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# LOCKING-FREE CG-TYPE FINITE ELEMENT SOLVERS FOR LINEAR ELASTICITY ON SIMPLICIAL MESHES

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**Abstract.** This paper presents numerical methods for solving linear elasticity on simplicial meshes based on enrichment of Lagrangian bilinear/trilinear finite elements. This is a renovated use of the classical 1st order Bernardi-Raugel spaces, which were originally designed for Stokes flow. A projection to the elementwise constant space is employed to handle the dilation (divergence of displacement) in the strain-div formulation. Mixed (both Dirichlet and Neumann) boundary conditions are considered for error estimates in the energy-norm and the $L_2$-norms of displacement and stress. Rigorous analysis and numerical experiments demonstrate that these methods are free of Poisson-locking. Renovation of other Stokes element pairs to linear elasticity is also examined.

**Key words.** Bernardi-Raugel spaces, enriched Lagrangian elements, linear elasticity, locking-free, simplicial meshes.

## 1. Introduction

This paper is concerned with finite element methods for linear elasticity in its usual form

$$\begin{cases} -\nabla \cdot \sigma = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ \mathbf{u}|_{\Gamma^D} = \mathbf{u}_D, \quad (\sigma \mathbf{n})|_{\Gamma^N} = \mathbf{t}_N, \end{cases} \tag{1}$$

where $\Omega$ is a two- or three-dimensional bounded domain occupied by a homogeneous and isotropic elastic material, $\mathbf{f}$ is a body force, $\mathbf{u}_D, \mathbf{t}_N$ are respectively Dirichlet and Neumann data, $\mathbf{n}$ is the outward unit normal vector on the domain boundary $\partial\Omega = \Gamma^D \cup \Gamma^N$. As usual, $\mathbf{u}$ is the solid displacement,

$$\varepsilon(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)$$

is the strain tensor, and

$$\sigma = 2\mu\, \varepsilon(\mathbf{u}) + \lambda(\nabla \cdot \mathbf{u})\mathbf{I}$$

is the Cauchy stress tensor, where $\mathbf{I}$ is the order- two or three identity matrix. The Lamé constants $\lambda, \mu$ are given by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}, \tag{2}$$

where $E > 0$ is the elasticity modulus and $\nu \in (0, \frac{1}{2})$ is the Poisson's ratio.

A main issue in the development of finite element methods for linear elasticity is the so-called *Poisson-locking*, which is often manifested as loss of convergence

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rates in displacement and/or other quantities when $\lambda \to \infty$ or $\nu \to \frac{1}{2}$, that is, the material becomes nearly incompressible. It is well known that the linear Lagrangian $P_1^d (d = 2, 3)$ elements suffer Poisson-locking [9, 11].

By design, the mixed finite element methods (MFEMs) based on the Hellinger-Reissner formulation overcome Poisson-locking. Various types of MFEMs can be found in the literature, e.g., [4, 5, 16, 21, 24, 25, 34]. However, the MFEMs involve more unknowns and result in saddle-point problems that are usually not easy to solve. Recently, novel weak Galerkin (WG) finite element methods have been developed for linear elasticity. These include (i) The lowest-order methods on various types of meshes that use constant vector approximants in element interiors and on inter-element boundaries [22, 36]; (ii) Higher order methods using polynomial approximants (degree 1 or higher) for general polygonal or polyhedral meshes [33]. These WG methods are developed based on the primal formulation but proven to be locking-free.

It is known there are similarities between linear elasticity and Stokes flow, when a pseudo-pressure is introduced to elasticity based on the divergence of displacement (dilation). There are efforts on reusing the Stokes elements for linear elasticity, e.g., [26, 27]. These locking-free finite element methods are developed based on the displacement-pressure mixed formulation, but a biorthogonal system can be established so that the pressure degrees of freedom can be statically condensed and the mixed finite element methods become much more efficient.

Therefore, it is natural to consider reusing stable Stokes element pairs for solving linear elasticity in the primal formulation. In [35], the Bernardi-Raugel elements for Stokes flow [8] were reused for the elasticity part in poroelasticity problems on triangular and tetrahedral meshes. The Darcy part was solved in [35] by a mixed method based on the Raviart-Thomas element. But the error analysis was conducted for the whole Biot system (poroelasticity). [23] presents an algorithm based on the Bernardi-Raugel elements for linear elasticity on quadrilateral and hexahedral meshes.

This paper intends to provide an independent and rigorous analysis on reusing Bernardi-Raugel elements and other Stokes elements to develop locking-free finite element solvers for linear elasticity on simplicial meshes. Our investigation reveals that to reuse a Stokes element for linear elasticity as presented in Scheme (15), the approximation space for Stokes velocity or elasticity displacement needs to satisfy the following property elementwise:

$$\overline{\nabla \cdot (\mathbf{v} - \Pi_h \mathbf{v})} = 0, \tag{3}$$

where $\mathbf{v} \in H^1(\Omega)^d$, $\Pi_h$ is the global projection operator from $H^1(\Omega)^d$ to the aforementioned approximation space, and the overline is the elementwise average for the divergence of a vector-valued function. For an accurate definition, see Equation (12). This will be further elaborated in Sections 2, 3, and 6.

This paper shares the same spirit as our previous work [23] in seeking simple locking-free CG-type methods for elasticity. But this paper focuses more on the analysis side and studies also other elements beyond Bernardi-Raugel.

The rest of this paper is organized as follows. Section 2 briefly reviews the definitions and properties of the first order Bernardi-Raugel elements that were originally designed for Stokes flow [8]. Section 3 presents finite element schemes for linear elasticity based on renovation of the first order Bernardi-Raugel elements on simplicial meshes. These schemes involve only the displacement unknowns and are in the general strain-div formulation. Section 4 presents rigorous error estimation in the energy-norm and $L^2$-norm for the finite element schemes. Section 5 performs numerical experiments on three widely tested examples to illustrate the theoretical estimates. Section 6 examines reuse of other Stokes element pairs for linear elasticity. Section 7 concludes the paper with some remarks.

## 2. Bernardi-Raugel Spaces on Triangles and Tetrahedra

This section briefly reviews the definitions and properties of the first order Bernardi-Raugel elements ($\mathrm{BR}_1$) constructed in the original paper [8] for triangles and tetrahedra.

**$\mathrm{BR}_1$ Spaces on Triangles**. Let $T$ be a triangle with vertices $a_i = (x_i, y_i), i = 1, 2, 3$. Let $e_i (i = 1, 2, 3)$ be the edge opposite to vertex $a_i$ and $\mathbf{n}_i$ be the outward unit normal vector on $e_i$. Let $\lambda_i\,(i = 1, 2, 3)$ be the barycentric coordinates. We consider three edge-based bubble functions

$$\mathbf{b}_1 = \mathbf{n}_1\lambda_2\lambda_3, \quad \mathbf{b}_2 = \mathbf{n}_2\lambda_3\lambda_1, \quad \mathbf{b}_3 = \mathbf{n}_3\lambda_1\lambda_2. \tag{4}$$

Let $P_1(T)^2$ be the space of vector-valued linear polynomials defined on $T$. We define

$$\mathrm{EP}_1(T) = \mathrm{BR}_1(T) = P_1(T)^2 + \mathrm{Span}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3). \tag{5}$$

This space may also be called $\mathrm{EP}_1$ as an enrichment of the classical $P_1(T)^2$ space to emphasize their connection and difference. This definition is further extended to a triangular mesh with considerations of (i) continuous piecewise linear polynomials on the whole mesh; (ii) consistency in edge normal vectors for two adjacent elements and domain boundaries.

It was shown in [8] that $\mathrm{BR}_1$ together with the piecewise constant space form a stable element pair for Stokes flow. This pair is usually denoted as $(\mathrm{BR}_1, P_0)$.

**$\mathrm{BR}_1$ Spaces on Tetrahedra**. This is very similar to that discussed in the previous paragraphs. Now let $T$ be a tetrahedron with vertices $a_i = (x_i, y_i, z_i), i = 1, 2, 3, 4$. Let $e_i\,(i = 1, 2, 3, 4)$ be the face opposite to vertex $a_i$ and $\mathbf{n}_i$ be the outward unit normal vector on face $e_i$. Let $\lambda_i\,(i = 1, 2, 3, 4)$ be the barycentric coordinates. We consider four face-based bubble functions

$$\mathbf{b}_1 = \mathbf{n}_1\lambda_2\lambda_3\lambda_4, \quad \mathbf{b}_2 = \mathbf{n}_2\lambda_3\lambda_4\lambda_1, \quad \mathbf{b}_3 = \mathbf{n}_3\lambda_4\lambda_1\lambda_2, \quad \mathbf{b}_4 = \mathbf{n}_4\lambda_1\lambda_2\lambda_3. \tag{6}$$

Let $P_1(T)^3$ be the space of vector-valued linear polynomials on tetrahedron $T$. Then the $\mathrm{BR}_1$ space on this tetrahedron is defined as

$$\mathrm{EP}_1(T) = \mathrm{BR}_1(T) = P_1(T)^3 + \mathrm{Span}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4). \tag{7}$$

For the rest of this paper, we unify the treatments for $\mathrm{BR}_1$ on triangles and tetrahedra. Distinction is made only when necessary.

