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A FULLY DISCRETE, DECOUPLED SCHEME WITH DIFFERENT TIME STEPS FOR APPROXIMATING NEMATIC LIQUID CRYSTAL FLOW

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Abstract. This paper designs a decoupled scheme for approximating nematic liquid crystal flow based on a fully discrete mixed finite element method, which allows different time steps for different physical fields. Besides, error estimates for velocity and macroscopic molecular orientation of the nematic liquid crystal flow are shown. Finally, numerical tests are provided to demonstrate efficiency of the scheme. It is found the presented scheme can save lots of computational time compared with common decoupled scheme.

Key words. Nematic liquid crystal flow, decoupled scheme, different time steps, error estimates.

1. Introduction

Liquid crystal is usually known as the fourth state of matter and is different to gas, liquid and solid. The simplest liquid crystal phase is the nematic liquid crystal. It is consisted of elongated rod-like molecules with similar size. The centers of mass of these molecules have no positional order, but tend to align along preferred direction. In recent decades, many studies are dealing with the nematic liquid crystal, due to the importance of related scientific and, engineering applications [2].

Ericksen-Leslie model, built by Ericksen [9, 10] and Leslie [18], can simulate the hydrodynamics of the nematic liquid crystal flow, and it is the macroscopic continuum description of the time evolution of both flow velocity and microscopic orientation. Further, a simplified Ericksen-Leslie model is derived by Lin [22] initially and its governing equations are written as follows [22, 1]:

$$
\begin{aligned}
&\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + \lambda\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d}) = \mathbf{f},\\
&\mathbf{d}_t - \gamma\Delta\mathbf{d} + (\mathbf{u}\cdot\nabla)\mathbf{d} = \gamma|\nabla\mathbf{d}|^2\mathbf{d},\\
&\nabla\cdot\mathbf{u} = 0,\ \ |\mathbf{d}| = 1,
\end{aligned}
\tag{1}
$$

for $(\mathbf{x},t)\in Q_T$, where $Q_T = \Omega\times(0,T)$ with a fixed $T\in(0,\infty)$. Here, $\mathbf{u}(\mathbf{x},t): Q_T\to\mathbb{R}^2$ and $p(\mathbf{x},t): Q_T\to\mathbb{R}$ denote the velocity field and the pressure of the flow, respectively. Besides, $\mathbf{d}(\mathbf{x},t): Q_T\to\mathbb{S}$ is the director, which represents the molecular orientation field of the nematic liquid crystal material and describes the average molecular alignment, where $\mathbb{S}\subset\mathbb{R}^2$ is a unit circle. In addition, $\mathbf{f}(\mathbf{x},t): Q_T\to\mathbb{R}^2$ represents a body force on the flow. Three parameters ν, λ and γ denote the kinematic viscosity, the competition between kinetic and potential energy, and the microscopic elastic relaxation time for the molecular orientation field, respectively. Hereafter, $|\nabla\mathbf{d}|$ or $|\mathbf{d}|$ denotes the Euclidean norm of $\nabla\mathbf{d}$ or $\mathbf{d}$, and $\nabla\mathbf{d}\odot\nabla\mathbf{d}$ is a 2×2 matrix whose (i,j)-the entry is written by $(\sum_{k=1}^{2}\frac{\partial d_k}{\partial x_i}\frac{\partial d_k}{\partial x_j})_{i,j}$. As in [1], in

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this paper the system (1) is considered in conjunction with the following initial and boundary conditions:

$$(2) \qquad \begin{aligned} &\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}), \quad \mathbf{d}(\mathbf{x},0)=\mathbf{d}_0(\mathbf{x}),\ \forall \mathbf{x}\in\Omega,\\ &\mathbf{u}|_{S_T}=0, \quad \partial_{\mathbf{n}}\mathbf{d}|_{S_T}=0, \end{aligned}$$

with $\nabla\cdot\mathbf{u}_0=0$ and $|\mathbf{d}_0|=1$, where $S_T=\partial\Omega\times(0,T)$ and $\mathbf{n}$ is the outer unit normal of $\partial\Omega$.

Although this simplified Ericksen-Leslie model neglects the Leslie stress in the Ericksen-Leslie model, it still retains some essential difficulties of the Ericksen-Leslie model and keeps the core of the mathematical structure, such as strong nonlinearities and constraints, as well as the physical structure, such as the anisotropic effect of the elasticity on the velocity field. Thus, the system (1)-(2) can be regarded as a nice initial step towards the theoretical and numerical analysis of the Ericksen-Leslie model.

Because the governing equations (1)-(2) of the simplified Ericksen-Leslie model include not only the incompressibility, the strong nonlinearity and the physical and nonconvex side constraint $|\mathbf{d}|=1$ but also the coupling between the harmonic map flow and the fluid equations of motion, which make it not easy to solve these equations effectively. Therefore, much effort has been throwing to the development of some efficient numerical methods for investigating this system [13, 6, 7, 27, 19, 16] and the references therein. Besides, Du et al. [8] have studied a Fourier-spectral method for the simplified Ericksen-Leslie system and established spectral accuracy. In [12], a linear fully discrete mixed scheme has been considered, using finite element method in space and a semi-implicit Euler scheme in time. In addition, Becker et al. [3] have constructed a fully discrete scheme, which uses low order finite elements and enjoys a discrete energy law. Based on explicit treatment of the unitary constraint for the director field, a fully splitting and decoupled in time linear algorithm has been designed [14]. Recently, An and Su [1] have shown optimal error estimates for an linearized semi-implicit Euler finite element scheme for the considered system.

In this paper, we design a fully discrete, decoupled finite element scheme for approximating the simplified Ericksen-Lesliel system (1)-(2). Since the system has many physical fields and is a multiphysics problem, we adopt different time step sizes for different physical fields. In fact, Ge and Ma [11] have proposed a multi-rate iterative scheme based on multiphysics discontinuous Galerkin method for a poroelasticity model, which is a fluid-solid interaction system at pore scale. Shi et al. [26, 25] have designed a multistep technique to overcome the instability mainly caused by the explicit treatment of the convection system and to enlarge the stability region such that the resulting scheme behaved like an unconditionally stable scheme. Besides, the differing time steps methods have been applied to the Stokes-Darcy model [24], the Navier-Stokes/Darcy model [17] and the Darcy-Brinkman problem [21].

2. A decoupled scheme with different time steps for the nematic liquid crystal flow

In this section, we describe some necessary definitions and inequalities, which will be frequently applied to following sections.

Firstly, we introduce standard notations for Lebesgue space $L^p(\Omega)$ and Sobolve space $W^{m,p}(\Omega)$, $1\le p\le\infty$, $m\in\mathbb{N}^+$. Then, their norms are denoted by $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{W^{m,p}(\Omega)}$, respectively. In particular, $H^m(\Omega)$ is used to represent the space $W^{m,2}(\Omega)$ and $\|\cdot\|_m$ denotes the norm in $H^m(\Omega)$. Besides, $L^2(\Omega)$ norm and its inner product are denoted by $\|\cdot\|_0$ and $(\cdot,\cdot)$. For X being a normed function space in Ω, $L^p(0,T;X)$ is the space of all functions defined on Q_T for which the norm

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u\|_X^p dt\right)^{\frac{1}{p}}, \quad p \in [1,\infty),$$

is finite.

Next, for the mathematical setting of the nematic liquid crystal model (1)-(2), we introduce the following function spaces:

$$\begin{aligned} \mathbf{V} &= H_0^1(\Omega)^2 = \{\mathbf{v} \in H^1(\Omega)^2 : \mathbf{v}|_{\partial\Omega} = 0\}, \quad \mathbf{W} = H^2(\Omega)^2, \\ M &= L_0^2(\Omega) = \{q \in L^2(\Omega) : (q,1) = 0\}. \end{aligned}$$

Then, as in [2], based on the above definitions of the function spaces, we have the following variational formulation of problem (1)-(2): Find $(\mathbf{u}(t), p(t), \mathbf{d}(t)) \in \mathbf{V} \times M \times \mathbf{W}$ such that, for all $(\mathbf{v}, q, \boldsymbol{\phi}) \in \mathbf{V} \times M \times \mathbf{W}$

$$\begin{aligned} &(\mathbf{u}_t, \mathbf{v}) + \nu(\nabla\mathbf{u}, \nabla\mathbf{v}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}) - (\nabla\cdot\mathbf{v}, p) + (\nabla\cdot\mathbf{u}, q) \\ &\quad - \lambda(\nabla\mathbf{d}\odot\nabla\mathbf{d}, \nabla\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \end{aligned} \tag{3}$$

$$(\mathbf{d}_t, \boldsymbol{\phi}) + \gamma(\nabla\mathbf{d}, \nabla\boldsymbol{\phi}) + ((\mathbf{u}\cdot\nabla)\mathbf{d}, \boldsymbol{\phi}) - \gamma(|\nabla\mathbf{d}|^2\mathbf{d}, \boldsymbol{\phi}) = 0. \tag{4}$$

Furthermore, we assume that the domain Ω is regular partitioned into a mesh K_h, which consists of triangle elements K. Denote $h = \max_{K\in K_h} h_K$, where h_K is the diameter of the element K. Accordingly, we define the following finite element subspaces on K_h by

$$\begin{aligned} \mathbf{V}_h &= \{\mathbf{v}_h \in C(\Omega)^2 \cap \mathbf{V},\ \mathbf{v}_h|_K \in P_1(K)^2,\ \forall K \in K_h\}, \\ \mathbf{W}_h &= \{\mathbf{w}_h \in C(\Omega)^2 \cap \mathbf{W},\ \mathbf{w}_h|_K \in P_2(K)^2,\ \forall K \in K_h\}, \\ M_h &= \{q_h \in L^2(\Omega) \cap M,\ q_h|_K \in P_0(K),\ \forall K \in K_h\}, \\ X_h &= \{v_h \in C(\Omega),\ v_h|_K \in P_1(K),\ \forall K \in K_h\}, \end{aligned}$$

where $P_i(K)$ $(i = 0,1,2)$ denote the space of the polynomials on K of degree at most i for every $K \in K_h$. Note that the lowest order conforming finite element pair $\mathbf{V}_h \times M_h$ does not satisfy the discrete inf-sup condition. Hence, in order to fulfill this condition, a stabilized bilinear term is used [4, 28]:

$$G(p_h, q_h) = (p_h - \Pi p_h, q_h - \Pi q_h), \quad \forall p_h, q_h \in M_h,$$

where Π is a projection operator from $L^2(\Omega)$ to X_h.

To derive fully discrete scheme of problem (1)-(2), we introduce a generalized bilinear form:

$$B(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h) = \nu(\nabla\mathbf{u}_h, \nabla\mathbf{v}_h) - (\nabla\cdot\mathbf{v}_h, p_h) + (\nabla\cdot\mathbf{u}_h, q_h) + G(p_h, q_h),$$

for all $(\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times M_h$, and a skew-symmetric trilinear form:

$$\begin{aligned} b(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h) &= ((\mathbf{u}_h\cdot\nabla)\mathbf{v}_h, \mathbf{w}_h) + \frac{1}{2}((\nabla\cdot\mathbf{u}_h)\mathbf{v}_h, \mathbf{w}_h) \\ &= \frac{1}{2}((\mathbf{u}_h\cdot\nabla)\mathbf{v}_h, \mathbf{w}_h) - \frac{1}{2}((\mathbf{u}_h\cdot\nabla)\mathbf{w}_h, \mathbf{v}_h), \end{aligned} \tag{5}$$

for all $\mathbf{u}_h \in \mathbf{V}_h$, and $\mathbf{v}_h, \mathbf{w}_h \in \mathbf{V}_h$ or $\mathbf{W}_h$, which satisfies following property [23]:

$$|b(\mathbf{u}_h, \mathbf{v}, \mathbf{w}_h)| \le C\|\mathbf{u}_h\|_0\|\mathbf{v}\|_2\|\nabla\mathbf{w}_h\|_0, \tag{6}$$

for all $\mathbf{u}_h, \mathbf{w}_h \in \mathbf{V}_h$ or $\mathbf{W}_h$ and $\mathbf{v} \in H^2(\Omega)^2$. Here and after, we denote C a general positive constant which is independent of h and may stand for different values at different occurrences. In addition, we need the following inverse inequality [5], which holds for $\mathbf{v}_h \in \mathbf{V}_h$ or $\mathbf{W}_h$,

$$\|\mathbf{v}_h\|_{W^{l,p}(\Omega)^2} \le Ch^{m-l+2\min\{0,\frac{1}{p}-\frac{1}{q}\}}\|\mathbf{v}_h\|_{W^{m,q}(\Omega)^2}, \tag{7}$$

where $1 \le p, q \le \infty$, $0 \le m \le l$.

Finally, we assume that on each time level s_k for the director field, there exists a subtime level t_{m_k}. For simplicity, we further assume uniform time levels, that is,

$$s_k = k\Delta s,\ k = 0, 1, \cdots, S,\ \Delta s = r\Delta t \text{ and } t_m = m\Delta t,\ m = 0, 1, \cdots, N,$$

where $\Delta t = \frac{T}{N}$ and $N = rS$, $r \ge 1$. So $m_k = kr$. For $t_m, t_{m_k} \in [0, T]$, $(\mathbf{u}_h^m, p_h^m, \mathbf{d}_h^{m_k})$ will denote the fully discrete approximation by the presented decoupled scheme to $(\mathbf{u}(t_m), p(t_m), \mathbf{d}(t_{m_k}))$. Setting $\mathbf{f}^{m+1} = \mathbf{f}(\mathbf{x}, t_{m+1})$, the fully discrete, decoupled scheme with different time steps for approximating nematic liquid crystal flow reads as:

Give $\mathbf{u}_h^0 \in \mathbf{V}_h$ and $\mathbf{d}_h^0 \in \mathbf{W}_h$.

