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COLLOCATION METHODS FOR A CLASS OF INTEGRO-DIFFERENTIAL ALGEBRAIC EQUATIONS

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Abstract. A class of index-1 integro-differential algebraic equations modeling a hydraulic circuit that feed a combustion process is considered. The existence, uniqueness and regularity are analyzed in detail. Two kinds of collocation methods are employed to solve the equation numerically. For the first one, the derivative and algebraic components are approximated in globally continuous and discontinuous polynomial spaces, respectively; and for another one, both the derivative and algebraic components are solved in globally continuous piecewise polynomial spaces. The convergence, global and local superconvergence are described for these two classes of collocation methods. Some numerical experiments are given to illustrate the obtained theoretical results.

Key words. Integro-differential algebraic equations, tractability index, regularity, collocation methods, convergence analysis.

1. Introduction

Integro-differential algebraic equations (IDAEs) arise in many mathematical modeling processes, for example, [6, 7] (Kirchhoff's laws); [10] (circuit simulation); [9] (the seat-occupant dynamic model); and [16] (hydraulic circuit that feeds a combustion process). Due to the rich applications, there are many researchers focus on this research area. In [12], the convergence properties of implicit Runge-Kutta methods of Pouzet-type for IDAEs that arise when solving singularly perturbed Volterra integro-differential equations are analyzed; in [1], the global and local superconvergence properties of piecewise polynomial collocation solutions for index-1 semi-explicit IDAEs are discussed; various aspects of the numerical treatment of IDAEs are studied in [2, 3, 4, 5] (existence and uniqueness of analytic solutions of certain IDAEs; convergence of the implicit Euler method and methods based on backward differentiation formulas (BDFs)); [18] (well-posedness results for non-autonomous integro-differential-algebraic evolutionary problems); and [17] (convergence of the Legendre spectral Tau-method); in [15], the tractability index of IDAEs are defined, the given IDAEs system of index 1 is decoupled into the inherent system of regular Volterra integro-differential equations and a system of second-kind Volterra integral equations, and the convergence, global and local superconvergence are studied for two kinds of collocation methods.

Motivated by [16], in this paper, we consider the following IDAE comes from a hydraulic circuit that feed a combustion process:

(1)
$$\begin{cases} y'(t) + b_{11}(t)y(t) + b_{12}(t)z(t) = f(t) + \displaystyle\int_0^t [K_{11}(t,s)y(s) + K_{12}(t,s)z(s)]\, ds, \\ b_{21}(t)y(t) + b_{22}(t)z(t) = g(t), \end{cases}$$

where $t \in I := [0,T]$, the given functions $b_{pq}, K_{1q} \in \mathbb{R},\ p,q = 1,2$, and $|b_{22}(t)| \geq b_0 > 0$. The system (1) is complemented by a given set of initial values $(y(0), z(0))^T =$

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$(y_0, z_0)^T$, which is assumed to be consistent, i.e.,

$$b_{21}(0)y(0) + b_{22}(0)z(0) = g(0).$$

This paper is aimed at the IDAE (1), and the outline is as follows. In Section 2, we recall the tractability index of IDAEs, and check the index for the IDAE (1). In Section 3, we analyze the existence, uniqueness and regularity of the analytic solution. The collocation scheme, convergence and global (local) superconvergence results in different collocation spaces are shown in Section 4, and the corresponding results in the same collocation space are shown in Section 5. In Section 6, we give some numerical examples to illustrate the theoretical results obtained in this paper.

2. The tractability index of IDAEs

In this section, we will check that (1) is index-1 tractable. For this purpose, we first review the definition of the tractability index induced in [15] for the following general linear IDAE:

$$A(t)x'(t) + B(t)x(t) + \int_0^t K(t,s)x(s)\,ds = F(t), \tag{2}$$

where $A, B, K \in \mathbb{R}^{d\times d}$ and $F \in \mathbb{R}^d$. Before stating the definition of the tractability index for IDAEs (2), we first recall the definition of the notion of ν-smoothing of the linear Volterra integral operator $\mathscr{V} : C(I) \to C(I)$ defined by

$$(\mathscr{V}x)(t) := \int_0^t K(t,s)x(s)\,ds,\ t \in I, \tag{3}$$

with the continuous matrix kernel

$$K(t,s) := [K_{pq}(t,s)] \in \mathbb{R}^{d\times d}.$$

Definition 1. *(see [13]) The Volterra integral operator $\mathscr{V}$ in (3) is said to be ν-smoothing if there exist integers $\nu_{pq} \geq 1$ with*

$$\nu := \max_{1\leq p,q\leq d}\{\nu_{pq}\},$$

such that

(a): $\left.\dfrac{\partial^j K_{pq}(t,s)}{\partial t^j}\right|_{s=t} = 0,\ t \in I,\ j = 0, 1, \ldots, \nu_{pq} - 2;$

(b): $\left.\dfrac{\partial^{\nu_{pq}-1} K_{pq}(t,s)}{\partial t^{\nu_{pq}-1}}\right|_{s=t} \neq 0,\ t \in I;$

(c): $\dfrac{\partial^{\nu_{pq}} K_{pq}(t,s)}{\partial t^{\nu_{pq}}} \in C(D).$

If $K_{pq}(t,s) \equiv 0$, we set $\nu_{pq} = 0$. The IDAE (2) is called a ν-smoothing problem if $\mathscr{V}$ is a ν-smoothing operator.

We assume that the matrix kernel $K(t,s)$ of (2) does not vanish identically. Let $i \geq 0$ be an integer, $K^i, K_i, A_i, B_i \in \mathbb{R}^{d\times d}$, and denote by $(K^i)_{pq}$ and $(K_i)_{pq}$ as the element (p,q) of the matrix K^i and K_i, respectively. Let

$$K^0(t,s) := K(t,s),\ K_0 = K := K(t,t),$$
$$A_0 := A,\ B_0 := B - A_0P_0',\ A_1 := A_0 + B_0Q_0. \tag{4}$$

If $(K_i)_{pq}(t,t) \neq 0\,(i \geq 0)$, set $(K^{i+1})_{pq}(t,s) := 0$; otherwise

$$(K^{i+1})_{pq}(t,s) := \frac{\partial^{i+1}((K^i)_{pq}(t,s))}{\partial t^{i+1}}.$$

Define $K_{i+1} = K_{i+1}(t,t) := (K^{i+1})_{pq}(t,s)|_{s=t}\,(p,q = 1,2,\dots,d)$ and

(5)
$$B_{i+1} = B_{i+1}(t) := B_iP_i - A_{i+1}P_0\Pi'_{i+1}\Pi_i,$$

(6)
$$A_{i+2} = A_{i+2}(t) := A_{i+1} + B_{i+1}Q_{i+1} + \begin{cases} \left(\sum_{l=0}^{i} K_l\Pi_{i-l-1}Q_{i-l}\right)Q_{i+1}, & 0 \le i \le \nu-1,\\ \left(\sum_{l=0}^{\nu} K_l\Pi_{i-l-1}Q_{i-l}\right)Q_{i+1}, & i \ge \nu. \end{cases}$$

Here $\Pi_{-1} := I_d$ is the identity matrix in $\mathbb{R}^{d\times d}$, $Q_j = Q_j(t)$ denotes a projector onto $\ker A_j$, $P_j = P_j(t) := I_d - Q_j$ and $\Pi_j := P_0P_1\dots P_j,\ j \ge 0$.

Definition 2. *(see* [15]*) Assume that the Voterra integral operator describing the IDAE system* (2) *is* $(\nu+1)$*-smoothing with* $\nu \ge 0$*. Then* (2) *is said to be index-*μ *tractable if all matrices* $A_j(t),\ t \in I\,(j = 0,\dots,\mu-1)$ *are singular with smooth null space, and* $A_\mu(t)$ *is nonsingular for all* $t \in I$*.*

Now we consider the tractability index of the IDAE (1). Obviously, (1) can be rewritten as

(7)
$$A(t)x'(t) + B(t)x(t) + \int_0^t K(t,s)x(s)\,ds = F(t),$$

with

$$A_0 = A = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\ B = \begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix},\ K(t,s) = \begin{bmatrix}-K_{11}(t,s) & -K_{12}(t,s)\\ 0 & 0\end{bmatrix},$$

$$X(s) = \begin{bmatrix}y(s)\\ z(s)\end{bmatrix},\ F(t) = \begin{bmatrix}f(t)\\ g(t)\end{bmatrix}.$$

Take $Q_0 = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$, then $P_0 = I - Q_0 = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, $B_0 = B - AP'_0 = B$, and

$$A_1 = A_0 + B_0Q_0 = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & b_{12}\\ 0 & b_{22}\end{bmatrix} = \begin{bmatrix}1 & b_{12}\\ 0 & b_{22}\end{bmatrix}.$$

Since $b_{22} \ne 0$, so $\det(A_1) = b_{22} \ne 0$, i.e., A_1 is nonsingular, and the tractability index $\mu = 1$.

3. Existence, uniqueness and regularity of the analytic solution

By the second equation of (1), we obtain

(8)
$$z(t) = B_1(t)g(t) + B_2(t)y(t),$$

where

$$B_1(t) := \frac{1}{b_{22}(t)},\quad B_2(t) := -\frac{b_{21}(t)}{b_{22}(t)}.$$

Integration of the first equation of (1) from 0 to t, then

(9)
$$y(t) = y(0) + \int_0^t f(v)\,dv + \int_0^t K^0_{11}(t,v)y(v)\,dv + \int_0^t K^0_{12}(t,v)z(v)\,dv,$$

where

$$K^0_{11}(t,v) := -b_{11}(v) + \int_v^t K_{11}(s,v)\,ds,\quad K^0_{12}(t,v) := -b_{12}(v) + \int_v^t K_{12}(s,v)\,ds.$$

By [1, Theorem 2.1.2], we know that there exists a resolvent kernel $R_{11}(t,s)$, which is associated with $K_{11}^0(t,s)$, such that

$$
\begin{aligned}
y(t) =& y(0) + \int_0^t f(v)\,dv + \int_0^t K_{12}^0(t,v)z(v)\,dv \\
&+ \int_0^t R_{11}(t,v)\left[y(0) + \int_0^v f(s)\,ds + \int_0^v K_{12}^0(v,s)z(s)\,ds\right] dv \\
=& A_1(t)y(0) + \int_0^t A_2(t,s)f(s)\,ds + \int_0^t A_3(t,s)z(s)\,ds, \qquad (10)
\end{aligned}
$$

where

$$
A_1(t) := 1 + \int_0^t R_{11}(t,v)\,dv, \quad A_2(t,s) := 1 + \int_s^t R_{11}(t,v)\,dv,
$$

$$
A_3(t,s) := K_{12}^0(t,s) + \int_s^t R_{11}(t,v)K_{12}^0(v,s)\,dv.
$$

Substituting (10) into (8), we get

$$
\begin{aligned}
z(t) &= B_1(t)g(t) + B_2(t)\left[A_1(t)y(0) + \int_0^t A_2(t,s)f(s)\,ds + \int_0^t A_3(t,s)z(s)\,ds\right] \\
&= B_1(t)g(t) + C_1(t)y(0) + \int_0^t C_2(t,s)f(s)\,ds + \int_0^t C_3(t,s)z(s)\,ds, \qquad (11)
\end{aligned}
$$

where

$$
C_1(t) := B_2(t)A_1(t), \quad C_2(t,s) := B_2(t)A_2(t,s), \quad C_3(t,s) := B_2(t)A_3(t,s).
$$

Again by [1, Theorem 2.1.2], we know that there exists a resolvent kernel $R_{22}(t,s)$, which is associated with $C_3(t,s)$, such that

$$
\begin{aligned}
z(t) =& B_1(t)g(t) + C_1(t)y(0) + \int_0^t C_2(t,s)f(s)\,ds \\
&+ \int_0^t R_{22}(t,s)\left[B_1(s)g(s) + C_1(s)y(0) + \int_0^s C_2(s,v)f(v)\,dv\right] ds \\
=& B_1(t)g(t) + D_1(t)y(0) + \int_0^t D_2(t,s)f(s)\,ds + \int_0^t D_3(t,s)g(s)\,ds, \qquad (12)
\end{aligned}
$$

where

$$
D_1(t) := C_1(t) + \int_0^t R_{22}(t,s)C_1(s)\,ds, \quad D_2(t,s) := C_2(t,s) + \int_s^t R_{22}(t,v)C_2(v,s)\,dv,
$$

$$
D_3(t,s) := R_{22}(t,s)B_1(s).
$$

Substituting (12) into (10), we get

$$
\begin{aligned}
y(t) =& A_1(t)y(0) + \int_0^t A_2(t,s)f(s)\,ds \\
&+ \int_0^t A_3(t,s)\left[B_1(s)g(s) + D_1(s)y(0) + \int_0^s D_2(s,v)f(v)\,dv \right. \\
&\left. + \int_0^s D_3(s,v)g(v)\,dv\right] ds \\
=& E_1(t)y(0) + \int_0^t E_2(t,v)f(v)\,dv + \int_0^t E_3(t,v)g(v)\,dv, \qquad (13)
\end{aligned}
$$

where

$$E_1(t) := A_1(t) + \int_0^t A_3(t,s)D_1(s)ds, \quad E_2(t,v) := A_2(t,v) + \int_v^t A_3(t,s)D_2(s,v)\,ds,$$

$$E_3(t,v) := A_3(t,v)B_1(v) + \int_v^t A_3(t,s)D_3(s,v)\,ds.$$

Therefore, we have the following theorem.

Theorem 1. *Let $d \geq 0$ and assume that*

(a): $b_{i1} \in C^d(I)$, $b_{i2} \in C^{d+1}(I)$, $i = 1,2$ *with* $|b_{22}(t)| \geq b_0 > 0$;
(b): $K_{11} \in C^d(D),\ K_{12} \in C^{d+1}(D)$;
(c): $f, g \in C^d(I)$.

