-order approximations for susceptible $\xi_s(t)$, exposed $\xi_e(t)$, infected $\xi_i(t)$ and recovered $\xi_r(t)$, respectively were calculated for $\alpha = 1$ and $\alpha = 0.9$.

Case 1: By using Mathematica software, we will examine the classical HIV epidemic model (3) along with (4) by setting $\alpha = 1$. The partial sums (38) are determined and in particular 7^{th}-order approximations for susceptibles $\xi_s(t)$, exposed $\xi_e(t)$, infected $\xi_i(t)$ and recovered $\xi_r(t)$, respectively were calculated, and they are presented below

$$\begin{aligned}
\phi_{\xi_s,7}(t) &= \sum_{m=0}^{6} \xi_{s,m}(t) \\
&= 20 + 22.2\hbar t + 55.5\hbar^2 t + 74.\hbar^3 t + 55.5\hbar^4 t + 22.2\hbar^5 t + 3.7\hbar^6 t \\
&\quad + 5.6115\hbar^2 t^2 + 14.964\hbar^3 t^2 + 16.8345\hbar^4 t^2 + 8.9784\hbar^5 t^2 + 1.8705\hbar^6 t^2 \\
&\quad + 0.510807\hbar^3 t^3 + 1.149321\hbar^4 t^3 + 0.919453\hbar^5 t^3 + 0.255404\hbar^6 t^3 \\
&\quad + 0.0202456\hbar^4 t^4 + 0.0323929\hbar^5 t^4 + 0.0134971\hbar^6 t^4 \\
&\quad + 0.000371781\hbar^5 t^5 + 0.000309818\hbar^6 t^5 + 2.89445 \times 10^{-5}\hbar^6 t^6, \\
\phi_{\xi_e,7}(t) &= \sum_{m=0}^{6} \xi_{e,m}(t) \\
&= 0.01 - 0.0924\hbar t - 0.23\hbar^2 t - 0.308\hbar^3 t - 0.23\hbar^4 t - 0.0924\hbar^5 t \\
&\quad - 0.0154\hbar^6 t - 0.11463\hbar^2 t^2 - 0.30568\hbar^3 t^2 - 0.34389\hbar^4 t^2 \\
&\quad - 0.183408\hbar^5 t^2 - 0.03821\hbar^6 t^2 - 0.0354428\hbar^3 t^3 - 0.0797463\hbar^4 t^3 \\
&\quad - 0.063797\hbar^5 t^3 - 0.0177214\hbar^6 t^3 - 0.00414724\hbar^4 t^4 \\
&\quad - 0.00663559\hbar^5 t^4 - 0.00276483\hbar^6 t^4 - 0.000200471\hbar^5 t^5 \\
&\quad - 0.000167059\hbar^6 t^5 - 3.39057 \times 10^{-6}\hbar^6 t^6, \\
\phi_{\xi_i,7}(t) &= \sum_{m=0}^{6} \xi_{i,m}(t) \\
&= 0.02 + 0.027\hbar t + 0.0675\hbar^2 t + 0.09\hbar^3 t + 0.0675\hbar^4 t + 0.0267\hbar^5 t \\
&\quad + 0.0045\hbar^6 t + 0.00891\hbar^2 t^2 + 0.02378\hbar^3 t^2 + 0.0267525\hbar^4 t^2 \\
&\quad + 0.014268\hbar^5 t^2 + 0.0029725\hbar^6 t^2 + 0.00142103\hbar^3 t^3 + 0.00319732\hbar^4 t^3
\end{aligned}$$

$$
\begin{aligned}
&+ 0.002557865\hbar^5t^3 + 0.000710517\hbar^6t^3 + 0.000127737\hbar^4t^4\\
&+ 0.00020438\hbar^5t^4 + 0.0000852\hbar^6t^4 + 5.66816\times10^{-5}\hbar^5t^5\\
&+ 4.72347\times10^{-6}\hbar^6t^5 + 9.18996\times10^{-8}\hbar^6t^6,\\
\phi_{\xi_r,7}(t) &= \sum_{m=0}^{6}\xi_{r,m}(t)\\
&= -0.0186\hbar t - 0.0465\hbar^2t - 0.062\hbar^3t - 0.0465\hbar^4t - 0.0186\hbar^5t\\
&- 0.0031\hbar^6t + 0.0232125\hbar^2t^2 + 0.0619\hbar^3t^2\\
&+ 0.0696375\hbar^4t^2 + 0.03714\hbar^5t^2 + 0.0077375\hbar^6t^2 + 0.0146811\hbar^3t^3\\
&+ 0.0330325\hbar^4t^3 + 0.026426\hbar^5t^3 + 0.00734055\hbar^6t^3 + 0.00220393\hbar^4t^4\\
&+ 0.003526294\hbar^5t^4 + 0.001469297\hbar^6t^4 + 0.000117901\hbar^5t^5\\
&+ 0.000098251\hbar^6t^5 + 2.04244\times10^{-6}\hbar^6t^6.
\end{aligned}
$$