For $k = 0, 1, \cdots, S-1$, do the following four steps.

Step 1: Find $(\mathbf{u}_h^{m+1}, p_h^{m+1}) \in (\mathbf{V}_h, M_h)$, with $m = m_k, m_k+1, \cdots, m_{k+1}-1$, such that for all $(\mathbf{v}_h, q_h) \in (\mathbf{V}_h, M_h)$,

$$\begin{aligned}&\left(\frac{\mathbf{u}_h^{m+1} - \mathbf{u}_h^m}{\Delta t}, \mathbf{v}_h\right) + B(\mathbf{u}_h^{m+1}, p_h^{m+1}; \mathbf{v}_h, q_h) + b(\mathbf{u}_h^m, \mathbf{u}_h^{m+1}, \mathbf{v}_h)\\ &\quad - \lambda(\nabla\mathbf{d}_h^{m_k} \odot \nabla\mathbf{d}_h^{m_k}, \nabla\mathbf{v}_h) = (\mathbf{f}^{m+1}, \mathbf{v}_h),\end{aligned} \tag{8}$$

with the small time step size Δt.

Step 2: Set $\mathbf{S}^{m_k} = \frac{1}{r}\sum_{i=m_k}^{m_{k+1}-1} \mathbf{u}_h^i$.

Step 3: Find $\mathbf{d}_h^{m_{k+1}} \in \mathbf{W}_h$, such that for all $\boldsymbol{\phi}_h \in \mathbf{W}_h$,

$$\begin{aligned}&\left(\frac{\mathbf{d}_h^{m_{k+1}} - \mathbf{d}_h^{m_k}}{\Delta s}, \boldsymbol{\phi}_h\right) + \gamma(\nabla\mathbf{d}_h^{m_{k+1}}, \nabla\boldsymbol{\phi}_h) - \gamma(|\nabla\mathbf{d}_h^{m_k}|^2\mathbf{d}_h^{m_k}, \boldsymbol{\phi}_h)\\ &\quad + b(\mathbf{S}^{m_k}, \mathbf{d}_h^{m_{k+1}}, \boldsymbol{\phi}_h) = 0,\end{aligned} \tag{9}$$

with the large time step size $\Delta s = r\Delta t$.

Step 4: Set $k = k + 1$ and repeat until $k = S - 1$.

3. Error estimates

In this section, we will state and prove the error estimates for the fully discrete, decoupled scheme with different time steps (8)-(9). In order to derive error estimates, we need to introduce the following projection [1, 5] $(\mathbf{R}_h, Q_h) : \mathbf{V} \times M \longrightarrow \mathbf{V}_h \times M_h$ defined by

$$B(\mathbf{R}_h\mathbf{v}, Q_hq; \mathbf{v}_h, q_h) = B(\mathbf{v}, q; \mathbf{v}_h, q_h) - G(q, q_h), \tag{10}$$

for $(\mathbf{v}, q) \in \mathbf{V} \times M$ and $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times M_h$, which satisfies following properties [1, 5, 15]

$$\|\mathbf{v}(t_m) - \mathbf{R}_h\mathbf{v}(t_m)\|_0 + h\|\nabla(\mathbf{v}(t_m) - \mathbf{R}_h\mathbf{v}(t_m))\|_0 \le Ch^2\|\mathbf{v}(t_m)\|_2, \tag{11}$$

$$\|q(t_m) - Q_hq(t_m)\|_0 \le Ch\|q(t_m)\|_1, \tag{12}$$

for $(\mathbf{v}, q) \in \mathbf{V} \cap H^2(\Omega)^2 \times M \cap H^1(\Omega)$ and $0 \le m \le N$. Further, for $0 \le k \le S$, we define the projection operator [20, 5] $\mathbf{P}_h : \mathbf{W} \to \mathbf{W}_h$ by

$$(\nabla(\mathbf{d}(t_{m_k}) - \mathbf{P}_h\mathbf{d}(t_{m_k})), \nabla\boldsymbol{\phi}_h) = 0, \quad \forall\mathbf{d} \in \mathbf{W},\ \boldsymbol{\phi}_h \in \mathbf{W}_h. \tag{13}$$

Then, the projection operator satisfies the following properties [20, 5, 1]

$$\|\mathbf{d}(t_{m_k}) - \mathbf{P}_h\mathbf{d}(t_{m_k})\|_0 + h\|\nabla(\mathbf{d}(t_{m_k}) - \mathbf{P}_h\mathbf{d}(t_{m_k}))\|_0 \le Ch^3\|\mathbf{d}(t_{m_k})\|_3, \tag{14}$$

for $\mathbf{d} \in H^3(\Omega)^2 \cap \mathbf{W}$ and $0 \le k \le S$.

Besides, we denote $(\tilde{\mathbf{u}}^m, \tilde{p}^m, \tilde{\mathbf{d}}^{m_k}) = (\mathbf{R}_h\mathbf{u}(t_m), Q_h p(t_m), \mathbf{P}_h\mathbf{d}(t_{m_k}))$, then we split errors as

$$\begin{aligned}
\mathbf{u}(t_m) - \mathbf{u}_h^m &= \mathbf{u}(t_m) - \tilde{\mathbf{u}}^m + \tilde{\mathbf{u}}^m - \mathbf{u}_h^m =: \mathbf{e}_c^m + \mathbf{e}^m,\\
p(t_m) - p_h^m &= p(t_m) - \tilde{p}^m + \tilde{p}^m - p_h^m =: \eta_c^m + \eta^m,\\
\mathbf{d}(t_{m_k}) - \mathbf{d}_h^{m_k} &= \mathbf{d}(t_{m_k}) - \tilde{\mathbf{d}}^{m_k} + \tilde{\mathbf{d}}^{m_k} - \mathbf{d}_h^{m_k} =: \boldsymbol{\epsilon}_c^{m_k} + \boldsymbol{\epsilon}^{m_k}.
\end{aligned}$$

Besides, we suppose that $\mathbf{e}^0 = 0$ and $\boldsymbol{\epsilon}^0 = 0$.

In order to obtain the error equations, set $(\mathbf{v}, q) = (\mathbf{v}_h, q_h)$ in (3) with $t = t_{m+1}$ and $\boldsymbol{\phi} = \boldsymbol{\phi}_h$ in (4) with $t = t_{m_{k+1}}$, respectively, and use (10) and (13) to get

$$\begin{aligned}
&\left(\frac{\tilde{\mathbf{u}}^{m+1} - \tilde{\mathbf{u}}^m}{\Delta t}, \mathbf{v}_h\right) + B(\tilde{\mathbf{u}}^{m+1}, \tilde{p}^{m+1}; \mathbf{v}_h, q_h) + b(\mathbf{u}(t_{m+1}), \mathbf{u}(t_{m+1}), \mathbf{v}_h)\\
&- \lambda(\nabla\mathbf{d}(t_{m+1}) \odot \nabla\mathbf{d}(t_{m+1}), \nabla\mathbf{v}_h) = (\boldsymbol{\omega}_u^{m+1}, \mathbf{v}_h) + (\mathbf{f}^{m+1}, \mathbf{v}_h),
\end{aligned} \tag{15}$$

$$\begin{aligned}
&\left(\frac{\tilde{\mathbf{d}}^{m_{k+1}} - \tilde{\mathbf{d}}^{m_k}}{\Delta s}, \boldsymbol{\phi}_h\right) + \gamma(\nabla\tilde{\mathbf{d}}^{m_{k+1}}, \nabla\boldsymbol{\phi}_h) - \gamma(|\nabla\mathbf{d}(t_{m_{k+1}})|^2\mathbf{d}(t_{m_{k+1}}), \boldsymbol{\phi}_h)\\
&+ b(\mathbf{u}(t_{m_{k+1}}), \mathbf{d}(t_{m_{k+1}}), \boldsymbol{\phi}_h) = (\boldsymbol{\omega}_d^{m_{k+1}}, \boldsymbol{\phi}_h),
\end{aligned} \tag{16}$$

where

$$\boldsymbol{\omega}_u^{m+1} = \frac{\tilde{\mathbf{u}}^{m+1} - \tilde{\mathbf{u}}^m}{\Delta t} - \mathbf{u}_t(t_{m+1}), \quad \boldsymbol{\omega}_d^{m_{k+1}} = \frac{\tilde{\mathbf{d}}^{m_{k+1}} - \tilde{\mathbf{d}}^{m_k}}{\Delta s} - \mathbf{d}_t(t_{m_{k+1}}).$$

In fact, according to Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|\boldsymbol{\omega}_u^{m+1}\|_0 &\le \left\|\frac{\tilde{\mathbf{u}}^{m+1} - \tilde{\mathbf{u}}^m}{\Delta t} - \frac{\mathbf{u}(t_{m+1}) - \mathbf{u}(t_m)}{\Delta t}\right\|_0 + \left\|\frac{\mathbf{u}(t_{m+1}) - \mathbf{u}(t_m)}{\Delta t} - \mathbf{u}_t(t_{m+1})\right\|_0\\
&\le \frac{1}{\Delta t^{1/2}}\left(\int_{t_m}^{t_{m+1}} \|\tilde{\mathbf{u}}_t - \mathbf{u}_t\|_0^2 \mathrm{d}t\right)^{1/2} + \Delta t^{1/2}\left(\int_{t_m}^{t_{m+1}} \|\mathbf{u}_{tt}\|_0^2 \mathrm{d}t\right)^{1/2},
\end{aligned} \tag{17}$$

as well as

$$\begin{aligned}
\|\boldsymbol{\omega}_d^{m_{k+1}}\|_0 &\le \left\|\frac{\tilde{\mathbf{d}}^{m_{k+1}} - \tilde{\mathbf{d}}^{m_k}}{\Delta s} - \frac{\mathbf{d}(t_{m_{k+1}}) - \mathbf{u}(t_{m_k})}{\Delta s}\right\|_0\\
&\quad + \left\|\frac{\mathbf{d}(t_{m_{k+1}}) - \mathbf{d}(t_{m_k})}{\Delta s} - \mathbf{d}_t(t_{m_{k+1}})\right\|_0\\
&\le \frac{1}{\Delta s^{1/2}}\left(\int_{t_{m_k}}^{t_{m_{k+1}}} \|\tilde{\mathbf{d}}_t - \mathbf{d}_t\|_0^2 \mathrm{d}t\right)^{1/2} + \Delta s^{1/2}\left(\int_{t_{m_k}}^{t_{m_{k+1}}} \|\mathbf{d}_{tt}\|_0^2 \mathrm{d}t\right)^{1/2}.
\end{aligned} \tag{18}$$

Moreover, subtracting (8) and (9) from (15) and (16), respectively, we arrive at

$$\begin{aligned}
&\left(\frac{\mathbf{e}^{m+1} - \mathbf{e}^m}{\Delta t}, \mathbf{v}_h\right) + B(\mathbf{e}^{m+1}, \eta^{m+1}; \mathbf{v}_h, q_h)\\
&+ b(\mathbf{u}(t_{m+1}), \mathbf{u}(t_{m+1}), \mathbf{v}_h) - b(\mathbf{u}_h^m, \mathbf{u}_h^{m+1}, \mathbf{v}_h)\\
&- \lambda(\nabla\mathbf{d}(t_{m+1}) \odot \nabla\mathbf{d}(t_{m+1}) - \nabla\mathbf{d}_h^{m_k} \odot \nabla\mathbf{d}_h^{m_k}, \nabla\mathbf{v}_h) = (\boldsymbol{\omega}_u^{m+1}, \mathbf{v}_h),
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
&\left(\frac{\boldsymbol{\epsilon}^{m_{k+1}} - \boldsymbol{\epsilon}^{m_k}}{\Delta s}, \boldsymbol{\phi}_h\right) + \gamma(\nabla\boldsymbol{\epsilon}^{m_{k+1}}, \nabla\boldsymbol{\phi}_h)\\
&- \gamma(|\nabla\mathbf{d}(t_{m_{k+1}})|^2\mathbf{d}(t_{m_{k+1}}) - |\nabla\mathbf{d}_h^{m_k}|^2\mathbf{d}_h^{m_k}, \boldsymbol{\phi}_h)\\
&+ b(\mathbf{u}(t_{m_{k+1}}), \mathbf{d}(t_{m_{k+1}}), \boldsymbol{\phi}_h) - b(\mathbf{S}^{m_k}, \mathbf{d}_h^{m_{k+1}}, \boldsymbol{\phi}_h) = (\boldsymbol{\omega}_d^{m_{k+1}}, \boldsymbol{\phi}_h).
\end{aligned} \tag{20}$$

As is known, the discrete Gronwall's inequality will paly an important rule in convergence's analysis, so we introduce it in the following lemma.