Then the IDAE (1) *possesses a unique solution* $x = (y,z)^T$ *on* I, *with* $y \in C^{d+1}(I)$, $z \in C^d(I)$. *In additions, there exist functions* $B_1, D_1 \in C^d(I),\ E_1 \in C^{d+1}(I)$, $D_2, D_3 \in C^d(D)$, $E_2, E_3 \in C^{d+1}(D)$, *such that*

$$\text{(14)} \quad \begin{cases} y(t) = E_1(t)y(0) + \int_0^t E_2(t,v)f(v)\,dv + \int_0^t E_3(t,v)g(v)\,dv, \\ z(t) = B_1(t)g(t) + D_1(t)y(0) + \int_0^t D_2(t,s)f(s)\,ds + \int_0^t D_3(t,s)g(s)\,ds. \end{cases}$$

On the other hand, if the functions b_{21}, $g \in C^{d+1}(I)$, *then* y, $z \in C^{d+1}(I)$, *with*

$$B_1, D_1, E_1 \in C^{d+1}(I),\ D_j, E_j \in C^{d+1}(D),\ j = 2,3.$$

4. Collocation by different piecewise polynomial spaces

4.1. The collocation scheme. Let $I_h := \{t_n := nh, n = 0,1,\ldots,N\,(t_N := T)\}$ be a given mesh on $I = [0,T]$. The solution $x = (y,\, z)^T$ can be approximated by elements $x_h = (y_h,\, z_h)^T$ with

$$\text{(15)} \qquad y_h \in S_m^{(0)}(I_h) := \{y \in C(I) : y|_{\overline{e}_h} \in \pi_m(0 \leq n \leq N-1)\},$$

and

$$\text{(16)} \qquad z_h \in S_{m-1}^{(-1)}(I_h) := \{z|_{e_h} \in \pi_{m-1}(0 \leq n \leq N-1)\},$$

where $e_h := (t_n, t_{n+1}]$, $\overline{e}_h := [t_n, t_{n+1}]$, and π_m denotes the set of real polynomials of degree not exceeding m.

For prescribed m collocation parameters $\{c_i\}$, the collocation points are given by $X_h := \{t = t_n + c_i h : 0 < c_1 < \cdots < c_m \leq 1 (0 \leq n \leq N-1)\}$. Then at $t \in X_h$, the collocation equation is

(17)

$$y_h'(t) + b_{11}(t)y_h(t) + b_{12}(t)z_h(t) = f(t) + \int_0^t [K_{11}(t,s)y_h(s) + K_{12}(t,s)z_h(s)]\,ds,$$

$$\text{(18)} \qquad b_{21}(t)y_h(t) + b_{22}(t)z_h(t) = g(t),$$

with the initial value $(y_h(0), z_h(0))^T = (y(0), z(0))^T$ satisfying $b_{21}(0)y_h(0) + b_{22}(0) \times z_h(0) = g(0)$.

Setting $U_{n,i} := y_h'(t_{n,i}),\ \widetilde{V}_{n,i} := z_h(t_{n,i})$, we can write

$$\text{(19)} \qquad y_h'(t_n + sh) = \sum_{j=1}^m L_j(s)U_{n,j},\ z_h(t_n + sh) = \sum_{j=1}^m L_j(s)\widetilde{V}_{n,j},\ s \in (0,1],$$

then

$$y_h(t_n + sh) = y_h(t_n) + h\sum_{j=1}^{m} \beta_j(s) U_{n,j}, \ \ s \in [0,1], \tag{20}$$

where $\beta_j(s) := \int_0^s L_j(v)\,dv$. By (17) and (18), we obtain

$$\begin{aligned} U_{n,i} &+ b_{11}(t_{n,i})\left[y_h(t_n) + h\sum_{j=1}^{m} a_{ij}U_{n,j}\right] + b_{12}(t_{n,i})\widetilde{V}_{n,i} \\ &- h\int_0^{c_i} K_{11}(t_{n,i}, t_n + sh)\left[y_h(t_n) + h\sum_{j=1}^{m}\beta_j(s)U_{n,j}\right] ds \\ &- h\int_0^{c_i} K_{12}(t_{n,i}, t_n + sh)\left[\sum_{j=1}^{m} L_j(s)\widetilde{V}_{n,j}\right] ds \\ =& h\sum_{l=0}^{n-1}\int_0^1 K_{11}(t_{n,i}, t_l + sh)\left[y_h(t_l) + h\sum_{j=1}^{m}\beta_j(s)U_{l,j}\right] ds \\ &+ h\sum_{l=0}^{n-1}\int_0^1 K_{12}(t_{n,i}, t_l + sh)\left[\sum_{j=1}^{m} L_j(s)\widetilde{V}_{l,j}\right] ds + f(t_{n,i}), \end{aligned}$$

and

$$b_{21}(t_{n,i})\left[y_h(t_n) + h\sum_{j=1}^{m} a_{ij}U_{n,j}\right] + b_{22}(t_{n,i})\widetilde{V}_{n,i} = g(t_{n,i}),$$

where $a_{ij} := \beta_j(c_i)$. In order to write these equations in a more transparent form, we introduce the following notations:

$$M^n := \begin{pmatrix} \int_0^{c_i} K_{11}(t_{n,i}, t_n + sh)\beta_j(s)\,ds \\ (i,j = 1,\dots,m) \end{pmatrix},$$

$$M^{n,l} := \begin{pmatrix} \int_0^{1} K_{11}(t_{n,i}, t_l + sh)\beta_j(s)\,ds \\ (i,j = 1,\dots,m) \end{pmatrix},$$

$$N_{1q}^n := \operatorname{diag}\begin{pmatrix} \int_0^{c_i} K_{1q}(t_{n,i}, t_n + sh)\,ds \\ (i = 1,\dots,m) \end{pmatrix},$$

$$N_{1q}^{n,l} := \operatorname{diag}\begin{pmatrix} \int_0^{1} K_{1q}(t_{n,i}, t_l + sh)\,ds \\ (i = 1,\dots,m) \end{pmatrix},$$

$$W^n := \begin{pmatrix} \int_0^{c_i} K_{12}(t_{n,i}, t_n + sh)L_j(s)\,ds \\ (i,j = 1,\dots,m) \end{pmatrix},$$

$$W^{n,l} := \begin{pmatrix} \int_0^{1} K_{12}(t_{n,i}, t_l + sh)L_j(s)\,ds \\ (i,j = 1,\dots,m) \end{pmatrix},$$

$$A := \begin{pmatrix} a_{ij} \\ (i,j=1,\ldots,m) \end{pmatrix}, \quad B_{pq}^n := \operatorname{diag}\begin{pmatrix} b_{pq}(t_{n,i}) \\ (i=1,\ldots,m) \end{pmatrix},$$

$$f_n := (f(t_{n,1}),\ldots,f(t_{n,m}))^T, \; g_n := (g(t_{n,1}),\ldots,g(t_{n,m}))^T,$$

$$e := (1,\ldots,1)^T, \quad U_n := (U_{n,1},\ldots,U_{n,m})^T, \quad \widetilde{V}_n := \left(\widetilde{V}_{n,1},\ldots,\widetilde{V}_{n,m}\right)^T,$$

where $0 \le l \le n-1$, $0 \le n \le N-1$, $p,q = 1,2$. Then we obtain

$$\begin{aligned} &\begin{bmatrix} I_m + hB_{11}^n A - h^2 M^n & B_{12}^n - hW^n \\ hB_{21}^n A & B_{22}^n \end{bmatrix} \begin{bmatrix} U_n \\ \widetilde{V}_n \end{bmatrix} \\ =& h\sum_{l=0}^{n-1} \begin{bmatrix} hM^{n,l} & W^{n,l} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_l \\ \widetilde{V}_l \end{bmatrix} \\ &+ \begin{bmatrix} -B_{11}^n e y_h(t_n) + hN_{11}^n e y_h(t_n) + h\sum_{l=0}^{n-1} N_{11}^{n,l} e y_h(t_l) + f_n \\ -B_{21}^n e y_h(t_n)) + g_n \end{bmatrix}. \end{aligned} \tag{21}$$

The determinant of the coefficient matrix on the left-hand side of this system has the form

$$\det(I_m)\det(B_{22}^n) + O(h^m) = \det(B_{22}^n) + O(h^m).$$

Due to $b_{22}(t) \ge b_0 > 0$, $\det(B_{22}^n) \ne 0$, so we have the following theorem.

Theorem 2. *There exists an $\overline{h} > 0$ so that for any given mesh I_h with mesh diameter $h > 0$ satisfying $h < \overline{h}$, each of the linear algebraic systems* (21) *has a unique solution* $\begin{pmatrix} U_n \\ \widetilde{V}_n \end{pmatrix} \in \mathbb{R}^{2m}$. *Hence the collocation equations* (17) *and* (18) *define a unique collocation solution $x_h = (y_h, z_h)^T$, with $y_h \in S_m^{(0)}(I_h)$, $z_h \in S_{m-1}^{(-1)}(I_h)$ for the IDAE* (1), *and its representation on the subinterval $(t_n, t_{n+1}]$ is given by* (19) *and* (20).

4.2. Convergence analysis.

Theorem 3. *Assume that:*

- **(a):** *the given functions in* (1) *satisfy the conditions of Theorem 1 so that $y \in C^{d+1}(I)$, $z \in C^d(I)$ with $d \ge m$;*
- **(b):** *$(y_h, z_h)^T$ is the collocation solution for the solution $(y,z)^T$ of the IDAE* (1), *with $y_h \in S_m^{(0)}(I_h)$ and $z_h \in S_{m-1}^{(-1)}(I_h)$;*
- **(c):** *$\overline{h} > 0$ is such that, for any $h \in (0, \overline{h})$, the linear algebraic systems* (21) *has a unique solution.*

Then for all uniform meshes I_h with $h \in (0,\overline{h})$, the collocation solution $(y_h,\, z_h)^T$ converges uniformly to $(y,\, z)^T$ on I, for any set X_h with distinct collocation parameters $0 < c_1 < \cdots < c_m \le 1$, and the attainable global orders of convergence are given by

$$\|y - y_h\|_\infty := \max_{t\in I}\|y(t) - y_h(t)\| \le C_1 h^m, \; \|y' - y_h'\|_\infty := \sup_{t\in I}\|y'(t) - y_h'(t)\| \le C_2 h^m,$$

and

$$\|z - z_h\|_\infty := \sup_{t\in I}\|z(t) - z_h(t)\| \le C_3 h^m,$$

where the constants C_1, C_2, C_3 depend on the collocation parameters, but not on h.

Proof. By the assumption, we have

$$y'(t_n+sh) = \sum_{j=1}^{m} L_j(s) y'(t_n+c_jh) + h^m \overline{R}^1_{m,n}(s),\ \ s\in(0,1], \tag{22}$$

$$z(t_n+sh) = \sum_{j=1}^{m} L_j(s) z(t_n+c_jh) + h^m \overline{R}^2_{m,n}(s),\ \ s\in(0,1], \tag{23}$$

where the remainder terms and the Peano kernel (see [1, Section 1.8]) are given by

$$\overline{R}^1_{m,n}(s) := \int_0^1 K_m(s,v) y^{(m+1)}(t_n+vh)\,dv,$$
$$\overline{R}^2_{m,n}(s) := \int_0^1 K_m(s,v) z^{(m)}(t_n+vh)\,dv,$$

and

$$K_m(s,v) := \frac{1}{(m-1)!}\left\{(s-v)_+^{m-1} - \sum_{k=1}^{m} L_k(s)(c_k-v)_+^{m-1}\right\},\ \ s\in(0,1].$$

Here, $(s-v)_+^{m-1} := 0$ for $s<v$, and $(s-v)_+^{m-1} := (s-v)^{m-1}$ for $s\geq v$.

