INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 18, Number 6, Pages 865–880

OPTIMALITY CRITERIA AND DUALITY FOR NONLINEAR FRACTIONAL CONTINUOUS-TIME PROGRAMMING

ALEKSANDAR JOVIĆ

Abstract. This paper addressed the fractional continuous-time programming problem. The necessary and sufficient optimality conditions under generalized concavity assumptions are established. Dual problem to the primal one and investigate duality relations between them are also addressed.

Key words. Fractional programming, continuous-time programming, optimality conditions.

1. Introduction

Continuous-time programming problem originated from Bellman's "bottleneck" problems, considered in [2]. Since its first detailed introduction into the literature by Tyndall [15], many authors have contributed to the subject. Optimality criteria and duality theory, both in linear and nonlinear cases, have been investigated extensively. Nonlinear problem was first investigated, in 1968 by Hanson [6], Hanson and Mond [5] and in 1974 by Farr and Hanson [4]. Since then, a comprehensive bibliography has been produced. For more information, the reader refer to [3, 12, 20, 21, 22, 23].

In this paper, we consider fractional continuous-time problem. The problem of maximizing (or minimizing) the ratio of two real-valued functionals, subject to a set of constraints, is known as a fractional optimal control problem in the area of control theory. This class of problems is important for modeling various decision processes in the field of economics, game theory and operational research. They also often appear in some other contexts such as numerical analysis, approximation problems, facility location, optimal engineering design and information theory. For the reasons mentioned, fractional continuous-time programming problems have received major attention in the past thirty years, resulting into a comprehensive literature, dealing with their various theoretical and computational aspects. For recent results in this field, the reader is referred to [7, 8, 11, 13, 14, 16, 17, 18, 19]. In [13], optimality criteria is obtained for fractional continuous-time programming problem. Charnes-Cooper transformation, convexity and perturbation functions play a key role in these results. In papers [11, 19] fundamental tools were results given in [3, 23]. In [1], Arutyunov et al. have pointed out that such results in [3, 23] are not valid. Therefore, some results from aforementioned papers, unfortunately, are also not valid. Most recently, in [10], the authors have provided new necessary optimality conditions for nonlinear continuous-time programming problems with scalar valued objective function. Numerical algorithms for linear fractional continuous-time problem have been proposed in [16, 17]. In [16], the authors have introduced the discrete approximation method to solve the primal and dual pair of parametric continuous-time linear programming problems by using the recurrence method and provided numerical examples. The main purpose in [17] was to develop a discrete approximation method to solve a class of linear fractional continuous-time

Received by the editors May 17, 2020 and, in revised form, July 15, 2021.
2000 *Mathematics Subject Classification.* 90C30, 90C32, 90C46, 46N10.

programming problems. Also, in mentioned paper, the authors have established an estimate of the error bound and have provided numerical examples to demonstrate the usefulness of this numerical algorithm.

Our aim in this paper is to provide necessary and sufficient optimality conditions for nonlinear fractional continuous-time problem. In addition, we shall construct duality model for the primal problem and prove suitable duality theorems.

The paper is organized in the following way. Some preliminaries about the problem are given in Section 2, where some important definitions are stated. In Section 3, necessary optimality conditions are obtained. Sufficient optimality conditions are obtained under concavity and generalized concavity assumptions. In Section 4 and 5, dual problems are presented and certain duality results are obtained.

2. Preliminaries

Consider the following fractional continuous-time programming problem:

$$\text{(FCTP)}\qquad \begin{aligned} &\max \ \frac{\int_0^T f(t,x(t))\,dt}{\int_0^T g(t,x(t))\,dt} \\ &\text{s. t.}\ \ h_i(t,x(t)) \geq 0,\ \ i \in I = \{1,\ldots,m\} \text{ a.e. in } [0,T], \\ &\qquad x \in L_\infty([0,T];\mathbb{R}^n), \end{aligned}$$

where $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, $g : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, $h_i : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, $i \in I$ are given functions. Here for each $t \in [0,T]$, $x_k(t)$ is the kth component of $x(t) \in \mathbb{R}^n$. All integrals are given in the Lebesgue sense. All vectors are column vectors. Inequality signs between vectors should be read componentwise. B denotes the open unit ball with centre at the origin, independently of the space or dimension. Let Ω_P be the set of feasible solutions of (FCTP),

$$\Omega_P = \{x \in L_\infty([0,T];\mathbb{R}^n) : h_i(t,x(t)) \geq 0,\ \ i \in I,\ \text{a.e. in } [0,T]\}\,.$$

Let $\varepsilon > 0$ and $\hat{x} \in \Omega_P$. Suppose the following assumptions are valid:

(i) Functions $f(t,\cdot)$ and $g(t,\cdot)$ are continuously differentiable on $\hat{x}(t)+\varepsilon\bar{B}$ a.e. in $[0,T]$. Functions $f(t,\cdot)$ and $g(t,\cdot)$ are Lebesgue measurable for each x, and there exist numbers $K_f > 0$ and $K_g > 0$ such that

$$\|\nabla f(t,\hat{x}(t))\| \leq K_f \ \text{ a.e in } [0,T],$$

$$\|\nabla g(t,\hat{x}(t))\| \leq K_g \ \text{ a.e in } [0,T];$$

(ii) For each $i \in I$, the function $h_i(t,\cdot)$ is continuously differentiable on $\hat{x}(t)+\varepsilon\bar{B}$ a.e. in $[0,T]$. For each $i \in I$, the function $h_i(\cdot,x(\cdot))$ is essentially bounded in $[0,T]$ for all $x \in L_\infty([0,T],\mathbb{R}^n)$ and there exists a number $K_h > 0$ such that

$$\|\nabla h_i(t,\hat{x}(t))\| \leq K_h,\ i \in I,\ \text{ a.e in } [0,T].$$

For $x \in \Omega_P$, we also assume that

$$\int_0^T f(t,x(t))\,dt \geq 0 \ \text{ and } \int_0^T g(t,x(t))\,dt > 0. \tag{1}$$

3. Optimality conditions

In this section, we discuss the necessary and sufficient optimality conditions for (FCTP). The following results do not appear in the literature when (FCTP) is defined in $L_\infty([0,T];\mathbb{R}^n)$.

Given $b > 0$ and $\hat{x} \in \Omega_P$, we will denote by $I_b(t)$ the index set of all the b-active constraints at $\hat{x} \in \Omega_P$, that is,

$$I_b(t) = \{\, i \in I : 0 \le h_i(t, \hat{x}(t)) \le b \,\},$$

for each $t \in [0,T]$. For all $i \in I$, let us define the function $\delta_i^b : [0,T] \to \mathbb{R}$ as

$$\delta_i^b(t) = \begin{cases} 1, \ i \in I_b(t) \\ 0, \text{ otherwise.} \end{cases}$$

We start with an optimality condition for auxiliary scalar continuous-time problem. For each $w \in \mathbb{R}_+$ where $\mathbb{R}_+$ denotes the positive orthant of $\mathbb{R}$, we consider the following auxiliary scalar continuous-time problem:

$$\begin{aligned} (\text{SCTP})_w \quad & \max \int_0^T \left(f(t,x(t)) - w g(t,x(t)) \right) dt \\ & \text{s. t.} \quad h_i(t,x(t)) \ge 0, \quad i \in I, \ \text{a.e. in } [0,T]. \end{aligned}$$

The following lemma shows the connection between (FCTP) and $(\text{SCTP})_w$, and plays a key role in proving the main result in this section.