Lemma 3.1. *Let a_k, b_k and d_k, for integers $k \geq 0$, be nonnegative numbers such that*

$$a_n + \Delta t \sum_{k=0}^{n} b_k \leq \Delta t \sum_{k=0}^{n} a_k d_k + C, \quad \forall n \geq 0,$$

suppose that $\Delta t d_k < 1$, for all k, and set $\sigma_k = (1 - \Delta t d_k)^{-1}$, then

$$a_n + \Delta t \sum_{k=0}^{n} b_k \leq \exp\left(\Delta t \sum_{k=0}^{n} d_k \sigma_k\right) C, \quad \forall n \geq 0.$$

We are now in a position to state and prove the error estimates of the velocity at the larger time step $\Delta s = r\Delta t$ and the director.

Theorem 3.1. *Assume the true solution is smooth and $\mathbf{e}^0 = 0$ and $\boldsymbol{\epsilon}^0 = 0$. If the time step and mesh width satisfy $\Delta t \leq Ch^2$, then there exists a positive constant h_0 such that when $h \leq h_0$ the following estimates for the error at the larger time steps hold,*

$$\|\boldsymbol{\epsilon}^{m_S}\|_0^2 + \|\mathbf{e}^{m_S}\|_0^2 + \Delta t \sum_{k=0}^{S-1} \sum_{m=m_k}^{m_{k+1}-1} \|\nabla \mathbf{e}^{m+1}\|_0^2 + \Delta s \sum_{k=0}^{S-1} \|\nabla \boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 \leq C(\Delta t^2 + h^4).$$

Proof. Taking $\mathbf{v}_h = \mathbf{e}^{m+1}$ and $q_h = \eta^{m+1}$ in (19), we get

$$\begin{aligned}
&\frac{1}{2\Delta t}\|\mathbf{e}^{m+1}\|_0^2 + \frac{1}{2\Delta t}\|\mathbf{e}^{m+1} - \mathbf{e}^m\|_0^2 - \frac{1}{2\Delta t}\|\mathbf{e}^m\|_0^2 \\
&+ \nu\|\nabla \mathbf{e}^{m+1}\|_0^2 + \|\eta^{m+1} - \Pi\eta^{m+1}\|_0^2 \\
= &(\boldsymbol{\omega}_u^{m+1}, \mathbf{e}^{m+1}) - b(\mathbf{u}(t_{m+1}), \mathbf{u}(t_{m+1}), \mathbf{e}^{m+1}) + b(\mathbf{u}_h^m, \mathbf{u}_h^{m+1}, \mathbf{e}^{m+1}) \\
&+ \lambda(\nabla \mathbf{d}(t_{m+1}) \odot \nabla \mathbf{d}(t_{m+1}) - \nabla \mathbf{d}_h^{m_k} \odot \nabla \mathbf{d}_h^{m_k}, \nabla \mathbf{e}^{m+1}).
\end{aligned} \tag{21}$$

Then, sum (21) over $m = m_k, m_k + 1, \cdots, m_{k+1} - 1$ to give

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\mathbf{e}^{m_{k+1}}\|_0^2 - \|\mathbf{e}^{m_k}\|_0^2) + \nu \sum_{m=m_k}^{m_{k+1}-1} \|\nabla \mathbf{e}^{m+1}\|_0^2 \leq \sum_{m=m_k}^{m_{k+1}-1} |(\boldsymbol{\omega}_u^{m+1}, \mathbf{e}^{m+1})| \\
&+ \sum_{m=m_k}^{m_{k+1}-1} |b(\mathbf{u}_h^m, \mathbf{u}_h^{m+1}, \mathbf{e}^{m+1}) - b(\mathbf{u}(t_{m+1}), \mathbf{u}(t_{m+1}), \mathbf{e}^{m+1})| \\
&+ \lambda \sum_{m=m_k}^{m_{k+1}-1} |(\nabla \mathbf{d}(t_{m+1}) \odot \nabla \mathbf{d}(t_{m+1}) - \nabla \mathbf{d}_h^{m_k} \odot \nabla \mathbf{d}_h^{m_k}, \nabla \mathbf{e}^{m+1})| =: \sum_{i=1}^{3} I_i.
\end{aligned} \tag{22}$$

We now estimate each terms of the right-hand side (RHS) of (22) separately. Applying the Cauchy-Schwarz and Young inequality, we have the following estimate

$$I_1 \leq \frac{1}{2} \sum_{m=m_k}^{m_{k+1}-1} (\|\mathbf{e}^{m+1}\|_0^2 + \|\boldsymbol{\omega}_u^{m+1}\|_0^2). \tag{23}$$

Besides, for I_2, adding and subtracting some trilinear terms, and combining (5) with the definitions of the errors, we arrive at the following estimate

$$
\begin{aligned}
I_2 \le & \sum_{m=m_k}^{m_{k+1}-1} \left| b(\mathbf{e}^m, \mathbf{u}_h^{m+1} - \mathbf{u}(t_{m+1}), \mathbf{e}^{m+1}) \right| + \sum_{m=m_k}^{m_{k+1}-1} \left| b(\mathbf{e}^m, \mathbf{u}(t_{m+1}), \mathbf{e}^{m+1}) \right| \\
& + \sum_{m=m_k}^{m_{k+1}-1} \left| b(\mathbf{e}_c^m, \mathbf{u}_h^{m+1} - \mathbf{u}(t_{m+1}), \mathbf{e}^{m+1}) \right| + \sum_{m=m_k}^{m_{k+1}-1} \left| b(\mathbf{e}_c^m, \mathbf{u}(t_{m+1}), \mathbf{e}^{m+1}) \right| \\
& + \sum_{m=m_k}^{m_{k+1}-1} \left| b(\mathbf{u}(t_m) - \mathbf{u}(t_{m+1}), \mathbf{u}(t_{m+1}), \mathbf{e}^{m+1}) \right| \\
& + \sum_{m=m_k}^{m_{k+1}-1} \left| b(\mathbf{u}(t_{m+1}), \mathbf{u}_h^{m+1} - \mathbf{u}(t_{m+1}), \mathbf{e}^{m+1}) \right| \\
& + \sum_{m=m_k}^{m_{k+1}-1} \left| b(\mathbf{u}(t_m) - \mathbf{u}(t_{m+1}), \mathbf{u}_h^{m+1} - \mathbf{u}(t_{m+1}), \mathbf{e}^{m+1}) \right| \\
\le & \sum_{m=m_k}^{m_{k+1}-1} \left| b(\mathbf{e}^m, \mathbf{e}_c^{m+1}, \mathbf{e}^{m+1}) \right| + \sum_{m=m_k}^{m_{k+1}-1} \left| b(\mathbf{e}^m, \mathbf{u}(t_{m+1}), \mathbf{e}^{m+1}) \right| \\
& + \sum_{m=m_k}^{m_{k+1}-1} \left| b(\mathbf{e}_c^m, \mathbf{e}_c^{m+1}, \mathbf{e}^{m+1}) \right| + \sum_{m=m_k}^{m_{k+1}-1} \left| b(\mathbf{e}_c^m, \mathbf{u}(t_{m+1}), \mathbf{e}^{m+1}) \right| \\
& + \sum_{m=m_k}^{m_{k+1}-1} \left| b(\mathbf{u}(t_m) - \mathbf{u}(t_{m+1}), \mathbf{u}(t_{m+1}), \mathbf{e}^{m+1}) \right| \\
& + \sum_{m=m_k}^{m_{k+1}-1} \left| b(\mathbf{u}(t_{m+1}), \mathbf{e}_c^{m+1}, \mathbf{e}^{m+1}) \right| \\
& + \sum_{m=m_k}^{m_{k+1}-1} \left| b(\mathbf{u}(t_m) - \mathbf{u}(t_{m+1}), \mathbf{e}_c^{m+1}, \mathbf{e}^{m+1}) \right| =: \sum_{i=1}^{7} I_2^i.
\end{aligned} \tag{24}
$$

Next, due to (6) and the Young inequality, we deduce that

$$
\begin{aligned}
I_2^1 + I_2^2 \le & \sum_{m=m_k}^{m_{k+1}-1} \|\mathbf{e}^m\|_0 \|\mathbf{e}_c^{m+1}\|_2 \|\nabla \mathbf{e}^{m+1}\|_0 \\
& + \sum_{m=m_k}^{m_{k+1}-1} \|\mathbf{e}^m\|_0 \|\mathbf{u}(t_{m+1})\|_2 \|\nabla \mathbf{e}^{m+1}\|_0 \\
\le & \frac{8}{\nu} \sum_{m=m_k}^{m_{k+1}-1} \|\mathbf{e}^m\|_0^2 (\|\mathbf{e}_c^{m+1}\|_2^2 + \|\mathbf{u}(t_{m+1})\|_2^2) + \frac{\nu}{16} \sum_{m=m_k}^{m_{k+1}-1} \|\nabla \mathbf{e}^{m+1}\|_0^2,
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
I_2^3+I_2^4 &\le \sum_{m=m_k}^{m_{k+1}-1}\|\mathbf{e}_c^m\|_0\|\mathbf{e}_c^{m+1}\|_2\|\nabla\mathbf{e}^{m+1}\|_0+\sum_{m=m_k}^{m_{k+1}-1}\|\mathbf{e}_c^m\|_0\|\mathbf{u}(t_{m+1})\|_2\|\nabla\mathbf{e}^{m+1}\|_0\\
&\le \frac{8}{\nu}\sum_{m=m_k}^{m_{k+1}-1}\|\mathbf{e}_c^m\|_0^2(\|\mathbf{e}_c^{m+1}\|_2^2+\|\mathbf{u}(t_{m+1})\|_2^2)+\frac{\nu}{16}\sum_{m=m_k}^{m_{k+1}-1}\|\nabla\mathbf{e}^{m+1}\|_0^2, \qquad (26)\\
I_2^5+I_2^7 &\le \sum_{m=m_k}^{m_{k+1}-1}\|\mathbf{u}(t_m)-\mathbf{u}(t_{m+1})\|_0(\|\mathbf{u}(t_{m+1})\|_2+\|\mathbf{e}_c^{m+1}\|_2)\|\nabla\mathbf{e}^{m+1}\|_0\\
&\le \frac{8}{\nu}\sum_{m=m_k}^{m_{k+1}-1}\|\mathbf{u}(t_m)-\mathbf{u}(t_{m+1})\|_0^2(\|\mathbf{u}(t_{m+1})\|_2^2+\|\mathbf{e}_c^{m+1}\|_2^2)\\
&\quad+\frac{\nu}{16}\sum_{m=m_k}^{m_{k+1}-1}\|\nabla\mathbf{e}^{m+1}\|_0^2, \qquad (27)
\end{aligned}
$$

as well as

$$
\begin{aligned}
I_2^6 &\le \sum_{m=m_k}^{m_{k+1}-1}\|\mathbf{u}(t_{m+1})\|_2\|\mathbf{e}_c^{m+1}\|_0\|\nabla\mathbf{e}^{m+1}\|_0\\
&\le \frac{4}{\nu}\sum_{m=m_k}^{m_{k+1}-1}\|\mathbf{u}(t_{m+1})\|_2^2\|\mathbf{e}_c^{m+1}\|_0^2+\frac{\nu}{16}\sum_{i=m_k}^{m_{k+1}-1}\|\nabla\mathbf{e}^{m+1}\|_0^2. \qquad (28)
\end{aligned}
$$

Then, by collecting (25)-(28), we conclude that

$$
\begin{aligned}
I_2 &\le \frac{8}{\nu}\sum_{m=m_k}^{m_{k+1}-1}(\|\mathbf{e}^m\|_0^2+\|\mathbf{e}_c^m\|_0^2+\|\mathbf{u}(t_m)-\mathbf{u}(t_{m+1})\|_0^2)(\|\mathbf{u}(t_{m+1})\|_2^2+\|\mathbf{e}_c^{m+1}\|_2^2)\\
&\quad+\frac{4}{\nu}\sum_{m=m_k}^{m_{k+1}-1}\|\mathbf{u}(t_{m+1})\|_2^2\|\mathbf{e}_c^{m+1}\|_0^2+\frac{\nu}{4}\sum_{m=m_k}^{m_{k+1}-1}\|\nabla\mathbf{e}^{m+1}\|_0^2\\
&\le \frac{C}{\nu}\sum_{m=m_k}^{m_{k+1}-1}\left(\|\mathbf{e}^m\|_0^2+h^4\|\mathbf{u}(t_m)\|_2^2+\Delta t\|\mathbf{u}_t(t)\|^2_{L^2(t_m,t_{m+1};L^2(\Omega)^2)}\right)\|\mathbf{u}(t_{m+1})\|_2^2\\
&\quad+\frac{Ch^4}{\nu}\sum_{m=m_k}^{m_{k+1}-1}\|\mathbf{u}(t_{m+1})\|_2^4+\frac{\nu}{4}\sum_{m=m_k}^{m_{k+1}-1}\|\nabla\mathbf{e}^{m+1}\|_0^2, \qquad (29)
\end{aligned}
$$

where we have used (11).

