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SPLITTING SCHEMES FOR SOME SECOND-ORDER EVOLUTIONARY EQUATIONS

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Abstract. We consider the Cauchy problem for a second-order evolutionary equation, in which the problem operator is the sum of two self-adjoint operators. The main feature of the problem is that one of the operators is represented in the form of the product of the operator A by its conjugate operator A^*. Time approximations are implemented so that the transition to a new level in time is associated with a separate solution of problems for operators A and A^*, not their products. The construction of unconditionally stable schemes is based on general results of the theory of stability (well-posedness) of operator-difference schemes in Hilbert spaces and is associated with the multiplicative perturbation of the problem operators, which lead to stable implicit schemes. As an example, the problem of the dynamics of a thin plate on an elastic foundation is considered.

Key words. Second-order evolutionary equation, Cauchy problem, explicit schemes, splitting schemes, vibrations of a thin plate.

1. Introduction

Many applied problems lead to the need for an approximate solution of the Cauchy problem for second-order evolutionary equations [2]. As a typical example, we note the dynamic problems of solid mechanics [3]. A class of problems can be distinguished, a characteristic feature of which is that the main part of the problem operator is the product of two operators. For example, when considering models of thin plates we have a biharmonic operator, the product of two Laplace operators.

Unconditionally stable schemes for these problems are built based on implicit approximations in time [5, 8]. In the theory of stability (well-posedness) of operator-difference schemes [9, 10] the most complete results were obtained on the stability of two-level and three-level schemes in Hilbert spaces. The computational complexity of solving the Cauchy problem at a new level in time using implicit schemes may be unacceptable. Therefore, various approaches are being developed to obtain computationally simpler problems when solving non-stationary problems.

Simplification of the problem at a new level is often implemented for evolutionary problems when the problem operator is represented in the form the sums of more simple operators. For such problems, additive operator-difference schemes are constructed, which are related to one or another inhomogeneous approximation in time for individual operator terms. The traditional approach is based on explicit-implicit approximations (IMEX methods) [1, 4] when one part of the problem operator is taken from the lower level in time (explicit approximation), and the other — from the upper one (implicit approximation). This idea of time-stepping is implemented most consistently when constructing splitting schemes [6, 17]. In this case, the transition to a new level in time is carried out by solving evolutionary problems for individual operator terms [19, 20].

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Another class of evolutionary problems can also be noted, in which the problem operator is represented as the product of two or more operators. An example is nonstationary problems with a variable weighting factor, the study of which is held in [10, 13]. Special time approximations are constructed to simplify the problem at a new time level. For example, paper [18] investigates schemes that are based on the solution of a discrete problem at a new time level with one operator factor.

In this paper, we consider the Cauchy problem for a second-order evolutionary equation in which the problem operator includes the product of operator A by its conjugate operator A^*. Unconditionally stable schemes are constructed based on a perturbation of both the A operator and the A^* operator. In this case, the computational implementation is associated with the separate solution of problems for operators A and A^*, not their products.

The article is organized as follows. The statement of the Cauchy problem for a second-order evolutionary equation, which includes the product of the operator A and A^*, is given in Section 2. Section 3 describes a general approach to constructing unconditionally stable schemes for second-order evolutionary equations based on multiplicative perturbation of the operators of the problem. Splitting schemes for the evolutionary problem, when the problem operator includes A^*A, are constructed in Section 4. In Section 5, our schemes are applied to the model problem of the dynamics of a thin plate on an elastic foundation. The results of our work are summarized in Section 6.

2. Problem statement

The Cauchy problem for a second-order evolutionary equation is considered in a finite-dimensional Hilbert space H. Omitting technical details, we restrict ourselves to the following homogeneous equation when

$$\frac{d^2 w}{dt^2} + A^*Aw + Bw = 0, \quad 0 < t \le T, \tag{1}$$

$$w(0) = w^0, \quad \frac{dw}{dt}(0) = \widetilde{w}^0. \tag{2}$$

Assume that the operators A and B in (1) are constant (do not depend on t), and the operator B is self-adjoint and non-negative:

$$B = B^* \ge 0. \tag{3}$$

We arrive at the problem (1)-(3), for example, after discretization by spatial variables in the numerical solution of initial-boundary value problems for hyperbolic equations. The key feature of the problem under consideration is associated with the operator A, so that it enters equation (1) as the product A^*A. An example of such a construction is the biharmonic operator ($A = A^*$).

The scalar product for $u, v \in H$ is (u, v), and the norm is $\|u\| = (u, u)^{1/2}$. Let us define a Hilbert space H_S with the scalar product and norm $(u, v)_S = (Su, v)$, $\|u\|_S = (u, v)_S^{1/2}$, which is generated by the self-adjoint and positive definite operator S.