Let $\mathcal{T}_h$ be a partition of the given domain $\Omega$ consisting of d-simplexes ($d = 2$ for triangles and $d = 3$ for tetrahedra). We use $\Gamma_h^D$ to denote the set of all edges or faces of $\mathcal{T}_h$ that are on the Dirichelt boundary $\Gamma^D$. We define approximation spaces

$$V_h = \{\mathbf{v} \in H^1(\Omega)^d : \mathbf{v}|_T \in \mathrm{BR}_1(T), \forall T \in \mathcal{T}_h\}, \tag{8}$$

$$V_h^{0,D} = \{\mathbf{v} \in V_h : \mathbf{v}|_{\Gamma_h^D} = \mathbf{0}\}, \tag{9}$$

where

$$\mathrm{BR}_1(T) = P_1(T)^d \oplus \mathrm{Span}\{\mathbf{b}_i, 1 \le i \le d+1\},$$
$$\mathbf{b}_i = \mathbf{n}_i \prod_{j=1, j\neq i}^{d+1} \lambda_j.$$

Now we consider the local and global projection operators defined in [8], which consists of two parts, a traditional interpolation operator to the $P_1$ space and a projection operator for the residual to the space of bubble functions defined in (4) or (6). For $T \in \mathcal{T}_h$, $\mathbf{v} \in H^1(T)^d$, the projection operator $\Pi_T : H^1(T)^d \to \mathrm{BR}_1(T)$ is defined as

$$\Pi_T \mathbf{v} = \widetilde{\Pi}_T \mathbf{v} + \sum_{i=1}^{d+1} \alpha_i \mathbf{b}_i,$$

where $\widetilde{\Pi}_T$ is actually the nodal interpolation operator

$$\widetilde{\Pi}_T \mathbf{v} = \sum_{i=1}^{d+1} \mathbf{v}(a_i) \lambda_i$$

and

$$\alpha_i = \left( \int_{e_i} (\mathbf{v} - \widetilde{\Pi}_T \mathbf{v}) \cdot \mathbf{n} \right) \Big/ \int_{e_i} \prod_{j=1, j\neq i}^{d+1} \lambda_j, \qquad 1 \le i \le d+1.$$

The projection operator $\Pi_T$ satisfies

$$(\Pi_T \mathbf{v})(a_i) = \mathbf{v}(a_i), \quad 1 \le i \le d+1, \tag{10}$$

$$\int_{e_i} (\mathbf{v} - \Pi_T \mathbf{v}) \cdot \mathbf{n} = 0, \quad 1 \le i \le d+1. \tag{11}$$

The global projection operator $\Pi_h : H^1(\Omega)^d \to V_h$ is defined as

$$(\Pi_h \mathbf{v})|_T = \Pi_T(\mathbf{v}|_T)$$

with necessary adjustments for signs associated with the edge/face normals.

For the dilation $\nabla \cdot \mathbf{v}$, its elementwise average is defined as

$$\overline{\nabla \cdot \mathbf{v}} = \frac{1}{|T|} \int_T \nabla \cdot \mathbf{v}, \tag{12}$$

where $|T|$ is the area or volume of the element.

From (11), we have, for any $\mathbf{v} \in H^1(\Omega)^d$,

$$\begin{aligned}\|\overline{\nabla\cdot(\mathbf{v}-\Pi_h\mathbf{v})}\|^2 &= \sum_{T\in\mathcal{T}_h}\int_T \overline{\nabla\cdot(\mathbf{v}-\Pi_h\mathbf{v})}\;\overline{\nabla\cdot(\mathbf{v}-\Pi_h\mathbf{v})}\\ &= \sum_{T\in\mathcal{T}_h}\overline{\nabla\cdot(\mathbf{v}-\Pi_h\mathbf{v})}\int_T \nabla\cdot(\mathbf{v}-\Pi_h\mathbf{v})\\ &= \sum_{T\in\mathcal{T}_h}\overline{\nabla\cdot(\mathbf{v}-\Pi_h\mathbf{v})}\int_{\partial T}(\mathbf{v}-\Pi_h\mathbf{v})\cdot\mathbf{n}\\ &=0.\end{aligned}$$

This is to say that on each element we have

$$\overline{\nabla\cdot(\mathbf{v}-\Pi_h\mathbf{v})} = 0. \tag{13}$$

## 3. Solving Linear Elasticity by Enriched Lagrangian Elements

The previously discussed enriched Lagrangian elements in combination with another projection operator can be used for solving linear elasticity.

For ease of presentation, we focus on triangular meshes.

Let $\Omega$ be a polygonal domain equipped with a triangular mesh $\mathcal{T}_h$. Let $\mathbf{v}_h \in V_h$ be as defined in (8). For a triangle $T \in \mathcal{T}_h$, it is known that in general $\mathrm{div}(\mathbf{v}_h)$ is not a constant on $T$. We consider its average $\overline{\mathrm{div}(\mathbf{v}_h)}$ on $T$, namely, the ***local projection into the space of constant scalars***. This technique is also called *reduced integration* [12, 17, 29].

For a triangle $T \in \mathcal{T}_h$ satisfying $T\cap\Gamma^D \neq \emptyset$ and an edge $e$ on $\partial T\cap\Gamma_h^D$, we define

$$(\mathbf{u}_{D,h})|_e = \widetilde{\Pi}_e(\mathbf{u}_D) + \left(\int_e(\mathbf{u}_D-\widetilde{\Pi}_e\mathbf{u}_D)\cdot\mathbf{n}\Big/\int_e\mathbf{b}_e\cdot\mathbf{n}\right)\mathbf{b}_e, \tag{14}$$

where $\widetilde{\Pi}_e$ is the interpolation operator onto $P_1(e)^d$, $\mathbf{n}$ is the outward unit normal vector on $e$, and $\mathbf{b}_e$ is the bubble function associated with edge $e$ such that $(\mathbf{b}_e)|_e \neq \mathbf{0}$.

We consider a finite element scheme in the strain-div formulation as follows. Find $\mathbf{u}_h \in V_h$ such that $\mathbf{u}_h|_{\Gamma_h^D} = \mathbf{u}_{D,h}$ and

$$\mathcal{A}_h(\mathbf{u}_h,\mathbf{v}_h) = \mathcal{F}_h(\mathbf{v}_h), \qquad \forall\mathbf{v}_h\in V_h^{0,D}, \tag{15}$$

where

$$\mathcal{A}_h(\mathbf{u}_h,\mathbf{v}_h) = 2\mu\sum_{T\in\mathcal{T}_h}(\varepsilon(\mathbf{u}_h),\varepsilon(\mathbf{v}_h))_T + \lambda\sum_{T\in\mathcal{T}_h}(\overline{\nabla\cdot\mathbf{u}_h},\overline{\nabla\cdot\mathbf{v}_h})_T \tag{16}$$

and

$$\mathcal{F}_h(\mathbf{v}_h) = \sum_{T\in\mathcal{T}_h}(\mathbf{f},\mathbf{v}_h)_T + \sum_{e\in\Gamma_h^N}\langle\mathbf{t}_N,\mathbf{v}_h\rangle_e. \tag{17}$$

**Enforcing boundary conditions**. It is clear that there are two sets of basis functions for the displacement: node-based and edge-based. Compatibility among these two types of functions needs to be maintained in enforcement or incorporation of boundary conditions. Here we discuss the treatments for 2-dim problems only, since 3-dim treatments are very similar.

(i) For a Dirichlet edge, one can enforce the Dirichlet condition at its two end nodes by a direct evaluation (interpolation) of the Dirichlet data. Then the difference between the original Dirichlet data and the interpolant is utilized to calculate the coefficient for the edge bubble function. See formula (14).

(ii) For a Neumann edge, the integrals of the Neumann data against the three basis functions (2 linear polynomials for the end nodes, 1 quadratic for the edge) are computed directly and assembled into the global right-hand side of the sparse discrete linear system. See the 2nd term in (17).

The discrete stress tensor $\sigma_h$ corresponding to $\mathbf{u}_h$ is given by

$$\sigma_h = 2\mu\varepsilon(\mathbf{u}_h) + \lambda\overline{\nabla\cdot\mathbf{u}_h}\mathbf{I}, \tag{18}$$

which is considered in the analysis and the numerical experiments. Since $\overline{\nabla\cdot\mathbf{u}_h}$ is the piece-wise average of $\nabla\cdot\mathbf{u}_h$, $\sigma_h$ is not continuous across edges/faces.

**Remarks**. Our finite element methods for elasticity are obviously different than those mixed methods studied in [13, 26, 27]. The schemes here are in the primal formulation, solving for displacement only in the $\mathrm{EP}_1 = \mathrm{BR}_1$ spaces. The resulting discrete linear systems are symmetric positive-definite (SPD). We call the solvers as $\mathrm{EP}_1$ also.