Now the only issue left is to estimate the last term of (22). In fact, one easily finds that

$$
\begin{aligned}
I_3 &\le \lambda\sum_{m=m_k}^{m_{k+1}-1}|(\nabla\mathbf{d}(t_{m+1})\odot(\nabla\mathbf{d}(t_{m+1})-\nabla\mathbf{d}(t_m)),\nabla\mathbf{e}^{m+1})|\\
&\quad+\lambda\sum_{m=m_k}^{m_{k+1}-1}|(\nabla\mathbf{d}(t_{m+1})-\nabla\mathbf{d}(t_m))\odot\nabla\mathbf{d}(t_m),\nabla\mathbf{e}^{m+1})|
\end{aligned}
$$

$$
\begin{aligned}
&+ \lambda \sum_{m=m_k}^{m_{k+1}-1} |(\nabla \mathbf{d}(t_m) - \nabla \mathbf{d}_h^{m_k}) \odot \nabla \mathbf{d}(t_m), \nabla \mathbf{e}^{m+1})| \\
&- \lambda \sum_{m=m_k}^{m_{k+1}-1} |(\nabla \mathbf{d}(t_m) - \nabla \mathbf{d}_h^{m_k}) \odot (\nabla \mathbf{d}(t_m) - \nabla \mathbf{d}_h^{m_k}), \nabla \mathbf{e}^{m+1})| \\
&+ \lambda \sum_{m=m_k}^{m_{k+1}-1} |(\nabla \mathbf{d}(t_m) \odot (\nabla \mathbf{d}(t_m) - \nabla \mathbf{d}_h^{m_k}), \nabla \mathbf{e}^{m+1})| \\
=: &\sum_{i=1}^{5} I_3^i.
\end{aligned}
$$

Then, making use of the Hölder and Young's inequalities, we arrive at

$$
\begin{aligned}
I_3^1 &\leq \lambda \sum_{m=m_k}^{m_{k+1}-1} \|\nabla \mathbf{d}(t_{m+1})\|_{L^\infty(\Omega)^2} \|\nabla \mathbf{d}(t_{m+1}) - \nabla \mathbf{d}(t_m)\|_0 \|\nabla \mathbf{e}^{m+1}\|_0 \\
&\leq \frac{C\lambda^2 \Delta t}{\nu} \sum_{m=m_k}^{m_{k+1}-1} \|\nabla \mathbf{d}(t_{m+1})\|^2_{L^\infty(\Omega)^2} \|\mathbf{d}_t\|^2_{L^2(t_m,t_{m+1};H^1(\Omega)^2)} \\
&\quad + \frac{\nu}{20} \sum_{m=m_k}^{m_{k+1}-1} \|\nabla \mathbf{e}^{m+1}\|_0^2, \qquad (30)
\end{aligned}
$$

as well as

$$
\begin{aligned}
I_3^2 &\leq \lambda \sum_{m=m_k}^{m_{k+1}-1} \|\nabla \mathbf{d}(t_m)\|_{L^\infty(\Omega)^2} \|\nabla \mathbf{d}(t_{m+1}) - \nabla \mathbf{d}(t_m)\|_0 \|\nabla \mathbf{e}^{m+1}\|_0 \\
&\leq \frac{C\lambda^2 \Delta t}{\nu} \sum_{m=m_k}^{m_{k+1}-1} \|\nabla \mathbf{d}(t_m)\|^2_{L^\infty(\Omega)^2} \|\mathbf{d}_t\|^2_{L^2(t_m,t_{m+1};H^1(\Omega)^2)} \\
&\quad + \frac{\nu}{20} \sum_{m=m_k}^{m_{k+1}-1} \|\nabla \mathbf{e}^{m+1}\|_0^2. \qquad (31)
\end{aligned}
$$

Besides, for I_3^3, applying (14), the Hölder and Young's inequalities, we obtain the following estimate

$$
\begin{aligned}
I_3^3 \leq \lambda &\sum_{m=m_k}^{m_{k+1}-1} |(\nabla \boldsymbol{\epsilon}_c^m \odot \nabla \mathbf{d}(t_m), \nabla \mathbf{e}^{m+1})| \\
&+ \lambda \sum_{m=m_k}^{m_{k+1}-1} |((\nabla \tilde{\mathbf{d}}^m - \nabla \tilde{\mathbf{d}}^{m_k}) \odot \nabla \mathbf{d}(t_m), \nabla \mathbf{e}^{m+1})| \\
&+ \lambda \sum_{m=m_k}^{m_{k+1}-1} |(\nabla \boldsymbol{\epsilon}^{m_k} \odot \nabla \mathbf{d}(t_m), \nabla \mathbf{e}^{m+1})| \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \lambda \sum_{m=m_k}^{m_{k+1}-1} \|\nabla \boldsymbol{\epsilon}_c^m\|_0 \|\nabla \mathbf{d}(t_m)\|_{L^\infty(\Omega)^2} \|\nabla \mathbf{e}^{m+1}\|_0 \\
&\quad + \lambda \sum_{m=m_k}^{m_{k+1}-1} \|\nabla \boldsymbol{\epsilon}^{m_k}\|_0 \|\nabla \mathbf{d}(t_m)\|_{L^\infty(\Omega)^2} \|\nabla \mathbf{e}^{m+1}\|_0 \\
&\quad + \lambda \sum_{m=m_k}^{m_{k+1}-1} \|\nabla \tilde{\mathbf{d}}^m - \nabla \tilde{\mathbf{d}}^{m_k}\|_0 \|\nabla \mathbf{d}(t_m)\|_{L^\infty(\Omega)^2} \|\nabla \mathbf{e}^{m+1}\|_0 \\
&\leq \frac{C\lambda^2}{\nu} \sum_{m=m_k}^{m_{k+1}-1} (h^4 \|\mathbf{d}(t_m)\|_3^2 + \|\nabla \tilde{\mathbf{d}}^m - \nabla \tilde{\mathbf{d}}^{m_k}\|_0^2 + \|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^2) \|\nabla \mathbf{d}(t_m)\|_{L^\infty(\Omega)^2}^2 \\
&\quad + \frac{\nu}{20} \sum_{m=m_k}^{m_{k+1}-1} \|\nabla \mathbf{e}^{m+1}\|_0^2,
\end{aligned}
$$

Further, according to the Hölder and Young's inequalities, and the inverse inequality (7), we have

$$
\begin{aligned}
I_3^4 &\leq \lambda \sum_{m=m_k}^{m_{k+1}-1} \|\nabla \mathbf{d}(t_m) - \nabla \mathbf{d}_h^{m_k}\|_{L^3(\Omega)^2} \|\nabla \mathbf{d}(t_m) - \nabla \mathbf{d}_h^{m_k}\|_{L^6(\Omega)^2} \|\nabla \mathbf{e}^{m+1}\|_0 \\
&\leq \frac{C\lambda^2 h^{-2}}{\nu} \sum_{m=m_k}^{m_{k+1}-1} \|\nabla \mathbf{d}(t_m) - \nabla \mathbf{d}_h^{m_k}\|_0^4 + \frac{\nu}{20} \sum_{m=m_k}^{m_{k+1}-1} \|\nabla \mathbf{e}^{m+1}\|_0^2 \\
&\leq \frac{C\lambda^2 h^{-2}}{\nu} \sum_{m=m_k}^{m_{k+1}-1} (h^8 \|\mathbf{d}(t_m)\|_3^4 + \|\nabla \tilde{\mathbf{d}}^m - \nabla \tilde{\mathbf{d}}^{m_k}\|_0^4 + \|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^4) \\
&\quad + \frac{\nu}{20} \sum_{m=m_k}^{m_{k+1}-1} \|\nabla \mathbf{e}^{m+1}\|_0^2,
\end{aligned} \tag{33}
$$

as well as

$$
\begin{aligned}
I_3^5 &\leq \lambda \sum_{m=m_k}^{m_{k+1}-1} \|\nabla \mathbf{d}(t_m)\|_{L^\infty} \|\nabla \mathbf{d}(t_m) - \nabla \mathbf{d}_h^{m_k}\|_0 \|\nabla \mathbf{e}^{m+1}\|_0 \\
&\leq \frac{C\lambda^2}{\nu} \sum_{m=m_k}^{m_{k+1}-1} (h^4 \|\mathbf{d}(t_m)\|_3^2 + \|\nabla \tilde{\mathbf{d}}^m - \nabla \tilde{\mathbf{d}}^{m_k}\|_0^2 + \|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^2) \|\nabla \mathbf{d}(t_m)\|_{L^\infty(\Omega)^2}^2 \\
&\quad + \frac{\nu}{20} \sum_{m=m_k}^{m_{k+1}-1} \|\nabla \mathbf{e}^{m+1}\|_0^2.
\end{aligned} \tag{34}
$$

Consequently, (30)-(34) imply that

$$
\begin{aligned}
I_3 \leq & \frac{\nu}{4}\sum_{m=m_k}^{m_{k+1}-1}\|\nabla \mathbf{e}^{m+1}\|_0^2 + \frac{C\lambda^2\Delta t}{\nu}\sum_{m=m_k}^{m_{k+1}-1}\|\nabla \mathbf{d}(t_{m+1})\|_{L^\infty(\Omega)^2}^2\|\mathbf{d}_t\|_{L^2(t_m,t_{m+1};H^1(\Omega)^2)}^2 \\
& + \frac{C\lambda^2}{\nu}\sum_{m=m_k}^{m_{k+1}-1}(h^4\|\mathbf{d}(t_m)\|_3^2 + \|\nabla\tilde{\mathbf{d}}^m - \nabla\tilde{\mathbf{d}}^{m_k}\|_0^2 + \|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^2)\|\nabla \mathbf{d}(t_m)\|_{L^\infty(\Omega)^2}^2 \\
& + \frac{C\lambda^2 h^{-2}}{\nu}\sum_{m=m_k}^{m_{k+1}-1}(h^8\|\mathbf{d}(t_m)\|_3^4 + \|\nabla\tilde{\mathbf{d}}^m - \nabla\tilde{\mathbf{d}}^{m_k}\|_0^4 + \|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^4).
\end{aligned} \tag{35}
$$

Finally, combining (23), (29) and (35) with (22), multiplying $2\Delta t$ and summing over $k = 0, 1, \ldots, S-1$, we obtain

$$
\begin{aligned}
& \|\mathbf{e}^{m_S}\|_0^2 + \nu\Delta t\sum_{k=0}^{S-1}\sum_{m=m_k}^{m_{k+1}-1}\|\nabla \mathbf{e}^{m+1}\|_0^2 \\
\leq & C\Delta t^2 + Ch^4 + C\Delta t^3 h^{-2} + \Delta t\sum_{k=0}^{S-1}\sum_{m=m_k}^{m_{k+1}-1}(\|\mathbf{e}^{m+1}\|_0^2 + \|\boldsymbol{\omega}_u^{m+1}\|_0^2) \\
& + \frac{C\Delta t}{\nu}\sum_{k=0}^{S-1}\sum_{m=m_k}^{m_{k+1}-1}\|\mathbf{e}^m\|_0^2 + \frac{C\lambda^2 r\Delta t}{\nu}\sum_{k=0}^{S-1}(\|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^2 + h^{-2}\|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^4),
\end{aligned} \tag{36}
$$

where we have observed that

$$
\begin{aligned}
\sum_{k=0}^{S-1}\sum_{m=m_k}^{m_{k+1}-1}\|\nabla\tilde{\mathbf{d}}^m - \nabla\tilde{\mathbf{d}}^{m_k}\|_0^2 \leq & C\sum_{k=0}^{S-1}\sum_{m=m_k}^{m_{k+1}-1}\|\nabla\tilde{\mathbf{d}}^m - \nabla\tilde{\mathbf{d}}^{m+1}\|_0^2 \\
\leq & C\Delta t\sum_{k=0}^{S-1}\sum_{m=m_k}^{m_{k+1}-1}\|\mathbf{d}_t\|_{L^2(t_m,t_{m+1};H^1(\Omega)^2)}^2.
\end{aligned}
$$

In addition, choosing $\boldsymbol{\phi}_h = \boldsymbol{\epsilon}^{m_{k+1}}$ in (20), we obtain

$$
\begin{aligned}
& \frac{1}{2\Delta s}\|\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 + \frac{1}{2\Delta s}\|\boldsymbol{\epsilon}^{m_{k+1}} - \boldsymbol{\epsilon}^{m_k}\|_0^2 - \frac{1}{2\Delta s}\|\boldsymbol{\epsilon}^{m_k}\|_0^2 + \gamma\|\nabla \boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 \\
\leq & |(\boldsymbol{\omega}_d^{m_{k+1}}, \boldsymbol{\epsilon}^{m_{k+1}})| \\
& + |b(\mathbf{u}(t_{m_{k+1}}), \mathbf{d}(t_{m_{k+1}}), \boldsymbol{\epsilon}^{m_{k+1}}) - b(\mathbf{S}^{m_k}, \mathbf{d}_h^{m_{k+1}}, \boldsymbol{\epsilon}^{m_{k+1}})| \\
& + \gamma|(|\nabla \mathbf{d}(t_{m_{k+1}})|^2\mathbf{d}(t_{m_{k+1}}) - |\nabla \mathbf{d}_h^{m_k}|^2\mathbf{d}_h^{m_k}, \boldsymbol{\epsilon}^{m_{k+1}})| =: \sum_{i=4}^{6} I_i.
\end{aligned} \tag{37}
$$

Firstly, it's easy to show that

$$
I_4 \leq \frac{1}{2}(\|\boldsymbol{\omega}_d^{m_{k+1}}\|_0^2 + \|\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2). \tag{38}
$$