The subject of our consideration is time-stepping for equation (1). We focus on unconditionally stable schemes for an approximation solution to the problem (1)-(3), which are convenient for computational implementation. When obtaining the corresponding stability estimates we compare them with a priori estimates for the differential problem.

We multiply the equation (1) scalarly in H by dw/dt and obtain

$$\frac{d}{dt}\left(\left\|\frac{dw}{dt}\right\|^2 + \|Aw\|^2 + \|w\|_B^2\right) = 0.$$

This equality implies the estimate

$$\left\|\frac{dw}{dt}(t)\right\|^2 + \|Aw(t)\|^2 + \|w(t)\|_B^2 = \|\widetilde{w}^0\|^2 + \|Aw^0\|^2 + \|w^0\|_B^2, \tag{4}$$

which ensures stability with respect to the initial data of the solution to the problem (1)–(3).

We will use a uniform, for simplicity, grid in time with step τ and notation $u^n = u(t^n)$, $t^n = n\tau$, $n = 0, \dots, N$, $N\tau = T$. As a basic scheme for the numerical solution of the problem (1)–(3) we will use a three-level scheme with weights ($\sigma =$ const):

$$\begin{gathered}\frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2} + (A^*A + B)(\sigma u^{n+1} + (1-2\sigma)u^n + \sigma u^{n-1}) = 0,\\ n = 1, \dots, N-1,\end{gathered} \tag{5}$$

with the initial conditions

$$u^0 = w^0, \quad u^1 = \overline{w}^1. \tag{6}$$

We have on the solutions of the problem (1), (2)

$$\begin{aligned} w(\tau) &= w(0) + \tau \frac{dw}{dt}(0) + \frac{\tau^2}{2}\frac{d^2w}{dt^2}(\tau) + \mathcal{O}(\tau^3) \\ &= w^0 + \tau\widetilde{w}^0 - \frac{\tau^2}{2}(A^*A + B)w(\tau) + \mathcal{O}(\tau^3). \end{aligned}$$

For the second initial condition, we put

$$\left(I + \frac{\tau^2}{2}(A^*A + B)\right)\overline{w}^1 = w^0 + \tau\widetilde{w}^0. \tag{7}$$

Difference scheme (5), (6) approximates (1), (2) with the second order in τ.

Note that explicit approximations of the type

$$\overline{w}^1 = \left(I - \frac{\tau^2}{2}(A^*A + B)\right)w^0 + \tau\widetilde{w}^0.$$

are unacceptable due to the overestimated requirements on the smoothness of the initial conditions. Therefore, we focus on the implicit approximations (7).

Our consideration is based on the use of general results in the theory of stability (well-posedness) of operator-difference schemes in Hilbert spaces [9, 10]. The main statement on the stability of three-level schemes for the problems under consideration is formulated as follows.

Lemma 1. *Let in the three-level scheme*

$$C\frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2} + Du^n = 0, \quad n = 1, \dots, N-1, \tag{8}$$

with the initial conditions (6), the operators

$$C = C^* > 0, \quad D = D^* > 0. \tag{9}$$

Then at

$$G = C - \frac{\tau^2}{4}D > 0 \tag{10}$$

the scheme (6)–(9) is stable and a priori equality

$$\left\| \frac{u^{n+1} - u^n}{\tau} \right\|_G^2 + \left\| \frac{u^{n+1} + u^n}{2} \right\|_D^2 = \left\| \frac{\overline{w}^1 - w^0}{\tau} \right\|_G^2 + \left\| \frac{\overline{w}^1 + w^0}{2} \right\|_D^2, \tag{11}$$

holds for all $n = 1, \dots, N-1$.

Proof. Taking into account

$$u^n = \frac{u^{n+1} + 2u^n + u^{n-1}}{4} - \frac{\tau^2}{4} \frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2},$$

rewrite (8) in the form

$$G \frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2} + D \frac{u^{n+1} + 2u^n + u^{n-1}}{4} = 0. \tag{12}$$

Let's introduce new variables

$$s^n = \frac{u^n + u^{n-1}}{2}, \quad r^n = \frac{u^n - u^{n-1}}{\tau},$$

and from (12) we arrive at the equation

$$G \frac{r^{n+1} - r^n}{\tau} + D \frac{s^{n+1} + s^n}{2} = 0.$$

Let's multiply it by

$$2(s^{n+1} - s^n) = \tau(r^{n+1} + r^n),$$

what gives

$$(Gr^{n+1}, r^{n+1}) + (Ds^{n+1}, s^{n+1}) = (Gr^n, r^n) + (Ds^n, s^n).$$

Returning to the original variables, we obtain that the equality (11) holds. □

Application of this Lemma to a weighted scheme (3), (5), (6) brings us to the next statement.