Integrating (22), we have

$$y(t_n+sh) = y(t_n) + h\sum_{j=1}^{m} \beta_j(s) y'(t_n+c_jh) + h^{m+1}\widetilde{R}^1_{m,n}(s),\ \ s\in[0,1], \tag{24}$$

where $\widetilde{R}^1_{m,n}(s) := \int_0^s \overline{R}^1_{m,n}(v)\,dv$. Setting

$$e_h(t_n+sh) := y(t_n+sh) - y_h(t_n+sh),\quad \widetilde{e}_h(t_n+sh) := z(t_n+sh) - z_h(t_n+sh),$$

and using (19), (20), (22), (23) and (24), we have

$$e_h'(t_n+sh) = \sum_{j=1}^{m} L_j(s) e_h'(t_{n,j}) + h^m \overline{R}^1_{m,n}(s),\ \ s\in(0,1], \tag{25}$$

$$e_h(t_n+sh) = e_h(t_n) + h\sum_{j=1}^{m} \beta_j(s) e_h'(t_{n,j}) + h^{m+1}\widetilde{R}^1_{m,n}(s),\ \ s\in[0,1], \tag{26}$$

$$\widetilde{e}_h(t_n+sh) = \sum_{j=1}^{m} L_j(s) \widetilde{e}_h(t_{n,j}) + h^m \overline{R}^2_{m,n}(s),\ \ s\in(0,1]. \tag{27}$$

For $t = t_{n,i}$, by (1), (17), (18), (26) and (27), the error equations have the form

$$\begin{aligned}
&e'_h(t_{n,i}) + b_{11}(t_{n,i}) \left[e_h(t_n) + h \sum_{j=1}^{m} a_{ij} e'_h(t_{n,j}) \right] + b_{12}(t_{n,i}) \widetilde{e}_h(t_{n,i}) \\
&- h \int_0^{c_i} K_{11}(t_{n,i}, t_n + sh) \left[e_h(t_n) + h \sum_{j=1}^{m} \beta_j(s) e'_h(t_{n,j}) \right] ds \\
&- h \int_0^{c_i} K_{12}(t_{n,i}, t_n + sh) \left[\sum_{j=1}^{m} L_j(s) \widetilde{e}_h(t_{n,j}) \right] ds \\
=& h \sum_{l=0}^{n-1} \int_0^1 K_{11}(t_{n,i}, t_l + sh) \left[e_h(t_l) + h \sum_{j=1}^{m} \beta_j(s) e'_h(t_{l,j}) \right] ds \\
&+ h \sum_{l=0}^{n-1} \int_0^1 K_{12}(t_{n,i}, t_l + sh) \left[\sum_{j=1}^{m} L_j(s) \widetilde{e}_h(t_{l,j}) \right] ds + h^m \overline{\rho}_{n,i}, \qquad (28)
\end{aligned}$$

and

$$b_{21}(t_{n,i}) \left[e_h(t_n) + h \sum_{j=1}^{m} a_{ij} e'_h(t_{n,j}) \right] + b_{22}(t_{n,i}) \widetilde{e}_h(t_{n,i}) = h^m \overline{\sigma}_{n,i}, \qquad (29)$$

where

$$\begin{aligned}
\overline{\rho}_{n,i} :=& - h b_{11}(t_{n,i}) \widetilde{R}^1_{m,n}(c_i) + h \int_0^{c_i} K_{11}(t_{n,i}, t_n + sh) \left[h \widetilde{R}^1_{m,n}(s) \right] ds \\
&+ h \int_0^{c_i} K_{12}(t_{n,i}, t_n + sh) \overline{R}^2_{m,n}(s) \, ds \\
&+ h \sum_{l=0}^{n-1} \int_0^1 K_{11}(t_{n,i}, t_l + sh) \left[h \widetilde{R}^1_{m,l}(s) \right] ds \\
&+ h \sum_{l=0}^{n-1} \int_0^1 K_{12}(t_{n,i}, t_l + sh) \overline{R}^2_{m,l}(s) \, ds, \\
\overline{\sigma}_{n,i} :=& - h b_{21}(t_{n,i}) \widetilde{R}^1_{m,n}(c_i).
\end{aligned}$$

Set

$$E_n := \left(e'_h(t_{n,1}), \ldots, e'_h(t_{n,m}) \right)^T, \quad \widetilde{E}_n := \left(\widetilde{e}_h(t_{n,1}), \ldots, \widetilde{e}_h(t_{n,m}) \right)^T,$$

$$\overline{\rho}_n := (\overline{\rho}_{n,1}, \ldots, \overline{\rho}_{n,m})^T, \quad \overline{\sigma}_n := (\overline{\sigma}_{n,1}, \ldots, \overline{\sigma}_{n,m})^T,$$

then we obtain that

$$\begin{aligned}
&\begin{bmatrix} I_m + h B^n_{11} A - h^2 M^n & B^n_{12} - h W^n \\ h B^n_{21} A & B^n_{22} \end{bmatrix} \begin{bmatrix} E_n \\ \widetilde{E}_n \end{bmatrix} \\
&= h \sum_{l=0}^{n-1} \begin{bmatrix} h M^{n,l} & W^{n,l} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_l \\ \widetilde{E}_l \end{bmatrix} \\
&+ \begin{bmatrix} -B^n_{11} e e_h(t_n) + h N^n_{11} e e_h(t_n) + h \sum_{l=0}^{n-1} N^{n,l}_{11} e e_h(t_l) + h^m \overline{\rho}_n \\ -B^n_{21} e e_h(t_n) + h^m \overline{\sigma}_n \end{bmatrix}. \qquad (30)
\end{aligned}$$

Since e_h is continuous on $[0,T]$, we have $e_h(t_0) = e_h(0) = 0$, and

$$
\begin{aligned}
e_h(t_n) = e_h(t_{n-1}+h) &= e_h(t_{n-1}) + h\sum_{j=1}^{m} b_j e_h'(t_{n-1,j}) + h^{m+1}\widetilde{R}^1_{m,n-1}(1) \\
&= \cdots = h\sum_{l=0}^{n-1}\sum_{j=1}^{m} b_j e_h'(t_{l,j}) + h^m \sum_{l=0}^{n-1} h\widetilde{R}^1_{m,l}(1) \\
&= h\sum_{l=0}^{n-1} b^T E_l + h^m \sum_{l=0}^{n-1} h\widetilde{R}^1_{m,l}(1), \qquad (31)
\end{aligned}
$$

with $b_j := \beta_j(1) = \int_0^1 L_j(s)\,ds,\ b := (b_1,\dots,b_m)^T$. Then

$$
\begin{aligned}
&\begin{bmatrix} I_m + hB^n_{11}A - h^2M^n & B^n_{12} - hW^n \\ hB^n_{21}A & B^n_{22} \end{bmatrix} \begin{bmatrix} E_n \\ \widetilde{E}_n \end{bmatrix} \\
=&h\sum_{l=0}^{n-1} \begin{bmatrix} hM^{n,l} + (hN^n_{11} - B^n_{11})eb^T & W^{n,l} \\ -B^n_{21}eb^T & 0 \end{bmatrix} \begin{bmatrix} E_l \\ \widetilde{E}_l \end{bmatrix} \\
&+ h\sum_{l=0}^{n-2} \begin{bmatrix} h\displaystyle\sum_{k=l+1}^{n-1} N^{n,k}_{11} eb^T & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_l \\ \widetilde{E}_l \end{bmatrix} + h^m \begin{bmatrix} \overline{\rho}^1_n \\ \overline{\sigma}^1_n \end{bmatrix}, \qquad (32)
\end{aligned}
$$

with

$$
\overline{\rho}^1_n := (hN^n_{11} - B^n_{11})\, e\sum_{l=0}^{n-1} h\widetilde{R}^1_{m,l}(1) + h\sum_{l=0}^{n-2} h\sum_{k=l+1}^{n-1} N^{n,k}_{11} e\widetilde{R}^1_{m,k}(1) + \overline{\rho}_n,
$$

$$
\overline{\sigma}^1_n := -B^n_{21} e\sum_{l=0}^{n-1} h\widetilde{R}^1_{m,l}(1) + \overline{\sigma}_n.
$$

Similarly to Theorem 2, for sufficiently small h, the coefficient matrix

$$
\begin{bmatrix} I_m & B^n_{12} \\ 0 & B^n_{22} \end{bmatrix} + O(h)
$$

is nonsingular, with the inverse

$$
\begin{bmatrix} I_m & -B^n_{12}\,(B^n_{22})^{-1} \\ 0 & (B^n_{22})^{-1} \end{bmatrix} + O(h).
$$

Then we have

$$
\begin{bmatrix} E_n \\ \widetilde{E}_n \end{bmatrix} = h\sum_{l=0}^{n-1} \begin{bmatrix} \overline{M}^{n,l}_{11} & \overline{M}^{n,l}_{12} \\ \overline{M}^{n,l}_{21} & \overline{M}^{n,l}_{22} \end{bmatrix} \begin{bmatrix} E_l \\ \widetilde{E}_l \end{bmatrix} + \begin{bmatrix} O(h^m) \\ O(h^m) \end{bmatrix}, \qquad (33)
$$

with obvious meanings of $\overline{M}^{n,l}_{ij}$. By the discrete Gronwall's inequality (see [1, Corollary 2.1.19]), we have

$$
\|E_n\|_\infty = O(h^m), \quad \|\widetilde{E}_n\|_\infty = O(h^m), \qquad (34)
$$

which together with (25), (26), (27) and (31) yield the desired result. □

Next, we will show that for certain particular choices of the collocation parameters c_i, the global or local superconvergence can be achieved (respectively on I and I_h). The key to the analysis is the representation of the collocation errors given in

Theorem 4 below. It is based on the defects (or residuals) $\delta_h(t)$ and $d_h(t)$ associated with the collocation equations (17) and (18). They are defined by

$$\delta_h(t) := -\,y_h'(t) - b_{11}(t)y_h(t) - b_{12}(t)z_h(t) + \int_0^t [K_{11}(t,s)y_h(s) + K_{12}(t,s)z_h(s)]\,ds + f(t),\ t \in I, \tag{35}$$

and

$$d_h(t) := -b_{21}(t)y_h(t) - b_{22}(t)z_h(t) + g(t),\ t \in I. \tag{36}$$

Therefore, the collocation errors e_h and $\widetilde{e}_h$ are

$$e_h'(t) + b_{11}(t)e_h(t) + b_{12}(t)\widetilde{e}_h(t) - \int_0^t [K_{11}(t,s)e_h(s) + K_{12}(t,s)\widetilde{e}_h(s)]\,ds = \delta_h(t),\ t \in I, \tag{37}$$

and

$$b_{21}(t)e_h(t) + b_{22}(t)\widetilde{e}_h(t) = d_h(t),\ t \in I. \tag{38}$$

It is obvious that

$$\delta_h(t) = 0 \text{ and } d_h(t) = 0,\ \text{for all } t \in X_h,$$

and it follows from Theorem 3 that $\|\delta_h(t)\| = O(h^m)$ and $\|d_h(t)\| = O(h^m)$.

Similarly to Theorem 1, we have the following result.

Theorem 4. *Let* (1) *be index-1 tractable, and assume that the given functions in* (1) *satisfy the conditions of Theorem 1 so that* $y \in C^{d+1}(I),\ z \in C^d(I)$, *then the system of error equations* (37) *and* (38) *has a unique solution* $(e_h,\ \widetilde{e}_h)^T$ *with* $e_h \in C^{d+1}(t_n,\ t_{n+1}],\ \widetilde{e}_h \in C^d(t_n,\ t_{n+1}]$ $(0 \le n \le N-1)$, *and there exist functions* $B_1 \in C^d(I),\ D_2,\ D_3 \in C^d(D),\ E_2,\ E_3 \in C^{d+1}(D)$, *such that the solution can be represented in the form*

$$e_h(t) = \int_0^t E_2(t,v)\delta_h(v)\,dv + \int_0^t E_3(t,v)d_h(v)\,dv,\ t \in I, \tag{39}$$

$$\widetilde{e}_h(t) = B_1(t)d_h(t) + \int_0^t D_2(t,s)\delta_h(s)\,ds + \int_0^t D_3(t,s)d_h(s)\,ds,\ t \in I. \tag{40}$$

Theorem 5. *Let* (1) *be index-1 tractable. Assume that the assumptions (b), (c) of Theorem 3 hold, and let (a) be replaced by the assumption* $y \in C^{d+1}(I),\ z \in C^d(I)$ *with* $d \ge m+1$. *If the* m *collocation parameters* c_i *are subject to the orthogonality condition*

$$J_0 := \int_0^1 \prod_{i=1}^m (s - c_i)\,ds = 0,$$

then the corresponding collocation solution $(y_h,\ z_h)^T$, *with* $y_h \in S_m^{(0)}(I_h)$ *and* $z_h \in S_{m-1}^{(-1)}(I_h)$, *satisfies*

$$\max_{t\in I}\|y(t) - y_h(t)\| \le C_4 h^{m+1}, \quad \sup_{t\in I}\|y'(t) - y_h'(t)\| \le C_5 h^m,$$

and

$$\sup_{t\in I}\|z(t) - z_h(t)\| \le C_6 h^m,$$

where the constants C_4, C_5, C_6 *depend on the collocation parameters, but not on* h.

Proof. By (39), for $t = t_n + vh$,

$$\int_0^t E_2(t,v)\delta_h(v)\,dv = h\sum_{l=0}^{n-1}\int_0^1 E_2(t,t_l+sh)\delta_h(t_l+sh)\,ds + h\int_0^v E_2(t,t_n+sh)\delta_h(t_n+sh)\,ds. \tag{41}$$

Suppose now that each of integrals over $[0,1]$ is approximated by the interpolatory m-points quadrature formula with abscissas c_i, then for $v \in [0,1]$ and $l < n$,

$$\int_0^1 E_2(t,t_l+sh)\delta_h(t_l+sh)\,ds = \sum_{j=1}^m b_j E_2(t,t_l+c_jh)\delta_h(t_l+c_jh) + E_{1,n}^{(l)}(v) = E_{1,n}^{(l)}(v). \tag{42}$$

Here, $E_{1,n}^{(l)}(v)$ denote the quadrature errors induced by these quadrature approximations. The orthogonality condition $J_0 = 0$ implies that each of these quadrature formula has degree of precision m, and thus the quadrature errors can be bounded by

$$\left|E_{1,n}^{(l)}(v)\right| \le Q_{1,l}h^{m+1},\ v \in [0,1]\ (l < n),$$

where $Q_{1,l}$ is bounded. Therefore, there exists a constant C_1', such that

$$\left|\int_0^t E_2(t,v)\delta_h(v)\,dv\right| \le C_1' h^{m+1}. \tag{43}$$

Similarly, there exists a constant C_2', such that

$$\left|\int_0^t E_3(t,v)d_h(v)\,dv\right| \le C_2' h^{m+1}. \tag{44}$$

The desired results follow from (43), (44), (37) and (40). □

Theorem 6. *Let (1) be index-1 tractable and assume that:*

- **(a):** *the given functions satisfy the conditions of Theorem 3 so that* $y \in C^{d+1}(I)$, $z \in C^d(I)$ *with* $d \ge m + \kappa$ *for some* κ *with* $1 \le \kappa \le m$ *specified in (b) below;*
- **(b):** *the* m *collocation parameters*c_i *are subject to the orthogonality conditions*

$$J_\nu := \int_0^1 s^\nu \prod_{i=1}^m (s - c_i)\,ds = 0,\ \nu = 0,\dots,\kappa-1,\ \textit{with}\ J_\kappa \ne 0.$$

Then for all uniform meshes I_h *with* $h \in (0,\overline{h})$, *the corresponding collocation solution* $(y_h,\, z_h)^T$ *with* $y_h \in S_m^{(0)}(I_h)$ *and* $z_h \in S_{m-1}^{(-1)}(I_h)$ *satisfies*

$$\max_{t\in I_h}\|y(t) - y_h(t)\| \le C_7 h^{m+\kappa},\ \sup_{t\in I_h}\|y'(t) - y_h'(t)\| \le C_8 h^m,$$

and

$$\sup_{t\in I_h}\|z(t) - z_h(t)\| \le C_9 h^m,$$

where the constants C_7, C_8, C_9 *depend on the collocation parameters, but not on* h.