Secondly, by use of (5) and (6), the Hölder and Young's inequalities, we deduce that

$$
\begin{aligned}
I_5 \leq & \left| b\left(\frac{1}{r} \sum_{m=m_k}^{m_{k+1}-1} \mathbf{e}^m, \mathbf{d}(t_{m_{k+1}}), \boldsymbol{\epsilon}^{m_{k+1}} \right) \right| \\
& + \left| b\left(\frac{1}{r} \sum_{m=m_k}^{m_{k+1}-1} (\tilde{\mathbf{u}}^m - \tilde{\mathbf{u}}^{m_k}), \boldsymbol{\epsilon}_c^{m_{k+1}}, \boldsymbol{\epsilon}^{m_{k+1}} \right) \right| \\
& + \left| b\left(\frac{1}{r} \sum_{m=m_k}^{m_{k+1}-1} \mathbf{e}^m, \boldsymbol{\epsilon}_c^{m_{k+1}}, \boldsymbol{\epsilon}^{m_{k+1}} \right) \right| \\
& + \left| b\left(\frac{1}{r} \sum_{m=m_k}^{m_{k+1}-1} (\tilde{\mathbf{u}}^m - \tilde{\mathbf{u}}^{m_k}), \mathbf{d}(t_{m_{k+1}}), \boldsymbol{\epsilon}^{m_{k+1}} \right) \right| \\
& + |b(\mathbf{e}_c^{m_k}, \boldsymbol{\epsilon}_c^{m_{k+1}}, \boldsymbol{\epsilon}^{m_{k+1}})| + \left| b(\mathbf{e}_c^{m_k}, \mathbf{d}(t_{m_{k+1}}), \boldsymbol{\epsilon}^{m_{k+1}}) \right| \\
& + |b(\mathbf{u}(t_{m_k}), \boldsymbol{\epsilon}_c^{m_{k+1}}, \boldsymbol{\epsilon}^{m_{k+1}})| + \left| b(\mathbf{u}(t_{m_k}) - \mathbf{u}(t_{m_{k+1}}), \mathbf{d}(t_{m_{k+1}}), \boldsymbol{\epsilon}^{m_{k+1}}) \right| \\
\leq & \frac{C}{r^2\gamma} \sum_{m=m_k}^{m_{k+1}-1} (\|\mathbf{e}^m\|_0^2 \|\mathbf{d}(t_{m_{k+1}})\|_2^2 + \|\tilde{\mathbf{u}}^m - \tilde{\mathbf{u}}^{m_k}\|_0^2 \|\boldsymbol{\epsilon}_c^{m_{k+1}}\|_2^2) \\
& + \frac{C}{r^2\gamma} \sum_{m=m_k}^{m_{k+1}-1} (\|\mathbf{e}^m\|_0^2 \|\boldsymbol{\epsilon}_c^{m_{k+1}}\|_2^2 + \|\tilde{\mathbf{u}}^m - \tilde{\mathbf{u}}^{m_k}\|_0^2 \|\mathbf{d}(t_{m_{k+1}})\|_2^2) \\
& + \frac{C}{\gamma} (\|\mathbf{e}_c^{m_k}\|_0^2 \|\boldsymbol{\epsilon}_c^{m_{k+1}}\|_2^2 + \|\mathbf{e}_c^{m_k}\|_0^2 \|\mathbf{d}(t_{m_{k+1}})\|_2^2) \\
& + \frac{C}{\gamma} (\|\mathbf{u}(t_{m_k})\|_2^2 \|\boldsymbol{\epsilon}_c^{m_{k+1}}\|_0^2 + \|\mathbf{u}(t_{m_k}) - \mathbf{u}(t_{m_{k+1}})\|_0^2 \|\mathbf{d}(t_{m_{k+1}})\|_2^2) \\
& + \frac{\gamma}{4} \|\nabla \boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 \\
\leq & \frac{\gamma}{4} \|\nabla \boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 + \frac{C}{r^2\gamma}(1+h^2) \sum_{m=m_k}^{m_{k+1}-1} \|\mathbf{e}^m\|_0^2 \\
& + \frac{C}{r^2\gamma}(1+h^2) \sum_{m=m_k}^{m_{k+1}-1} \|\tilde{\mathbf{u}}^m - \tilde{\mathbf{u}}^{m_k}\|_0^2 \\
& + Ch^4 + \frac{C\Delta t}{\gamma} \|\mathbf{u}_t(t)\|_{L^2(t_{m_k}, t_{m_{k+1}}; L^2(\Omega)^2)}^2. \qquad (39)
\end{aligned}
$$

Thirdly, we now aim to estimate I_6. Rewrite $|\nabla \mathbf{d}(t_{m_{k+1}})|^2 \mathbf{d}(t_{m_{k+1}}) - |\nabla \mathbf{d}_h^{m_k}|^2 \mathbf{d}_h^{m_k}$ as

$$
\begin{aligned}
& |\nabla \mathbf{d}(t_{m_{k+1}})|^2 \mathbf{d}(t_{m_{k+1}}) - |\nabla \mathbf{d}_h^{m_k}|^2 \mathbf{d}_h^{m_k} \\
= & (\nabla \mathbf{d}(t_{m_{k+1}}) - \nabla \mathbf{d}(t_{m_k}))(\nabla \mathbf{d}(t_{m_{k+1}}) + \nabla \mathbf{d}(t_{m_k}))\mathbf{d}(t_{m_{k+1}}) \\
& + |\nabla \mathbf{d}(t_{m_k})|^2 (\mathbf{d}(t_{m_{k+1}}) - \mathbf{d}(t_{m_k})) + |\nabla \mathbf{d}(t_{m_k})|^2 (\mathbf{d}(t_{m_k}) - \mathbf{d}_h^{m_k}) \\
& + 2\nabla(\mathbf{d}(t_{m_k}) - \mathbf{d}_h^{m_k}) \nabla \mathbf{d}(t_{m_k}) \mathbf{d}(t_{m_k}) \\
& - 2\nabla(\mathbf{d}(t_{m_k}) - \mathbf{d}_h^{m_k}) \nabla \mathbf{d}(t_{m_k}) (\mathbf{d}(t_{m_k}) - \mathbf{d}_h^{m_k}) \\
& + |\nabla(\mathbf{d}(t_{m_k}) - \mathbf{d}_h^{m_k})|^2 (\mathbf{d}(t_{m_k}) - \mathbf{d}_h^{m_k}) - |\nabla(\mathbf{d}(t_{m_k}) - \mathbf{d}_h^{m_k})|^2 \mathbf{d}(t_{m_k}),
\end{aligned}
$$

which implies

$$
\begin{aligned}
I_6 \le\ & \gamma\|\nabla \mathbf{d}(t_{m_{k+1}}) + \nabla \mathbf{d}(t_{m_k})\|_{L^\infty(\Omega)^2}\|\mathbf{d}(t_{m_{k+1}})\|_{L^\infty(\Omega)^2} \\
& \times \|\nabla \mathbf{d}(t_{m_{k+1}}) - \nabla \mathbf{d}(t_{m_k})\|_0 \|\boldsymbol{\epsilon}^{m_{k+1}}\|_0 \\
& + \gamma\|\nabla \mathbf{d}(t_{m_k})\|^2_{L^\infty(\Omega)^2}\|\mathbf{d}(t_{m_{k+1}}) - \mathbf{d}(t_{m_k})\|_0\|\boldsymbol{\epsilon}^{m_{k+1}}\|_0 \\
& + \gamma\|\nabla \mathbf{d}(t_{m_k})\|^2_{L^\infty(\Omega)^2}\|\boldsymbol{\epsilon}^{m_k} + \boldsymbol{\epsilon}_c^{m_k}\|_0\|\boldsymbol{\epsilon}^{m_{k+1}}\|_0 \\
& + 2\gamma\|\nabla \mathbf{d}(t_{m_k})\|_{L^\infty(\Omega)^2}\|\mathbf{d}(t_{m_k})\|_{L^\infty(\Omega)^2}\|\nabla\boldsymbol{\epsilon}^{m_k} + \nabla\boldsymbol{\epsilon}_c^{m_k}\|_0\|\boldsymbol{\epsilon}^{m_{k+1}}\|_0 \\
& + 2\gamma\|\nabla \mathbf{d}(t_{m_k})\|_{L^\infty(\Omega)^2}\|\nabla\boldsymbol{\epsilon}^{m_k} + \nabla\boldsymbol{\epsilon}_c^{m_k}\|_0\|\boldsymbol{\epsilon}^{m_k} + \boldsymbol{\epsilon}_c^{m_k}\|_{L^\infty(\Omega)^2}\|\boldsymbol{\epsilon}^{m_{k+1}}\|_0 \\
& + \gamma\|\boldsymbol{\epsilon}^{m_k} + \boldsymbol{\epsilon}_c^{m_k}\|_{L^\infty(\Omega)^2}\|\nabla\boldsymbol{\epsilon}^{m_k} + \nabla\boldsymbol{\epsilon}_c^{m_k}\|_0 \\
& \times \|\nabla\boldsymbol{\epsilon}^{m_k} + \nabla\boldsymbol{\epsilon}_c^{m_k}\|_{L^3(\Omega)^2}\|\boldsymbol{\epsilon}^{m_{k+1}}\|_{L^6(\Omega)^2} \\
& + \gamma\|\mathbf{d}(t_{m_k})\|_{L^\infty(\Omega)^2}\|\nabla\boldsymbol{\epsilon}^{m_k} + \nabla\boldsymbol{\epsilon}_c^{m_k}\|_0 \\
& \times \|\nabla\boldsymbol{\epsilon}^{m_k} + \nabla\boldsymbol{\epsilon}_c^{m_k}\|_{L^3(\Omega)^2}\|\boldsymbol{\epsilon}^{m_{k+1}}\|_{L^6(\Omega)^2} \\
=:\ & \sum_{i=1}^{7} I_6^i. \qquad (40)
\end{aligned}
$$

In what follows we bound (40). To this end, we will apply the Cauchy-Schwarz and Young's inequalities and (14) to get

$$
\begin{aligned}
I_6^1 + I_6^2 &\le C\gamma\|\nabla \mathbf{d}(t_{m_{k+1}}) - \nabla \mathbf{d}(t_{m_k})\|_0\|\boldsymbol{\epsilon}^{m_{k+1}}\|_0 \\
&\le \frac{C\Delta t}{\gamma}\|\mathbf{d}_t(t)\|^2_{L^2(t_{m_k},t_{m_{k+1}};H^1(\Omega)^2)} + \frac{\gamma}{16}\|\nabla\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2, \\
I_6^3 + I_6^4 &\le C\gamma\|\nabla\boldsymbol{\epsilon}^{m_k} + \nabla\boldsymbol{\epsilon}_c^{m_k}\|_0^2 + \frac{\gamma}{16}\|\nabla\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 \\
&\le C\gamma(h^4\|\mathbf{d}(t_{m_k})\|_3^2 + \|\nabla\boldsymbol{\epsilon}^{m_k}\|_0^2) + \frac{\gamma}{16}\|\nabla\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2, \\
I_6^5 + I_6^7 &\le C\gamma\|\nabla\boldsymbol{\epsilon}^{m_k} + \nabla\boldsymbol{\epsilon}_c^{m_k}\|_0^2\|\nabla\boldsymbol{\epsilon}^{m_{k+1}}\|_0 \\
&\le C\gamma\|\nabla\boldsymbol{\epsilon}^{m_k} + \nabla\boldsymbol{\epsilon}_c^{m_k}\|_0^4 + \frac{\gamma}{16}\|\nabla\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 \\
&\le C\gamma(h^8\|\mathbf{d}(t_{m_k})\|_3^4 + \|\nabla\boldsymbol{\epsilon}^{m_k}\|_0^4) + \frac{\gamma}{16}\|\nabla\boldsymbol{\epsilon}^{m_{k+1}}\|_0, \\
I_6^6 &\le C\gamma\|\nabla\boldsymbol{\epsilon}^{m_k} + \nabla\boldsymbol{\epsilon}_c^{m_k}\|_0^3\|\nabla\boldsymbol{\epsilon}^{m_{k+1}}\|_0 \\
&\le C\gamma(h^{12}\|\mathbf{d}(t_{m_k})\|_3^6 + \|\nabla\boldsymbol{\epsilon}^{m_k}\|_0^6) + \frac{\gamma}{16}\|\nabla\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
I_6 \le\ & \frac{\gamma}{4}\|\nabla\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 + \frac{C\Delta t}{\gamma}\|\mathbf{d}_t(t)\|^2_{L^2(t_{m_k},t_{m_{k+1}};H^1(\Omega)^2)} + Ch^4 \\
& + C\gamma\|\nabla\boldsymbol{\epsilon}^{m_k}\|_0^2 + C\gamma\|\nabla\boldsymbol{\epsilon}^{m_k}\|_0^4 + C\gamma\|\nabla\boldsymbol{\epsilon}^{m_k}\|_0^6. \qquad (41)
\end{aligned}
$$