## 4. Error Analysis

This section presents error analysis for the proposed finite element schemes. In [11, 12], similar estimates were established for pure displacement problems in the grad-div formulation. In [23], brief proofs are given for a homogeneous Dirichlet boundary condition on the quadrilateral and hexahedral meshes. In this paper, we consider more general mixed boundary conditions, for which we need to use the strain-div formulation. Moreover, the conclusions may be extended to higher order cases by replacing the average of the dilation in the numerical scheme with a $L_2$-projection for the dilation.

For convenience, we use $A \lesssim B$ to simplify an inequality $A \le CB$, where $C > 0$ is a constant that may take different values at different occasions but is independent of $\lambda$ and $h$.

First, it is assumed that the elasticity boundary value problem has a unique solution for sufficiently smooth boundary data $\mathbf{u}_D$ and $\mathbf{t}_N$. Hypotheses 1 & 2 are about regularity.

**Hypothesis 1**. There exists $\mathbf{z} \in H^2(\Omega)^d$ $(d = 2, 3)$ such that

$$\begin{cases} -\nabla\cdot\sigma(\mathbf{z}) = \mathbf{0}, \quad \mathbf{x}\in\Omega, \\ \mathbf{z}|_{\Gamma^D} = \mathbf{u}_D, \quad \sigma(\mathbf{z})\mathbf{n}|_{\Gamma^N} = \mathbf{0}, \end{cases} \tag{19}$$

and

$$\|\mathbf{z}\|_{H^2(\Omega)} \lesssim \|\mathbf{u}_D\|_{H^{\frac{3}{2}}(\Gamma^D)}. \tag{20}$$

**Hypothesis 2**. Let $\Omega \subset \mathbb{R}^d\,(d = 2, 3)$ be a bounded convex polygonal or polyhedral domain and $\mathbf{f} \in L^2(\Omega)^d$. Then the elasticity boundary value problem (1) has a unique solution $\mathbf{u} \in H^2(\Omega)^d$ such that

$$\|\mathbf{u}\|_{H^2(\Omega)} + \lambda \|\nabla \cdot \mathbf{u}\|_{H^1(\Omega)} \lesssim \|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{u}_D\|_{H^{\frac{3}{2}}(\Gamma^D)} + \|\mathbf{t}_N\|_{H^{\frac{1}{2}}(\Gamma^N)}. \tag{21}$$

Moreover, these norms of $\mathbf{f}$, $\mathbf{u}_D$, and $\mathbf{t}_N$ are assumed to be bounded when $\lambda \to \infty$.

**4.1. Energy-norm Error Estimate.** Since $V_h^{0,D} \subset H^1(\Omega)^d$ and $\mathbf{v}_h|_{\Gamma_h^D} = \mathbf{0}$ for any $\mathbf{v}_h \in V_h^{0,D}$, it follows from [10] that

$$\|\mathbf{v}_h\|^2_{H^1(\Omega)} \le \|\mathbf{v}_h\|_h^2 := \mathcal{A}_h(\mathbf{v}_h, \mathbf{v}_h).$$

In other words, $\|\cdot\|_h$ is a norm on $V_h^{0,D}$.

**Theorem 1.** *Let $\mathbf{u} \in H^2(\Omega)^d$ be the exact solution of (1) and $\mathbf{u}_h \in V_h$ be the finite element solution obtained from (15). There holds*

$$\|\mathbf{u} - \mathbf{u}_h\|_h \lesssim h \left( \|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{u}_D\|_{H^{\frac{3}{2}}(\Gamma^D)} + \|\mathbf{t}_N\|_{H^{\frac{1}{2}}(\Gamma^N)} \right). \tag{22}$$

*Proof.* Unlike [23], the result of this theorem is for the mixed boundary conditions. Let $\mathbf{u}_h^* \in V_h$ be the $\mathcal{A}_h(\cdot,\cdot)$-orthogonal projection of $\mathbf{u}$ [12, 18, 32] such that $\mathbf{u}_h^*|_{\Gamma_h^D} = \mathbf{u}_{D,h}$ and

$$\mathcal{A}_h(\mathbf{u} - \mathbf{u}_h^*, \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in V_h^{0,D}. \tag{23}$$

For any $\mathbf{v}_h \in V_h$, it follows from the definition of $\|\cdot\|_h$ that

$$\begin{aligned}\|\mathbf{u} - \mathbf{u}_h^*\|_h^2 =& \mathcal{A}_h(\mathbf{u} - \mathbf{u}_h^*, \mathbf{u} - \mathbf{u}_h^*)\\ =& \mathcal{A}_h(\mathbf{u} - \mathbf{u}_h^*, \mathbf{u} - \mathbf{v}_h) + \mathcal{A}_h(\mathbf{u} - \mathbf{u}_h^*, \mathbf{v}_h - \mathbf{u}_h^*).\end{aligned}$$

Let $\mathbf{v}_h|_{\Gamma_h^D} = \mathbf{u}_h^*|_{\Gamma_h^D} = \mathbf{u}_{D,h}$. Since $\mathbf{v}_h - \mathbf{u}_h^* \in V_h^{0,D}$, we derive from (23) that

$$\|\mathbf{u} - \mathbf{u}_h^*\|_h \le \inf_{\mathbf{v}_h \in V_h, \mathbf{v}_h|_{\Gamma_h^D} = \mathbf{u}_{D,h}} \|\mathbf{u} - \mathbf{v}_h\|_h.$$

It is clear that

$$\begin{aligned}\|\mathbf{u} - \mathbf{u}_h\|_h &\le \|\mathbf{u} - \mathbf{u}_h^*\|_h + \|\mathbf{u}_h^* - \mathbf{u}_h\|_h\\ &\le \inf_{\substack{\mathbf{v}_h \in V_h\\ \mathbf{v}_h|_{\Gamma_h^D} = \mathbf{u}_{D,h}}} \|\mathbf{u} - \mathbf{v}_h\|_h + \sup_{\mathbf{w}_h \in V_h^{0,D} \backslash \{\mathbf{0}\}} \frac{|\mathcal{A}_h(\mathbf{u}_h^* - \mathbf{u}_h, \mathbf{w}_h)|}{\|\mathbf{w}_h\|_h}\\ &\le \inf_{\substack{\mathbf{v}_h \in V_h\\ \mathbf{v}_h|_{\Gamma_h^D} = \mathbf{u}_{D,h}}} \|\mathbf{u} - \mathbf{v}_h\|_h + \sup_{\mathbf{w}_h \in V_h^{0,D} \backslash \{\mathbf{0}\}} \frac{|\mathcal{A}_h(\mathbf{u}, \mathbf{w}_h) - \mathcal{F}(\mathbf{w}_h)|}{\|\mathbf{w}_h\|_h}. \quad (24)\end{aligned}$$

To estimate the first term, we utilize the nice property stated in (13) about the interpolant $\Pi_h \mathbf{u}$ and the approximation property

$$\|\varepsilon(\mathbf{u} - \Pi_h \mathbf{u})\|_{L^2(\Omega)} \lesssim h \|\mathbf{u}\|_{H^2(\Omega)}$$

to obtain

$$\begin{aligned}\|\mathbf{u}-\Pi_h\mathbf{u}\|_h^2 =&2\mu\|\varepsilon(\mathbf{u}-\Pi_h\mathbf{u})\|_{L^2(\Omega)}^2+\lambda\|\overline{\nabla\cdot(\mathbf{u}-\Pi_h\mathbf{u})}\|_{L^2(\Omega)}^2\\ =&2\mu\|\varepsilon(\mathbf{u}-\Pi_h\mathbf{u})\|_{L^2(\Omega)}^2\lesssim h^2\|\mathbf{u}\|_{H^2(\Omega)}^2.\end{aligned}$$

Then we have

$$\begin{aligned}\inf_{\mathbf{v}_h\in V_h,\mathbf{v}_h|_{\Gamma_h^D}=\mathbf{u}_{D,h}}\|\mathbf{u}-\mathbf{v}_h\|_h &\le\|\mathbf{u}-\Pi_h\mathbf{u}\|_h\lesssim h\|\mathbf{u}\|_{H^2(\Omega)}\\ &\lesssim h\left(\|\mathbf{f}\|_{L^2(\Omega)}+\|\mathbf{u}_D\|_{H^{\frac{3}{2}}(\Gamma^D)}+\|\mathbf{t}_N\|_{H^{\frac{1}{2}}(\Gamma^N)}\right).\end{aligned}$$

For the second term, we proceed as follows. By integration by parts, we have

$$\begin{aligned}\mathcal{F}_h(\mathbf{w}_h) =&(\mathbf{f},\mathbf{w}_h)+\text{Neumann boundary condition}\\ =&(-\nabla\cdot\sigma(\mathbf{u}),\mathbf{w}_h)+\text{Neumann boundary condition}\\ =&(\sigma(\mathbf{u}),\nabla\mathbf{w}_h)+0\\ =&2\mu(\varepsilon(\mathbf{u}),\varepsilon(\mathbf{w}_h))+\lambda(\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{w}_h).\end{aligned}$$