If in addition we choose $c_m = 1$ *(implying* $\kappa \le m-1$*), the collocation solutions* $y_h \in S_m^{(0)}(I_h)$, $z_h \in S_{m-1}^{(-1)}(I_h)$ *have the properties*

$$\sup_{t\in I_h}\|y'(t) - y_h'(t)\| = O(h^{m+\kappa}),\quad \sup_{t\in I_h}\|z(t) - z_h(t)\| = O(h^{m+\kappa}).$$

Proof. By (39), for $t = t_n$,

$$\int_0^{t_n} E_2(t_n, v)\delta_h(v)\,dv = h\sum_{l=0}^{n-1}\int_0^1 E_2(t_n, t_l + sh)\delta_h(t_l + sh)\,ds. \tag{45}$$

The following proof is similar to the proof of Theorem 5.

For $c_m = 1$, note that $t_n = t_{n-1} + h$ and now $\delta_h(t_n) = 0,\ d_h(t_n) = 0$, then the second part for the local superconvergence for y' and z follows. □

Remark 1. *Assume that the conditions of Theorem 6 hold.*

- *If the collocation points are chosen as the (shifted) Gauss points in* $(0, 1)$*, the local convergence orders at the mesh points become*

$$\max_{t\in I_h}\|y(t) - y_h(t)\| = O(h^{2m}),\ \sup_{t\in I_h}\|y'(t) - y_h'(t)\| = O(h^m),$$

and

$$\sup_{t\in I_h}\|z(t) - z_h(t)\| = O(h^m).$$

- *If the collocation points are chosen as the (shifted) Radau II points in* $(0, 1]$*, then*

$$\max_{t\in I_h\setminus\{0\}}\|y(t) - y_h(t)\| = O(h^{2m-1}),\quad \sup_{t\in I_h\setminus\{0\}}\|y'(t) - y_h'(t)\| = O(h^{2m-1}),$$

and

$$\sup_{t\in I_h\setminus\{0\}}\|z(t) - z_h(t)\| = O(h^{2m-1}).$$

5. Collocation by the same piecewise polynomial space

5.1. The collocation scheme. Now both the components of the solution $x = (y,\, z)^T$ of (1) are approximated by the same collocation space, i.e., $y_h, z_h \in S_m^{(0)}(I_h)$, the collocation equations for $t \in X_h$ are

(46)

$$y_h'(t) + b_{11}(t)y_h(t) + b_{12}(t)z_h(t) = f(t) + \int_0^t [K_{11}(t, s)y_h(s) + K_{12}(t, s)z_h(s)]\,ds,$$

$$b_{21}(t)y_h(t) + b_{22}(t)z_h(t) = g(t), \tag{47}$$

with the initial value $(y_h(0), z_h(0))^T = (y(0), z(0))^T$ satisfying $b_{21}(0)y_h(0)+b_{22}(0)z_h(0) = g(0)$.

Setting $U_{n,i} := y_h'(t_{n,i})$, $V_{n,i} := z_h'(t_{n,i})$, we can write

$$y_h'(t_n + sh) = \sum_{j=1}^m L_j(s)U_{n,j},\quad z_h'(t_n + sh) = \sum_{j=1}^m L_j(s)V_{n,j},\ s \in (0, 1], \tag{48}$$

then

$$y_h(t_n + sh) = y_h(t_n) + h\sum_{j=1}^m \beta_j(s)U_{n,j},$$

$$z_h(t_n + sh) = z_h(t_n) + h\sum_{j=1}^m \beta_j(s)V_{n,j},\ s \in [0, 1]. \tag{49}$$

For $t = t_n + c_i h$, by (46) and (47), we obtain

$$\begin{aligned}
&U_{n,i} + b_{11}(t_{n,i}) \left[y_h(t_n) + h \sum_{j=1}^{m} a_{ij} U_{n,j} \right] + b_{12}(t_{n,i}) \left[z_h(t_n) + h \sum_{j=1}^{m} a_{ij} V_{n,j} \right] \\
&- h \int_0^{c_i} K_{11}(t_{n,i}, t_n + sh) \left[y_h(t_n) + h \sum_{j=1}^{m} \beta_j(s) U_{n,j} \right] ds \\
&- h \int_0^{c_i} K_{12}(t_{n,i}, t_n + sh) \left[z_h(t_n) + h \sum_{j=1}^{m} \beta_j(s) V_{n,j} \right] ds \\
=&h \sum_{l=0}^{n-1} \int_0^1 K_{11}(t_{n,i}, t_l + sh) \left[y_h(t_l) + h \sum_{j=1}^{m} \beta_j(s) U_{l,j} \right] ds \\
&+ h \sum_{l=0}^{n-1} \int_0^1 K_{12}(t_{n,i}, t_l + sh) \left[z_h(t_l) + h \sum_{j=1}^{m} \beta_j(s) V_{l,j} \right] ds + f(t_{n,i}),
\end{aligned}$$

and

$$b_{21}(t_{n,i}) \left[y_h(t_n) + h \sum_{j=1}^{m} a_{ij} U_{n,j} \right] + b_{22}(t_{n,i}) \left[z_h(t_n) + h \sum_{j=1}^{m} a_{ij} V_{n,j} \right] = g(t_{n,i}).$$

Let

$$V_n := (V_{n,1}, \ldots, V_{n,m})^T,$$

$$\widetilde{W}^n := \begin{pmatrix} \int_0^{c_i} K_{12}(t_{n,i}, t_n + sh)\beta_j(s)\, ds \\ (i, j = 1, \ldots, m) \end{pmatrix},$$

$$\widetilde{W}^{n,l} := \begin{pmatrix} \int_0^{1} K_{12}(t_{n,i}, t_l + sh)\beta_j(s) ds \\ (i, j = 1, \ldots, m) \end{pmatrix},$$

where $0 \le l \le n-1$, $0 \le n \le N-1$. Then

$$\begin{aligned}
&\begin{bmatrix} I_m + hB_{11}^n A - h^2 M^n & hB_{12}^n A - h^2 \widetilde{W}^n \\ hB_{21}^n A & hB_{22}^n A \end{bmatrix} \begin{bmatrix} U_n \\ V_n \end{bmatrix} \\
=&h \sum_{l=0}^{n-1} \begin{bmatrix} hM^{n,l} & h\widetilde{W}^{n,l} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_l \\ V_l \end{bmatrix} + h \sum_{l=0}^{n-1} \begin{bmatrix} N_{11}^{n,l} & N_{12}^{n,l} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} ey_h(t_l) \\ ez_h(t_l) \end{bmatrix} \\
&+ \begin{bmatrix} -B_{11}^n ey_h(t_n) - B_{12}^n ez_h(t_n) + hN_{11}^n ey_h(t_n) + hN_{12}^n ez_h(t_n) + f_n \\ -B_{21}^n ey_h(t_n) - B_{22}^n ez_h(t_n) + g_n \end{bmatrix}.
\end{aligned} \tag{50}$$

The determinant of the coefficient matrix on the left-hand side has the form

$$\det(I_m) h^m \det(B_{22}^n A) + O(h^{2m}) = h^m \det(B_{22}^n) \det(A) + O(h^{2m}),$$

which is nonzero for sufficiently small h, due to $|b_{22}(t)| \ge b_0 > 0$ and $\det(A) \ne 0$. Thus we have the following theorem:

Theorem 7. *There exists an $\overline{h} > 0$ so that for any given mesh I_h with mesh diameter $h > 0$ satisfying $h < \overline{h}$, each of the linear algebraic systems* (50) *has a unique solution $\begin{pmatrix} U_n \\ V_n \end{pmatrix} \in \mathbb{R}^{2m}$. Hence the collocation equations* (46) *and* (47) *define* *a unique collocation solution* $x_h = (y_h, z_h)^T$*, with* $y_h, z_h \in S_m^{(0)}(I_h)$ *for the IDAE* (1)*, and its representation on the subinterval* $(t_n, t_{n+1}]$ *is given by* (49).

5.2. Convergence analysis.

Theorem 8. *Assume that*

- **(a):** *the given functions in* (1) *satisfy the conditions of Theorem 1 so that* $y, z \in C^{d+2}(I)$ *with* $d \geq m$*;*
- **(b):** $(y_h, z_h)^T$ *is the collocation solution for the solution* $(y, z)^T$ *of the IDAE* (1)*, with* $y_h, z_h \in S_m^{(0)}(I_h)$*;*
- **(c):** $\overline{h} > 0$ *is such that, for any* $h \in (0, \overline{h})$*, the linear algebraic systems* (50) *has a unique solution.*

Then for all uniform meshes I_h *with* $h \in (0, \overline{h})$*, the collocation solution* (y_h, z_h) *converges uniformly to* (y, z) *on* I*, if, and only if, the collocation parameters* $\{c_i\}$ *are subject to the condition*

$$-1 \leq \rho_m := (-1)^m \prod_{i=1}^{m} \frac{1-c_i}{c_i} \leq 1, \tag{51}$$

and the attainable global orders of convergence are

$$\|y - y_h\|_\infty := \max_{t \in I} \|y(t) - y_h(t)\| \leq \bar{C}_1 h^m, \quad \|y' - y_h'\|_\infty := \sup_{t \in I} \|y'(t) - {y_h}'(t)\| \leq \bar{C}_2 h^m,$$

and

$$\|z - z_h\|_\infty := \max_{t \in I} \|z(t) - z_h(t)\| \leq \bar{C}_3 h^m.$$

Here, $\bar{C}_1, \bar{C}_2$ *and* $\bar{C}_3$ *denote general constants that depend on the collocation parameters* c_i*, but are independent of* h*, and the exponent* m *of* h *cannot in general be replaced by* $m+1$.

Proof. By the assumption, we have (see [1, Section 1.8])

$$y'(t_n + sh) = \sum_{j=1}^{m} L_j(s) y'(t_n + c_j h) + h^m R_{m,n}^1(s), \; s \in (0, 1], \tag{52}$$

$$z'(t_n + sh) = \sum_{j=1}^{m} L_j(s) z'(t_n + c_j h) + h^m R_{m,n}^2(s), \; s \in (0, 1], \tag{53}$$

where

$$R_{m,n}^1(s) := \int_0^1 K_m(s, v) y^{(m+1)}(t_n + vh)\, dv,$$
$$R_{m,n}^2(s) := \int_0^1 K_m(s, v) z^{(m+1)}(t_n + vh)\, dv.$$

Integration of (52) and (53) yields

$$y(t_n + sh) = y(t_n) + h \sum_{j=1}^{m} \beta_j(s) y'(t_n + c_j h) + h^{m+1} \widetilde{R}_{m,n}^1(s), \; s \in [0, 1], \tag{54}$$

$$z(t_n + sh) = z(t_n) + h \sum_{j=1}^{m} \beta_j(s) z'(t_n + c_j h) + h^{m+1} \widetilde{R}_{m,n}^2(s), \; s \in [0, 1], \tag{55}$$

with

$$\widetilde{R}_{m,n}^i(s) := \int_0^s R_{m,n}^i(v)\, dv \; (i = 1, 2).$$

Setting

$e_h(t_n+sh) := y(t_n+sh) - y_h(t_n+sh), \quad \widetilde{e}_h(t_n+sh) := z(t_n+sh) - z_h(t_n+sh),$

and using (49), (54) and (55), we can write

$$e_h(t_n+sh) = e_h(t_n) + h\sum_{j=1}^{m}\beta_j(s)e'_h(t_{n,j}) + h^{m+1}\widetilde{R}^1_{m,n}(s),\ s\in(0,1], \tag{56}$$

$$\widetilde{e}_h(t_n+sh) = \widetilde{e}_h(t_n) + h\sum_{j=1}^{m}\beta_j(s)\widetilde{e}'_h(t_{n,j}) + h^{m+1}\widetilde{R}^2_{m,n}(s),\ s\in(0,1]. \tag{57}$$