Combining (38), (39) and (41) with (37), multiplying $2\Delta s$ and summing the ensuing inequality over $k = 0, 1, \cdots, S-1$, yield

$$\begin{aligned}
&\|\boldsymbol{\epsilon}^{m_S}\|_0^2 + \gamma\Delta s \sum_{k=0}^{S-1} \|\nabla\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 \\
&\le \frac{C\Delta t}{r\gamma} \sum_{k=0}^{S-1} \sum_{m=m_k}^{m_{k+1}-1} \|\mathbf{e}^m\|_0^2 + \frac{C\Delta t^2}{r\gamma} \sum_{k=0}^{S-1} \sum_{m=m_k}^{m_{k+1}-1} \|\mathbf{u}_t\|^2_{L^2(t_m,t_{m+1};L^2(\Omega)^2)} \\
&\quad + Ch^4 + \frac{Cr\Delta t^2}{\gamma} \sum_{k=0}^{S-1} \|\mathbf{u}_t(t)\|^2_{L^2(t_{m_k},t_{m_{k+1}};L^2(\Omega)^2)} + \Delta s \sum_{k=0}^{S-1} \|\boldsymbol{\omega}_d^{m_{k+1}}\|_0^2 \\
&\quad + \Delta s \sum_{k=0}^{S-1} \|\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 + \frac{Cr\Delta t^2}{\gamma} \sum_{k=0}^{S-1} \|\mathbf{d}_t(t)\|^2_{L^2(t_{m_k},t_{m_{k+1}};H^1(\Omega)^2)} \\
&\quad + C\gamma\Delta s \sum_{k=0}^{S-1} (\|\nabla\boldsymbol{\epsilon}^{m_k}\|_0^2 + \|\nabla\boldsymbol{\epsilon}^{m_k}\|_0^4 + \|\nabla\boldsymbol{\epsilon}^{m_k}\|_0^6), \qquad (42)
\end{aligned}$$

where we have noticed that

$$\begin{aligned}
\sum_{k=0}^{S-1} \sum_{m=m_k}^{m_{k+1}-1} \|\tilde{\mathbf{u}}^m - \tilde{\mathbf{u}}^{m_k}\|_0^2 &\le C \sum_{k=0}^{S-1} \sum_{m=m_k}^{m_{k+1}-1} \|\tilde{\mathbf{u}}^m - \tilde{\mathbf{u}}^{m+1}\|_0^2 \\
&\le C\Delta t \sum_{k=0}^{S-1} \sum_{m=m_k}^{m_{k+1}-1} \|\mathbf{u}_t\|^2_{L^2(t_m,t_{m+1};L^2(\Omega)^2)}.
\end{aligned}$$

Furthermore, taking $\boldsymbol{\phi}_h = \frac{1}{\Delta s}(\boldsymbol{\epsilon}^{m_{k+1}} - \boldsymbol{\epsilon}^{m_k}) =: d_s\boldsymbol{\epsilon}^{m_{k+1}}$ in (20), we have

$$\begin{aligned}
&\|d_s\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 + \frac{\gamma}{2\Delta s}\|\nabla\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 \\
&\quad - \frac{\gamma}{2\Delta s}\|\nabla\boldsymbol{\epsilon}^{m_k}\|_0^2 + \frac{\gamma}{2\Delta s}\|\nabla\boldsymbol{\epsilon}^{m_{k+1}} - \nabla\boldsymbol{\epsilon}^{m_k}\|_0^2 \\
&\le |(\boldsymbol{\omega}_d^{m_{k+1}}, d_s\boldsymbol{\epsilon}^{m_{k+1}})| \\
&\quad + |b(\mathbf{u}(t_{m_{k+1}}), \mathbf{d}(t_{m_{k+1}}), d_s\boldsymbol{\epsilon}^{m_{k+1}}) - b(\mathbf{S}^{m_k}, \mathbf{d}_h^{m_{k+1}}, d_s\boldsymbol{\epsilon}^{m_{k+1}})| \\
&\quad + \gamma|(|\nabla\mathbf{d}(t_{m_{k+1}})|^2\mathbf{d}(t_{m_{k+1}}) - |\nabla\mathbf{d}_h^{m_k}|^2\mathbf{d}_h^{m_k}, d_s\boldsymbol{\epsilon}^{m_{k+1}})| =: \sum_{i=7}^{9} I_i. \qquad (43)
\end{aligned}$$

Moreover, it is easy to see that

$$I_7 \le C\|\boldsymbol{\omega}_d^{m_{k+1}}\|_0^2 + \frac{1}{6}\|d_s\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2. \qquad (44)$$

By the similar arguments for (39), I_8 is bounded by

$$
\begin{aligned}
I_8 \leq & \left| b\left(\frac{1}{r}\sum_{m=m_k}^{m_{k+1}-1}\mathbf{e}^m, \mathbf{d}(t_{m_{k+1}}), d_s\boldsymbol{\epsilon}^{m_{k+1}}\right)\right| \\
& + \left| b\left(\frac{1}{r}\sum_{m=m_k}^{m_{k+1}-1}(\tilde{\mathbf{u}}^m - \tilde{\mathbf{u}}^{m_k}), \boldsymbol{\epsilon}_c^{m_{k+1}} + \boldsymbol{\epsilon}^{m_{k+1}}, d_s\boldsymbol{\epsilon}^{m_{k+1}}\right)\right| \\
& + \left| b\left(\frac{1}{r}\sum_{m=m_k}^{m_{k+1}-1}\mathbf{e}^m, \boldsymbol{\epsilon}_c^{m_{k+1}}, d_s\boldsymbol{\epsilon}^{m_{k+1}}\right)\right| \\
& + \left| b\left(\frac{1}{r}\sum_{m=m_k}^{m_{k+1}-1}\mathbf{e}^m, \boldsymbol{\epsilon}^{m_{k+1}} - \boldsymbol{\epsilon}^{m_k}, d_s\boldsymbol{\epsilon}^{m_{k+1}}\right)\right| \\
& + \left| b\left(\frac{1}{r}\sum_{m=m_k}^{m_{k+1}-1}\mathbf{e}^m, \boldsymbol{\epsilon}^{m_k}, d_s\boldsymbol{\epsilon}^{m_{k+1}}\right)\right| \\
& + \left| b\left(\frac{1}{r}\sum_{m=m_k}^{m_{k+1}-1}(\tilde{\mathbf{u}}^m - \tilde{\mathbf{u}}^{m_k}), \mathbf{d}(t_{m_{k+1}}), d_s\boldsymbol{\epsilon}^{m_{k+1}}\right)\right| \\
& + |b(\mathbf{e}_c^{m_k}, \boldsymbol{\epsilon}_c^{m_{k+1}} + \boldsymbol{\epsilon}^{m_{k+1}}, d_s\boldsymbol{\epsilon}^{m_{k+1}})| + \left|b(\mathbf{e}_c^{m_k}, \mathbf{d}(t_{m_{k+1}}), d_s\boldsymbol{\epsilon}^{m_{k+1}})\right| \\
& + |b(\mathbf{u}(t_{m_k}), \boldsymbol{\epsilon}_c^{m_{k+1}} + \boldsymbol{\epsilon}^{m_{k+1}}, d_s\boldsymbol{\epsilon}^{m_{k+1}})| \\
& + \left|b(\mathbf{u}(t_{m_k}) - \mathbf{u}(t_{m_{k+1}}), \mathbf{d}(t_{m_{k+1}}), d_s\boldsymbol{\epsilon}^{m_{k+1}})\right| \\
\leq & \frac{C}{r^2}\sum_{m=m_k}^{m_{k+1}-1}(\|\mathbf{e}^m\|_0^2\|\mathbf{d}(t_{m_{k+1}})\|_{W^{1,\infty}(\Omega)^2}^2 + \|\nabla\tilde{\mathbf{u}}^m - \nabla\tilde{\mathbf{u}}^{m_k}\|_0^2\|\boldsymbol{\epsilon}_c^{m_{k+1}} + \boldsymbol{\epsilon}^{m_{k+1}}\|_2^2) \\
& + \frac{C}{r^2}\sum_{m=m_k}^{m_{k+1}-1}(\|\mathbf{e}^m\|_0^2\|\boldsymbol{\epsilon}_c^{m_{k+1}}\|_{W^{1,\infty}(\Omega)^2}^2 + \|\mathbf{e}^m\|_0^2\|\boldsymbol{\epsilon}^{m_k}\|_{W^{1,\infty}(\Omega)^2}^2) \\
& + \frac{C}{r^2}\sum_{m=m_k}^{m_{k+1}-1}\|\nabla\tilde{\mathbf{u}}^m - \nabla\tilde{\mathbf{u}}^{m_k}\|_0^2\|\mathbf{d}(t_{m_{k+1}})\|_2^2 + \frac{1}{6}\|d_s\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 \\
& + C(\|\mathbf{e}_c^{m_k}\|_2^2\|\nabla\boldsymbol{\epsilon}_c^{m_{k+1}} + \nabla\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 + \|\mathbf{e}_c^{m_k}\|_0^2\|\mathbf{d}(t_{m_{k+1}})\|_{W^{1,\infty}(\Omega)^2}^2) \\
& + C(\|\mathbf{u}(t_{m_k})\|_2^2\|\nabla\boldsymbol{\epsilon}_c^{m_{k+1}} + \nabla\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 + \|\nabla\mathbf{u}(t_{m_k}) - \nabla\mathbf{u}(t_{m_{k+1}})\|_0^2\|\mathbf{d}(t_{m_{k+1}})\|_2^2) \\
\leq & \frac{C}{r^2}\sum_{m=m_k}^{m_{k+1}-1}\|\mathbf{e}^m\|_0^2 + \frac{Ch^{-2}}{r^2}\sum_{m=m_k}^{m_{k+1}-1}(\|\nabla\boldsymbol{\epsilon}_c^{m_{k+1}}\|_0^2 + \|\nabla\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2)\|\nabla\tilde{\mathbf{u}}^m - \nabla\tilde{\mathbf{u}}^{m_k}\|_0^2 \\
& + \frac{Ch^{-2}}{r^2}\sum_{m=m_k}^{m_{k+1}-1}\|\mathbf{e}^m\|_0^2(\|\nabla\boldsymbol{\epsilon}_c^{m_{k+1}}\|_0^2 + \|\nabla\boldsymbol{\epsilon}^{m_k}\|_0^2) + \frac{C}{r^2}\sum_{m=m_k}^{m_{k+1}-1}\|\nabla\tilde{\mathbf{u}}^m - \nabla\tilde{\mathbf{u}}^{m_k}\|_0^2 \\
& + Ch^4 + C(\|\nabla\boldsymbol{\epsilon}_c^{m_{k+1}}\|_0^2 + \|\nabla\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2) \\
& + C\Delta t\|\mathbf{u}_t\|_{L^2(t_{m_k}, t_{m_{k+1}}; H^1(\Omega)^2)}^2 + \frac{1}{6}\|d_s\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 \\
\leq & \frac{C}{r^2}(1+h^2)\sum_{m=m_k}^{m_{k+1}-1}\|\mathbf{e}^m\|_0^2 + \frac{C}{r^2}(1+h^2)\sum_{m=m_k}^{m_{k+1}-1}\|\nabla\tilde{\mathbf{u}}^m - \nabla\tilde{\mathbf{u}}^{m_k}\|_0^2 \\
& + \frac{Ch^{-2}}{r^2}\sum_{m=m_k}^{m_{k+1}-1}\|\nabla\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2\|\nabla\tilde{\mathbf{u}}^m - \nabla\tilde{\mathbf{u}}^{m_k}\|_0^2 + \frac{Ch^{-2}}{r^2}\sum_{m=m_k}^{m_{k+1}-1}\|\mathbf{e}^m\|_0^2\|\nabla\boldsymbol{\epsilon}^{m_k}\|_0^2 \\
& + Ch^4 + C\|\nabla\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 + C\Delta t\|\mathbf{u}_t\|_{L^2(t_{m_k}, t_{m_{k+1}}; H^1(\Omega)^2)}^2 + \frac{1}{6}\|d_s\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2.
\end{aligned} \tag{45}
$$

In addition, using the same arguments as (40) once more, we immediately find that

$$
\begin{aligned}
I_9 \leq & \gamma \|\nabla \mathbf{d}(t_{m_{k+1}}) + \nabla \mathbf{d}(t_{m_k})\|_{L^\infty(\Omega)^2} \\
& \times \|\mathbf{d}(t_{m_{k+1}})\|_{L^\infty(\Omega)^2} \|\nabla \mathbf{d}(t_{m_{k+1}}) - \nabla \mathbf{d}(t_{m_k})\|_0 \|d_s \boldsymbol{\epsilon}^{m_{k+1}}\|_0 \\
& + \gamma \|\nabla \mathbf{d}(t_{m_k})\|^2_{L^\infty(\Omega)^2} \|\mathbf{d}(t_{m_{k+1}}) - \mathbf{d}(t_{m_k})\|_0 \|d_s \boldsymbol{\epsilon}^{m_{k+1}}\|_0 \\
& + \gamma \|\nabla \mathbf{d}(t_{m_k})\|^2_{L^\infty(\Omega)^2} \|\boldsymbol{\epsilon}^{m_k} + \boldsymbol{\epsilon}_c^{m_k}\|_0 \|d_s \boldsymbol{\epsilon}^{m_{k+1}}\|_0 \\
& + 2\gamma \|\nabla \mathbf{d}(t_{m_k})\|_{L^\infty(\Omega)^2} \|\mathbf{d}(t_{m_k})\|_{L^\infty(\Omega)^2} \|\nabla \boldsymbol{\epsilon}^{m_k} + \nabla \boldsymbol{\epsilon}_c^{m_k}\|_0 \|d_s \boldsymbol{\epsilon}^{m_{k+1}}\|_0 \\
& + 2\gamma \|\nabla \mathbf{d}(t_{m_k})\|_{L^\infty(\Omega)^2} \|\nabla \boldsymbol{\epsilon}^{m_k} + \nabla \boldsymbol{\epsilon}_c^{m_k}\|_0 \|\boldsymbol{\epsilon}^{m_k} + \boldsymbol{\epsilon}_c^{m_k}\|_{L^\infty(\Omega)^2} \|d_s \boldsymbol{\epsilon}^{m_{k+1}}\|_0 \\
& + \gamma \|\boldsymbol{\epsilon}^{m_k} + \boldsymbol{\epsilon}_c^{m_k}\|_{L^\infty(\Omega)^2} \|\nabla \boldsymbol{\epsilon}^{m_k} + \nabla \boldsymbol{\epsilon}_c^{m_k}\|_0 \\
& \times \|\nabla \boldsymbol{\epsilon}^{m_k} + \nabla \boldsymbol{\epsilon}_c^{m_k}\|_{L^3(\Omega)^2} \|d_s \boldsymbol{\epsilon}^{m_{k+1}}\|_{L^6(\Omega)^2} \\
& + \gamma \|\mathbf{d}(t_{m_k})\|_{L^\infty(\Omega)^2} \|\nabla \boldsymbol{\epsilon}^{m_k} + \nabla \boldsymbol{\epsilon}_c^{m_k}\|_0 \\
& \times \|\nabla \boldsymbol{\epsilon}^{m_k} + \nabla \boldsymbol{\epsilon}_c^{m_k}\|_{L^3(\Omega)^2} \|d_s \boldsymbol{\epsilon}^{m_{k+1}}\|_{L^6(\Omega)^2} \\
& =: \sum_{i=1}^{7} I_9^i.
\end{aligned}
$$