Lemma 3.1. *If a point $\hat{x} \in \Omega_P$ is an optimal solution for* (FCTP) *then $\hat{x}$ solves* $(\text{SCTP})_{\hat{w}}$, *where*

$$\hat{w} = \max_{x \in \Omega_P} \frac{\int_0^T f(t,x(t))\,dt}{\int_0^T g(t,x(t))\,dt} = \frac{\int_0^T f(t,\hat{x}(t))\,dt}{\int_0^T g(t,\hat{x}(t))\,dt}.$$

Proof. Assume that $\hat{x}$ maximizes

$$\frac{\int_0^T f(t,x(t))\,dt}{\int_0^T g(t,x(t))\,dt}$$

but it does not maximizes

$$\int_0^T \left(f(t,x(t)) - \hat{w} g(t,x(t)) \right) dt.$$

Then there exists $\bar{x} \in \Omega$ such that

$$0 = \int_0^T \left(f(t,\hat{x}(t)) - \hat{w} g(t,\hat{x}(t)) \right) dt < \int_0^T \left(f(t,\bar{x}(t)) - \hat{w} g(t,\bar{x}(t)) \right) dt.$$

Hence,

$$\frac{\int_0^T f(t,\bar{x}(t))\,dt}{\int_0^T g(t,\bar{x}(t))\,dt} > \hat{w}.$$

This means that $\hat{w}$ is not the maximum value. Consequently, this contradicts the hypothesis that it is the maximum value. Hence, $\hat{x}$ also maximizes

$$\int_0^T \left(f(t,x(t)) - \hat{w} g(t,x(t)) \right) dt.$$

Thus, the proof is complete. □

Let

$$
(2) \qquad \begin{aligned}
\phi_0(t,\xi) &= -\int_0^T \left(\nabla f(t,\hat{x}(t)) - \hat{w}\nabla g(t,\hat{x}(t))\right)^T \xi \, dt < 0,\\
\phi_i(t,\xi) &= -\, h_i(t,\hat{x}(t)) - \delta_i^b(t)\nabla h_i(t,\hat{x}(t))^T\xi \le 0, \quad i \in I,\\
&\quad \xi \in \mathbb{R}^n,
\end{aligned}
$$

be a system corresponding to the problem (SCTP), $K = \{0\} \sqcup I$, and

$$
\mathcal{I}(t,\xi) = \left\{i : \phi_i(t,\xi) = \max_{l\in K} \phi_l(t,\xi)\right\}, \quad t \in [0,T], \quad \xi \in \mathbb{R}^n.
$$

Definition 3.1. (see [1]) *System* (2) *is said to be regular when there exist a function* $\bar{x}(\cdot) \in L_\infty([0,T];\mathbb{R}^n)$*, real numbers* $R \ge 0$ *and* $\alpha > 0$ *such that for a.e.* $t \in [0,1]$ *and for all* $\xi \in \mathbb{R}^n$ *with* $\|\xi - \bar{x}(t)\| \ge R$*, there exists a unit vector* $e = e(t,\xi) \in \mathbb{R}^n$*, satisfying*

$$
\langle \partial_\xi \phi_i(t,\xi), e\rangle \ge \alpha \quad \forall i \in \mathcal{I}(t,\xi),
$$

where $\partial_\xi \phi_i$ *denotes the partial subdifferential of* ϕ_i *at* (t,ξ) *in the sense of convex analysis.*

Let us recall the following constraint qualification from [10].

Remark 3.1. *We say that* $\hat{x}$ *satisfies the constraint qualification* (MFCQ)*, if there exists* $\bar{\gamma} \in L_\infty([0,T];\mathbb{R}^n)$ *and* $\hat{b} > 0$ *such that, for almost every* $t \in [0,T]$*,*

$$
\nabla h_i^T(t,\hat{x}(t))\bar{\gamma}(t) \ge \beta, \quad i \in I_{\hat{b}}(t),
$$

for some $\beta > 0$.

Note that (MFCQ) is a continuous-time version of the Mangasarian-Fromovitz constraint qualification. Now, we give necessary optimality conditions for (FCTP).

Theorem 3.1. *Let* $\hat{x}$ *be an optimal slution for* (FCTP)*. Assume that* (i)*,* (ii) *and* (MFCQ) *are satisfied at* $\hat{x}$ *and that system* (2) *is regular. Then, there exists* $\hat{\lambda} \in L_\infty([0,T];\mathbb{R}^m)$ *such that, for almost every* $t \in [0,T]$*,*

$$
(3) \qquad \begin{aligned}
\int_0^T g(t,\hat{x}(t))\, dt \nabla f(t,\hat{x}(t)) - \int_0^T f(t,\hat{x}(t))\, dt \nabla g(t,\hat{x}(t))&\\
+ \sum_{i\in I} \hat{\lambda}_i(t)\nabla h_i(t,\hat{x}(t)) = 0&,
\end{aligned}
$$

$$
(4) \qquad \hat{\lambda}_i(t) h_i(t,\hat{x}(t)) = 0, \ \hat{\lambda}_i(t) \ge 0, \quad i \in I.
$$

Proof. Since $\hat{x}$ is an optimal solution of (FCTP), then by Lemma 3.1, $\hat{x}$ solves $\text{SCTP}_{\hat{w}}$. Hence, by Theorem 3.5 in [10], there exists $\hat{\mu} \in L_\infty([0,T];\mathbb{R}^m)$ such that

$$
(5) \qquad \nabla f(t,\hat{x}(t)) - \hat{w}\nabla g(t,\hat{x}(t)) + \sum_{i\in I} \hat{\mu}_i(t)\nabla h_i(t,\hat{x}(t)) = 0 \quad \text{a.e. in } [0,T],
$$

$$
(6) \qquad \hat{\mu}_i(t) h_i(t,\hat{x}(t)) = 0, \quad i \in I, \ \text{a.e. in } [0,T],
$$

$$
(7) \qquad \hat{\mu}_i(t) \ge 0, \quad i \in I, \ \text{a.e. in } [0,T],
$$

where

$$
\hat{w} = \frac{\int_0^T f(t,\hat{x}(t))\, dt}{\int_0^T g(t,\hat{x}(t))\, dt}.
$$