Theorem 2. *Three-level scheme (3), (5), (6) is unconditionally stable at* $\sigma \geq 1/4$. *Under these constraints, for an approximate solution of the problem, we have a priori equality (11), wherein*

$$G = I + \left(\sigma - \frac{1}{4} \right) \tau^2 D, \quad D = A^* A + B,$$

and I *is the identity operator.*

Proof. We write (5) in the form (8) for

$$C = I + \sigma \tau^2 D.$$

Conditions (9) for $\sigma \geq 0$ and (3) are satisfied, and the inequality (10) results in $\sigma \geq 1/4$ constraints. Thus, all conditions of the Lemma 1 are fulfilled. □

When using the scheme (5), (6) at a new $n+1$ level, the problem is solved

$$(I + \sigma \tau^2 (A^* A + B)) u^{n+1} = \varphi^n$$

with the known right-hand side φ^n. The computational complexity of this problem may be unacceptable and therefore it is necessary to simplify the problem at a new level in time by using special time approximations. In our case, we want to ensure the transition to a new level in time by solving individual problems for operators A and A^*, avoiding solving a more complex problem with the product of these operators.

3. Unconditionally stable schemes with multiplicative regularization

The principle of regularization of difference schemes provides great opportunities for constructing difference schemes of a given quality [7, 17]. Results of the theory of regularization for difference schemes are used to improve the quality of the difference scheme due to introducing regularizers into the operators of the original difference scheme [12, 16]. The regularization principle for constructing unconditionally stable difference schemes is implemented as follows:

(1) for the problem under consideration, we introduce the simplest difference scheme (generating difference scheme), not possessing the necessary properties, that is, the scheme is conditionally stable or even absolutely unstable;
(2) the difference scheme is written in a unified (canonical) form, for which stability conditions are known;
(3) the quality of the difference scheme (its stability) improves via the perturbation of the difference scheme operators.

Thus, the principle of regularization of difference schemes is based on the use of already known general stability conditions, which are given by the theory of stability (well-posedness) of operator-difference schemes.

Consider the model Cauchy problem for the equation

$$\frac{d^2w}{dt^2} + Qw = 0, \quad 0 < t \le T, \tag{13}$$

with a constant, self-adjoint, and positive definite in H linear operator Q. Following the regularization principle, we first choose some difference scheme for the problem (2), (13), from which we will start. As such a generating scheme, it is natural to consider the simplest explicit scheme

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2} + Qu^n = 0, \quad n = 1, \dots, N-1, \tag{14}$$

with the initial conditions (6).

To use Lemma 1, we write the difference scheme (14) in the form (8) with the operators $C = I,\ D = Q$. Taking into account that $Q \le \|Q\| I$, from (10) we get a time step constraint

$$\tau \le \tau_0 = \frac{2}{\|Q\|^{1/2}}$$

for the stability of the scheme (2), (13).

By (10), an increase in the stability of the difference scheme (8) can be achieved twofold. In the first case, via increasing the energy (Cy, y) of the operator C or by reducing the account energy of the operator D. The first possibility of constructing stable difference schemes is based on using additive regularization: increasing operator C or/and decreasing operator D due to additional terms. The second possibility is related to the multiplicative perturbation of the operators of the generating scheme.

With the multiplicative regularization of the operator C, for example, we will replace $C \longmapsto C(I + \mu R)$ or $C \longmapsto (I + \mu R)C$, where R is a regularizing operator and μ is a regularization parameter. With such a perturbation, we remain in the class of schemes with self-adjoint operators if $RC = CR^*$. An example of a more complex regularization is given by the transformation

$$C \longmapsto (I + \mu R^*)C(I + \mu R).$$

The multiplicative regularization is carried out similarly using the perturbation operator D. Taking into account the inequality (10), we can implement transformation $D \longmapsto D(I+\mu R)^{-1}$ or $D \longmapsto (I+\mu R)^{-1}D$. For the simplest two-level schemes, such a regularization can be treated as a new edition of the regularization of the operator C. To stay in the class of schemes with self-adjoint operators, it is enough to choose $R = R(D)$. We have great opportunities for regularization

$$D \longmapsto (I+\mu R^*)^{-1} D (I+\mu R)^{-1}.$$

In this case, the regularizing operator R may not be directly bind with operator D.

Under perturbation of the operator Q from (14), we arrive at the scheme

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2} + \widetilde{Q} u^n = 0, \quad n = 1, \ldots, N-1. \tag{15}$$

For multiplicative regularization, we have, for example, $\widetilde{Q} = \widetilde{R} Q$. In the simplest case $\widetilde{R} = (I + \mu Q)^{-1}$, from (15), we obtain a regularized scheme

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2} + (I+\mu Q)^{-1} Q u^n = 0, \quad n = 1, \ldots, N-1. \tag{16}$$

The scheme (16) is related to the additive regularization of the operator with the time derivative: $C \longmapsto C + \mu Q, \ C = I$.