In order to bound I_9, we need the Cauchy-Schwarz and Young's inequalities, the inverse inequality (7) and (14). Then

$$
\begin{aligned}
I_9^1 + I_9^2 &\leq C\gamma^2 \Delta t \|\mathbf{d}_t(t)\|^2_{L^2(t_{m_k}, t_{m_{k+1}}; H^1(\Omega)^2)} + \frac{1}{30}\|d_s \boldsymbol{\epsilon}^{m_{k+1}}\|_0^2, \\
I_9^3 + I_9^4 &\leq C\gamma^2 (h^4 \|\mathbf{d}(t_{m_k})\|_3^2 + \|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^2) + \frac{1}{30}\|d_s \boldsymbol{\epsilon}^{m_{k+1}}\|_0^2, \\
I_9^5 &\leq C\gamma^2 (h^8 \|\mathbf{d}(t_{m_k})\|_3^4 + \|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^4) + \frac{1}{30}\|d_s \boldsymbol{\epsilon}^{m_{k+1}}\|_0, \\
I_9^6 &\leq C\gamma h^{-1} \|\nabla \boldsymbol{\epsilon}^{m_k} + \nabla \boldsymbol{\epsilon}_c^{m_k}\|_0^3 \|d_s \boldsymbol{\epsilon}^{m_{k+1}}\|_0 \\
&\leq C\gamma^2 h^{-2} (h^{12} \|\mathbf{d}(t_{m_k})\|_3^6 + \|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^6) + \frac{1}{30}\|d_s \boldsymbol{\epsilon}^{m_{k+1}}\|_0^2, \\
I_9^7 &\leq C\gamma h^{-1} \|\nabla \boldsymbol{\epsilon}^{m_k} + \nabla \boldsymbol{\epsilon}_c^{m_k}\|_0^2 \|d_s \boldsymbol{\epsilon}^{m_{k+1}}\|_0 \\
&\leq C\gamma^2 h^{-2} (h^8 \|\mathbf{d}(t_{m_k})\|_3^4 + \|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^4) + \frac{1}{30}\|d_s \boldsymbol{\epsilon}^{m_{k+1}}\|_0^2.
\end{aligned}
$$

Hence, I_9 is bounded by

$$
\begin{aligned}
I_9 \leq & \frac{1}{6}\|d_s \boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 + C\gamma^2 \Delta t \|\mathbf{d}_t(t)\|^2_{L^2(t_{m_k}, t_{m_{k+1}}; H^1(\Omega)^2)} + Ch^4 \\
& + C\gamma^2 (\|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^2 + \|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^4 + h^{-2}\|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^4 + h^{-2}\|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^6).
\end{aligned}
\tag{46}
$$

Finally, combining (44), (45) and (46) with (43), multiplying $2\Delta s$ and summing over $k = 0, 1, \cdots, S-1$, yield

$$
\begin{aligned}
\Delta s \sum_{k=0}^{S-1} \|d_s \boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 + \gamma \|\nabla \boldsymbol{\epsilon}^{m_S}\|_0^2 \leq C\Delta s \sum_{k=0}^{S-1} \|\boldsymbol{\omega}_d^{m_{k+1}}\|_0^2 + \frac{C\Delta t}{r} \sum_{k=0}^{S-1} \sum_{m=m_k}^{m_{k+1}-1} \|\mathbf{e}^m\|_0^2 \\
+ \frac{C\Delta t^2}{r} \sum_{k=0}^{S-1} \sum_{m=m_k}^{m_{k+1}-1} \|\mathbf{u}_t\|^2_{L^2(t_m, t_{m+1}; H^1(\Omega)^2)} + C\Delta s \sum_{k=0}^{S-1} \|\nabla \boldsymbol{\epsilon}^{m_{k+1}}\|_0^2
\end{aligned}
$$

$$+\frac{C\Delta t^2 h^{-2}}{r}\sum_{k=0}^{S-1}\sum_{m=m_k}^{m_{k+1}-1}\|\mathbf{u}_t\|^2_{L^2(t_m,t_{m+1};H^1(\Omega)^2)}\|\nabla\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2$$
$$+\frac{C\Delta t h^{-2}}{r}\sum_{k=0}^{S-1}\sum_{m=m_k}^{m_{k+1}-1}\|\mathbf{e}^m\|_0^2\|\nabla\boldsymbol{\epsilon}^{m_k}\|_0^2 + Ch^4$$
$$+Cr\Delta t^2\sum_{k=0}^{S-1}\|\mathbf{u}_t\|^2_{L^2(t_{m_k},t_{m_{k+1}};H^1(\Omega)^2)}$$
$$+C\gamma^2 r\Delta t^2\sum_{k=0}^{S-1}\|\mathbf{d}_t(t)\|^2_{L^2(t_{m_k},t_{m_{k+1}};H^1(\Omega)^2)}$$
$$+C\gamma^2\Delta s\sum_{k=0}^{S-1}(\|\nabla\boldsymbol{\epsilon}^{m_k}\|_0^2+\|\nabla\boldsymbol{\epsilon}^{m_k}\|_0^4+h^{-2}\|\nabla\boldsymbol{\epsilon}^{m_k}\|_0^4+h^{-2}\|\nabla\boldsymbol{\epsilon}^{m_k}\|_0^6)$$
$$\le C\Delta s\sum_{k=0}^{S-1}\|\boldsymbol{\omega}_d^{m_{k+1}}\|_0^2+\frac{C\Delta t}{r}\sum_{k=0}^{S-1}\sum_{m=m_k}^{m_{k+1}-1}\|\mathbf{e}^m\|_0^2+C\Delta t^2+Ch^4$$
$$+C\Delta s\left(1+\frac{\Delta t h^{-2}}{r^2}\right)\sum_{k=0}^{S-1}\|\nabla\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2$$
$$+\frac{C\Delta t h^{-2}}{r}\sum_{k=0}^{S-1}\sum_{m=m_k}^{m_{k+1}-1}\|\mathbf{e}^m\|_0^2\|\nabla\boldsymbol{\epsilon}^{m_k}\|_0^2$$
$$+C\gamma^2\Delta s\sum_{k=0}^{S-1}(\|\nabla\boldsymbol{\epsilon}^{m_k}\|_0^2+\|\nabla\boldsymbol{\epsilon}^{m_k}\|_0^4+h^{-2}\|\nabla\boldsymbol{\epsilon}^{m_k}\|_0^4+h^{-2}\|\nabla\boldsymbol{\epsilon}^{m_k}\|_0^6). \tag{47}$$

Note that the following bound holds

$$\Delta t\sum_{k=0}^{S-1}\sum_{m=m_k}^{m_{k+1}-1}\|\boldsymbol{\omega}_u^{m+1}\|_0^2+C\Delta s\sum_{k=0}^{S-1}\|\boldsymbol{\omega}_d^{m_{k+1}}\|_0^2$$
$$\le C\Delta t\sum_{k=0}^{S-1}\sum_{m=m_k}^{m_{k+1}-1}\left(\frac{1}{\Delta t}\int_{t_m}^{t_{m+1}}\|\tilde{\mathbf{u}}_t-\mathbf{u}_t\|_0^2\mathrm{d}t+\Delta t\int_{t_m}^{t_{m+1}}\|\mathbf{u}_{tt}\|_0^2\mathrm{d}t\right)$$
$$+C\Delta s\sum_{k=0}^{S-1}\left(\frac{1}{\Delta s}\int_{t_{m_k}}^{t_{m_{k+1}}}\|\tilde{\mathbf{d}}_t-\mathbf{d}_t\|_0^2\mathrm{d}t+\Delta s\int_{t_{m_k}}^{t_{m_{k+1}}}\|\mathbf{d}_{tt}\|_0^2\mathrm{d}t\right)$$
$$\le C\int_0^T\|\tilde{\mathbf{u}}_t-\mathbf{u}_t\|_0^2\mathrm{d}t+C\Delta t^2\int_0^T\|\mathbf{u}_{tt}\|_0^2\mathrm{d}t$$
$$+C\int_0^T\|\tilde{\mathbf{d}}_t-\mathbf{d}_t\|_0^2\mathrm{d}t+\Delta s^2\int_0^T\|\mathbf{d}_{tt}\|_0^2\mathrm{d}t\le C(\Delta t^2+h^4). \tag{48}$$

Here, (48) is obtained by applying (17) and (18) with approximation properties. Summing up (36), (42) and (47), we have

$$\|\mathbf{e}^{m_S}\|_0^2+\|\boldsymbol{\epsilon}^{m_S}\|_0^2+\gamma\|\nabla\boldsymbol{\epsilon}^{m_S}\|_0^2+\nu\Delta t\sum_{k=0}^{S-1}\sum_{m=m_k}^{m_{k+1}-1}\|\nabla\mathbf{e}^{m+1}\|_0^2$$
$$+\gamma\Delta s\sum_{k=0}^{S-1}\|\nabla\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2+\Delta s\sum_{k=0}^{S-1}\|d_s\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2$$

$$
\begin{aligned}
&\le C\Delta t^3 h^{-2} + C\Delta t^2 + Ch^4 + \Delta t \sum_{k=0}^{S-1} \sum_{m=m_k}^{m_{k+1}-1} \|\mathbf{e}^{m+1}\|_0^2 \\
&\quad + \frac{C\Delta t}{\nu} \sum_{k=0}^{S-1} \sum_{m=m_k}^{m_{k+1}-1} \|\mathbf{e}^m\|_0^2 + \frac{C\Delta t}{\gamma r} \sum_{k=0}^{S-1} \sum_{m=m_k}^{m_{k+1}-1} \|\mathbf{e}^m\|_0^2 + \frac{C\Delta t}{r} \sum_{k=0}^{S-1} \sum_{m=m_k}^{m_{k+1}-1} \|\mathbf{e}^m\|_0^2 \\
&\quad + \Delta s \sum_{k=0}^{S-1} \|\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 + C\Delta s \left(1 + \frac{\Delta t h^{-2}}{r^2}\right) \sum_{k=0}^{S-1} \|\nabla \boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 \\
&\quad + \frac{C\lambda^2 r \Delta t}{\nu} \sum_{k=0}^{S-1} (\|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^2 + h^{-2}\|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^4) \\
&\quad + \Delta t \sum_{k=0}^{S-1} \sum_{m=m_k}^{m_{k+1}-1} \|\boldsymbol{\omega}_u^{m+1}\|_0^2 + C\Delta s \sum_{k=0}^{S-1} \|\boldsymbol{\omega}_d^{m_{k+1}}\|_0^2 \\
&\quad + C\gamma \Delta s \sum_{k=0}^{S-1} (\|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^2 + \|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^4 + \|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^6) + \frac{C\Delta t h^{-2}}{r} \sum_{k=0}^{S-1} \|\mathbf{e}^m\|_0^2 \|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^2 \\
&\quad + C\gamma^2 \Delta s \sum_{k=0}^{S-1} (\|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^2 + \|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^4 + h^{-2}\|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^4 + h^{-2}\|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^6) \\
&\le C\Delta t^2 + Ch^4 + C\Delta t^3 h^{-2} + C\Delta t \sum_{k=0}^{S-1} \sum_{m=m_k}^{m_{k+1}-1} \|\mathbf{e}^{m+1}\|_0^2 + \Delta s \sum_{k=0}^{S-1} \|\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 \\
&\quad + C\Delta s \left(1 + \frac{\Delta t h^{-2}}{r^2}\right) \sum_{k=0}^{S-1} \|\nabla \boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 + \frac{C\Delta t h^{-2}}{r} \sum_{k=0}^{S-1} \sum_{m=m_k}^{m_{k+1}-1} \|\mathbf{e}^m\|_0^2 \|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^2 \\
&\quad + C\Delta s \sum_{k=0}^{S-1} (\|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^2 + \|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^4 + \|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^6 + h^{-2}\|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^4 + h^{-2}\|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^6).
\end{aligned}
\tag{49}
$$

Now we prove that $\|\nabla \boldsymbol{\epsilon}^{m_k}\|_0 \le h$ for $0 \le k \le S$ by using mathematical induction method. Clearly, this inequality holds for $k = 0$. If we assume that this inequality holds for $k \le S-1$ and $\Delta t \le Ch^2$. Then, we rewrite (49) as

$$
\begin{aligned}
&\|\mathbf{e}^{m_S}\|_0^2 + \|\boldsymbol{\epsilon}^{m_S}\|_0^2 + \gamma \|\nabla \boldsymbol{\epsilon}^{m_S}\|_0^2 \le C\Delta t^2 + Ch^4 \\
&\quad + C\Delta t \sum_{k=0}^{S-1} \sum_{m=m_k}^{m_{k+1}-1} \|\mathbf{e}^{m+1}\|_0^2 + C\Delta s \sum_{k=0}^{S-1} \|\nabla \boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 + \Delta s \sum_{k=0}^{S-1} \|\boldsymbol{\epsilon}^{m_{k+1}}\|_0^2.
\end{aligned}
\tag{50}
$$

Hence, by applying Lemma 3.1, we derive that

$$
\|\nabla \boldsymbol{\epsilon}^{m_S}\|_0^2 \le Ch^4 \le h^2, \quad \text{if} \quad h \le \frac{1}{\sqrt{C}} = h_0,
$$

which completes the induction.