Now, multiplying all terms in (5) and (6) by

$$\int_0^T g(t,\hat{x}(t))\,dt$$

and setting

$$\hat{\lambda}_i(t) = \hat{\mu}_i(t)\int_0^T g(t,\hat{x}(t))\,dt,$$

we obtain

$$\int_0^T g(t,\hat{x}(t))\,dt\nabla f(t,\hat{x}(t)) - \int_0^T f(t,\hat{x}(t))\,dt\nabla g(t,\hat{x}(t)) + \sum_{i\in I}\hat{\lambda}_i(t)\nabla h_i^T(t,\hat{x}(t)) = 0 \quad \text{a.e. in } [0,T],$$

and

$$\hat{\lambda}_i(t)h_i(t,\hat{x}(t)) = 0,\ \hat{\lambda}_i(t) \geq 0, \quad i \in I,\ \text{a.e. in } [0,T].$$

Thus, the proof is complete. □

Remark 3.2. *Assume that $h_i(t,\cdot)$ is a concave function almost everywhere in $[0,T]$, $i \in I$. We say that the constraint qualification* (SCQ) *holds, if there exist $x \in \Omega_P$ and $\hat{b} > 0$ such that, for almost every $t \in [0,T]$,*

$$h_i(t,x(t)) \geq \beta,\ \ i \in I_{\hat{b}}(t),$$

for some $\beta > 0$.

The qualification in the Remark above can be seen as a continuous-time version of the Slater constraint qualification. In [10], Monte and Oliveira have showed that (SCQ) is a sufficient condition for (MFCQ) under concavity assumption.

Corollary 3.1. *Let $\hat{x}$ be an optimal slution for* (FCTP). *Assume that $h_i(t,\cdot)$ is a concave function almost everywhere in $[0,T]$, $i \in I$,* (i), (ii) *are satisfied at $\hat{x}$ and that system* (2) *is regular. If* (SCQ) *holds, then there exists $\hat{\lambda} \in L_\infty([0,T];\mathbb{R}^m)$ such that, for almost every $t \in [0,T]$,*

$$\int_0^T g(t,\hat{x}(t))\,dt\nabla f(t,\hat{x}(t)) - \int_0^T f(t,\hat{x}(t))\,dt\nabla g(t,\hat{x}(t)) + \sum_{i\in I}\hat{\lambda}_i(t)\nabla h_i(t,\hat{x}(t)) = 0, \tag{8}$$

$$\hat{\lambda}_i(t)h_i(t,\hat{x}(t)) = 0,\ \hat{\lambda}_i(t) \geq 0, \quad i \in I. \tag{9}$$

As an illustration, we will consider the following example.

Example 3.1.

$$\max\ \frac{\int_0^1 f(t,x(t))\,dt}{\int_0^1 g(t,x(t))\,dt}$$

$$\text{(P)}\qquad \begin{aligned} h_1(t,x(t)) &\geq 0 \text{ a.e. in } [0,1],\\ h_2(t,x(t)) &\geq 0 \text{ a.e. in } [0,1],\\ x &\in L_\infty([0,1];\mathbb{R}), \end{aligned}$$

where $f(t,x(t)) := 2t+2-x^2(t)$, $g(t,x(t)) := e^{x(t)}$, $h_1(t,x(t)) := x(t)$, $h_2(t,x(t)) := t+1-x(t)$. It can be easily verified that $\hat{x}(t) = 0$ is an optimal solution for (P) and $I_{\hat{b}}(t) = \{1\}$ for $\hat{b} = \frac{1}{2}$. It is obvious that

$$\nabla f(t,\hat{x}(t)) = 0,\ \nabla g(t,\hat{x}(t)) = 1,\ \nabla h_1(t,\hat{x}(t)) = 1,\ \nabla h_2(t,\hat{x}(t)) = -1.$$

Take $\bar{\gamma}(t) = \frac{1}{2}$ a.e. in $[0,1]$ and $\beta = \frac{1}{5}$. It follows

$$\nabla h_1^T(t,\hat{x}(t))\bar{\gamma}(t) = \frac{1}{2} \geq \beta \text{ a.e. in } [0,1],$$

i.e., constraint qualification (MFCQ) holds. Now, show that corresponding system to (FCTP) is regular. Given $\xi \in \mathbb{R}$, for almost everywhere in $[0,1]$,

$$\phi_0(t,\xi) = \xi,\ \phi_1(t,\xi) = -\xi,\ \phi_2(t,\xi) = -t-1.$$

We have $\mathcal{I}(t,\xi) = \{1\}$ for $\xi < 0$, $\mathcal{I}(t,\xi) = \{0\}$ for $\xi > 0$ and $\{2\} \notin \mathcal{I}(t,\xi)$. The regularity of the system

$$\begin{aligned} \phi_0(t,\xi) &= 3\xi < 0,\\ \phi_1(t,\xi) &= -\xi \leq 0,\\ \phi_2(t,\xi) &= -t-1 \leq 0, \end{aligned} \tag{10}$$

is verified with $\bar{x} \equiv 0$, $R > 0$ and $\alpha = \frac{1}{2}$, for a.e. t in $[0,1]$, $e = 1$ for $\xi > 0$ and $e = -1$ for $\xi < 0$. Indeed,

(1) $\langle \partial_\xi \phi_0(t,\xi), e\rangle = 3 \geq \alpha$, for $\xi > 0$,

(2) $\langle \partial_\xi \phi_1(t,\xi), e\rangle = 1 \geq \alpha$, for $\xi < 0$.

Further, the necessary optimality conditions are satisfied for $\hat{\lambda}_1(t) = 3$ and $\hat{\lambda}_2(t) = 0$.

The next result establishes a sufficient optimality conditions for (FCTP). The proofs of the main theorems will be based primarily on the concavity, convexity and generalized concavity assumptions imposed on the functions involved, and will not require the regularity condition. In the sequel we will use some of the basic properties of quasiconcave functions. For these, the reader is referred to Mangasarian [9].

Theorem 3.2. *Assume that there exist a feasible solution $\hat{x}$ for* (FCTP) *and $\hat{\lambda} \in L_\infty([0,T];\mathbb{R}^m)$ such that, for almost every $t \in [0,T]$,*

$$\begin{aligned} \int_0^T g(t,\hat{x}(t))\,dt\nabla f(t,\hat{x}(t)) - \int_0^T f(t,\hat{x}(t))\,dt\nabla g(t,\hat{x}(t))&\\ +\sum_{i\in I}\hat{\lambda}_i(t)\nabla h_i(t,\hat{x}(t)) = 0&, \end{aligned} \tag{11}$$

$$\hat{\lambda}_i(t)h_i(t,\hat{x}(t)) = 0,\ \hat{\lambda}_i(t) \geq 0, \quad i \in I. \tag{12}$$

If the function $f(t,\cdot)$ is concave in its second argument at $\hat{x}(t)$ almost everywhere in $[0,T]$, and $g(t,\cdot)$ is convex in its second argument at $\hat{x}(t)$ almost everywhere in $[0,T]$ and $\sum_{i\in I}\hat{\lambda}_i(t)h_i(t,\cdot)$ is quasiconcave in its second argument at $\hat{x}(t)$ almost everywhere in $[0,T]$, then $\hat{x}$ is an optimal solution for (FCTP).