Applying the fact that $\overline{\nabla\cdot\mathbf{w}_h}$ is an elementwise constant, we have

$$\begin{aligned}|\mathcal{A}_h(\mathbf{u},\mathbf{w}_h)-\mathcal{F}_h(\mathbf{w}_h)| =&\left|\lambda(\overline{\nabla\cdot\mathbf{u}},\overline{\nabla\cdot\mathbf{w}_h})-\lambda(\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{w}_h)\right|\\ =&\lambda\left|(\overline{\nabla\cdot\mathbf{u}}-\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{w}_h)\right|\\ \le&\lambda\|\overline{\nabla\cdot\mathbf{u}}-\nabla\cdot\mathbf{u}\|_{L^2(\Omega)}\|\nabla\cdot\mathbf{w}_h\|_{L^2(\Omega)}\\ \lesssim&\lambda h\|\nabla\cdot\mathbf{u}\|_{H^1(\Omega)}\|\nabla\cdot\mathbf{w}_h\|_{L^2(\Omega)}.\end{aligned}$$

Here for the last two lines, we have used the Cauchy-Schwarz inequality and the approximation property

$$\|\overline{\nabla\cdot\mathbf{u}}-\nabla\cdot\mathbf{u}\|_{L^2(\Omega)}\lesssim h\|\nabla\cdot\mathbf{u}\|_{H^1(\Omega)}.$$

Since

$$\|\nabla\cdot\mathbf{w}_h\|_{L^2(\Omega)}\le\|\varepsilon(\mathbf{w}_h)\|_{L^2(\Omega)}\lesssim\|\mathbf{w}_h\|_h,$$

we have

$$\begin{aligned}\sup_{\mathbf{w}_h\in V_h^{0,D}\setminus\{\mathbf{0}\}}\frac{|\mathcal{A}_h(\mathbf{u},\mathbf{w}_h)-\mathcal{F}_h(\mathbf{w}_h)|}{\|\mathbf{w}\|_h}&\lesssim\sup_{\mathbf{w}_h\in V_h^{0,D}\setminus\{\mathbf{0}\}}\frac{\lambda h\|\nabla\cdot\mathbf{u}\|_{H^1(\Omega)}\|\nabla\cdot\mathbf{w}_h\|_{L^2(\Omega)}}{\|\mathbf{w}_h\|_h}\\ &\le h\lambda\|\nabla\cdot\mathbf{u}\|_{H^1(\Omega)}\lesssim h\left(\|\mathbf{f}\|_{L^2(\Omega)}+\|\mathbf{u}_D\|_{H^{\frac{3}{2}}(\Gamma^D)}+\|\mathbf{t}_N\|_{H^{\frac{1}{2}}(\Gamma^N)}\right).\end{aligned}$$

Here for the last step, we have used Hypothesis 2 (regularity of the exact solution).

Combining the estimates for these two terms gives the desired result in the theorem. □

**4.2. $L^2$-norm Error Estimate for Displacement.** This subsection presents an $L^2$-norm error estimate for the numerical displacement based on a duality argument. We conduct a complete analysis for a general elasticity boundary value problem that has both Dirichlet and Neumann conditions. This involves details that are usually not found in the literature.

**Theorem 2.** *Let* $\mathbf{u} \in H^2(\Omega)^d$ *be the exact solution of (1) and* $\mathbf{u}_h \in V_h$ *be the finite element solution obtained from (15). There holds*

$$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)} \lesssim h^2 \left(\|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{u}_D\|_{H^{\frac{3}{2}}(\Gamma^D)} + \|\mathbf{t}_N\|_{H^{\frac{1}{2}}(\Gamma^N)}\right).$$

*Proof.* Assume Hypotheses 1 & 2 are satisfied. Then there exists $\mathbf{z} \in H^2(\Omega)^d$ satisfying (19) and (20). It is obvious that $\tilde{\mathbf{u}} = \mathbf{u}-\mathbf{z}$ is the solution to the boundary value problem that involves only a homogeneous Dirichlet boundary condition. We consider also $\tilde{\mathbf{u}}_h = \mathbf{u}_h - \Pi_h \mathbf{z}$. It is clear that

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)} \le& \|\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h\|_{L^2(\Omega)} + \|\mathbf{z}-\Pi_h\mathbf{z}\|_{L^2(\Omega)}\\ \lesssim& \|\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h\|_{L^2(\Omega)} + h^2\|\mathbf{z}\|_{H^2(\Omega)}.\end{aligned} \tag{25}$$

On the other hand, by (13), Theorem 1, and (20), we have,

$$\begin{aligned}&\|\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h\|_h\\ \le& \|\mathbf{u}-\mathbf{u}_h\|_h + \|\mathbf{z}-\Pi_h\mathbf{z}\|_h\\ \lesssim& \|\mathbf{u}-\mathbf{u}_h\|_h + \|\varepsilon(\mathbf{z}-\Pi_h\mathbf{z})\|_{L^2(\Omega)} + \sqrt{\lambda}\|\overline{\nabla\cdot(\mathbf{z}-\Pi_h\mathbf{z})}\|_{L^2(\Omega)}\\ \lesssim& \|\mathbf{u}-\mathbf{u}_h\|_h + h\|\mathbf{z}\|_{H^2(\Omega)}\\ \lesssim& h\left(\|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{u}_D\|_{H^{\frac{3}{2}}(\Gamma^D)} + \|\mathbf{t}_N\|_{H^{\frac{1}{2}}(\Gamma^N)}\right).\end{aligned} \tag{26}$$

Assume that $\zeta \in H^2(\Omega)^d$ is the solution of the following dual problem

$$\begin{aligned}-\nabla\cdot(2\mu\varepsilon(\zeta)+\lambda(\nabla\cdot\zeta)\mathbf{I}) =& \tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h \quad \text{in } \Omega,\\ \zeta =& \mathbf{0} \quad \text{on } \Gamma^D,\\ (2\mu\varepsilon(\zeta)+\lambda\nabla\cdot\zeta\mathbf{I})\mathbf{n} =& \mathbf{0} \quad \text{on } \Gamma^N\end{aligned}$$

with dual regularity

$$\|\zeta\|_{H^2(\Omega)} + \lambda\|\nabla\cdot\zeta\|_{H^1(\Omega)} \lesssim \|\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h\|_{L^2(\Omega)}. \tag{27}$$

For convenience, we define

$$\mathcal{A}(\zeta, \tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h) = 2\mu(\varepsilon(\zeta), \varepsilon(\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h)) + \lambda(\nabla\cdot\zeta, \nabla\cdot(\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h)).$$

We use the fact that $(\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h)|_{\Gamma_h^D} = \mathbf{0}$ to obtain

$$\begin{aligned}\|\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h\|^2_{L^2(\Omega)} =& (-\nabla\cdot(2\mu\varepsilon(\zeta)+\lambda(\nabla\cdot\zeta)\mathbf{I}), \tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h)\\ =& \mathcal{A}(\zeta, \tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h).\end{aligned}$$

Accordingly, we split the latter into three group terms as follows

$$\begin{aligned}\|\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h\|^2_{L^2(\Omega)} =& \Big(\mathcal{A}(\zeta, \tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h) - \mathcal{A}_h(\zeta, \tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h)\Big)\\ &+ \mathcal{A}_h(\zeta-\Pi_h\zeta, \tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h) + \mathcal{A}_h(\Pi_h\zeta, \tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h)\\ =&: I + II + III.\end{aligned} \tag{28}$$

Next we estimate each of these three terms.

For Term I, it follows from the projection inequality, (26), and (27) that

$$
\begin{aligned}
&\mathcal{A}(\zeta, \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h) - \mathcal{A}_h(\zeta, \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h) \\
=&2\mu(\varepsilon(\zeta), \varepsilon(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h)) + \lambda(\nabla\cdot\zeta, \nabla\cdot(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h)) \\
&- 2\mu(\varepsilon(\zeta), \varepsilon(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h)) - \lambda(\overline{\nabla\cdot\zeta}, \overline{\nabla\cdot(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h)}) \\
=&\lambda(\nabla\cdot\zeta - \overline{\nabla\cdot\zeta}, \nabla\cdot(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h)) \\
\lesssim&\lambda\, h\|\nabla\cdot\zeta\|_{H^1(\Omega)}\|\varepsilon(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h)\|_{L^2(\Omega)} \\
\lesssim&\lambda\, h\|\nabla\cdot\zeta\|_{H^1(\Omega)}\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_h \\
\lesssim&h^2\left(\|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{u}_D\|_{H^{\frac{3}{2}}(\Gamma^D)} + \|\mathbf{t}_N\|_{H^{\frac{1}{2}}(\Gamma^N)}\right)\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_{L^2(\Omega)}.
\end{aligned}
\tag{29}
$$