Checking the inequality (10) gives that for

$$\mu = \sigma \tau^2, \quad \sigma \geq \frac{1}{4},$$

the regularized scheme (16), (6) is stable. This scheme is directly related to the conventional weighted scheme for equation (13):

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2} + Q(\sigma u^{n+1} + (1-2\sigma) u^n + \sigma u^{n-1}) = 0,$$

whose stability conditions are well known [8, 9].

In the case of an additive representation of the operator Q, stable splitting schemes can be constructed based on the perturbation of the operator terms. Let in the equation (13)

$$Q = \sum_{\alpha=1}^{p} Q_\alpha, \quad Q_\alpha^* = Q_\alpha \geq 0, \quad \alpha = 1, \ldots, p.$$

Similarly to (15), (16), we will use the scheme

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2} + \sum_{\alpha=1}^{p} \widetilde{Q}_\alpha u^n = 0, \quad n = 1, \ldots, N-1, \tag{17}$$

wherein

$$\widetilde{Q}_\alpha = (I + \mu_\alpha Q_\alpha)^{-1} Q_\alpha, \quad \alpha = 1, \ldots, p.$$

In the simplest case of equal weights $\mu_\alpha, \ \alpha = 1, \ldots, p$, this additive scheme will be stable when

$$\mu_\alpha = \sigma_\alpha \tau^2, \quad \sigma_\alpha = \sigma \geq \frac{p}{4}, \quad \alpha = 1, \ldots, p.$$

Thus, stability is ensured by increasing the weighting factors.

The implementation of the scheme (17) can be carried out based on solving independent problems

$$\frac{u_\alpha^{n+1} - 2u^n + u^{n-1}}{p\tau^2} + \widetilde{Q}_\alpha u^n = 0, \quad \alpha = 1, \ldots, p,$$

and determining the solution at a new level in time according to the rule

$$u^{n+1} = \frac{1}{p} \sum_{\alpha=1}^{p} u_\alpha^{n+1}, \quad n = 1, \ldots, N-1.$$

Such an organization of computations corresponds to the use of an additive-averaged scheme [17].

We separately note the possibilities of multiplicative regularization for problems with the product of operators. Let in the equation (13) $Q = A^*A > 0$ and $A > 0$. Standard multiplicative regularization when in (15)

$$\widetilde{Q} = (I + \mu A^*A)^{-1} A^*A,$$

maybe unacceptable due to the need to solve the problem with the operator $(I + \mu_\alpha A^*A)$. Therefore, it seems reasonable to consider the option with perturbation of each operator factor in $Q = A^*A$. For example, put

$$\widetilde{Q} = (I + \mu A^*)^{-1} A^*A (I + \mu A)^{-1}. \tag{18}$$

Under the conditions (18), the inequality (10) for $C = I$, $D = \widetilde{Q}$ is satisfied if

$$(I + \mu A^*)(I + \mu A) \geq \frac{\tau^2}{4} A^*A.$$

Thus, it suffices to put

$$\mu = \sigma\tau, \quad \sigma \geq \frac{1}{2}. \tag{19}$$

The main potential drawback of the regularization (18), (19) is related to the fact that

$$\widetilde{Q} = Q + (A^* + A)\, \mathcal{O}(\tau).$$

In case (16) we have

$$\widetilde{Q} = Q + A^*A\, \mathcal{O}(\tau^2),$$

that is, the perturbation is associated with the second order in τ.

4. Regularized scheme

Now we can construct an unconditionally stable scheme based on the multiplicative regularization for our problem (1)–(3). It is associated with the perturbation of the operators A^*A and B and has the form

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2} + \widetilde{A^*A} u^n + \widetilde{B} u^n = 0, \quad n = 1, \ldots, N-1. \tag{20}$$

This scheme is written in the form (8) with

$$C = I, \quad D = \widetilde{A^*A} + \widetilde{B}.$$

According to Lemma 1, stability will be ensured, in particular, for

$$I \geq \frac{\tau^2}{2} \widetilde{A^*A}, \quad I \geq \frac{\tau^2}{2} \widetilde{B}. \tag{21}$$

For the regularizing operator $\widetilde{B}$ put

$$\widetilde{B} = (I + \sigma_B \tau^2 B)^{-1} B, \quad \sigma_B > 0. \tag{22}$$

We will perturb the operator A^*A according to (18):

$$\widetilde{A^*A} = (I + \sigma_A \tau A^*)^{-1} A^*A (I + \sigma_A \tau A)^{-1}, \quad \sigma_A > 0. \tag{23}$$

For such $\widetilde{A^*A}$ and $\widetilde{B}$ the inequalities (21) will hold for the following restrictions on weight parameters:

$$\sigma_A^2 \geq \frac{1}{2}, \quad \sigma_B \geq \frac{1}{2}. \tag{24}$$

The result of our consideration is the following statement.