Proof. From $x \in \Omega_P$ and (12), we have

$$\hat{\lambda}_i(t)h_i(t,x(t)) \geq \hat{\lambda}_i(t)h_i(t,\hat{x}(t)) = 0, \quad i \in I, \quad \forall x \in \Omega_P \text{ a.e. in } [0,T],$$

i.e.

$$\sum_{i\in I}\hat{\lambda}_i(t)h_i(t,x(t)) \geq \sum_{i\in I}\hat{\lambda}_i(t)h_i(t,\hat{x}(t)), \quad \forall x \in \Omega_P \text{ a.e. in } [0,T]. \tag{13}$$

Since $\sum_{i\in I}\hat{\lambda}_i(t)h_i(t,\cdot)$ is quasiconcave at $x(t) = \hat{x}(t)$ almost everywhere in $[0,T]$, (13) yields

$$\sum_{i\in I}\hat{\lambda}_i(t)\nabla h_i^T(t,\hat{x}(t))(x(t)-\hat{x}(t)) \geq 0, \quad \forall x \in \Omega_P, \text{ a.e. in } [0,T]. \tag{14}$$

From (11) and (14), we obtain

$$\left(\int_0^T g(t,\hat{x}(t))\,dt\nabla f(t,\hat{x}(t)) - \int_0^T f(t,\hat{x}(t))\,dt\nabla g(t,\hat{x}(t))\right)^T (x(t)-\hat{x}(t)) \leq 0,$$

$\forall x \in \Omega_P$, a.e. in $[0,T]$.

Setting

$$\psi(\hat{x}) = \int_0^T g(t,\hat{x}(t))\,dt, \ \varphi(\hat{x}) = \int_0^T f(t,\hat{x}(t))\,dt,$$

previous inequality can be written as

$$\big(\psi(\hat{x})\nabla f(t,\hat{x}(t)) - \varphi(\hat{x})\nabla g(t,\hat{x}(t))\big)^T (x(t)-\hat{x}(t)) \leq 0, \ \forall x \in \Omega_P, \text{ a.e. in } [0,T].$$

Since $f(t,\cdot)$ and $-g(t,\cdot)$ are concave at $\hat{x}(t)$ almost everywhere in $[0,T]$, $\psi(\hat{x}) > 0$ and $\varphi(\hat{x}) \geq 0$ it follows

$$\psi(\hat{x})f(t,\cdot) - \varphi(\hat{x})g(t,\cdot)$$

is concave at $\hat{x}(t)$ almost everywhere in $[0,T]$. Therefore,

$$\psi(\hat{x})f(t,x(t)) - \varphi(\hat{x})g(t,x(t)) - \psi(\hat{x})f(t,\hat{x}(t)) + \varphi(\hat{x})g(t,\hat{x}(t)) \leq 0,$$

$\forall x \in \Omega_P$, a.e. in $[0,T]$.

Integrating the previous inequality from 0 to T, we have

$$\int_0^T \left(\psi(\hat{x})f(t,x(t)) - \varphi(\hat{x})g(t,x(t))\right)\,dt \leq 0, \ \forall x \in \Omega_P,$$

i.e.

$$\psi(\hat{x})\int_0^T f(t,x(t))\,dt \leq \varphi(\hat{x})\int_0^T g(t,x(t))\,dt, \ \forall x \in \Omega_P.$$

It follows

$$\frac{\int_0^T f(t,x(t))\,dt}{\int_0^T g(t,x(t))\,dt} \leq \frac{\int_0^T f(t,\hat{x}(t))\,dt}{\int_0^T g(t,\hat{x}(t))\,dt}, \ \forall x \in \Omega_P.$$

Thus $\hat{x}$ must be an optimal solution for (FCTP). □

Remark 3.3. *Theorem 3.2 also holds under concavity assumption of*

$$\sum_{i\in I}\hat{\lambda}_i(t)h_i(t,\cdot).$$

In Example 3.1 note that $\hat{x}(t) = 0$ a.e. $t \in [0,1]$ and $(\hat{\lambda}_1(t), \hat{\lambda}_2(t)) = (3,0)$ a.e. $t \in [0,1]$ satisfy (11)-(12). Also, functions $f(t,\cdot)$, $g(t,\cdot)$ and $\sum_{i\in I} \hat{\lambda}_i(t) h_i(t,\cdot)$ satisfy assumptions of Theorem 3.2. Therefore, from Theorem 3.2 we see that $\hat{x}$ is optimal solution for the problem (P).

We define, for almost every $t \in [0,T]$, the index set of all the binding inequality constraints at $\hat{x} \in \Omega_P$ as

$$A(t) = \{ i \in I \,:\, h_i(t, \hat{x}(t)) = 0 \} .$$

Following the same approach, we obtain sufficient conditions for (FCTP) without complementary slackness condition.

Theorem 3.3. *Let $\hat{x}$ be a feasible solution for* (FCTP)*. Assume that $h_i(t,\cdot)$ is quasiconcave for all $i \in A(t)$ in its second argument at $\hat{x}(t)$ almost everywhere in $[0,T]$ and there exists $\hat{\lambda} \in L_\infty([0,T];\mathbb{R}^m)$ such that, for almost every $t \in [0,T]$,*

$$\int_0^T g(t,\hat{x}(t))\,dt \nabla f(t,\hat{x}(t)) - \int_0^T f(t,\hat{x}(t))\,dt \nabla g(t,\hat{x}(t)) + \sum_{i\in I} \hat{\lambda}_i(t) \nabla h_i(t,\hat{x}(t)) = 0, \tag{15}$$

$$\hat{\lambda}_i(t) \geq 0,\ i \in A(t),\ \hat{\lambda}_i(t) = 0,\ i \in I \setminus A(t). \tag{16}$$

If the function $f(t,\cdot)$ is concave in its second argument at $\hat{x}(t)$ almost everywhere in $[0,T]$ and $g(t,\cdot)$ is convex in its second argument at $\hat{x}(t)$ almost everywhere in $[0,T]$, then $\hat{x}$ is an optimal solution for (FCTP).