For $t = t_{n,i}$, by (1), (46) and (47), the error equations have the form

$$\begin{aligned}
&e'_h(t_{n,i}) + b_{11}(t_{n,i})\left[e_h(t_n) + h\sum_{j=1}^{m}a_{ij}e'_h(t_{n,j})\right]\\
&+ b_{12}(t_{n,i})\left[\widetilde{e}_h(t_n) + h\sum_{j=1}^{m}a_{ij}\widetilde{e}'_h(t_{n,j})\right]\\
&- h\int_0^{c_i}K_{11}(t_{n,i},t_n+sh)\left[e_h(t_n) + h\sum_{j=1}^{m}\beta_j(s)e'_h(t_{n,j})\right]ds\\
&- h\int_0^{c_i}K_{12}(t_{n,i},t_n+sh)\left[\widetilde{e}_h(t_n) + h\sum_{j=1}^{m}\beta_j(s)\widetilde{e}'_h(t_{n,j})\right]ds\\
=&h\sum_{l=0}^{n-1}\int_0^1 K_{11}(t_{n,i},t_l+sh)\left[e_h(t_l) + h\sum_{j=1}^{m}\beta_j(s)e'_h(t_{l,j})\right]ds\\
&+ h\sum_{l=0}^{n-1}\int_0^1 K_{12}(t_{n,i},t_l+sh)\left[\widetilde{e}_h(t_l) + h\sum_{j=1}^{m}\beta_j(s)\widetilde{e}'_h(t_{l,j})\right]ds + h^{m+1}\widetilde{\rho}_{n,i},
\end{aligned} \tag{58}$$

and

$$\begin{aligned}
&b_{21}(t_{n,i})\left[e_h(t_n) + h\sum_{j=1}^{m}a_{ij}e'_h(t_{n,j})\right]\\
&+ b_{22}(t_{n,i})\left[\widetilde{e}_h(t_n) + h\sum_{j=1}^{m}a_{ij}\widetilde{e}'_h(t_{n,j})\right] = h^{m+1}\sigma_{n,i},
\end{aligned} \tag{59}$$

where

$$\begin{aligned}
\widetilde{\rho}_{n,i} :=& -b_{11}(t_{n,i})\widetilde{R}^1_{m,n}(c_i) - b_{12}(t_{n,i})\widetilde{R}^2_{m,n}(c_i) + h\int_0^{c_i}K_{11}(t_{n,i},t_n+sh)\widetilde{R}^1_{m,n}(s)\,ds\\
&+ h\int_0^{c_i}K_{12}(t_{n,i},t_n+sh)\widetilde{R}^2_{m,n}(s)\,ds + h\sum_{l=0}^{n-1}\int_0^1 K_{11}(t_{n,i},t_l+sh)\widetilde{R}^1_{m,n}(s)\,ds\\
&+ h\sum_{l=0}^{n-1}\int_0^1 K_{12}(t_{n,i},t_l+sh)\widetilde{R}^2_{m,n}(s)\,ds,\\
\sigma_{n,i} :=& -b_{21}(t_{n,i})\widetilde{R}^1_{m,n}(c_i) - b_{22}(t_{n,i})\widetilde{R}^2_{m,n}(c_i).
\end{aligned}$$

Set

$$E_n := (e_h'(t_{n,1}), \ldots, e_h'(t_{n,m}))^T, \quad \varepsilon_n := (\widetilde{e}_h'(t_{n,1}), \ldots, \widetilde{e}_h'(t_{n,m}))^T,$$
$$\widetilde{\rho}_n := (\widetilde{\rho}_{n,1}, \ldots, \widetilde{\rho}_{n,m})^T, \quad \sigma_n := (\sigma_{n,1}, \ldots, \sigma_{n,m})^T.$$

Now we rewrite (59) with n replaced by $n-1$ and $i = m$, then subtract this equation from (59). This yields

$$\begin{aligned} & b_{21}(t_{n,i}) \left[e_h(t_n) + h \sum_{j=1}^{m} a_{ij} e_h'(t_{n,j}) \right] + b_{22}(t_{n,i}) \left[\widetilde{e}_h(t_n) + h \sum_{j=1}^{m} a_{ij} \widetilde{e}_h'(t_{n,j}) \right] \\ =& b_{21}(t_{n-1,m}) \left[e_h(t_{n-1}) + h \sum_{j=1}^{m} a_{mj} e_h'(t_{n-1,j}) \right] \\ & + b_{22}(t_{n-1,m}) \left[\widetilde{e}_h(t_{n-1}) + h \sum_{j=1}^{m} a_{mj} \widetilde{e}_h'(t_{n-1,j}) \right] + h^{m+1} \left[\sigma_{n,i} - \sigma_{n-1,m} \right]. \end{aligned} \tag{60}$$

Notice that

$$b_{2j}(t_{n-1,m}) = b_{2j}(t_{n,i}) + [c_m - (c_i + 1)] h b_{2j}'(\cdot), \ j = 1, 2,$$

where $\cdot$ is between $t_{n-1,m}$ and $t_{n,i}$. Then (60) becomes

$$\begin{aligned} & b_{21}(t_{n,i}) \left[e_h(t_n) - e_h(t_{n-1}) \right] \\ & + h b_{21}(t_{n,i}) \left[\sum_{j=1}^{m} a_{ij} e_h'(t_{n,j}) - \sum_{j=1}^{m} a_{mj} e_h'(t_{n-1,j}) \right] \\ & + b_{22}(t_{n,i}) \left[\widetilde{e}_h(t_n) - \widetilde{e}_h(t_{n-1}) \right] \\ & + h b_{22}(t_{n,i}) \left[\sum_{j=1}^{m} a_{ij} \widetilde{e}_h'(t_{n,j}) - \sum_{j=1}^{m} a_{mj} \widetilde{e}_h'(t_{n-1,j}) \right] \\ =& O(h) \left[e_h(t_{n-1}) + h \sum_{j=1}^{m} a_{mj} e_h'(t_{n-1,j}) \right] \\ & + O(h) \left[\widetilde{e}_h(t_{n-1}) + h \sum_{j=1}^{m} a_{mj} \widetilde{e}_h'(t_{n-1,j}) \right] + h^{m+1} (\sigma_{n,i} - \sigma_{n-1,m}). \end{aligned} \tag{61}$$

Since now e_h and $\widetilde{e}_h$ are continuous on $[0, T]$, by (31), we have

$$e_h(t_n) = h \sum_{l=0}^{n-1} \sum_{j=1}^{m} b_j e_h'(t_{l,j}) + h^m \sum_{l=0}^{n-1} h \widetilde{R}_{m,l}^1(1), \tag{62}$$

and similarly,

$$\widetilde{e}_h(t_n) = h \sum_{l=0}^{n-1} \sum_{j=1}^{m} b_j \widetilde{e}_h'(t_{l,j}) + h^m \sum_{l=0}^{n-1} h \widetilde{R}_{m,l}^2(1). \tag{63}$$

Therefore,

$$\frac{e_h(t_n) - e_h(t_{n-1})}{h} = b^T E_{n-1} + h^m \widetilde{R}_{m,n-1}^1(1), \tag{64}$$

$$\frac{\widetilde{e}_h(t_n) - \widetilde{e}_h(t_{n-1})}{h} = b^T \varepsilon_{n-1} + h^m \widetilde{R}_{m,n-1}^2(1). \tag{65}$$

Divide both sides of the equation (61) by h, we get

$$
\begin{aligned}
&b_{21}(t_{n,i})b^T E_{n-1} + b_{21}(t_{n,i})\left[\sum_{j=1}^{m} a_{ij}e_h'(t_{n,j}) - \sum_{j=1}^{m} a_{mj}e_h'(t_{n-1,j})\right] \\
&\quad + b_{22}(t_{n,i})b^T \varepsilon_{n-1} + b_{22}(t_{n,i})\left[\sum_{j=1}^{m} a_{ij}\widetilde{e}_h'(t_{n,j}) - \sum_{j=1}^{m} a_{mj}\widetilde{e}_h'(t_{n-1,j})\right] \\
&= O(1)\left[e_h(t_{n-1}) + h\sum_{j=1}^{m} a_{mj}e_h'(t_{n-1,j})\right] \\
&\quad + O(1)\left[\widetilde{e}_h(t_{n-1}) + h\sum_{j=1}^{m} a_{mj}\widetilde{e}_h'(t_{n-1,j})\right] + h^m\widetilde{\sigma}_{n,i}, \qquad (66)
\end{aligned}
$$

with $\widetilde{\sigma}_{n,i} := \sigma_{n,i} - \sigma_{n-1,m} - b_{21}(t_{n,i})\widetilde{R}^1_{m,n-1}(1) - b_{22}(t_{n,i})\widetilde{R}^2_{m,n-1}(1)$.

By (58) and (66), we have

$$
\begin{aligned}
&\begin{bmatrix} I_m + hB_{11}^n A - h^2 M^n & hB_{12}^n A - h^2\widetilde{W}^n \\ B_{21}^n A & B_{22}^n A \end{bmatrix}\begin{bmatrix} E_n \\ \varepsilon_n \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ B_{21}^n e\left(e_m^T A - b^T\right) + O(h) & B_{22}^n e\left(e_m^T A - b^T\right) + O(h) \end{bmatrix}\begin{bmatrix} E_{n-1} \\ \varepsilon_{n-1} \end{bmatrix} \\
&\quad + h\sum_{l=0}^{n-1}\begin{bmatrix} hM^{n,l} & h\widetilde{W}^{n,l} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} E_l \\ \varepsilon_l \end{bmatrix} \\
&\quad + \begin{bmatrix} -B_{11}^n e e_h(t_n) - B_{12}^n e\widetilde{e}_h(t_n) + hN_{11}^n e e_h(t_n) + hN_{12}^n e\widetilde{e}_h(t_n) \\ O(1)e_h(t_{n-1}) + O(1)\widetilde{e}_h(t_{n-1}) \end{bmatrix} \\
&\quad + \begin{bmatrix} h\sum_{l=0}^{n-1} N_{11}^{n,l} e e_h(t_l) + h\sum_{l=0}^{n-1} N_{12}^{n,l} e\widetilde{e}_h(t_l) + h^{m+1}\widetilde{\rho}_n \\ h^m\widetilde{\sigma}_n, \end{bmatrix}, \\
&= \begin{bmatrix} 0 & 0 \\ B_{21}^n e\left(e_m^T A - b^T\right) + O(h) & B_{22}^n e\left(e_m^T A - b^T\right) + O(h) \end{bmatrix}\begin{bmatrix} E_{n-1} \\ \varepsilon_{n-1} \end{bmatrix} \\
&\quad + h\sum_{l=0}^{n-1}\begin{bmatrix} hM^{n,l} + \left(hN_{11}^n - B_{11}^n\right)eb^T & h\widetilde{W}^{n,l} + \left(hN_{12}^n - B_{12}^n\right)eb^T \\ 0 & 0 \end{bmatrix}\begin{bmatrix} E_l \\ \varepsilon_l \end{bmatrix} \qquad (67) \\
&\quad + h\sum_{l=0}^{n-2}\begin{bmatrix} h\sum_{k=l+1}^{n-1} N_{11}^{n,k}eb^T & h\sum_{k=l+1}^{n-1} N_{12}^{n,k}eb^T \\ O(1) & O(1) \end{bmatrix}\begin{bmatrix} E_l \\ \varepsilon_l \end{bmatrix} + \begin{bmatrix} h^m\widetilde{\rho}_n^2 \\ h^m\widetilde{\sigma}_n^2 \end{bmatrix},
\end{aligned}
$$

where $\widetilde{\sigma}_n := (\widetilde{\sigma}_{n,1}, \dots, \widetilde{\sigma}_{n,m})^T$, and

$$\widetilde{\rho}_n^2 := (hN_{11}^n - B_{11}^n)\, eh \sum_{l=0}^{n-1} \widetilde{R}_{m,l}^1(1) + (hN_{12}^n - B_{12}^n)\, eh \sum_{l=0}^{n-1} \widetilde{R}_{m,l}^2(1) + h\sum_{l=0}^{n-2} h \sum_{k=l+1}^{n-1} N_{11}^{n,k} e\widetilde{R}_{m,l}^1(1) + h\sum_{l=0}^{n-2} h \sum_{k=l+1}^{n-1} N_{12}^{n,k} e\widetilde{R}_{m,l}^2(1) + h\widetilde{\rho}_n,$$

$$\widetilde{\sigma}_n^2 := O(h) \sum_{l=0}^{n-2} \widetilde{R}_{m,l}^1(1) + O(h) \sum_{l=0}^{n-2} \widetilde{R}_{m,l}^2(1) + \widetilde{\sigma}_n.$$

We divide the proof into the following two cases:

Case I: $c_m = 1$. Then

$$e_m^T A - b^T = (a_{m1} - b_1, \dots, a_{mm} - b_m)^T = (0, \dots, 0)^T,$$

and

$$\begin{bmatrix} I_m + hB_{11}^n A - h^2 M^n & hB_{12}^n A - h^2 \widetilde{W}^n \\ B_{21}^n A & B_{22}^n A \end{bmatrix} \begin{bmatrix} E_n \\ \varepsilon_n \end{bmatrix} = h \sum_{l=0}^{n-1} \begin{bmatrix} hM^{n,l} + (hN_{11}^n - B_{11}^n)\, eb^T & h\widetilde{W}^{n,l} + (hN_{12}^n - B_{12}^n)\, eb^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_l \\ \varepsilon_l \end{bmatrix} + h\sum_{l=0}^{n-2} \begin{bmatrix} h\displaystyle\sum_{k=l+1}^{n-1} N_{11}^{n,k} eb^T & h\displaystyle\sum_{k=l+1}^{n-1} N_{12}^{n,k} eb^T \\ O(1) & O(1) \end{bmatrix} \begin{bmatrix} E_l \\ \varepsilon_l \end{bmatrix} + \begin{bmatrix} h^m \widetilde{\rho}_n^2 \\ h^m \widetilde{\sigma}_n^2 \end{bmatrix}. \tag{68}$$

For sufficiently small h, the coefficient matrix

$$\begin{bmatrix} I_m & 0 \\ B_{21}^n A & B_{22}^n A \end{bmatrix} + O(h),$$

is nonsingular, with the inverse

$$\begin{bmatrix} I_m & 0 \\ -A^{-1} (B_{22}^n)^{-1} B_{21}^n A & A^{-1} (B_{22}^n)^{-1} \end{bmatrix} + O(h).$$

Therefore,

$$\begin{bmatrix} E_n \\ \varepsilon_n \end{bmatrix} = h \sum_{l=0}^{n-1} \begin{bmatrix} \widetilde{M}_{11}^{n,l} & \widetilde{M}_{12}^{n,l} \\ \widetilde{M}_{21}^{n,l} & \widetilde{M}_{22}^{n,l} \end{bmatrix} \begin{bmatrix} E_l \\ \varepsilon_l \end{bmatrix} + h^m \begin{bmatrix} \bar{\widetilde{\rho}}_n^2 \\ \bar{\widetilde{\sigma}}_n^2 \end{bmatrix}, \tag{69}$$

with obvious meanings of $\widetilde{M}_{ij}^{n,l}$, $\bar{\widetilde{\rho}}_n^2$ and $\bar{\widetilde{\sigma}}_n^2$. By the discrete Gronwall's inequality (see [1, Corollary 2.1.19]), we have

$$\|E_n\| = O(h^m), \quad \|\varepsilon_n\| = O(h^m),$$

which together with (56), (57), (62) and (63) yield that

$$\|y - y_h\|_\infty = O(h^m), \quad \|y' - y_h'\|_\infty = O(h^m), \quad \|z - z_h\|_\infty = O(h^m).$$