Theorem 3. *The regularized scheme (6), (20), (22), (23) is unconditionally stable for the constraints (24).*

When using the proposed splitting scheme, it is necessary to use the splitting scheme when specifying the second initial condition (6). Instead of (7) we can use

$$\overline{w}^1 = \left(I - \frac{\tau^2}{2} (\widetilde{A^*A} + \widetilde{B}) \right) w^0 + \tau \widetilde{w}^0$$

under the constraints (24).

Similarly to (17), schemes with additional splitting are constructed. The simplest variant is associated with splitting operator B. Let in (3)

$$B = \sum_{\alpha=1}^{p} B_\alpha, \quad B_\alpha^* = B_\alpha \geq 0, \quad \alpha = 1, \dots, p.$$

In the scheme (20), (23) we define

$$\widetilde{B} = \sum_{\alpha=1}^{p} (I + \sigma_B \tau^2 B_\alpha)^{-1} B_\alpha. \tag{25}$$

The stability of the scheme (6), (20), (23), (25) takes place, for example, for

$$\sigma_A^2 \geq \frac{1}{2}, \quad \sigma_B \geq \frac{p}{2}.$$

Similarly, we consider the case of splitting the operator A^*A, when

$$A^*A = \sum_{\alpha=1}^{p} A_\alpha^* A_\alpha, \quad \widetilde{A^*A} = \sum_{\alpha=1}^{p} (I + \sigma_A \tau A_\alpha^*)^{-1} A_\alpha^* A_\alpha (I + \sigma_A \tau A_\alpha)^{-1}.$$

In this case, the scheme (6), (20), (22) is unconditionally stable for

$$\sigma_A^2 \geq \frac{p}{2}, \quad \sigma_B \geq \frac{1}{2}.$$

The variant of splitting the operator's A and A^* deserves special attention when

$$A = \sum_{\alpha=1}^{p} A_\alpha, \quad A^* = \sum_{\alpha=1}^{p} A_\alpha^*, \quad \alpha = 1, \dots, p.$$

For $\widetilde{A^*A}$ put

$$\widetilde{A^*A} = \sum_{\alpha=1}^{p} \widetilde{A}_\alpha^* \sum_{\alpha=1}^{p} \widetilde{A}_\alpha, \quad \widetilde{A}_\alpha = (I + \sigma_A \tau A_\alpha)^{-1} A_\alpha, \quad \alpha = 1, \dots, p. \tag{26}$$

In this case, we have

$$(\widetilde{A^*A} u, u) = \left(\Big(\sum_{\alpha=1}^{p} \widetilde{A}_\alpha u \Big)^2, 1 \right) \leq p \sum_{\alpha=1}^{p} \left(\Big(\widetilde{A}_\alpha u \Big)^2, 1 \right) = p \sum_{\alpha=1}^{p} \Big(\widetilde{A}_\alpha^* \widetilde{A}_\alpha u, u \Big).$$

With this in mind, inequalities (21) will be satisfied with

$$\sigma_A^2 \geq \frac{p^2}{2}, \quad \sigma_B \geq \frac{1}{2},$$

which ensures the stability of the scheme (6), (20), (22), (26).

5. Numerical experiments

The possibilities of using the constructed splitting schemes will be illustrated by the results of the numerical solution of a model problem. We consider the most evident test problem in a two-dimensional regular computational domain under trivial boundary conditions with a basic biharmonic operator. We use difference approximations on a uniform grid in space, when the operators A and B are easy to construct, and we can write down the approximate solution itself explicitly. The calculations performed have only methodological significance; the calculated data complement our general theoretical consideration. We can use the developed computational splitting technology for more significant applied problems; these problems are characterized, in particular, by complex computational domains, equations with variable coefficients, more general boundary conditions, and finite element approximation in space.

Assume that the computational domain is a rectangle

$$\Omega = \{\boldsymbol{x} \mid \boldsymbol{x} = (x_1, x_2),\ 0 < x_\alpha < l_\alpha,\ \alpha = 1, 2\},$$

with the boundary $\partial\Omega$. We need to find a solution $v(\boldsymbol{x}, t)$ of the equation

$$\frac{\partial^2 v}{\partial t^2} + \triangle^2 v + \gamma_1 v - \gamma_2 \triangle v = 0, \quad \boldsymbol{x} \in \Omega, \quad 0 < t \le T, \tag{27}$$

where $\gamma_1 = \text{const} > 0$, $\gamma_2 = \text{const} > 0$, and $\triangle = \operatorname{div}\operatorname{grad}$ is the Laplace operator. Equation (27) is supplemented with the following boundary and initial conditions:

$$v(\boldsymbol{x}, t) = 0, \quad \triangle v(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in \partial\Omega, \tag{28}$$

$$v(\boldsymbol{x}, 0) = v^0(\boldsymbol{x}), \quad \frac{\partial v}{\partial t}(\boldsymbol{x}, 0) = 0, \quad \boldsymbol{x} \in \Omega. \tag{29}$$