Proof. For any feasible x,

$$h_i(t,x(t)) \geq h_i(t,\hat{x}(t)) = 0,\ i \in A(t),\ \text{ a.e. in } [0,T].$$

By the quasiconcavity $h_i(t,\cdot)$, we have

$$\nabla h_i^T(t,\hat{x}(t))(x(t) - \hat{x}(t)) \geq 0,\ i \in A(t),\ \text{a.e. in } [0,T].$$

Since $\hat{\lambda}_i(t) \geq 0$, $i \in A(t)$, a.e. in $[0,T]$, we obtain

$$\sum_{i\in A(t)} \hat{\lambda}_i(t) \nabla h_i^T(t,\hat{x}(t))(x(t) - \hat{x}(t)) \geq 0,\ \ \forall x \in \Omega_P,\ \text{a.e. in } [0,T]. \tag{17}$$

From (15) and (17), we have

$$\left(\int_0^T g(t,\hat{x}(t))\,dt \nabla f(t,\hat{x}(t)) - \int_0^T f(t,\hat{x}(t))\,dt \nabla g(t,\hat{x}(t)) \right)^T (x(t) - \hat{x}(t)) \leq 0,$$

$\forall x \in \Omega_P$, a.e. in $[0,T]$, i.e.,

$$\big(\psi(\hat{x}) \nabla f(t,\hat{x}(t)) - \varphi(\hat{x}) \nabla g(t,\hat{x}(t))\big)^T (x(t) - \hat{x}(t)) \leq 0,\ \forall x \in \Omega_P,\ \text{a.e. in } [0,T].$$

Since $f(t,\cdot)$ and $-g(t,\cdot)$ are concave at $\hat{x}(t)$ almost everywhere in $[0,T]$, $\psi(\hat{x}) > 0$ and $\varphi(\hat{x}) \geq 0$ it follows $\psi(\hat{x}) f(t,\cdot) - \varphi(\hat{x}) g(t,\cdot)$ is concave at $\hat{x}(t)$ almost everywhere in $[0,T]$. As in the proof of Theorem 3.2, we conclude that

$$\psi(\hat{x}) f(t,x(t)) - \varphi(\hat{x}) g(t,x(t)) - \psi(\hat{x}) f(t,\hat{x}(t)) + \varphi(\hat{x}) g(t,\hat{x}(t)) \leq 0,$$

$\forall x \in \Omega_P$ a.e. in $[0,T]$.

Integrating the previous inequality from 0 to T, we obtain

$$\frac{\int_0^T f(t,x(t))\,dt}{\int_0^T g(t,x(t))\,dt} \leq \frac{\int_0^T f(t,\hat{x}(t))\,dt}{\int_0^T g(t,\hat{x}(t))\,dt},\ \ \forall x \in \Omega_P.$$

Thus $\hat{x}$ must be an optimal solution for (FCTP). ☐

4. Wolfe type dual model

In this section, we introduce first dual model and prove appropriate duality theorems. Let

$$\Omega_{D_1} = \Bigg\{(u,\lambda) \in L_\infty([0,T];\mathbb{R}^n \times \mathbb{R}^m) \,:\, \int_0^T g(t,u(t))\,dt \nabla f(t,u(t)) - \int_0^T f(t,u(t))\,dt \nabla g(t,u(t)) + \sum_{i\in I} \lambda_i(t)\nabla h_i(t,u(t)) = 0,$$
$$\lambda_i(t)h_i(t,u(t)) = 0,\ \lambda_i(t) \geq 0,\ \ i\in I,\ \ \text{a.e. in } [0,T]\Bigg\}.$$

We consider the following Wolf type dual problem of (FCTP):

$$\text{(DFCTP1)} \quad \min \ \frac{\int_0^T f(t,u(t))\,dt}{\int_0^T g(t,u(t))\,dt}$$
$$\text{subject to } (u,\lambda) \in \Omega_{D_1},$$

where Ω_{D_1} is feasible set for (DFCTP1).

Theorem 4.1. *Let x and (u,λ) be feasible solutions for* (FCTP) *and* (DFCTP1)*. Further, assume that the function $f(t,\cdot)$ is concave in its second argument at $u(t)$ almost everywhere in $[0,T]$, $g(t,\cdot)$ is convex in its second argument at $u(t)$ almost everywhere in $[0,T]$ and $\sum_{i\in I}\lambda_i(t)h_i(t,\cdot)$ is quasiconcave in its second argument at $u(t)$ almost everywhere in $[0,T]$. Then,*

$$\frac{\int_0^T f(t,x(t))\,dt}{\int_0^T g(t,x(t))\,dt} \leq \frac{\int_0^T f(t,u(t))\,dt}{\int_0^T g(t,u(t))\,dt}.$$

Proof. Since x and (u,λ) are feasible for (FCTP) and (DFCTP1), respectively,

$$\lambda_i(t)h_i(t,x(t)) \geq \lambda_i(t)h_i(t,u(t)) = 0,\ \ i\in I,\ \ \text{a.e. in } [0,T],$$

i.e.

$$\sum_{i\in I}\lambda_i(t)h_i(t,x(t)) \geq \sum_{i\in I}\lambda_i(t)h_i(t,u(t)),\ \ \text{a.e. in } [0,T]. \tag{18}$$

Since

$$\sum_{i\in I}\lambda_i(t)h_i(t,\cdot)$$

is quasiconcave at $u(t)$ almost everywhere in $[0,T]$, (18) yields

$$\sum_{i\in I}\lambda_i(t)\nabla h_i^T(t,u(t))(x(t)-u(t)) \geq 0,\ \ \text{a.e. in } [0,T]. \tag{19}$$

Inequality (19) and $(u,\lambda)\in\Omega_{D_1}$ imply

$$\left(\int_0^T g(t,u(t))\,dt\nabla f(t,u(t)) - \int_0^T f(t,u(t))\,dt\nabla g(t,u(t))\right)^T (x(t)-u(t)) \leq 0,$$

a.e. in $[0,T]$.

Setting

$$\psi(u) = \int_0^T g(t,u(t))\,dt,\ \ \varphi(u) = \int_0^T f(t,u(t))\,dt,$$

previous inequality can be written as

$$\big(\psi(u)\nabla f(t,u(t)) - \varphi(u)\nabla g(t,u(t))\big)^T (x(t)-u(t)) \le 0, \quad \text{a.e. in } [0,T].$$

Since $f(t,\cdot)$ and $-g(t,\cdot)$ are concave at $u(t)$ almost everywhere in $[0,T]$, $\psi(u) > 0$ and $\varphi(u) \ge 0$, it follows

$$\psi(u) f(t,\cdot) - \varphi(u) g(t,\cdot)$$

is concave at $u(t)$ almost everywhere in $[0,T]$.
Therefore,

$$\psi(u)f(t,x(t)) - \varphi(u)g(t,x(t)) - \psi(u)f(t,u(t)) + \varphi(u)g(t,u(t)) \le 0,$$

a.e. in $[0,T]$.