For Term II, we use (13), the projection inequality, (26), and (27) to obtain

$$
\begin{aligned}
&\mathcal{A}_h(\zeta - \Pi_h\zeta, \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h) \\
=&2\mu(\varepsilon(\zeta - \Pi_h\zeta), \varepsilon(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h)) + \lambda(\overline{\nabla\cdot(\zeta - \Pi_h\zeta)}, \overline{\nabla\cdot(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h)}) \\
\lesssim&h\,\|\zeta\|_{H^2(\Omega)}\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_h \\
\lesssim&h^2\left(\|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{u}_D\|_{H^{\frac{3}{2}}(\Gamma^D)} + \|\mathbf{t}_N\|_{H^{\frac{1}{2}}(\Gamma^N)}\right)\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_{L^2(\Omega)}.
\end{aligned}
\tag{30}
$$

For Term III, we test (1) with $\Pi_h\zeta$ and use the fact $(\Pi_h\zeta)|_{\Gamma_h^D} = \mathbf{0}$ to obtain

$$
\mathcal{F}_h(\Pi_h\zeta) = 2\mu(\varepsilon(\mathbf{u}), \varepsilon(\Pi_h\zeta)) + \lambda(\nabla\cdot\mathbf{u}, \nabla\cdot(\Pi_h\zeta)).
$$

Then we use (13), the projection inequality, (20), Hypothesis 2, and (27) to derive

$$
\begin{aligned}
&\mathcal{A}_h(\Pi_h\zeta, \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h) \\
=&\mathcal{A}_h(\Pi_h\zeta, \mathbf{u}) - \mathcal{F}_h(\Pi_h\zeta) + \mathcal{A}_h(\Pi_h\zeta, \Pi_h\mathbf{z} - \mathbf{z}) \\
=&\lambda(\overline{\nabla\cdot\mathbf{u}}, \overline{\nabla\cdot(\Pi_h\zeta)}) - \lambda(\nabla\cdot\mathbf{u}, \nabla\cdot(\Pi_h\zeta)) \\
&+ 2\mu(\varepsilon(\Pi_h\zeta), \varepsilon(\Pi_h\mathbf{z} - \mathbf{z})) \\
=&\lambda(\overline{\nabla\cdot\mathbf{u}} - \nabla\cdot\mathbf{u}, \nabla\cdot(\Pi_h\zeta) - \overline{\nabla\cdot(\Pi_h\zeta)}) \\
&+ 2\mu(\varepsilon(\Pi_h\zeta - \overline{\Pi_h\zeta}), \varepsilon(\Pi_h\mathbf{z} - \mathbf{z})) \\
\lesssim&\lambda\, h^2\|\nabla\cdot\mathbf{u}\|_{H^1(\Omega)}\|\zeta\|_{H^2(\Omega)} + h^2\|\mathbf{z}\|_{H^2(\Omega)}\|\zeta\|_{H^2(\Omega)} \\
\lesssim&h^2\left(\|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{u}_D\|_{H^{\frac{3}{2}}(\Gamma^D)} + \|\mathbf{t}_N\|_{H^{\frac{1}{2}}(\Gamma^N)}\right)\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}.
\end{aligned}
\tag{31}
$$

Finally, the result stated in the theorem follows from (20), (25), and (28)-(31). □

**4.3. Error Estimates for Dilation and Stress.** This subsection presents error estimates for the dilation and stress for problems with mixed boundary conditions.

**Theorem 3.** *Let* $\mathbf{u} \in H^2(\Omega)^d$ *be the exact solution of (1) and* $\mathbf{u}_h \in V_h$ *be the numerical solution of (15). Let* $\sigma_h$ *be the numerical stress defined by (18), then we* *have*

$$\lambda\|\nabla\cdot\mathbf{u}-\overline{\nabla\cdot\mathbf{u}_h}\|_{L^2(\Omega)}\lesssim h\left(\|\mathbf{f}\|_{L^2(\Omega)}+\|\mathbf{u}_D\|_{H^{\frac{3}{2}}(\Gamma^D)}+\|\mathbf{t}_N\|_{H^{\frac{1}{2}}(\Gamma^N)}\right),\tag{32}$$

$$\|\sigma-\sigma_h\|_{L^2(\Omega)}\lesssim h\left(\|\mathbf{f}\|_{L^2(\Omega)}+\|\mathbf{u}_D\|_{H^{\frac{3}{2}}(\Gamma^D)}+\|\mathbf{t}_N\|_{H^{\frac{1}{2}}(\Gamma^N)}\right).\tag{33}$$

*Proof.* The variational formulation for (1) implies that for any $\mathbf{v}_h\in V_h^{0,D}$,

$$2\mu(\varepsilon(\mathbf{u}),\varepsilon(\mathbf{v}_h))+\lambda(\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{v}_h)=\mathcal{F}_h(\mathbf{v}_h).$$

Recall the numerical scheme (15), for any $\mathbf{v}_h\in V_h^{0,D}$

$$2\mu(\varepsilon(\mathbf{u}_h),\varepsilon(\mathbf{v}_h))+\lambda(\overline{\nabla\cdot\mathbf{u}_h},\overline{\nabla\cdot\mathbf{v}_h})=\mathcal{F}_h(\mathbf{v}_h).$$

Combining the above two equations, we get

$$\lambda(\nabla\cdot\mathbf{u}-\overline{\nabla\cdot\mathbf{u}_h},\nabla\cdot\mathbf{v}_h)=2\mu(\varepsilon(\mathbf{u}_h-\mathbf{u}),\varepsilon(\mathbf{v}_h)),\quad\forall\mathbf{v}_h\in V_h^{0,D}.\tag{34}$$

For $\overline{\nabla\cdot\mathbf{u}}-\overline{\nabla\cdot\mathbf{u}_h}\in L^2(\Omega)$, there is a function $w\in H^2(\Omega)$ such that [11]

$$\begin{aligned}-\Delta w=&\overline{\nabla\cdot\mathbf{u}}-\overline{\nabla\cdot\mathbf{u}_h},\quad\mathbf{x}\in\Omega,\\ \nabla w|_{\Gamma^D}\cdot\mathbf{n}=&0,\\ w|_{\Gamma^N}=&0,\end{aligned}$$

and

$$\|w\|_{H^2(\Omega)}\lesssim\|\overline{\nabla\cdot\mathbf{u}}-\overline{\nabla\cdot\mathbf{u}_h}\|_{L^2(\Omega)}.$$

Let $\mathbf{v}_1=-\nabla w\in H^1(\Omega)^d$, then

$$\nabla\cdot\mathbf{v}_1=\overline{\nabla\cdot\mathbf{u}}-\overline{\nabla\cdot\mathbf{u}_h},\quad\mathbf{x}\in\Omega,\tag{35}$$

$$\mathbf{v}_1|_{\Gamma^D}\cdot\mathbf{n}=0,\tag{36}$$

$$\|\mathbf{v}_1\|_{H^1(\Omega)}\lesssim\|\overline{\nabla\cdot\mathbf{u}}-\overline{\nabla\cdot\mathbf{u}_h}\|_{L^2(\Omega)}.\tag{37}$$

The trace theorem [1, 11] implies that there is a function $\phi\in H^2(\Omega)$ such that

$$\phi|_{\partial\Omega}=0,\tag{38}$$

$$\nabla\phi|_{\partial\Omega}\cdot\mathbf{n}=\mathbf{v}_1|_{\partial\Omega}\cdot\mathbf{t},\tag{39}$$

$$\|\phi\|_{H^2(\Omega)}\lesssim\|\mathbf{v}_1\|_{H^1(\Omega)},\tag{40}$$

where $\mathbf{t}$ is the unit tangential vector. Let $\mathbf{v}_2=\nabla\times\phi$, then

$$\begin{aligned}\mathbf{v}_2|_{\partial\Omega}\cdot\mathbf{n}=&\nabla\phi|_{\partial\Omega}\cdot\mathbf{t}=0,\\ \mathbf{v}_2|_{\partial\Omega}\cdot\mathbf{t}=&-\nabla\phi|_{\partial\Omega}\cdot\mathbf{n}=-\mathbf{v}_1|_{\partial\Omega}\cdot\mathbf{t},\end{aligned}$$

which implies that

$$\mathbf{v}_2|_{\Gamma^D}=-\mathbf{v}_1|_{\Gamma^D}.$$

Let $\mathbf{v}=\mathbf{v}_1+\mathbf{v}_2$, then we have

$$\nabla\cdot\mathbf{v}=\nabla\cdot\mathbf{v}_1=\overline{\nabla\cdot\mathbf{u}}-\overline{\nabla\cdot\mathbf{u}_h},\tag{41}$$

it follows from (40) and (37) that

$$\begin{aligned}\|\mathbf{v}\|_{H^1(\Omega)} \leq& \|\mathbf{v}_1\|_{H^1(\Omega)} + \|\mathbf{v}_2\|_{H^1(\Omega)}\\ \leq& \|\mathbf{v}_1\|_{H^1(\Omega)} + \|\phi\|_{H^2(\Omega)}\\ \lesssim& \|\mathbf{v}_1\|_{H^1(\Omega)}\\ \lesssim& \|\overline{\nabla\cdot\mathbf{u}} - \overline{\nabla\cdot\mathbf{u}_h}\|_{L^2(\Omega)},\end{aligned} \tag{42}$$

since $\mathbf{v}|_{\Gamma^D} = \mathbf{0}$, $\Pi_h\mathbf{v} \in V_h^{0,D}$. Using (41), (13), (34), (42), Theorem 1 and Hypothesis 2, we obtain