Boundary value problem (27)–(29) describes (see details, for example, in the [14, 15, 21]) displacement of the plate on the elastic base. In this case, $v(\boldsymbol{x}, t)$ is the normal displacement of the plate, $v^0(\boldsymbol{x})$ defines the displacement at the start time. The boundary conditions (28) correspond to hinge fastening. In the framework of two-dimensional elastic models, the parameter γ_1 is associated with the elastic foundation reaction modulus (Winkler model), and the γ_2 parameter — with the tension action of a thin elastic membrane in the Filonenko-Borodich model and with the shear action among the spring elements in the Pasternak model.

On the set of sufficiently smooth functions $w(\boldsymbol{x}) = 0$, $\boldsymbol{x} \in \partial\Omega$, we define the operator

$$\mathcal{A} w = -\triangle w, \quad \boldsymbol{x} \in \Omega.$$

Let us write the problem (27)–(29) in the form of the Cauchy problem for a second-order evolutionary equation. The solution $v(t) = v(\cdot, t)$ is determined from the equation

$$\frac{d^2 v}{dt^2} + \mathcal{A}^2 v + \gamma_1 v + \gamma_2 \mathcal{A} v = 0, \quad 0 < t \le T. \tag{30}$$

Taking into account (29), it is supplemented with the initial conditions

$$v(0) = v^0, \quad \frac{dv}{dt}(0) = 0. \tag{31}$$

To solve numerically the problem (27)–(29), we will use the standard difference approximations in space [8]. We will introduce in the region Ω a uniform rectangular grid

$$\overline{\omega} = \{\boldsymbol{x} \mid \boldsymbol{x} = (x_1, x_2), \quad x_\alpha = i_\alpha h_\alpha, \quad i_\alpha = 0, 1, ..., N_\alpha, \quad N_\alpha h_\alpha = l_\alpha, \ \alpha = 1, 2\},$$

where $\overline{\omega} = \omega \cup \partial\omega$, ω is the set of interior mesh nodes, and $\partial\omega$ is the set of boundary mesh nodes. For grid functions $w(\boldsymbol{x})$ such that $w(\boldsymbol{x}) = 0, \ \boldsymbol{x} \notin \omega$, we define the Hilbert space $H = L_2(\omega)$, in which the dot product and norm are

$$(w, u) = \sum_{\boldsymbol{x} \in \omega} w(\boldsymbol{x}) u(\boldsymbol{x}) h_1 h_2, \quad \|w\| = (w, w)^{1/2}.$$

For $u(\boldsymbol{x}) = 0, \ \boldsymbol{x} \notin \omega$, we define the grid Laplace operator $-A$ on the usual five-point stencil:

$$\begin{aligned} Au = & -\frac{1}{h_1^2}(u(x_1 + h_1, x_2) - 2u(\boldsymbol{x}) + u(x_1 - h_1, x_2)) \\ & -\frac{1}{h_2^2}(u(x_1, x_2 + h_2) - 2u(\boldsymbol{x}) + u(x_1, x_2 - h_2)), \quad \boldsymbol{x} \in \omega. \end{aligned}$$

For this grid operator (see, for example, [8]), we have

$$A = A^* \geq \delta I, \quad \delta > 0. \tag{32}$$

On sufficiently smooth functions, the operator A approximates the differential operator $\mathcal{A}$ with an error $\mathcal{O}\left(|h|^2\right)$, $|h|^2 = h_1^2 + h_2^2$.

The finite-difference approximation in the space of the problem (30), (31) leads us to equation (1), which is complemented by the initial conditions

$$w(0) = w^0, \quad \frac{dw}{dt}(0) = 0, \tag{33}$$

when $w^0 = v^0(\boldsymbol{x}), \ \boldsymbol{x} \in \omega$. For the operator B we have

$$B = \gamma_1 I + \gamma_2 A. \tag{34}$$

We carry out numerical experiments based on the exact solution of the problem (1), (32)–(34). Consider the grid spectral problem

$$A\psi = \lambda\psi.$$

For eigenfunctions and eigenvalues we have (see, for example, [11]):

$$\psi_{k_1,k_2}(\boldsymbol{x}) = \prod_{\beta=1}^{2} \sqrt{\frac{2}{l_\beta}} \sin\Big(\frac{k_\beta \pi x_\beta}{l_\beta}\Big), \quad \boldsymbol{x} \in \omega,$$

$$\lambda_{k_1,k_2} = \sum_{\beta=1}^{2} \frac{4}{h_\beta^2} \sin^2 \frac{k_\beta \pi}{2N_\beta}, \quad k_\alpha = 1, 2, ..., N_\alpha - 1, \quad \alpha = 1, 2.$$