Integrating the previous inequality from 0 to T, we have

$$\int_0^T g(t,u(t))dt \int_0^T f(t,x(t))dt - \int_0^T f(t,u(t))dt \int_0^T g(t,x(t))\,dt \le 0,$$

i.e.

$$\frac{\int_0^T f(t,x(t))\,dt}{\int_0^T g(t,x(t))\,dt} \le \frac{\int_0^T f(t,u(t))\,dt}{\int_0^T g(t,u(t))\,dt}.$$

□

Theorem 4.2. *Let $\hat{x}$ be an optimal slution for* (FCTP)*. Assume that* (i)*,* (ii)*,* (MFCQ) *are satisfied at $\hat{x}$, system* (2) *is regular and let all the conditions of Theorem 4.1 be fulfilled for all feasible solutions of* (FCTP)*. Then, there exists $\hat{\lambda} \in L_\infty([0,T];\mathbb{R}^m)$ such that $(\hat{x},\hat{\lambda})$ is an optimal solution for* (DFCTP1) *and*

$$\max_{x\in\Omega_P} \frac{\int_0^T f(t,x(t))\,dt}{\int_0^T g(t,x(t))\,dt} = \min_{(u,\lambda)\in\Omega_{D_1}} \frac{\int_0^T f(t,u(t))\,dt}{\int_0^T g(t,u(t))\,dt}.$$

Proof. Since $\hat{x}$ is an optimal solution of (FCTP), we conclude from Theorem 3.1 that there exists $\hat{\lambda} \in L_\infty([0,T];\mathbb{R}^m)$ such that, for almost every $t \in [0,T]$,

$$\begin{aligned} \int_0^T g(t,\hat{x}(t))\,dt \nabla f(t,\hat{x}(t)) - \int_0^T f(t,\hat{x}(t))\,dt \nabla g(t,\hat{x}(t)) \\ + \sum_{i\in I} \hat{\lambda}_i(t)\nabla h_i(t,\hat{x}(t)) = 0, \end{aligned} \tag{20}$$

$$\hat{\lambda}_i(t) h_i(t,\hat{x}(t)) = 0, \ \hat{\lambda}_i(t) \ge 0, \quad i \in I. \tag{21}$$

We conclude that $(\hat{x},\hat{\lambda}) \in \Omega_{D_1}$. Hence, from Theorem 4.1 that the feasible solution of (FCTP) becomes optimal solution for (DFCTP1). Thus the optimal values of (FCTP) and (DFCTP1) are equal. □

Theorem 4.3. *Let $\hat{x}$ and $(\hat{u},\hat{\lambda})$ be optimal solutions for* (FCTP) *and* (DFCTP1)*, respectively. Assume that the function $f(t,\cdot)$ is concave in its second argument at $\hat{u}(t)$ almost everywhere in $[0,T]$, $g(t,\cdot)$ is convex in its second argument at $\hat{u}(t)$ almost everywhere in $[0,T]$ and $\sum_{i\in I}\hat{\lambda}_i(t)h_i(t,\cdot)$ is quasiconcave in its second argument at $\hat{u}(t)$ almost everywhere in $[0,T]$. Then,* (FCTP)*,* (DFCTP1) *have the same optimal values and $\hat{x} = \hat{u}$.*

Proof. From Theorem 4.2 it follows that there exists $\hat{\lambda}$ such that $(\hat{x}, \hat{\lambda})$ constitutes an optimal solution of (DFCTP1) and

$$\frac{\int_0^T f(t, \hat{x}(t))\, dt}{\int_0^T g(t, \hat{x}(t))\, dt} = \frac{\int_0^T f(t, \hat{u}(t))\, dt}{\int_0^T g(t, \hat{u}(t))\, dt}. \tag{22}$$

Suppose on the contrary that

$$\hat{x} \neq \hat{u}. \tag{23}$$

Then from Theorem 4.1, for all feasible points x, (u, λ) for (FCTP) and (DFCTP1), respectively, we obtain

$$\frac{\int_0^T f(t, x(t))\, dt}{\int_0^T g(t, x(t))\, dt} \leq \frac{\int_0^T f(t, u(t))\, dt}{\int_0^T g(t, u(t))\, dt}.$$

By hypothesis (23), it must be

$$\frac{\int_0^T f(t, \hat{x}(t))\, dt}{\int_0^T g(t, \hat{x}(t))\, dt} < \frac{\int_0^T f(t, \hat{u}(t))\, dt}{\int_0^T g(t, \hat{u}(t))\, dt}.$$

This inequality contradicts (22). Therefore, $\hat{x} = \hat{u}$. □

Example 4.1. As an illustration, we will consider the following dual model (D1) for (P) :

$$\min \ \frac{\int_0^1 \left(2t + 2 - u^2(t)\right)\, dt}{\int_0^1 e^{u(t)}\, dt}$$

$$\text{s.t.} - 2u(t) \int_0^1 e^{u(t)} dt - e^{u(t)} \int_0^1 (2t + 2 - u^2(t)) dt + \lambda_1(t) - \lambda_2(t) = 0, \ \text{a.e. in } [0, 1],$$

$$\lambda_1(t) u(t) = 0, \quad \text{a.e. in } [0, 1],$$

$$\lambda_2(t)(t + 1 - u(t)) = 0, \quad \text{a.e. in } [0, 1],$$

$$\lambda_i(t) \geq 0, \ i = 1, 2, \quad \text{a.e. in } [0, 1],$$

$$u \in L_\infty([0, 1]; \mathbb{R}), \ (\lambda_1, \lambda_2) \in L_\infty([0, 1]; \mathbb{R}^2).$$

All conditions of Theorem 4.2 are fulfilled. It can be easily verified that $(\hat{u}, \hat{\lambda}_1, \hat{\lambda}_2) = (0, 3, 0)$ *is optimal solution of preceding problem. Also,* (P) *and* (D1) *have the same optimal values.*

5. Lagrangian type dual model

In this section, we introduce a second dual model and prove appropriate duality theorems. Let

$$\Omega_{D_2} = \Bigg\{ (u,\lambda) \in L_\infty([0,T];\mathbb{R}^n \times \mathbb{R}^m) \, : \\ \int_0^T g(t,u(t))\,dt \left(\nabla f(t,u(t)) + \sum_{i\in I} \lambda_i(t) \nabla h_i(t,u(t)) \right) \\ - \int_0^T \left(f(t,u(t)) - \sum_{i\in I} \lambda_i(t) h_i(t,u(t)) \right) dt \nabla g(t,u(t)) \le 0, \\ \lambda_i(t) h_i(t,u(t)) \le 0, \; \lambda_i(t) \ge 0, \, i \in I, \quad \text{a.e. in } [0,T] \Bigg\}.$$

We consider the following second Lagrangian type dual problem of (FCTP):

$$\text{(DFCTP2)} \quad \min \; \frac{\int_0^T \left(f(t,u(t)) - \sum_{i\in I} \lambda_i(t) h_i(t,u(t)) \right) dt}{\int_0^T g(t,u(t))\,dt}$$

$$\text{subject to} \quad (u,\lambda) \in \Omega_{D_2},$$

where Ω_{D_2} is feasible set for (DFCTP2).