Case II: $c_m < 1$. Let $M := e\left(e_m^T A - b^T\right)$. Then (69) now becomes

$$\begin{bmatrix} E_n \\ \varepsilon_n \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ A^{-1} (B_{22}^n)^{-1} B_{21}^n M & A^{-1} M \end{bmatrix} \begin{bmatrix} E_{n-1} \\ \varepsilon_{n-1} \end{bmatrix} + h \sum_{l=0}^{n-1} \begin{bmatrix} \widetilde{M}_{11}^{n,l} & \widetilde{M}_{12}^{n,l} \\ \widetilde{M}_{21}^{n,l} & \widetilde{M}_{22}^{n,l} \end{bmatrix} \begin{bmatrix} E_l \\ \varepsilon_l \end{bmatrix} + h^m \begin{bmatrix} \bar{\widetilde{\rho}}_n^2 \\ \bar{\widetilde{\sigma}}_n^2 \end{bmatrix}. \tag{70}$$

Let

$$G := \begin{bmatrix} 0 & 0 \\ A^{-1}\left(B_{22}^n\right)^{-1} B_{21}^n M & A^{-1}M \end{bmatrix}.$$

Since

$$M = \begin{bmatrix} a_{m1}-b_1 & a_{m2}-b_2 & \dots & a_{mm}-b_m \\ a_{m1}-b_1 & a_{m2}-b_2 & \dots & a_{mm}-b_m \\ \dots & \dots & \dots & \dots \\ a_{m1}-b_1 & a_{m2}-b_2 & \dots & a_{mm}-b_m \end{bmatrix},$$

the rank of the matrix M is one, implying that the rank of the matrix $A^{-1}M$ is also one. This means that this matrix has exactly one nonzero eigenvalue. Setting $A^{-1} := (\nu_{ij})_{m\times m}$, we have

$$A^{-1}M = \begin{bmatrix} (a_{m1}-b_1)\sum_{j=1}^m \nu_{1j} & (a_{m2}-b_2)\sum_{j=1}^m \nu_{1j} & \dots & (a_{mm}-b_m)\sum_{j=1}^m \nu_{1j} \\ (a_{m1}-b_1)\sum_{j=1}^m \nu_{2j} & (a_{m2}-b_2)\sum_{j=1}^m \nu_{2j} & \dots & (a_{mm}-b_m)\sum_{j=1}^m \nu_{2j} \\ \dots & \dots & \dots & \dots \\ (a_{m1}-b_1)\sum_{j=1}^m \nu_{mj} & (a_{m2}-b_2)\sum_{j=1}^m \nu_{mj} & \dots & (a_{mm}-b_m)\sum_{j=1}^m \nu_{mj} \end{bmatrix},$$

and the nonzero eigenvalue is

$$\lambda\left(A^{-1}M\right) = \sum_{i=1}^m (a_{mi}-b_i)\sum_{j=1}^m \nu_{ij} = 1 - b^T A^{-1} e.$$

By Proposition 3.8 and Theorem 3.10 of [8], we obtain

$$1 - b^T A^{-1} e = (-1)^m \prod_{i=1}^m \frac{1-c_i}{c_i} = \rho_m.$$

Then, it is easy to verify that its eigenvalues of G are $\underbrace{0,\dots,0}_{m};\ \rho_m,\ \underbrace{0,\dots,0}_{m-1}.$ and the eigenvalue 0 of multiplicity $2m-1$ has $2m-1$ linearly independent eigenvectors. Therefore, G is diagonalizable, and there exists a nonsingular matrix T such that

$$T^{-1}GT = \operatorname{diag}(\underbrace{0,\dots,0}_{m}, \rho_m,\ \underbrace{0,\dots,0}_{m-1}) =: H.$$

Defining $Z_n := T^{-1}\begin{bmatrix} E_n \\ \varepsilon_n \end{bmatrix}$, and recalling (70), we obtain

$$Z_n = HZ_{n-1} + h\sum_{l=0}^{n-2} \widetilde{G}^{n,l} Z_l + O(h^m), \tag{71}$$

with obvious meanings of $\widetilde{G}^{n,l}$.

Now, we divide into the following three cases.

Case 1: $-1 < \rho_m < 1$. By [8, Lemma 6], we have

$$\|E_n\| = O(h^m), \quad \|\varepsilon_n\| = O(h^m),$$

which together with (56), (57), (62) and (63) yield the desired result.

Case 2: $\rho_m = -1$. Rewrite (70) with n replaced by $n-1$ and subtract it from (70), and notice that

$$B_{21}^n(t_{n-1,i}) = B_{21}^n(t_{n,i}) + O(h),\ \left(B_{22}^n\right)^{-1}(t_{n-1,i}) = \left(B_{22}^n\right)^{-1}(t_{n,i}) + O(h),$$
$$\widetilde{M}_{ij}^{n,l}(t_{n-1,i}) = \widetilde{M}_{ij}^{n,l}(t_{n,i}) + O(h),$$

and $\bar{\bar{\rho}}_n^2 - \bar{\bar{\rho}}_{n-1}^2 = O(h)$, $\bar{\bar{\sigma}}_n^2 - \bar{\bar{\sigma}}_{n-1}^2 = O(h)$ for $y, z \in C^{m+2}$, then we have

$$\begin{bmatrix} E_n \\ \varepsilon_n \\ E_{n-1} \\ \varepsilon_{n-1} \end{bmatrix}$$
$$= \begin{bmatrix} I_m + O(h) & O(h) & 0 & 0 \\ A^{-1}\left(B_{22}^n\right)^{-1} B_{21}^n M + O(h) & I_m + A^{-1}M + O(h) & -A^{-1}\left(B_{22}^n\right)^{-1} B_{21}^n M & -A^{-1}M \\ I_m & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \end{bmatrix}$$

(72)

$$\cdot \begin{bmatrix} E_{n-1} \\ \varepsilon_{n-1} \\ E_{n-2} \\ \varepsilon_{n-2} \end{bmatrix} + \sum_{l=0}^{n-2} \begin{bmatrix} O(h^2) & O(h^2) & 0 & 0 \\ O(h^2) & O(h^2) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_l \\ \varepsilon_l \\ E_{l-1} \\ \varepsilon_{l-1} \end{bmatrix} + \begin{bmatrix} O(h^{m+1}) \\ O(h^{m+1}) \\ 0 \\ 0 \end{bmatrix}.$$

Similarly to the proof for Case II of [14, Theorem 2.1], we obtain that the eigenvalues of the matrix

$$D := \begin{bmatrix} I_m & 0 & 0 & 0 \\ A^{-1}\left(B_{22}^n\right)^{-1} B_{21}^n M & I_m + A^{-1}M & -A^{-1}\left(B_{22}^n\right)^{-1} B_{21}^n M & -A^{-1}M \\ I_m & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \end{bmatrix}$$

are $\underbrace{1, \ldots, 1}_{2m}$; -1, $\underbrace{0, \ldots, 0}_{2m-1}$, and the eigenvalue 1 of multiplicity $2m$ has $2m$ linearly independent eigenvectors, while to the eigenvalue 0 of multiplicity $2m-1$ there correspond $2m-1$ linearly independent eigenvectors. Therefore, D is diagonalizable, and there exists a nonsingular matrix P such that

$$P^{-1}DP = \operatorname{diag}\left(\underbrace{1, \ldots, 1}_{2m},\ -1,\ \underbrace{0, \ldots, 0}_{2m-1}\right) =: F.$$

Defining $Y_n := P^{-1}\begin{bmatrix} E_n \\ \varepsilon_n \\ E_{n-1} \\ \varepsilon_{n-1} \end{bmatrix}$, and recalling (72), we obtain

$$Y_n = (F + O(h))\, Y_{n-1} + O(h^2) \sum_{l=0}^{n-2} \widetilde{D}^{n,l} Y_l + O(h^{m+1}), \tag{73}$$

with obvious meanings of $\widetilde{D}^{n,l}$. Therefore, there exist constants $\tilde{C}_1$, $\tilde{C}_2$ and $\tilde{C}_3$, such that

$$\|Y_n\|_1 \le (1 + \tilde{C}_1 h)\|Y_{n-1}\|_1 + \tilde{C}_2 h^2 \sum_{l=0}^{n-2} \|Y_l\|_1 + \tilde{C}_3 h^{m+1}.$$

An induction argument then leads to

$$
\begin{aligned}
\|Y_n\|_1 \leq & (1+\tilde{C}_1 h)^n \|Y_0\|_1 + \tilde{C}_2 h^2 \sum_{k=0}^{n-1} (1+\tilde{C}_1 h)^k \sum_{l=0}^{n-k-2} \|Y_l\|_1 \\
& + \tilde{C}_3 h^{m+1} \sum_{k=0}^{n-1} (1+\tilde{C}_1 h)^k \\
= & (1+\tilde{C}_1 h)^n \|Y_0\|_1 + \tilde{C}_2 h^2 \sum_{l=0}^{n-2} \left(\sum_{k=0}^{n-l-2} (1+\tilde{C}_1 h)^k \right) \|Y_l\|_1 \\
& + \tilde{C}_3 h^{m+1} \sum_{k=0}^{n-1} (1+\tilde{C}_1 h)^k \\
= & (1+\tilde{C}_1 h)^n \|Y_0\|_1 + \tilde{C}_2 h^2 \sum_{l=0}^{n-2} \frac{(1+\tilde{C}_1 h)^{n-l-1} - 1}{\tilde{C}_1 h} \|Y_l\|_1 \\
& + \tilde{C}_3 h^{m+1} \frac{(1+\tilde{C}_1 h)^n - 1}{\tilde{C}_1 h}.
\end{aligned}
$$

Therefore, by the discrete Gronwall's inequality (see [1, Corollary 2.1.19]), we get that

$$\|Y_n\|_1 = O(h^m), \tag{74}$$

i.e., $\|E_n\| = O(h^m)$, $\|\varepsilon_n\| = O(h^m)$ and the desired result follows from (56), (57), (62) and (63).

Case 3: $\rho_m = 1$. For this case, using the technique of [11], we write the collocation approximation y_h, z_h in the form

$$y_h'(t_n + sh) = \sum_{j=1}^{m} L_j(s) y_h'(t_{n,j}), \; s \in (0,1], \tag{75}$$

$$z_h(t_n + sh) = \sum_{j=1}^{m} L_j(s) z_h(t_{n,j}) + h^m \frac{z_h^{(m)}(\eta_n)}{m!} \prod_{i=1}^{m} (s - c_i), \; s \in (0,1], \tag{76}$$

with $\eta_n \in (t_n, t_{n+1})$, then

$$y_h(t_n + sh) = y_h(t_n) + h \sum_{j=1}^{m} \beta_j(s) y_h'(t_{n,j}), \; s \in [0,1]. \tag{77}$$

So (25), (26) and

$$\widetilde{e}_h(t_n + sh) = \sum_{j=1}^{m} L_j(s) \widetilde{e}_h(t_{n,j}) + h^m \hat{R}_{m,n}(s), \; s \in [0,1] \tag{78}$$

hold, where $\hat{R}_{m,n}(s) := \overline{R}^2_{m,n}(s) - \frac{z_h^{(m)}(\eta_n)}{m!}\prod_{i=1}^m (s-c_i)$. Then we have the error equations

$$
\begin{aligned}
&e_h'(t_{n,i}) + b_{11}(t_{n,i})\left[e_h(t_n) + h\sum_{j=1}^m a_{ij}e_h'(t_{n,j})\right] + b_{12}(t_{n,i})\widetilde{e}_h(t_{n,i}) \\
&- h\int_0^{c_i} K_{11}(t_{n,i}, t_n + sh)\left[e_h(t_n) + h\sum_{j=1}^m \beta_j(s)e_h'(t_{n,j})\right] ds \\
&- h\int_0^{c_i} K_{12}(t_{n,i}, t_n + sh)\left[\sum_{j=1}^m L_j(s)\widetilde{e}_h(t_{n,j})\right] ds \\
=&h\sum_{l=0}^{n-1}\int_0^1 K_{11}(t_{n,i}, t_l + sh)\left[e_h(t_l) + h\sum_{j=1}^m \beta_j(s)e_h'(t_{l,j})\right] ds \\
&+ h\sum_{l=0}^{n-1}\int_0^1 K_{12}(t_{n,i}, t_l + sh)\left[\sum_{j=1}^m L_j(s)\widetilde{e}_h(t_{l,j})\right] ds + h^m\hat{\rho}_{n,i},
\end{aligned}
\tag{79}
$$

and (29), where

$$
\begin{aligned}
\hat{\rho}_{n,i} :=& -hb_{11}(t_{n,i})\widetilde{R}^1_{m,n}(c_i) + h\int_0^{c_i} K_{11}(t_{n,i}, t_n + sh)\left[h\widetilde{R}^1_{m,n}(s)\right] ds \\
&+ h\int_0^{c_i} K_{12}(t_{n,i}, t_n + sh)\hat{R}_{m,n}(s)\, ds \\
&+ h\sum_{l=0}^{n-1}\int_0^1 K_{11}(t_{n,i}, t_l + sh)\left[h\widetilde{R}^1_{m,l}(s)\right] ds \\
&+ h\sum_{l=0}^{n-1}\int_0^1 K_{12}(t_{n,i}, t_l + sh)\hat{R}_{m,l}(s)\, ds.
\end{aligned}
$$

Similarly to the proof of Theorem 3, we have $\|E_n\| = O(h^m)$, $\|\widetilde{E}_n\| = O(h^m)$, so (56), (57), (62) and (63) yield the desired result. Obviously, the collocation solutions $y_h,\ z_h$ are divergent if $|\rho_m| > 1$. The proof is completed. □

Similarly to Theorem 4, we have the following result.