$$\begin{aligned}&\lambda\|\overline{\nabla\cdot\mathbf{u}} - \overline{\nabla\cdot\mathbf{u}_h}\|^2\\ \leq&\lambda(\overline{\nabla\cdot\mathbf{u}} - \overline{\nabla\cdot\mathbf{u}_h}, \nabla\cdot\mathbf{v})\\ \leq&\lambda(\overline{\nabla\cdot\mathbf{u}} - \overline{\nabla\cdot\mathbf{u}_h}, \nabla\cdot\Pi_h\mathbf{v})\\ \leq&\lambda(\nabla\cdot\mathbf{u} - \overline{\nabla\cdot\mathbf{u}_h}, \nabla\cdot\Pi_h\mathbf{v}) + \lambda(\overline{\nabla\cdot\mathbf{u}} - \nabla\cdot\mathbf{u}, \nabla\cdot\Pi_h\mathbf{v})\\ =&2\mu(\varepsilon(\mathbf{u}_h - \mathbf{u}), \varepsilon(\Pi_h\mathbf{v})) + \lambda(\overline{\nabla\cdot\mathbf{u}} - \nabla\cdot\mathbf{u}, \nabla\cdot\Pi_h\mathbf{v})\\ \lesssim&\|\mathbf{u}_h - \mathbf{u}\|_h\|\mathbf{v}\|_{H^1(\Omega)} + h\lambda\|\nabla\cdot\mathbf{u}\|_{H^1(\Omega)}\|\mathbf{v}\|_{H^1(\Omega)}\\ \lesssim&h(\|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{u}_D\|_{H^{\frac{3}{2}}(\Gamma^D)} + \|\mathbf{t}_N\|_{H^{\frac{1}{2}}(\Gamma^N)})\|\overline{\nabla\cdot\mathbf{u}} - \overline{\nabla\cdot\mathbf{u}_h}\|_{L^2(\Omega)}.\end{aligned}$$

Together with the projection inequality and Hypothesis 2, we get

$$\begin{aligned}\lambda\|\nabla\cdot\mathbf{u} - \overline{\nabla\cdot\mathbf{u}_h}\|_{L^2(\Omega)} \leq&\lambda\|\nabla\cdot\mathbf{u} - \overline{\nabla\cdot\mathbf{u}}\|_{L^2(\Omega)} + \lambda\|\overline{\nabla\cdot\mathbf{u}} - \overline{\nabla\cdot\mathbf{u}_h}\|_{L^2(\Omega)}\\ \lesssim&h\lambda\|\nabla\cdot\mathbf{u}\|_{H^1(\Omega)} + \lambda\|\overline{\nabla\cdot\mathbf{u}} - \overline{\nabla\cdot\mathbf{u}_h}\|_{L^2(\Omega)}\\ \lesssim&h(\|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{u}_D\|_{H^{\frac{3}{2}}(\Gamma^D)} + \|\mathbf{t}_N\|_{H^{\frac{1}{2}}(\Gamma^N)}).\end{aligned}$$

From Theorem 1 and (32), we get

$$\begin{aligned}\|\sigma - \sigma_h\|_{L^2(\Omega)} =&\|2\mu\varepsilon(\mathbf{u}) + \lambda\nabla\cdot\mathbf{u}\mathbf{I} - 2\mu\varepsilon(\mathbf{u}_h) - \lambda\overline{\nabla\cdot\mathbf{u}_h}\mathbf{I}\|_{L^2(\Omega)}\\ \leq&2\mu\|\varepsilon(\mathbf{u} - \mathbf{u}_h)\|_{L^2(\Omega)} + d\lambda\|\nabla\cdot\mathbf{u} - \overline{\nabla\cdot\mathbf{u}_h}\|_{L^2(\Omega)}\\ \lesssim&\|\mathbf{u} - \mathbf{u}_h\|_h + d\lambda\|\nabla\cdot\mathbf{u} - \overline{\nabla\cdot\mathbf{u}_h}\|_{L^2(\Omega)}\\ \lesssim&h(\|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{u}_D\|_{H^{\frac{3}{2}}(\Gamma^D)} + \|\mathbf{t}_N\|_{H^{\frac{1}{2}}(\Gamma^N)}).\end{aligned}$$

□

## 5. Numerical Experiments

This section presents numerical experiments on three frequently tested examples to illustrate the theoretical estimates established in the previous section and demonstrate the *locking-free* property of our finite element schemes. These schemes have been implemented respectively in our code packages `DarcyLite` (Matlab code for 2-dim problems) and `Darcy+` (C++ code for 3-dim problems). For Matlab implementation, we use those data structures and techniques discussed in [28] and `iFEM` [15].

**Example 1 (Locking-free)**. This example is adopted from Example 1 in [14] with some modifications for the divergence of displacement. It was tested in [22] by weak Galerkin finite element methods. A similar example was also tested in [30].

Here the domain is $\Omega = (0,1)^2$. We set $E = 1$ and test the example with different $\nu$ (or $\lambda$) values on a set of triangular meshes to demonstrate that the classical solver CG.$P_1^2$ suffers Poisson-locking, whereas the new solver CG.EP$_1$ is locking-free. A Neumann condition is posed on the right boundary of the domain and a Dirichlet boundary condition is specified on the other three sides. A known exact solution for the displacement is

$$\mathbf{u}(x,y) = \begin{bmatrix} (\pi/2)\sin^2(\pi x)\sin(2\pi y) \\ -(\pi/2)\sin(2\pi x)\sin^2(\pi y) \end{bmatrix} + \frac{1}{\lambda}\begin{bmatrix} \sin(\pi x)\sin(\pi y) \\ \sin(\pi x)\sin(\pi y) \end{bmatrix}.$$

TABLE 1. Example 1: CG.$P_1^2$ solver suffers locking (uniform triangular meshes).

| $h$ | $\Vert\mathbf{u}-\mathbf{u}_h\Vert_h$ | Rate | $\Vert\mathbf{u}-\mathbf{u}_h\Vert$ | Rate | $\Vert\sigma-\sigma_h\Vert$ | Rate |
|---|---|---|---|---|---|---|
| $\nu = 0.3$ or $\lambda = 0.57692$ | | | | | | |
| 1/8 | 1.9835E+00 | — | 1.8537E-01 | — | 2.2218E+00 | — |
| 1/16 | 1.0656E+00 | 0.896 | 5.6565E-02 | 1.712 | 1.2336E+00 | 0.848 |
| 1/32 | 5.4497E-01 | 0.967 | 1.5142E-02 | 1.901 | 6.3875E-01 | 0.949 |
| 1/64 | 2.7415E-01 | 0.991 | 3.8626E-03 | 1.970 | 3.2249E-01 | 0.985 |
| 1/128 | 1.3729E-01 | 0.997 | 9.7106E-04 | 1.991 | 1.6165E-01 | 0.996 |
| $\nu = 0.499$ or $\lambda = 1.6644 * 10^2$ | | | | | | |
| 1/8 | 3.7984E+00 | — | 8.5189E-01 | — | 2.2306E+01 | — |
| 1/16 | 3.3542E+00 | 0.179 | 6.6993E-01 | 0.346 | 3.1099E+01 | -0.479 |
| 1/32 | 2.6129E+00 | 0.360 | 4.2599E-01 | 0.653 | 3.2322E+01 | -0.055 |
| 1/64 | 1.7563E+00 | 0.573 | 2.0725E-01 | 1.039 | 2.6426E+01 | 0.290 |
| 1/128 | 1.0228E+00 | 0.780 | 7.5007E-02 | 1.466 | 1.7311E+01 | 0.610 |

TABLE 2. Example 1: CG.EP$_1$ solver is locking-free (uniform triangular meshes).

| $h$ | $\frac{\Vert\mathbf{u}-\mathbf{u}_h\Vert_h}{\Vert\mathbf{f}\Vert_{L_2(\Omega)}}$ | Rate | $\frac{\Vert\mathbf{u}-\mathbf{u}_h\Vert}{\Vert\mathbf{f}\Vert_{L_2(\Omega)}}$ | Rate | $\frac{\Vert\sigma-\sigma_h\Vert}{\Vert\mathbf{f}\Vert_{L_2(\Omega)}}$ | Rate |
|---|---|---|---|---|---|---|
| $\nu = 0.499$ or $\lambda = 1.6644 * 10^2$ | | | | | | |
| 1/8 | 3.3072E-02 | — | 1.4847E-03 | — | 3.1804E-02 | — |
| 1/16 | 1.6410E-02 | 1.011 | 3.6629E-04 | 2.019 | 1.5861E-02 | 1.003 |
| 1/32 | 8.1876E-03 | 1.003 | 9.1252E-05 | 2.005 | 7.9240E-03 | 1.001 |
| 1/64 | 4.0915E-03 | 1.000 | 2.2795E-05 | 2.001 | 3.9609E-03 | 1.000 |
| 1/128 | 2.0454E-03 | 1.000 | 5.6980E-06 | 2.000 | 1.9803E-03 | 1.000 |
| $\nu = 0.499999999$ or $\lambda = 1.6667 * 10^8$ | | | | | | |
| 1/8 | 3.3110E-02 | — | 1.4870E-03 | — | 3.1834E-02 | — |
| 1/16 | 1.6429E-02 | 1.011 | 3.6687E-04 | 2.019 | 1.5876E-02 | 1.003 |
| 1/32 | 8.1971E-03 | 1.003 | 9.1362E-05 | 2.005 | 7.9315E-03 | 1.001 |
| 1/64 | 4.0963E-03 | 1.000 | 2.2702E-05 | 2.008 | 3.9647E-03 | 1.000 |
| 1/128 | 2.0478E-03 | 1.000 | 5.3180E-06 | 2.093 | 1.9822E-03 | 1.000 |