Theorem 5.1. *Let x and (u,λ) be feasible solutions for* (FCTP) *and* (DFCTP2). *Further, assume that the function $f(t,\cdot)$ is concave in its second argument at $u(t)$ almost everywhere in $[0,T]$, $g(t,\cdot)$ is convex in its second argument at $u(t)$ almost everywhere in $[0,T]$ and $\sum_{i\in I} \lambda_i(t) h_i(t,\cdot)$ is quasiconcave in its second argument at $u(t)$ almost everywhere in $[0,T]$, then*

$$\frac{\int_0^T f(t,x(t))\,dt}{\int_0^T g(t,x(t))\,dt} \le \frac{\int_0^T \left(f(t,u(t)) - \sum_{i\in I} \lambda_i(t) h_i(t,u(t)) \right) dt}{\int_0^T g(t,u(t))\,dt}.$$

Proof. Since x and (u,λ) are feasible for (FCTP) and (DFCTP2), respectively,

$$\lambda_i(t) h_i(t,x(t)) \ge 0 \ge \lambda_i(t) h_i(t,u(t)), \quad i \in I, \quad \text{a.e. in } [0,T],$$

i.e.

$$\sum_{i\in I} \lambda_i(t) h_i(t,x(t)) \ge \sum_{i\in I} \lambda_i(t) h_i(t,u(t)), \quad \text{a.e. in } [0,T]. \tag{24}$$

Put

$$\psi(u) = \int_0^T g(t,u(t))\,dt, \; \varphi(u) = \int_0^T f(t,u(t))\,dt$$

and

$$\eta(u) = -\int_0^T \sum_{i\in I} \lambda_i(t) h_i(t,u(t))\,dt \ge 0.$$

Since

$$\sum_{i\in I} \lambda_i(t) h_i(t,\cdot)$$

is quasiconcave at $u(t)$ almost everywhere in $[0,T]$, $\psi(u)>0$ and (24) imply

$$\psi(u)\sum_{i\in I}\lambda_i(t)\nabla h_i^T(t,u(t))(x(t)-u(t))\geq 0, \text{ a.e. in } [0,T]. \tag{25}$$

From (25) and $(u,\lambda)\in\Omega_{D_2}$, we obtain

$$\left(\psi(u)\nabla f(t,u(t))-\varphi(u)\nabla g(t,u(t))-\eta(u)\nabla g(t,u(t))\right)^T(x(t)-u(t))\leq 0,$$

a.e. in $[0,T]$, i.e.,

$$\left(\psi(u)\nabla f(t,u(t))-\left(\varphi(u)+\eta(u)\right)\nabla g(t,u(t))\right)^T(x(t)-u(t))\leq 0,$$

a.e. in $[0,T]$. Since $f(t,\cdot)$ and $-g(t,\cdot)$ are concave at $u(t)$ almost everywhere in $[0,T]$, $\psi(u)>0$ and $\eta(u)\geq 0$, it follows

$$\psi(u)f(t,\cdot)-\left(\varphi(u)+\eta(u)\right)g(t,\cdot) \text{ is concave at } u(t) \text{ and } \quad \text{a.e. in } [0,T].$$

Therefore,

$$\psi(u)f(t,x(t))-(\varphi(u)+\eta(u))g(t,x(t))-\psi(u)f(t,u(t))+(\varphi(u)$$
$$+\eta(u))g(t,u(t))\leq 0, \text{ a.e. in } [0,T].$$

Integrating the previous inequality from 0 to T, we have

$$\begin{aligned}&\int_0^T g(t,u(t))\,dt\int_0^T f(t,x(t))\,dt\\&-\int_0^T\left(f(t,u(t))-\sum_{i\in I}\lambda_i(t)h_i(t,u(t))\right)dt\int_0^T g(t,x(t))\,dt\\&\qquad-\int_0^T\sum_{i\in I}\lambda_i(t)h_i(t,u(t))\,dt\int_0^T g(t,x(t))\,dt\leq 0.\end{aligned}$$

Inequalities

$$\int_0^T\sum_{i=1}^m\lambda_i(t)h_i(t,u(t))\,dt\leq 0$$

and (1) also imply

$$\begin{aligned}&\int_0^T g(t,u(t))\,dt\int_0^T f(t,x(t))\,dt\\&-\int_0^T\left(f(t,u(t))-\sum_{i\in I}\lambda_i(t)h_i(t,u(t))\right)dt\int_0^T g(t,x(t))\,dt\leq 0.\end{aligned}\tag{26}$$

Hence

$$\frac{\int_0^T f(t,x(t))\,dt}{\int_0^T g(t,x(t))\,dt}\leq\frac{\int_0^T\left(f(t,u(t))-\sum_{i\in I}\lambda_i(t)h_i(t,u(t))\right)dt}{\int_0^T g(t,u(t))\,dt}.$$

Thus, the proof is complete. □

Theorem 5.2. *Let $\hat{x}$ be an optimal solution for* (FCTP). *Assume that* (i), (ii), (MFCQ) *are satisfied at $\hat{x}$, system* (2) *is regular and let all the conditions of Theorem 5.1 be fulfilled for all feasible solutions of* (FCTP). *Then, there exists $\hat{\lambda}\in L_\infty([0,T];\mathbb{R}^m)$ such that $(\hat{x},\hat{\lambda})$ is an optimal solution for* (DFCTP2) *and*

$$\max_{x\in\Omega_P}\frac{\int_0^T f(t,x(t))\,dt}{\int_0^T g(t,x(t))\,dt}=\min_{(u,\lambda)\in\Omega_{D_2}}\frac{\int_0^T\left(f(t,u(t))-\sum_{i\in I}\lambda_i(t)h_i(t,u(t))\right)dt}{\int_0^T g(t,u(t))\,dt}.$$

Proof. Since $\hat{x}$ is an optimal solution of (FCTP), we conclude from Theorem 3.1 that there exists $\hat{\lambda} \in L_\infty([0,T];\mathbb{R}^m)$ such that, for almost every $t \in [0,T]$,

(27)
$$\nabla f(t,\hat{x}(t)) - \frac{\int_0^T f(t,\hat{x}(t))\,dt}{\int_0^T g(t,\hat{x}(t))\,dt}\nabla g(t,\hat{x}(t)) + \sum_{i\in I}\hat{\lambda}_i(t)\nabla h_i(t,\hat{x}(t)) = 0 \quad \text{a.e. in } [0,T],$$

(28)
$$\int_0^T \sum_{i=1}^m \hat{\lambda}_i(t) h_i(t,\hat{x}(t))\,dt\,\nabla g(t,\hat{x}(t)) = 0, \quad \text{a.e. in } [0,T],$$

(29)
$$\hat{\lambda}_i(t) \geq 0, \quad i \in I, \text{ a.e. in } [0,T].$$

Now, multiplying all terms in (27) by

$$\int_0^T g(t,\hat{x}(t))\,dt > 0$$

and summing by (28) we conclude that $(\hat{x},\hat{\lambda}) \in \Omega_{D_2}$. Hence, from Theorem 5.1 that the feasible solution of (FCTP) becomes optimal solution for (DFCTP2). Thus the optimal values of (FCTP) and (DFCTP2) are equal. □

Theorem 5.3. *Let $\hat{x}$ and $(\hat{u},\hat{\lambda})$ be optimal solutions of problems* (FCTP) *and* (D-FCTP2)*, respectively. Further, assume that the function $f(t,\cdot)$ is concave in its second argument at $\hat{u}(t)$ almost everywhere in $[0,T]$, $g(t,\cdot)$ is convex in its second argument at $\hat{u}(t)$ almost everywhere in $[0,T]$ and $\sum_{i\in I}\hat{\lambda}_i(t)h_i(t,\cdot)$ is quasiconcave in its second argument at $\hat{u}(t)$ almost everywhere in $[0,T]$. Then,* (FCTP), (DFCT-P2) *have the same optimal values and $\hat{x} = \hat{u}$.*

The proof of Theorem 5.3 is similar to the one of Theorem 4.3.