Theorem 9. *Let* (1) *be index-1 tractable. Assume that the given functions in* (1) *satisfy the conditions of Theorem 1 such that* $y,\, z \in C^{d+1}(I)$, *and the* m *collocation parameters* c_i *are subject to the condition* (51), *then the system of error equations* (37) *and* (38) *has a unique solution* $(e_h,\, \widetilde{e}_h)^T$ *with* $e_h,\ \widetilde{e}_h \in C^{d+1}(t_n,\ t_{n+1}]$ $(0 \le n \le N-1)$, *and there exist functions* $B_1 \in C^{d+1}(I)$, $D_2,\ D_3,\ E_2,\ E_3 \in C^{d+1}(D)$, *such that the solution can be represented in the form* (39) *and* (40).

Theorem 10. *Let* (1) *be index-1 tractable. Assume that the assumptions (b), (c) of Theorem 8 hold, and let (a) be replaced by the assumption* $y, z \in C^{d+1}(I)$ *with* $d \ge m+2$. *If the* m *collocation parameters* c_i *are subject to the condition* (51) *and to the orthogonality condition* $J_0 := \int_0^1 \prod_{i=1}^m (s-c_i)\, ds = 0$, *then the corresponding collocation solution* $(y_h,\, z_h)^T$ *with* $y_h,\, z_h \in S_m^{(0)}(I_h)$ *satisfies*

$$\max_{t\in I}\|y(t) - y_h(t)\| \le \bar{C}_4 h^{m+1},\ \sup_{t\in I}\|y'(t) - y_h'(t)\| \le \bar{C}_5 h^m,$$

and

$$\sup_{t\in I}\|z(t)-z_h(t)\| \le \bar{C}_6 \begin{cases} h^{m+1}, & \text{if } -1 \le \rho_m < 1, \\ h^m, & \text{if } \rho_m = 1, \end{cases}$$

where the constants $\bar{C}_4, \bar{C}_5, \bar{C}_6$ *depend on the collocation parameters, but not on* h.

Proof. Similarly to the proof of Theorem 5, we can obtain the estimations for y. For $-1 \le \rho_m < 1$, by (59), we have $\|\widetilde{e}_h(t_n)\| = O(h^{m+1})$, further, we have $\|\widetilde{e}_h(t_n+sh)\| = O(h^{m+1})$ by (57). □

Similarly to the proof of Theorem 6 and noticing (59), we get the following theorem.

Theorem 11. *Let* (1) *be index-1 tractable and assume that:*

- **(a):** *the given functions satisfy the conditions of Theorem 8 so that* y, $z \in C^{d+1}(I)$ *with* $d+1 \ge m+\kappa$ *for some* κ *with* $1 \le \kappa \le m$*;*
- **(b):** *the* m *collocation parameters* c_i *are subject to the condition* (51) *and to the orthogonality conditions*

$$J_\nu := \int_0^1 s^\nu \prod_{i=1}^m (s-c_i)\,ds = 0, \ \nu = 0,\dots,\kappa-1, \ \text{with } J_\kappa \ne 0.$$

Then for all uniform meshes I_h *with* $h \in (0,\overline{h})$*, the corresponding collocation solution* $(y_h,\, z_h)^T$ *with* $y_h,\, z_h \in S_m^{(0)}(I_h)$ *has the properties*

$$\max_{t\in I_h}\|y(t)-y_h(t)\| \le \bar{C}_7 h^{m+\kappa}, \quad \sup_{t\in I_h}\|y'(t)-y_h'(t)\| \le \bar{C}_8 h^m,$$

and

$$\sup_{t\in I_h}\|z(t)-z_h(t)\| \le \begin{cases} h^{m+1}, & \text{if } -1 \le \rho_m < 1, \\ h^m, & \text{if } \rho_m = 1, \end{cases}$$

where the constants $\bar{C}_7, \bar{C}_8, \bar{C}_9$ *depend on the collocation parameters, but not on* h.

If in addition we choose $c_m = 1$ *(implying* $\kappa \le m-1$*), the collocation solutions* $y_h,\, z_h \in S_m^{(0)}(I_h)$ *satisfy*

$$\sup_{t\in I_h}\|y'(t)-y_h'(t)\| = O(h^{m+\kappa}), \quad \sup_{t\in I_h}\|z(t)-z_h(t)\| = O(h^{m+\kappa}).$$

Remark 2. *Assume that the conditions of Theorem 11 hold.*

- *If the collocation points are chosen as the (shifted) Gauss points in* $(0,1)$*, the local convergence orders at the mesh points become*

  $$\max_{t\in I_h}\|y(t)-y_h(t)\| = O(h^{2m}), \quad \sup_{t\in I_h}\|y'(t)-y_h'(t)\| = O(h^m),$$

  and

  $$\sup_{t\in I_h}\|z(t)-z_h(t)\| = \begin{cases} O(h^{m+1}), & \text{if } m \text{ is odd}, \\ O(h^m), & \text{if } m \text{ is even}. \end{cases}$$

- *If the collocation points are chosen as the (shifted) Radau II points in* $(0,1]$*, then*

  $$\max_{t\in I_h\setminus\{0\}}\|y(t)-y_h(t)\| = O(h^{2m-1}), \quad \sup_{t\in I_h\setminus\{0\}}\|y'(t)-y_h'(t)\| = O(h^{2m-1}),$$

  and

  $$\sup_{t\in I_h\setminus\{0\}}\|z(t)-z_h(t)\| = O(h^{2m-1}).$$

TABLE 1. The errors of $y_h \in S_m^{(0)}(I_h)$ for Example 1 with $m = 2$.

N	Gauss ($\rho_m = 1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{4}, \frac{5}{6})$ ($\rho_m = \frac{3}{5}$)	$(\frac{1}{3}, \frac{2}{3})$ ($\rho_m = 1$)	$(\frac{1}{6}, \frac{1}{2})$ ($\rho_m = 5$)
2^4	2.5813e-08	5.3157e-06	1.3519e-06	2.2891e-05	3.7772e-05
2^5	1.6125e-09	6.6399e-07	1.6744e-07	5.7382e-06	9.0153e-06
2^6	1.0077e-10	8.2972e-08	2.0833e-08	1.4344e-06	2.2024e-06
2^7	6.2969e-12	1.0370e-08	2.5981e-09	3.5858e-07	5.4425e-07
Order	4.0003	3.0002	3.0033	2.0001	2.0168

TABLE 2. The errors of $z_h \in S_{m-1}^{(-1)}(I_h)$ for Example 1 with $m = 2$.

N	Gauss ($\rho_m = 1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{4}, \frac{5}{6})$ ($\rho_m = \frac{3}{5}$)	$(\frac{1}{3}, \frac{2}{3})$ ($\rho_m = 1$)	$(\frac{1}{6}, \frac{1}{2})$ ($\rho_m = 5$)
2^4	3.2856e-04	5.3157e-06	2.4631e-04	4.3185e-04	8.1442e-04
2^5	8.1782e-05	6.6399e-07	6.1317e-05	1.0823e-04	2.0356e-04
2^6	2.0397e-05	8.2972e-08	1.5295e-05	2.7092e-05	5.0878e-05
2^7	5.0928e-06	1.0370e-08	3.8192e-06	6.7773e-06	1.2718e-05
Order	2.0018	3.0002	2.0017	1.9991	2.0002

TABLE 3. The errors of $y_h \in S_m^{(0)}(I_h)$ for Example 1 with $m = 3$.

N	Gauss ($\rho_m = -1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{3}, \frac{1}{2}, \frac{2}{3})$ ($\rho_m = -1$)	$(\frac{1}{3}, \frac{2}{3}, \frac{3}{4})$ ($\rho_m = \frac{2}{3}$)	$(\frac{1}{9}, \frac{1}{3}, \frac{1}{2})$ ($\rho_m = 16$)
2^2	4.8095e-09	9.6445e-08	2.5678e-06	5.2938e-06	6.1070e-05
2^3	7.5815e-11	3.0729e-09	1.5745e-07	8.2217e-07	7.1010e-06
2^4	1.1876e-12	9.6495e-11	9.7932e-09	1.1248e-07	8.5105e-07
2^5	1.8652e-14	3.0192e-12	6.1133e-10	1.4648e-08	1.0398e-07
Order	5.9925	4.9982	4.0017	2.9408	3.0329

TABLE 4. The errors of $z_h \in S_{m-1}^{(-1)}(I_h)$ for Example 1 with $m = 3$.

N	Gauss ($\rho_m = -1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{3}, \frac{1}{2}, \frac{2}{3})$ ($\rho_m = -1$)	$(\frac{1}{3}, \frac{2}{3}, \frac{3}{4})$ ($\rho_m = \frac{2}{3}$)	$(\frac{1}{9}, \frac{1}{3}, \frac{1}{2})$ ($\rho_m = 16$)
2^2	1.0418e-04	9.6445e-08	2.2884e-04	1.7935e-04	5.4306e-04
2^3	1.3368e-05	3.0729e-09	2.9546e-05	2.3131e-05	7.1351e-05
2^4	1.6918e-06	9.6495e-11	3.7496e-06	2.9339e-06	9.1284e-06
2^5	2.1274e-07	3.0191e-12	4.7215e-07	3.6934e-07	1.1539e-06
Order	2.9913	4.9982	2.9894	2.9898	2.9838

6. Numerical experiments

We give some numerical examples to illustrate the theoretical result on the attainable order of the collocation method in this paper. The underlying collocation spaces are $S_m^{(0)}(I_h)$ and $S_{m-1}^{(-1)}(I_h)$ with $m = 2$ and $m = 3$, and we use, in addition to the Gauss and Radau II collocation parameters ($m = 2:\ c_1 = \frac{3-\sqrt{3}}{6},\ c_2 = \frac{3+\sqrt{3}}{6}$; $c_1 = \frac{1}{3},\ c_2 = 1$; $m = 3:\ c_1 = \frac{5-\sqrt{15}}{10},\ c_2 = \frac{1}{2},\ c_3 = \frac{5+\sqrt{15}}{10}$; $c_1 = \frac{4-\sqrt{6}}{10},\ c_2 = \frac{4+\sqrt{6}}{10},\ c_3 = 1$), some additional sets ($m = 2:\ c_1 = \frac{1}{4},\ c_2 = \frac{5}{6}\,(J_0 = 0)$; $c_1 =$

$\frac{1}{3},\ c_2 = \frac{2}{3};\ c_1 = \frac{1}{3},\ c_2 = \frac{2}{3};\ m = 3:\ c_1 = \frac{1}{3},\ c_2 = \frac{1}{2},\ c_3 = \frac{2}{3}\,(J_0 = 0);\ c_1 = \frac{1}{3},\ c_2 = \frac{1}{2},\ c_3 = \frac{3}{4};\ c_1 = \frac{1}{9},\ c_2 = \frac{1}{3},\ c_3 = \frac{1}{2}$). The errors are calculated for $\max_{0\le n\le N-1} |y(t_n) - y_h(t_n)|$ and $\max_{0\le n\le N-1} |z(t_n) - z_h(t_n)|$.

TABLE 5. The errors of $y_h \in S_m^{(0)}(I_h)$ for Example 1 with $m = 2$.

N	Gauss ($\rho_m = 1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{4}, \frac{5}{6})$ ($\rho_m = \frac{3}{5}$)	$(\frac{1}{3}, \frac{2}{3})$ ($\rho_m = 1$)	$(\frac{1}{6}, \frac{1}{2})$ ($\rho_m = 5$)
2^4	2.9843e-09	7.7072e-07	1.8492e-07	6.4901e-05	1.5432e+02
2^5	1.8673e-10	9.8676e-08	2.4181e-08	1.6225e-05	7.4473e+11
2^6	1.1674e-11	1.2478e-08	3.0894e-09	4.0562e-06	5.4490e+32
2^7	7.2919e-13	1.5687e-09	3.9031e-10	1.0140e-06	9.2560e+75
Order	4.0009	2.9918	2.9847	2.0000	–

TABLE 6. The errors of $z_h \in S_m^{(0)}(I_h)$ for Example 1 with $m = 2$.

N	Gauss ($\rho_m = 1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{4}, \frac{5}{6})$ ($\rho_m = \frac{3}{5}$)	$(\frac{1}{3}, \frac{2}{3})$ ($\rho_m = 1$)	$(\frac{1}{6}, \frac{1}{2})$ ($\rho_m = 5$)
2^4	2.9954e-05	7.7072e-07	7.8468e-06	3.9973e-05	5.9400e+05
2^5	7.5026e-06	9.8676e-08	1.1724e-06	9.9895e-06	1.1827e+16
2^6	1.8758e-06	1.2478e-08	1.6535e-07	2.4971e-06	3.5158e+37
2^7	4.6897e-07	1.5687e-09	2.2272e-08	6.2427e-07	2.4075e+81
Order	1.9999	2.9918	2.8922	2.0000	–

TABLE 7. The errors of $y_h \in S_m^{(0)}(I_h)$ for Example 1 with $m = 3$.