Finally, combining Lemma 3.1 and (50), we arrive at

$$
\begin{aligned}
&\|\boldsymbol{\epsilon}^{m_S}\|_0^2 + \|\mathbf{e}^{m_S}\|_0^2 + \Delta t \sum_{k=0}^{S-1} \sum_{m=m_k}^{m_{k+1}-1} \|\nabla \mathbf{e}^{m+1}\|_0^2 + \Delta s \sum_{k=0}^{S-1} \|\nabla \boldsymbol{\epsilon}^{m_{k+1}}\|_0^2 \\
&\le C(\Delta t^2 + h^4).
\end{aligned}
\tag{51}
$$

□

We are now in a position to state and prove the error estimate of the velocity at the smaller time step Δt.

Theorem 3.2. *Under the assumption of Theorem 3.1, the following estimate holds: for $J = 1,\ 2, \cdots,\ r-1$, and $k = 0,\ 1, \cdots,\ S-1$,*

$$\|\mathbf{e}^{m_k+J+1}\|_0^2 + \nu\Delta t \sum_{m=m_k}^{m_k+J} \|\nabla \mathbf{e}^{m+1}\|_0^2 \le C(\Delta t^2 + h^4).$$

Proof. Summing (21) over $m = m_k, m_k+1, \cdots, m_k+J$ yield

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\mathbf{e}^{m_k+J+1}\|_0^2 - \|\mathbf{e}^{m_k}\|_0^2) + \nu \sum_{m=m_k}^{m_k+J} \|\nabla \mathbf{e}^{m+1}\|_0^2 \\
&\le \sum_{m=m_k}^{m_k+J} |(\boldsymbol{\omega}_u^{m+1}, \mathbf{e}^{m+1})| + \sum_{m=m_k}^{m_k+J} |b(\mathbf{u}_h^m, \mathbf{u}_h^{m+1}, \mathbf{e}^{m+1}) - b(\mathbf{u}(t_{m+1}), \mathbf{u}(t_{m+1}), \mathbf{e}^{m+1})| \\
&\quad + \lambda \sum_{m=m_k}^{m_k+J} |(\nabla \mathbf{d}(t_{m+1}) \odot \nabla \mathbf{d}(t_{m+1}) - \nabla \mathbf{d}_h^{m_k} \odot \nabla \mathbf{d}_h^{m_k}, \nabla \mathbf{e}^{m+1})|.
\end{aligned} \tag{52}$$

Using similar arguments to (23), (29) and (35), and multiplying $2\Delta t$, we obtain

$$\begin{aligned}
&\|\mathbf{e}^{m_k+J+1}\|_0^2 - \|\mathbf{e}^{m_k}\|_0^2 + \nu\Delta t \sum_{m=m_k}^{m_k+J} \|\nabla \mathbf{e}^{m+1}\|_0^2 \le \Delta t \sum_{m=m_k}^{m_k+J} (\|\mathbf{e}^{m+1}\|_0^2 + \|\boldsymbol{\omega}_u^{m+1}\|_0^2) \\
&\quad + \frac{C\Delta t}{\nu} \sum_{m=m_k}^{m_k+J} \Big(\|\mathbf{e}^m\|_0^2 + h^4\|\mathbf{u}(t_m)\|_2^2 + \Delta t\|\mathbf{u}_t(t)\|_{L^2(t_m,t_{m+1};L^2(\Omega)^2)}^2\Big) \|\mathbf{u}(t_{m+1})\|_2^2 \\
&\quad + \frac{C\Delta t h^4}{\nu} \sum_{m=m_k}^{m_k+J} \|\mathbf{u}(t_{m+1})\|_2^4 \\
&\quad + \frac{C\lambda^2\Delta t^2}{\nu} \sum_{m=m_k}^{m_k+J} \|\nabla \mathbf{d}(t_{m+1})\|_{L^\infty(\Omega)^2}^2 \|\mathbf{d}_t\|_{L^2(t_m,t_{m+1};H^1(\Omega)^2)}^2 \\
&\quad + \frac{C\Delta t\lambda^2}{\nu} \sum_{m=m_k}^{m_k+J} (h^4\|\mathbf{d}(t_m)\|_3^2 + \|\nabla \tilde{\mathbf{d}}^m - \nabla \tilde{\mathbf{d}}^{m_k}\|_0^2 + \|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^2) \|\nabla \mathbf{d}(t_m)\|_{L^\infty(\Omega)^2}^2 \\
&\quad + \frac{C\Delta t\lambda^2 h^{-2}}{\nu} \sum_{m=m_k}^{m_k+J} (h^8\|\mathbf{d}(t_m)\|_3^4 + \|\nabla \tilde{\mathbf{d}}^m - \nabla \tilde{\mathbf{d}}^{m_k}\|_0^4 + \|\nabla \boldsymbol{\epsilon}^{m_k}\|_0^4) \\
&\le C\Delta t^2 + Ch^4 + \Delta t \sum_{m=m_k}^{m_k+J} \|\mathbf{e}^{m+1}\|_0^2 + \frac{C\Delta t}{\nu} \sum_{m=m_k}^{m_k+J} \|\mathbf{e}^m\|_0^2,
\end{aligned}$$

where we have applied (51).

Hence, employing Theorem 3.1 and Poincáre inequality, we arrive at

$$\begin{aligned}
&\|\mathbf{e}^{m_k+J+1}\|_0^2 + \nu\Delta t \sum_{m=m_k}^{m_k+J} \|\nabla \mathbf{e}^{m+1}\|_0^2 \\
&\le C(\Delta t^2 + h^4) + C\Delta t \sum_{m=m_k}^{m_k+J} \|\mathbf{e}^{m+1}\|_0^2 \le C(\Delta t^2 + h^4).
\end{aligned}$$

□

TABLE 1. Numerical errors and convergence rates at T=0.1 for $\Delta t = 0.2h^2$ with $r = 5$.

h	$\|\mathbf{u}(t_m) - \mathbf{u}_h^m\|_0$	Rate	$\|\nabla(\mathbf{d}(t_m) - \mathbf{d}_h^m)\|_0$	Rate
1/30	0.0093657	—	0.0371636	—
1/60	0.0022947	2.0290813	0.0095850	1.9550399
1/90	0.0010135	2.0154445	0.0039397	2.1927838

TABLE 2. Numerical errors and CPU time for $\Delta t = 0.1h^2$ at $T = 0.1$ with $r = 1$.

h	$\|\mathbf{u}(t_m) - \mathbf{u}_h^m\|_0$	$\|\nabla(\mathbf{u}(t_m) - \mathbf{u}_h^m)\|_0$	$\|\nabla(\mathbf{d}(t_m) - \mathbf{d}_h^m)\|_0$	CPU
1/20	0.0186602	0.1275500	0.0283859	5.219
1/30	0.0078589	0.0505034	0.0127069	32.668
1/40	0.0051803	0.0295617	0.0078813	115.991

TABLE 3. Numerical errors and CPU time for $\Delta t = 0.1h^2$ at $T = 0.1$ with $r = 5$.

h	$\|\mathbf{u}(t_m) - \mathbf{u}_h^m\|_0$	$\|\nabla(\mathbf{u}(t_m) - \mathbf{u}_h^m)\|_0$	$\|\nabla(\mathbf{d}(t_m) - \mathbf{d}_h^m)\|_0$	CPU
1/20	0.0219266	0.1134930	0.0227466	3.922
1/30	0.0084534	0.0541689	0.0106532	19.129
1/40	0.0049239	0.0307126	0.0056939	66.144

4. Numerical experiments

In this section, we assess the numerical performance of the fully discrete, decoupled scheme with different time steps for the nematic liquid crystal flow. All computations are carried out in the unit circle $\Omega = \{(x, y) : x^2 + y^2 < 1\}$.

On one hand, the initial data and the body force are taken as $\mathbf{u}_0 = \mathbf{0}$, $\mathbf{d}_0 = (\sin(a), \cos(a))$, $a = \pi(x^2 + y^2)^2$ and $\mathbf{f} = 0$, respectively.

The main goal of the experiment is to verify the convergence rates for the considered method. The exact solution of this problem is unknown. Thus, we take the numerical solution by the standard Galerkin method computed on a very fine mesh ($h = 1/150$) as the "exact" solution for the purpose of comparison. Parameters are set as $\lambda = \nu = \gamma = 1$. Besides, the time step $\tau = Ch^2$. We display the convergence orders and errors of the considered method at the final time $T = 0.1$ in Table 1. From this table, it can be easily to see that the presented method works well and keeps the convergence rates just like the theoretical analysis.

Moreover, to investigate the effectiveness of the decoupled scheme with different time steps, we compare the numerical results by the considered method with $r = 1$ and $r = 5$. Note that when $r = 1$ the method has the same time step for the velocity and the director. In fact, it becomes the standard Galerkin method.

The numerical errors and corresponding CPU time are listed in Table 2 and 3. As expected, the considered method with $r = 5$ spends less CPU time than the considered method with $r = 1$ to achieve nearly the same relative error.

On other hand, the initial data and the body force are taken as $\mathbf{u}_0 = \mathbf{0}$, $\mathbf{d}_0 = (\sin(a), \cos(a))$, $a = 4\pi(x^4 - y^4)^2$ and $\mathbf{f} = 0$, respectively.

In the following example, the parameters are set as $\lambda = \gamma = 1$ and $\nu = 2$. Besides, we choose the mesh size $h = 80$ and $\Delta t = h^2$. In Figure 1 and 2, we plot

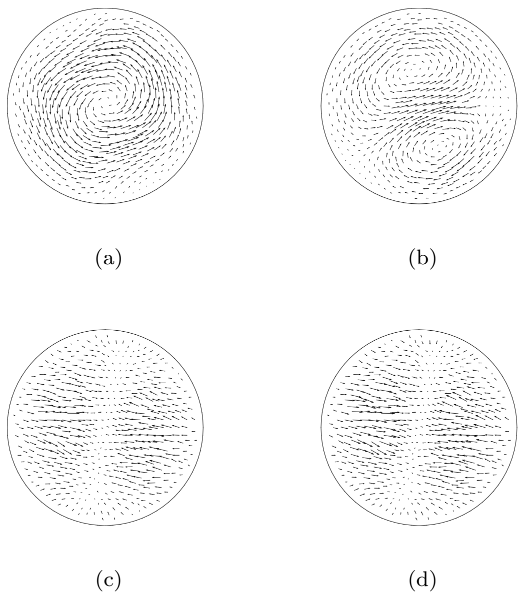

FIGURE 1. Evolution of velocity fields: $T = 0.1(a)$, $T = 0.5(b)$, $T = 1.5(c)$ and $T = 3.0(d)$.

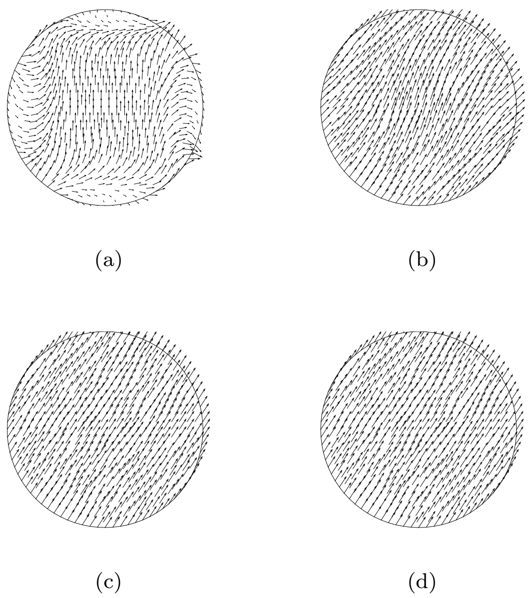

FIGURE 2. Evolution of director fields: $T = 0.01(a)$, $T = 0.5(b)$, $T = 1.5(c)$ and $T = 3.0(d)$.

the evolution of the velocity fields and the director fields by the presented method. From these figures, one can find that the velocity and director have almost the same trend after the final time $T = 1.5$.

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