Case 2: By using Mathematica software, we will examine the fractional-order HIV epidemic model (3) along with (4) by setting $\alpha = 0.9$, and similar results are obtained for $\alpha = 0.8, 0.7$. The partial sums (38) are determined and in particular 7^{th}-order approximations for susceptibles $\xi_s(t)$, exposed $\xi_e(t)$, infected $\xi_i(t)$ and recovered $\xi_r(t)$, respectively were calculated, and they are presented below

$$
\begin{aligned}
\phi_{\xi_s,7}(t) &= \sum_{m=0}^{6}\xi_{s,m}(t)\\
&=20 + 23.0825\hbar t^{0.9} + 57.7064\hbar^2t^{0.9} + 76.9418\hbar^3t^{0.9} + 57.7064\hbar^4t^{0.9}\\
&+ 23.0825\hbar^5t^{0.9} + 3.84709\hbar^6t^{0.9} + 6.69434\hbar^2t^{1.8} + 17.8516\hbar^3t^{1.8}\\
&+ 20.283\hbar^4t^{1.8} + 10.7109\hbar^5t^{1.8} + 2.23145\hbar^6t^{1.8} + 0.734111\hbar^3t^{2.7}\\
&+ 1.65175\hbar^4t^{2.7} + 1.3214\hbar^5t^{2.7} + 0.367055\hbar^6t^{2.7} + 0.0360963\hbar^4t^{3.6}\\
&+ 0.0577541\hbar^5t^{3.6} + 0.0240642\hbar^6t^{3.6} + 0.0008327\hbar^5t^{4.5}\\
&+ 0.000693983\hbar^6t^{4.5} + 8.05757\times10^{-6}\hbar^6t^{5.4},\\
\phi_{\xi_e,7}(t) &= \sum_{m=0}^{6}\xi_{e,m}(t)\\
&=0.01 - 0.0960733\hbar t^{0.9} - 0.240183\hbar^2t^{0.9} - 0.320244\hbar^3t^{0.9}\\
&- 0.240183\hbar^4t^{0.9} - 0.0960733\hbar^5t^{0.9} - 0.0160122\hbar^6t^{0.9} - 0.13675\hbar^2t^{1.8}\\
&- 0.364666\hbar^3t^{1.8} - 0.41025\hbar^4t^{1.8} - 0.2188\hbar^5t^{1.8} - 0.0455833\hbar^6t^{1.8}\\
&- 0.050240\hbar^3t^{2.7} - 0.11304\hbar^4t^{2.7} - 0.090432\hbar^5t^{2.7} - 0.02512\hbar^6t^{2.7}\\
&- 0.00717766\hbar^4t^{3.6} - 0.0114843\hbar^5t^{3.6} - 0.00478511\hbar^6t^{3.6}\\
&- 0.000432176\hbar^5t^{4.5} - 0.000360147\hbar^6t^{4.5} - 9.2256\times10^{-6}\hbar^6t^{5.4},\\
\phi_{\xi_i,7}(t) &= \sum_{m=0}^{6}\xi_{i,m}(t)\\
&=0.02 + 0.0280734\hbar t^{0.9} + 0.0701834\hbar^2t^{0.9} + 0.0935779\hbar^3t^{0.9}\\
&+ 0.0701834\hbar^4t^{0.9} + 0.0280734\hbar^5t^{0.9} + 0.00467889\hbar^6t^{0.9}
\end{aligned}
$$

$$
\begin{aligned}
&+ 0.0106383\hbar^2 t^{1.8} + 0.0283688\hbar^3 t^{1.8} + 0.0319149\hbar^4 t^{1.8} \\
&+ 0.0170213\hbar^5 t^{1.8} + 0.0035461\hbar^6 t^{1.8} + 0.00204433\hbar^3 t^{2.7} \\
&+ 0.00459975\hbar^4 t^{2.7} + 0.0036798\hbar^5 t^{2.7} + 0.0010221\hbar^6 t^{2.7} \\
&+ 0.000227353\hbar^4 t^{3.6} + 0.000363765\hbar^5 t^{3.6} + 0.000151569\hbar^6 t^{3.6} \\
&+ 0.000012687\hbar^5 t^{4.5} + 0.0000105725\hbar^6 t^{4.5} + 2.62249 \times 10^{-7}\hbar^6 t^{5.4},
\end{aligned}
$$

$$
\begin{aligned}
\phi_{\xi_r,7}(t) &= \sum_{m=0}^{6} \xi_{r,m}(t) \\
&= 0 - 0.019339\hbar t^{0.9} - 0.0483486\hbar^2 t^{0.9} - 0.0644648\hbar^3 t^{0.9} \\
&\quad - 0.0483486\hbar^4 t^{0.9} - 0.0193394\hbar^5 t^{0.9} - 0.00322324\hbar^6 t^{0.9} \\
&\quad + 0.0276918\hbar^2 t^{1.8} + 0.0738447\hbar^3 t^{1.8} + 0.0830753\hbar^4 t^{1.8} \\
&\quad + 0.0443068\hbar^5 t^{1.8} + 0.00923059\hbar^6 t^{1.8} + 0.0211206\hbar^3 t^{2.7} \\
&\quad + 0.0475213\hbar^4 t^{2.7} + 0.0380171\hbar^5 t^{2.7} + 0.0105603\hbar^6 t^{2.7} \\
&\quad + 0.0039091\hbar^4 t^{3.6} + 0.00625456\hbar^5 t^{3.6} + 0.00260606\hbar^6 t^{3.6} \\
&\quad + 0.000262745\hbar^5 t^{4.5} + 0.000218954\hbar^6 t^{4.5} + 5.80343 \times 10^{-6}\hbar^6 t^{5.4}.
\end{aligned}
$$

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