Example 5.1. Consider the following dual model (D2) for (P):

$$\min \ \frac{\int_0^1 \left(2t+2-u^2(t)-\lambda_1(t)u(t)-\lambda_2(t)(t+1-u(t))\right)\,dt}{\int_0^1 e^{u(t)}\,dt}$$

$$\text{s.t.}\ (\lambda_1(t)-\lambda_2(t)-2u(t))\int_0^1 e^{u(t)}dt$$
$$+\,e^{u(t)}\int_0^1 \left(u^2(t)+\lambda_1(t)u(t)+\lambda_2(t)(t+1-u(t))-2t-2\right)\,dt \leq 0, \ \text{a.e. in}[0,1],$$
$$\lambda_1(t)u(t) \leq 0, \quad \text{a.e. in}[0,1],$$
$$\lambda_2(t)(t+1-u(t)) \leq 0, \quad \text{a.e. in } [0,1],$$
$$\lambda_i(t) \geq 0, \ i=1,2, \quad \text{a.e. in } [0,1],$$
$$u \in L_\infty([0,1];\mathbb{R}), \ (\lambda_1,\lambda_2) \in L_\infty([0,1];\mathbb{R}^2).$$

All conditions of Theorem 5.2 are fulfilled. It can be easily verified that $(\hat{u},\hat{\lambda}_1,\hat{\lambda}_2) = (0,3,0)$ is optimal solution of preceding problem. Also, (P) *and* (D2) *have the same optimal values.*

6. Conclusions and future directions

The auxiliary tool employed in the derivation of necessary optimality criteria and duality, is a new version of Karush-Kuhn-Tucker-type optimality conditions for scalar problem. Sufficient optimality conditions were given under concavity and generalized concavity assumptions. It would be also of interest to see how the similar approach can be extended to investigate optimality conditions and duality theory for the nonsmooth fractional continuous-time problems. Unfortunately, some of the main results from bibliography for multiobjective-programming problems are not valid. For more information, see [1]. Therefore, an important direction of future work is to develop optimality criteria and duality results for multiobjective continuous-time problems. Also, there does not exist a numerical algorithm to solve the nonlinear continuous-time fractional programming problem. To this end, the necessary and sufficient conditions in this paper are good starting points.

Acknowledgments

This work was supported by Ministry of Education, Science and Technological Development of Republic of Serbia through project No. 174015.

References

[1] A. V. Arutyunov, S. E. Zhukovskiy and B. Marinkovic, Theorems of the alternative for systems of convex inequalities. Set-Valued and Variational Analysis, 27 (2019), 51-70.

[2] R. Bellman, Bottleneck problems and dynamic programming. Proc. Natl. Acad. Sci. 39 (1953), 947-951.

[3] A. J. V. Brandao, M. A. Rojas-Medar and G. N. Silva, Nonsmooth continouous-time optimization problems: necessary conditions. Comput. Math. Appl. 41 (2001), 1447-1456.

[4] W.H. Farr and M.A. Hanson, Continuous time programming with nonlinear constraints, J. Math. Anal. Appl. 45 (1974), 96-115.

[5] M.A. Hanson and B. Mond, A class of continuous convex programming problems, J. Math. Anal. Appl. 22 (1968), 427-437.

[6] M.A.Hanson, Duality for a class of infinite programming problems, SIAM J. Appl. Math. 16 (1968), 318-323.

[7] I. Husain and Z. Jabeen, Continuous-time fractional minmax programming, Math. Comput. Model. 42 (2005), 701-710.

[8] J. C. Lee and H. C. Lai, Parameter-free dual models for fractional programming with generalized invexity, Ann. Oper. Res. 133 (2005), 47-61.

[9] O.L. Mangasarian, Nonlinear programming. McGraw-Hill, New York (1969).

[10] M. R. C. do Monte and V.A. de Oliveira, Necessary conditions for continuous-time optimization under the Mangasarian-Fromovitz constraint qualification. Optimization, 69 (2020), 777-798.

[11] S. Nobakhtian and M.R. Pouryayevali, Optimality conditions and duality for nonsmooth fractional continuous-time problems. J. Optimiz. Theory App. 152 (2012), 245-255.

[12] M.A. Rojas-Medar, A.J.V. Brandao and G.N. Silva, Nonsmooth continouous-time optimization problems, sufficient conditions, J. Math. Anal. Appl. 227 (1998), 305-318.

[13] S. Suneja, C. Singh and R.N. Kaul, Optimality and duality in continuous-time nonlinear fractional programming. J. Austral. Math. Soc. Ser. B, 34 (1992), 2, 229-244.

[14] I.M. Stancu-Minasian and S. Tigan, Continuous time linear-fractional programming: the minimum-risk approach. Rairo Oper. Res. 34 (2000), 397-409.

[15] W.F. Tyndall, A duality theorem for a class of continuous linear programming problems, SIAM J. Appl. Math. 13 (1965), 644-666.

[16] C.F. Wen and H.C. Wu, Using the Dinkelbach-type algorithm to solve the continuous-time linear fractional programming problems, J. Global Opt. 49 (2011) 237-263.

[17] C.F. Wen and H.C. Wu, Approximate solutions and duality theorems for continuous-time linear fractional programming problems, Numer. Func. Anal. Opt. 33, 1 (2012), 80-129.

[18] G.J. Zalmai, Optimality conditions and duality models for a class of nonsmooth continuous-time generalized fractional programming problems, Optimization, 51 (2002), 353-399.

[19] G.J. Zalmai, Duality for a class of continuous-time homogeneous fractional programming problems. Z. Oper. Res. Ser. A-B 30 (1986) 43-48.
[20] G.J. Zalmai, A continuous time generalization of Gordan's Transposition Theorem. J. Math. Anal. Appl. 110 (1985) , 130-140.
[21] G.J. Zalmai, Optimality conditions and Lagrangian duality in continuous-time nonlinear programming, J. Math. Anal. Appl. 109 (1985), 426-452.
[22] G.J. Zalmai, Sufficient optimality conditions in continuous-time nonlinear programming, J. Math. Anal. Appl. 111 (1985), 130-147.
[23] G.J. Zalmai, The Fritz John and Kuhn-Tucker optimality conditions in continuous-time programming. J. Math. Anal. Appl. 110 (1985), 503-518.

Faculty of Mathematics, University of Belgrade, Studentski trg 16, 11000, Belgrade, Serbia
E-mail: ajovic@matf.bg.ac.rs