Because of this

$$\delta = \lambda_{1,1} = \sum_{\beta=1}^{2} \frac{4}{h_\beta^2} \sin^2 \frac{\pi}{2N_\beta} < 8\Big(\frac{1}{l_1^2} + \frac{1}{l_2^2}\Big).$$

Eigenfunctions ψ_{k_1,k_2}, $\|\psi_{k_1,k_2}\| = 1$, form a basis in H. Therefore, for any grid function $u \in H$, the representation takes place

$$u = \sum_{k_1=1}^{N_1-1} \sum_{k_2=1}^{N_2-1} (u, \psi_{k_1,k_2}) \psi_{k_1,k_2}.$$

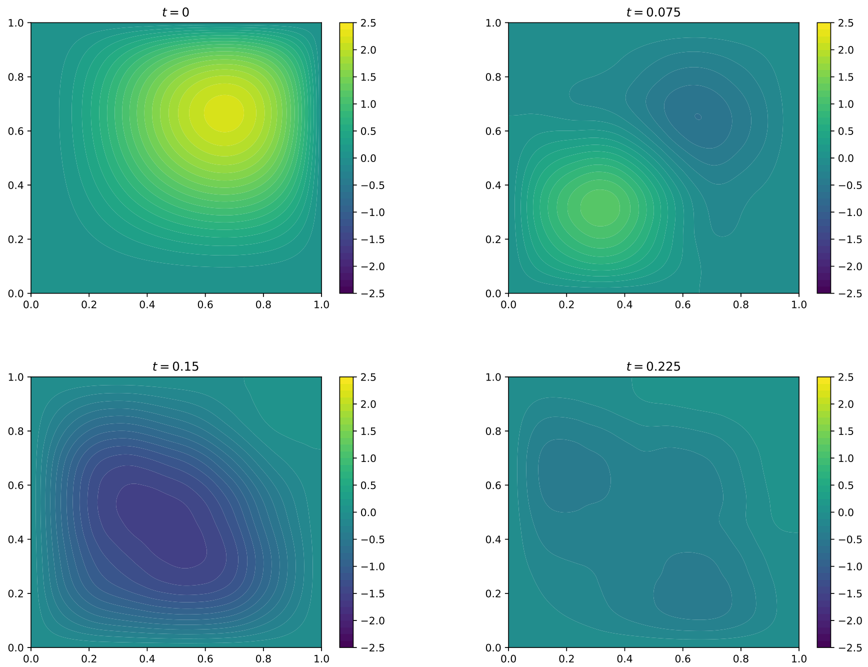

FIGURE 1. Exact solution of the problem at different time-moments.

To solve the problem (1), (32)–(34), we get

$$
w(\boldsymbol{x}, t) = \sum_{k_1=1}^{N_1-1} \sum_{k_2=1}^{N_2-1} (w^0, \psi_{k_1,k_2}) \cos(r_{k_1,k_2}^{1/2} t) \psi_{k_1,k_2}(\boldsymbol{x}),
\quad r_{k_1,k_2} = \gamma_1 + \gamma_2 \lambda_{k_1,k_2} + \lambda_{k_1,k_2}^2. \tag{35}
$$

The calculation results presented below were obtained for the problem with

$$
l_1 = l_2 = 1, \quad N_1 = N_2 = 256, \quad \gamma_1 = 1, \quad \gamma_2 = 0.05.
$$

Comparison of approximations in time is carried out on the problem with the initial condition

$$
w^0(\boldsymbol{x}) = 100 x_1^2 (1 - x_1) x_2^2 (1 - x_2).
$$

The solution to the test problem at different time-moments is shown in Fig. 1. The plate displacement dynamics at characteristic points are shown in Fig. 2. Complex vibrations of the plate are observed.

It is natural to compare the accuracy of the constructed splitting scheme with the accuracy usually weighted scheme (5), (6). At separate time-moments $t = t^n$ we define the error norms of the solution in $C(\omega)$ and $L_2(\omega)$:

$$
\varepsilon_\infty(t^n) = \max_{\boldsymbol{x} \in \omega} |u^n(\boldsymbol{x}) - w(\boldsymbol{x}, t^n)|, \quad \varepsilon_2(t^n) = \|u^n(\boldsymbol{x}) - w(\boldsymbol{x}, t^n)\|, \quad n = 0, \ldots, N.
$$

For the initial displacement of the plate, we have

$$
\max_{\boldsymbol{x} \in \omega} |w(\boldsymbol{x}, 0)| \approx 2.195, \quad \|w(\boldsymbol{x}, 0)\| \approx 0.9524.
$$

The error in the approximate solution of the problem (27)–(29) is shown in Fig. 3 when using the scheme (5), (6) with $\sigma = 0.25$. For the considered initial data

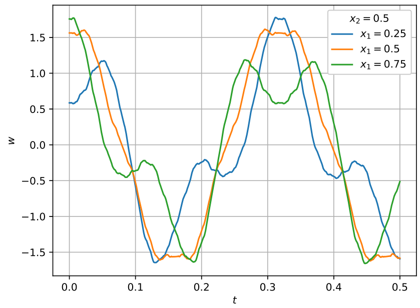

FIGURE 2. Displacement of the plate at individual points.