N	Gauss ($\rho_m = -1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{3}, \frac{1}{2}, \frac{2}{3})$ ($\rho_m = -1$)	$(\frac{1}{3}, \frac{2}{3}, \frac{3}{4})$ ($\rho_m = \frac{2}{3}$)	$(\frac{1}{9}, \frac{1}{3}, \frac{1}{2})$ ($\rho_m = 16$)
2^2	6.6871e-09	1.7767e-07	2.6578e-06	1.3455e-05	2.4457e-04
2^3	1.0519e-10	5.6816e-09	1.6827e-07	1.5747e-06	3.2001e-01
2^4	1.6451e-12	1.7937e-10	1.0551e-08	1.8911e-07	2.1120e+07
2^5	2.4036e-14	5.6304e-12	6.6000e-10	2.3128e-08	6.0371e+24
Order	6.0968	4.9936	3.9988	3.0315	–

TABLE 8. The errors of $z_h \in S_m^{(0)}(I_h)$ for Example 1 with $m = 3$.

N	Gauss ($\rho_m = -1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{3}, \frac{1}{2}, \frac{2}{3})$ ($\rho_m = -1$)	$(\frac{1}{3}, \frac{2}{3}, \frac{3}{4})$ ($\rho_m = \frac{2}{3}$)	$(\frac{1}{9}, \frac{1}{3}, \frac{1}{2})$ ($\rho_m = 16$)
2^2	1.5256e-05	1.7767e-07	3.2931e-05	2.8117e-05	3.5625e-01
2^3	9.8544e-07	5.6816e-09	2.1500e-06	2.0033e-06	1.4913e+03
2^4	6.2585e-08	1.7937e-10	1.3766e-07	2.1809e-07	4.0459e+11
2^5	3.9426e-09	5.6305e-12	8.7139e-09	2.4939e-08	4.6890e+29
Order	3.9886	4.9935	3.9816	3.1284	–

TABLE 9. The errors of $y_h \in S_m^{(0)}(I_h)$ for Case 1 of Example 2 with $m = 2$.

N	Gauss ($\rho_m = 1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{4}, \frac{5}{6})$ ($\rho_m = \frac{3}{5}$)	$(\frac{1}{3}, \frac{2}{3})$ ($\rho_m = 1$)	$(\frac{1}{6}, \frac{1}{2})$ ($\rho_m = 5$)
2^4	1.4289e-07	1.9664e-05	4.9974e-06	7.0177e-04	1.0671e-03
2^5	1.8145e-08	2.4963e-06	6.3635e-07	1.7532e-04	2.6479e-04
2^6	2.0865e-09	3.1607e-07	8.0553e-08	4.3817e-05	6.5952e-05
2^7	2.2679e-10	3.9926e-08	1.0156e-08	1.0953e-05	1.6458e-05
Order	3.2017	2.9848	2.9877	2.0002	2.0026

TABLE 10. The errors of $z_h \in S_{m-1}^{(-1)}(I_h)$ for Case 1 of Example 2 with $m = 2$.

N	Gauss ($\rho_m = 1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{4}, \frac{5}{6})$ ($\rho_m = \frac{3}{5}$)	$(\frac{1}{3}, \frac{2}{3})$ ($\rho_m = 1$)	$(\frac{1}{6}, \frac{1}{2})$ ($\rho_m = 5$)
2^4	1.2430e-03	1.9664e-05	9.3580e-04	9.6897e-04	2.0592e-03
2^5	3.0798e-04	2.4963e-06	2.3144e-04	2.3913e-04	5.0749e-04
2^6	7.6645e-05	3.1607e-07	5.7541e-05	5.9392e-05	1.2595e-04
2^7	1.9117e-05	3.9926e-08	1.4345e-05	1.4799e-05	3.1373e-05
Order	2.0033	2.9848	2.0040	2.0048	2.0053

TABLE 11. The errors of $y_h \in S_m^{(0)}(I_h)$ for Case 2 of Example 2 with $m = 2$.

N	Gauss ($\rho_m = 1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{4}, \frac{5}{6})$ ($\rho_m = \frac{3}{5}$)	$(\frac{1}{3}, \frac{2}{3})$ ($\rho_m = 1$)	$(\frac{1}{6}, \frac{1}{2})$ ($\rho_m = 5$)
2^4	3.1717e-07	5.1150e-05	1.2501e-05	1.4328e-03	2.1682e-03
2^5	2.0090e-08	6.3868e-06	1.5785e-06	3.5822e-04	5.3973e-04
2^6	1.2682e-09	7.9789e-07	1.9832e-07	8.9557e-05	1.3464e-04
2^7	7.9914e-11	9.9707e-08	2.4854e-08	2.2389e-05	3.3622e-05
Order	3.9882	3.0004	2.9963	2.0000	2.0016

TABLE 12. The errors of $z_h \in S_{m-1}^{(-1)}(I_h)$ for Case 2 of Example 2 with $m = 2$.

N	Gauss ($\rho_m = 1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{4}, \frac{5}{6})$ ($\rho_m = \frac{3}{5}$)	$(\frac{1}{3}, \frac{2}{3})$ ($\rho_m = 1$)	$(\frac{1}{6}, \frac{1}{2})$ ($\rho_m = 5$)
2^4	2.8703e-03	5.1150e-05	2.1642e-03	2.3956e-03	5.0259e-03
2^5	7.1503e-04	6.3868e-06	5.3769e-04	5.9524e-04	1.2503e-03
2^6	1.7840e-04	7.9789e-07	1.3398e-04	1.4832e-04	3.1169e-04
2^7	4.4553e-05	9.9707e-08	3.3437e-05	3.7016e-05	7.7802e-05
Order	2.0015	3.0004	2.0025	2.0025	2.0022

Example 1. *We consider the following IDAE system:*

$$\left\{ \begin{array}{l} y'(t) + y(t) + tz(t) + \displaystyle\int_0^t [(t-s)y(s) + \exp(t-s)z(s)]\, ds = f(t), \\ \qquad\qquad y(t) + z(t) = g(t), \end{array} \right. \quad t \in I := [0,1], \tag{80}$$

TABLE 13. The errors of $y_h \in S_m^{(0)}(I_h)$ for Case 3 of Example 2 with $m = 2$.

N	Gauss ($\rho_m = 1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{4}, \frac{5}{6})$ ($\rho_m = \frac{3}{5}$)	$(\frac{1}{3}, \frac{2}{3})$ ($\rho_m = 1$)	$(\frac{1}{6}, \frac{1}{2})$ ($\rho_m = 5$)
2^4	8.9259e-07	1.0380e-04	2.5023e-05	2.3771e-03	3.5889e-03
2^5	5.5754e-08	1.2951e-05	3.1798e-06	5.9453e-04	8.9478e-04
2^6	3.4838e-09	1.6171e-06	4.0067e-07	1.4865e-04	2.2335e-04
2^7	2.1772e-10	2.0203e-07	5.0281e-08	3.7163e-05	5.5792e-05
Order	4.0001	3.0008	2.9943	2.0000	2.0012

TABLE 14. The errors of $z_h \in S_{m-1}^{(-1)}(I_h)$ for Case 3 of Example 2 with $m = 2$.

N	Gauss ($\rho_m = 1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{4}, \frac{5}{6})$ ($\rho_m = \frac{3}{5}$)	$(\frac{1}{3}, \frac{2}{3})$ ($\rho_m = 1$)	$(\frac{1}{6}, \frac{1}{2})$ ($\rho_m = 5$)
2^4	5.1105e-03	1.0380e-04	3.8600e-03	4.4401e-03	9.1717e-03
2^5	1.2803e-03	1.2951e-05	9.6359e-04	1.1128e-03	2.3048e-03
2^6	3.2030e-04	1.6171e-06	2.4064e-04	2.7843e-04	5.7729e-04
2^7	8.0095e-05	2.0203e-07	6.0122e-05	6.9631e-05	1.4443e-04
Order	1.9997	3.0008	2.0009	1.9995	1.9989

TABLE 15. The errors of $y_h \in S_m^{(0)}(I_h)$ for Case 4 of Example 2 with $m = 2$.

N	Gauss ($\rho_m = 1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{4}, \frac{5}{6})$ ($\rho_m = \frac{3}{5}$)	$(\frac{1}{3}, \frac{2}{3})$ ($\rho_m = 1$)	$(\frac{1}{6}, \frac{1}{2})$ ($\rho_m = 5$)
2^4	4.1133e-07	3.2421e-05	8.4691e-06	1.4826e-03	1.6317e+02
2^5	2.6109e-08	3.9868e-06	1.0248e-06	3.7066e-04	2.8529e+11
2^6	1.6512e-09	4.9344e-07	1.2544e-07	9.2666e-05	7.7553e+31
2^7	1.0416e-10	6.1391e-08	1.5485e-08	2.3167e-05	4.9624e+74
Order	3.9867	3.0068	3.0181	2.0000	–

TABLE 16. The errors of $z_h \in S_m^{(0)}(I_h)$ for Case 4 of Example 2 with $m = 2$.

N	Gauss ($\rho_m = 1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{4}, \frac{5}{6})$ ($\rho_m = \frac{3}{5}$)	$(\frac{1}{3}, \frac{2}{3})$ ($\rho_m = 1$)	$(\frac{1}{6}, \frac{1}{2})$ ($\rho_m = 5$)
2^4	3.2749e-03	3.2421e-05	7.8332e-04	2.8844e-03	6.2809e+05
2^5	8.1899e-04	3.9868e-06	1.0721e-04	7.2135e-04	4.5308e+15
2^6	2.0476e-04	4.9344e-07	1.4010e-05	1.8035e-04	5.0038e+36
2^7	5.1192e-05	6.1391e-08	1.7901e-06	4.5089e-05	1.2907e+80
Order	2.0000	3.0068	2.9683	2.0000	–

with $f(t) = \frac{1}{2}e^t + e^{-t} + \frac{1}{2}\sin(t) + (t - \frac{1}{2})\cos(t) + t + 1,\ g(t) = e^{-t} + \cos(t),$ *and initial values* $y(0) = 1,\ z(0) = 1$. *It can be easily check that the (unique) exact solution is* $y(t) = e^{-t}$, $z(t) = \cos(t)$.

TABLE 17. The errors of $y_h \in S_m^{(0)}(I_h)$ for Case 5 of Example 2 with $m = 2$.

N	Gauss ($\rho_m = 1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{4}, \frac{5}{6})$ ($\rho_m = \frac{3}{5}$)	$(\frac{1}{3}, \frac{2}{3})$ ($\rho_m = 1$)	$(\frac{1}{6}, \frac{1}{2})$ ($\rho_m = 5$)
2^4	1.0549e-06	8.0472e-05	2.0275e-05	2.4235e-03	3.3080e+01
2^5	6.5913e-08	9.8946e-06	2.5050e-06	6.0613e-04	2.8889e+10
2^6	4.1189e-09	1.2264e-06	3.0941e-07	1.5155e-04	3.9224e+30
2^7	2.5745e-10	1.5264e-07	3.8366e-08	3.7888e-05	1.2536e+73
Order	3.9999	3.0062	3.0116	2.0000	–

TABLE 18. The errors of $z_h \in S_m^{(0)}(I_h)$ for Case 5 of Example 2 with $m = 2$.

N	Gauss ($\rho_m = 1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{4}, \frac{5}{6})$ ($\rho_m = \frac{3}{5}$)	$(\frac{1}{3}, \frac{2}{3})$ ($\rho_m = 1$)	$(\frac{1}{6}, \frac{1}{2})$ ($\rho_m = 5$)
2^4	5.1880e-03	8.0472e-05	1.5798e-03	4.4946e-03	1.2734e+05
2^5	1.2980e-03	9.8946e-06	2.2696e-04	1.1246e-03	4.5880e+14
2^6	3.2456e-04	1.2264e-06	3.0495e-05	2.8120e-04	2.5308e+35
2^7	8.1143e-05	1.5264e-07	3.9540e-06	7.0303e-05	3.2607e+78
Order	1.9999	3.0062	2.9472	1.9999	–

In Tables 1-4, we list the errors of y and z components in different polynomial collocation spaces. In Tables 5-8, we list the errors in the same polynomial collocation space. It is observed that the numerical results are consistent with the theoretical ones in Sections 4.2 and 5.2, respectively. We also observe that the numerical solutions are divergent for the last sets of collocation parameters due to $|\rho_m| > 1$.

Example 2. *In order to illustrate the theoretical convergence order with the exact regularity, we consider the same IDAE system as Example 1 with initial values* $y(0) = 0$, $z(0) = 1$, *and take functions* $f(t)$ *and* $g(t)$ *such that the (unique) exact solution are as follows:*

Case 1: $y(t) = t^{3+1/3} - \frac{1}{2}t^2,\ z(t) = t^{2+1/2} + 1\quad (y \in C^3(I),\ z \in C^2(I))$;
Case 2: $y(t) = t^{4+1/3} - \frac{1}{2}t^2,\ z(t) = t^{3+1/2} + 1\quad (y \in C^4(I),\ z \in C^3(I))$;
Case 3: $y(t) = t^{5+1/3} - \frac{1}{2}t^2,\ z(t) = t^{4+1/2} + 1\quad (y \in C^5(I),\ z \in C^4(I))$;
Case 4: $y(t) = t^{4+1/3} - \frac{1}{2}t^2,\ z(t) = t^{4+1/2} + 1\quad (y, z \in C^4(I))$;
Case 5: $y(t) = t^{5+1/3} - \frac{1}{2}t^2,\ z(t) = t^{5+1/2} + 1\quad (y, z \in C^5(I))$.

In Tables 9-10, 11-12 and 13-14, we list the errors of y and z components in different polynomial collocation spaces with $m = 2$ for Cases 1, 2 and 3, respectively. It is observed that the numerical results are consistent with Theorems 3, 5 and 6 respectively. It is noticed that in Table 9, due to $y \notin C^4(I)$, order 4 is not reached for the component y at Gauss points.

In Tables 15-16 and 17-18, we list the errors in the same polynomial collocation space with $m = 2$ for Cases 4 and 5, respectively. It is observed that the numerical results are consistent with Theorems 8, 10 and 11, respectively. We also observed that the numerical solutions are divergent for the last sets of collocation parameters due to $|\rho_m| > 1$.

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