It is clear that

$$\nabla\cdot\mathbf{u} = \frac{\pi}{\lambda}\sin(\pi(x+y)) = \frac{(1+\nu)(1-2\nu)}{E\nu}\pi\sin(\pi(x+y)).$$

Therefore, $\nabla\cdot\mathbf{u}\to 0$ as $\nu\to\frac{1}{2}$. The traction value on the right boundary is

$$\mathbf{t}_N(x,y) = \begin{bmatrix} -4\mu\pi x\sin(2\pi x)\sin(2\pi y) - \dfrac{2\mu}{\lambda}\pi\cos(\pi x)\sin(\pi y) - \pi\sin(\pi(x+y)) \\ -\mu\pi^2\left(\sin^2(\pi x)\cos(2\pi y) - \cos(2\pi x)\sin^2(\pi y)\right) - \dfrac{\mu}{\lambda}\pi\sin(\pi(x+y)) \end{bmatrix}.$$

Table 2 reports the numerical results for the new CG.EP$_1$ solver. For this particular problem, $\mathbf{u}_D = \mathbf{0}$, and $\|\mathbf{t}_N\|_{\frac{1}{2}}$ is a bounded quantity with respect to $\lambda$. So in Table 2 we use $\|\mathbf{f}\|_2$ to replace the bound

$$\|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{u}_D\|_{H^{\frac{3}{2}}(\Gamma^D)} + \|\mathbf{t}_N\|_{H^{\frac{1}{2}}(\Gamma^N)}$$

shown in Theorem 1,2,3. It is clear that the energy norm $\|\cdot\|_h$ exhibits 1st order convergence. The $L^2$-norm of the displacement error exhibits 2nd order convergence. In addition, the $L^2$-norm of the stress error exhibits 1st order convergence. It is also clear that these convergence rates do not deteriorate as $\lambda$ gets larger.

TABLE 3. Example 2 with $\nu = 0.3$: Errors of EP$_1$ solver on uniform triangular meshes.

| $h$ | $\|\mathbf{u}-\mathbf{u}_h\|_h$ | Rate | $\|\mathbf{u}-\mathbf{u}_h\|$ | Rate | $\|\sigma-\sigma_h\|$ | Rate |
|---|---|---|---|---|---|---|
| 1/16 | 1.4693E-03 | — | 2.1724E-07 | — | 4.7062E-01 | — |
| 1/32 | 1.0088E-03 | 0.542 | 9.4023E-08 | 1.208 | 3.2305E-01 | 0.542 |
| 1/64 | 6.9225E-04 | 0.543 | 4.1237E-08 | 1.189 | 2.2165E-01 | 0.543 |
| 1/128 | 4.7486E-04 | 0.543 | 1.8260E-08 | 1.175 | 1.5204E-01 | 0.543 |
| 1/256 | 3.2567E-04 | 0.544 | 8.1407E-09 | 1.165 | 1.0427E-01 | 0.544 |
| 1/512 | 2.2333E-04 | 0.544 | 3.6465E-09 | 1.158 | 7.1499E-02 | 0.544 |
| 1/1024 | 1.5313E-04 | 0.544 | 1.6389E-09 | 1.153 | 4.9026E-02 | 0.544 |

**Example 2 (Low regularity)**. This example is the same as Example 2 in our recent work [22]. It is derived from [3]. This example is similar to the example posed in [2] and tested in [36] (Section 9.3 therein). In particular, we consider a $\Gamma$-shaped domain $\Omega = (-1,1)^2 \setminus ([0,1]\times[-1,0])$. The physical parameters are $E = 10^5$, $\nu = 0.3$, and hence $\lambda = 57692$. The body force is $\mathbf{f} = \mathbf{0}$. A known analytical solution for the displacement is

$$\mathbf{u} = \Big[A\cos\theta - B\sin\theta,\ A\sin\theta + B\cos\theta\Big]^T, \tag{43}$$

where $(r,\theta)$ are the polar coordinates and

$$\begin{cases} A = \dfrac{r^\alpha}{2\mu}\Big(-(1+\alpha)\cos((1+\alpha)\theta) + C_1(C_2-1-\alpha)\cos((1-\alpha)\theta)\Big), \\ B = \dfrac{r^\alpha}{2\mu}\Big((1+\alpha)\sin((1+\alpha)\theta) - C_1(C_2-1+\alpha)\sin((1-\alpha)\theta)\Big). \end{cases} \tag{44}$$

Here $\alpha \approx 0.544483737$ is the so-called *critical exponent*. A Dirichlet boundary condition is posed on the whole boundary using the data derived from the exact solution.

It is known from [7, 36] that the exact solution has low regularity

$$\mathbf{u} \in H^{1+\alpha-\varepsilon}(\Omega)^2, \qquad \sigma \in H^{\alpha-\varepsilon}(\Omega)^{2\times 2}$$

for any small $\varepsilon > 0$. Furthermore, we have (for the same small $\varepsilon > 0$)

$$\mathbf{u}_D \in H^{\alpha+\frac{1}{2}-\varepsilon}(\partial\Omega)^2.$$

It can be clearly observed from Table 3 that the stress errors measured in the $L^2$-norm and the errors in the $h$-norm both have convergence order about 0.544, which is close to $\alpha$. But the displacement errors measured in the $L^2$-norm has convergence order about 1.15, which is close to $2\alpha$. This is because the domain is not convex and the solution does not have full regularity.

TABLE 4. Example 3 with $\lambda = 1$: Errors of $\mathrm{EP}_1$ on uniform tetrahedral meshes.

| $h$ | $\\|\mathbf{u} - \mathbf{u}_h\\|$ | Rate | $\\|\sigma - \sigma_h\\|$ | Rate | $\lambda\\|\nabla \cdot \mathbf{u} - \overline{\nabla \cdot \mathbf{u}_h}\\|$ | Rate |
|---|---|---|---|---|---|---|
| 1/4 | 9.840E-5 | — | 6.221E-3 | — | 1.050E-3 | — |
| 1/5 | 7.418E-4 | -9.05 | 1.649E-2 | -4.36 | 9.224E-4 | 0.580 |
| 1/8 | 3.541E-4 | 1.573 | 1.132E-2 | 0.800 | 5.997E-4 | 0.916 |
| 1/10 | 2.386E-4 | 1.769 | 9.263E-3 | 0.898 | 4.637E-4 | 1.152 |
| 1/16 | 9.878E-5 | 1.876 | 5.930E-3 | 0.948 | 2.622E-4 | 1.213 |
| 1/20 | 6.409E-5 | 1.938 | 4.769E-3 | 0.976 | 2.006E-4 | 1.200 |
| 1/32 | 2.540E-5 | 1.969 | 2.997E-3 | 0.988 | 1.165E-4 | 1.156 |
| 1/40 | 1.631E-5 | 1.985 | 2.400E-3 | 0.995 | 9.091E-5 | 1.111 |

TABLE 5. Example 3 with $\lambda = 10^3$: Errors of $\mathrm{EP}_1$ on uniform tetrahedral meshes.

| $h$ | $\\|\mathbf{u} - \mathbf{u}_h\\|$ | Rate | $\\|\sigma - \sigma_h\\|$ | Rate | $\lambda\\|\nabla \cdot \mathbf{u} - \overline{\nabla \cdot \mathbf{u}_h}\\|$ | Rate |
|---|---|---|---|---|---|---|
| 1/4 | 9.905E-4 | — | 2.160E-2 | — | 5.773E-3 | — |
| 1/5 | 7.473E-4 | 1.262 | 1.861E-2 | 0.667 | 4.836E-3 | 0.793 |
| 1/8 | 3.557E-4 | 1.579 | 1.236E-2 | 0.870 | 2.772E-3 | 1.184 |
| 1/10 | 2.389E-4 | 1.783 | 9.951E-3 | 0.971 | 2.032E-3 | 1.391 |
| 1/16 | 9.840E-5 | 1.887 | 6.221E-3 | 0.999 | 1.050E-3 | 1.404 |
| 1/20 | 6.373E-5 | 1.946 | 4.968E-3 | 1.007 | 7.777E-4 | 1.345 |
| 1/32 | 2.520E-5 | 1.974 | 3.094E-3 | 1.007 | 4.291E-4 | 1.265 |
| 1/40 | 1.617E-5 | 1.988 | 2.471E-3 | 1.007 | 3.290E-4 | 1.190 |