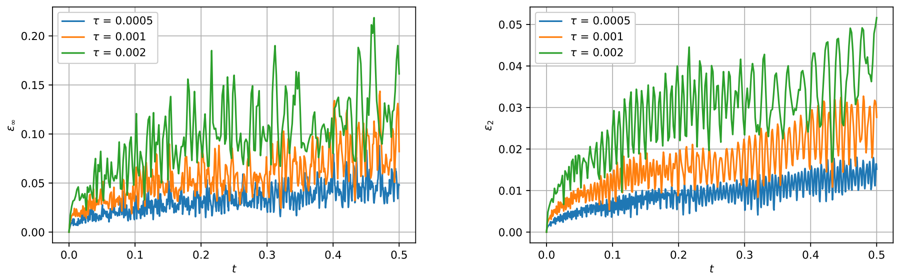

FIGURE 3. The solution error ε_∞ (left) and ε_2 (right) for the weighted scheme at $\sigma = 0.25$.

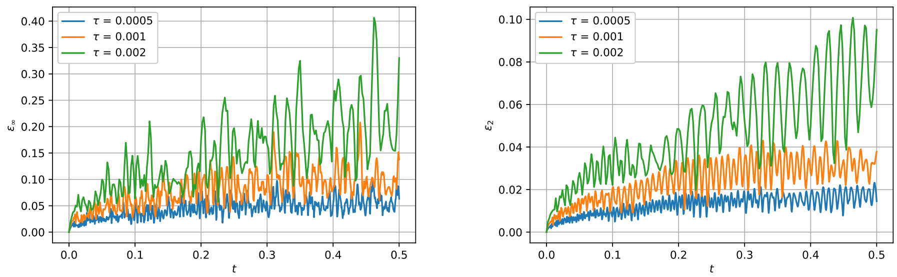

FIGURE 4. The solution error ε_∞ (left) and ε_2 (right) for the weighted scheme at $\sigma = 0.5$.

and time steps, the theoretical asymptotic dependence of the time step accuracy (second-order) is not visible. With increasing weight σ the errors grow — see Fig. 4.

When using the splitting scheme (6), (20), (22), (23) with the constraints (24), we set

$$\sigma_A^2 = \frac{1}{2}, \quad \sigma_B = \frac{1}{2}.$$

Time-histories of the error of the approximate solution on time for this case is shown in Fig. 5. Evidently, as the time step decreases, the accuracy increases, but,

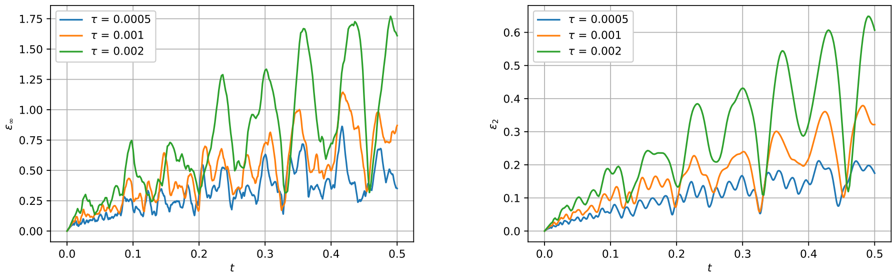

FIGURE 5. The solution error ε_∞ (left) and ε_2 (right) for the splitting scheme with $\sigma_A^2 = 0.5, \sigma_B = 0.5$.

as you would expect, errors in comparison with the usual scheme with weights (see Fig. 3, 4) are much larger.

6. Conclusions

Some applied models, in particular, of the theory of plates, lead to the necessity of solving the initial-boundary value problems for partial differential equations that include fourth-order elliptic operators. The paper discusses the problem of reducing the computational complexity of the implementation of unconditionally stable implicit schemes for these problems using special approximations in time. In the present work:

(1) A class of Cauchy problems for a second-order evolutionary equation is highlighted, in which the problem operator is the sum of two self-adjoint operators. Wherein one of the operators is represented as the product of the operator A by its conjugate operator A^*.
(2) Conditions for the absolute stability of three-level schemes with weights formulated using general results the theory of stability (well-posedness) of operator-difference schemes.
(3) Splitting schemes are constructed and investigated for which the transition to a new level in time is associated with a separate solution of problems for operators A and A^*, not their products.
(4) The efficiency of the proposed splitting schemes is demonstrated by the results of calculations of the dynamics of a thin square plate on an elastic foundation.

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