**Example 3 (A 3-dim problem)**. This example is adopted from [31] with some interesting modifications. Here we consider the unit cube $\Omega = (0,1)^3$. For

convenience, we introduce three auxiliary functions:

$$b_0(s) = (1-s)^2 s^2, \qquad b_1(s) = b_0'(s) = 2(1-s)s(1-2s),$$

and

$$c(x,y,z) = (1-6x+6x^2)(1-y)y(1-z)z - 3(1-x)^2x^2\Big((1-y)y + (1-z)z\Big).$$

Then we specify the displacement as

$$\mathbf{u}(x,y,z) = A \begin{bmatrix} 2\, b_0(x)\, b_1(y)\, b_1(z) \\ -\, b_1(x)\, b_0(y)\, b_1(z) \\ -\, b_1(x)\, b_1(y)\, b_0(z) \end{bmatrix} + \frac{B}{\lambda} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{45}$$

where $A, B$ are parameters for adjusting the magnitudes of the two parts in the displacement expression. Clearly, the first part is divergence-free. The second part generates a constant divergence that decays to zero as $\lambda \to \infty$. Accordingly, the body force is

$$\mathbf{f}(x,y,z) = \mu A \begin{bmatrix} -16\, c(x,y,z)(1-2y)(1-2z) \\ 8\, c(y,z,x)(1-2z)(1-2x) \\ 8\, c(z,x,y)(1-2x)(1-2y) \end{bmatrix}. \tag{46}$$

In our numerical experiments, we set $\mu = 1$ and consider $\lambda = 1$ and $\lambda = 10^3$, respectively. We set $A = B = 1$ for simplicity. A Dirichlet boundary condition is specified on the whole boundary using the known exact solution for displacement. Uniform tetrahedral meshes are used for tests. Shown in Table 4 and Table 5 are the numerical results obtained from using the $\mathrm{EP}_1$ elements. By enforcing the Dirichlet boundary conditions in a certain way, we can maintain the symmetry in the large-size sparse linear systems, so a conjugate gradient type linear solver can still be used. For simplicity, we set the maximal number of iterations as 10000 and both threshold and tolerance as $10^{-18}$. In Table 4 and Table 5, for two consecutive rows with step sizes $h_1, h_2$ and corresponding errors $E_1, E_2$, we use

$$\alpha = \log_2(E_1/E_2) \,/\, \log_2(h_1/h_2) \tag{47}$$

to calculate the convergence rate. As we refine the tetrahedral meshes, it can be observed from Table 4 and Table 5 that the convergence rates for numerical displacement, dilation (divergence of displacement), and stress are close to $2, 1, 1$, respectively. These rates are maintained as $\lambda$ is increased from 1 to $10^3$.

Note that for this example, $\|\mathbf{f}\|_2$ does not dependent on $\lambda$, $\|\mathbf{u}_D\|_{\frac{3}{2}}$ is a bounded quantity with respect to $\lambda$, and $\mathbf{t}_N$ is empty. So we ignore the bound on the RHS of the error estimates in Theorems 1, 2, & 3 when organizing the data in Table 4 & 5.

We also want to point out that as $\lambda$ gets larger, the condition number of the sparse discrete linear system gets larger and hence more iterations are needed to reach a specified accuracy. Design of efficient linear solvers and preconditioners for 3-dim nearly incompressible elasticity problems will be an interesting topic for further research.

## 6. Renovation of Other Stokes Elements for Linear Elasticity

This section examines applicability of the renovation approach developed in Section 3 to other Stokes elements. We consider four cases for triangular meshes. This allows us to put $(BR_1, P_0)$ in perspective.

*(I) The simple pair $(P_1^2, P_0)$ is unstable for Stokes problems and hence not considered for renovation.* This pair uses continuous piecewise linear vector-valued polynomials for approximation of velocity and (discontinuous) piecewise constants for approximation of pressure. This pair is known to be unstable [20] but also serves as a starting point for various enrichments.

*(II) The 1st order Bernardi-Raugel element pair $(BR_1, P_0)$ is stable for Stokes problems and its renovation for linear elasticity is the main topic of this paper.* By enriching the $P_1^2$ space with vector-valued edge bubble functions (quadratics), we obtain $\mathrm{BR}_1$. The corresponding approximation space for pressure is still $P_0$. The increment in degrees of freedom is just the number of edges in the mesh, but we get a stable pair. After renovation, it works well for linear elasticity (Theorems 1 and 2).

TABLE 6. Convergent numerical results obtained from applying Scheme (15) with Crouzeix-Raviart $(P_2^2, P_0)$ elements to Example 1 (with Dirichlet boundary conditions) on uniform triangular meshes.

| $h$ | $\|\mathbf{u}-\mathbf{u}_h\|_h$ | Rate | $\|\mathbf{u}-\mathbf{u}_h\|$ | Rate |
|---|---|---|---|---|
| $\nu = 0.499$ or $\lambda = 1.6644 * 10^2$ | | | | |
| 1/8 | 4.6164e-01 | — | 2.0411e-02 | — |
| 1/16 | 2.1957e-01 | 1.072 | 5.2398e-03 | 1.961 |
| 1/32 | 1.0869e-01 | 1.014 | 1.3346e-03 | 1.973 |
| 1/64 | 5.4336e-02 | 1.000 | 3.3687e-04 | 1.986 |
| 1/128 | 2.7199e-02 | 0.998 | 8.4618e-05 | 1.993 |
| $\nu = 0.499999999$ or $\lambda = 1.6667 * 10^8$ | | | | |
| 1/8 | 4.6173e-01 | — | 2.0420e-02 | — |
| 1/16 | 2.1963e-01 | 1.072 | 5.2434e-03 | 1.961 |
| 1/32 | 1.0872e-01 | 1.014 | 1.3357e-03 | 1.972 |
| 1/64 | 5.4354e-02 | 1.000 | 3.3717e-04 | 1.986 |
| 1/128 | 2.7208e-02 | 0.998 | 8.5772e-05 | 1.974 |

*(III) The Crouzeix-Raviart element pair $(P_2^2, P_0)$ is stable for Stokes problems and its renovation also works for linear elasticity.* This pair was discussed in [18]. Continuous elementwise quadratic polynomials are used for approximating velocity, whereas discontinuous piecewise constants are used for approximating pressure. As derscribed in Section 3, one takes the elementwise averages for the divergences of the twelve $P_2^2$ basis functions and applies the finite element scheme (15) to linear elasticity.

Note that the displacement is approximated by quadratic polynomials but the dilation is approximated by piecewise constants, one can expect only a 1st order convergence in the energy norm and a 2nd order convergence in the $L^2$-norm. This is reflected in Table 6 by the numerical results obtained from testing this renovated pair on Example 1.

Next we provide a brief theoretical explanation why the renovated Crouzeix-Raviart pair $(P_2^2, P_0)$ works for linear elasticity. Let $T$ be a triangle with vertices $a_i\,(i = 1, 2, 3)$ and $a_{i,j}$, $1 \le i < j \le 3$ be the midpoints of the edges connecting vertices $a_i$ and $a_j$. Let $\lambda_i\,(i = 1, 2, 3)$ be the barycentric coordinates. We can choose the following $P_2(T)^2$ basis functions for vertices and edges, respectively,

$$\begin{aligned}\mathbf{p}_i =& \lambda_i(2\lambda_i - 1), \qquad 1 \le i \le 3,\\ \mathbf{p}_{i,j} =& 4\lambda_i\lambda_j, \qquad 1 \le i < j \le 3.\end{aligned}$$

A local projection operator $\Pi_T : H^2(T)^2 \to P_2(T)^2$ is given by

$$(\Pi_T \mathbf{v})(a_i) = \mathbf{v}(a_i), \qquad i = 1, 2, 3, \tag{48}$$

$$\int_{[a_i,a_j]} \Pi_T \mathbf{v} = \int_{[a_i,a_j]} \mathbf{v}, \qquad 1 \le i < j \le 3. \tag{49}$$

For a triangular mesh $\mathcal{T}_h$, we define

$$V_h = \{\mathbf{v}_h \in C^0(\Omega) : \mathbf{v}_h|_T \in P_2(T)^2,\ \forall T \in \mathcal{T}_h\}.$$

Accordingly, the global projection operator $\Pi_h : H^2(\Omega)^2 \to V_h$ is defined by

$$(\Pi_h \mathbf{v})|_T = \Pi_T \mathbf{v}, \qquad \forall T \in \mathcal{T}_h.$$

Applying (49) and Green's formula, we obtain

$$\int_T \nabla \cdot (\Pi_h \mathbf{v}) = \int_{\partial T} (\Pi_h \mathbf{v}) \cdot \mathbf{n} = \int_{\partial T} \mathbf{v} \cdot \mathbf{n} = \int_T (\nabla \cdot \mathbf{v}).$$

For any $\mathbf{v} \in H^1(\Omega)^2$, it follows from the above identity that

$$\begin{aligned}\|\overline{\nabla \cdot (\mathbf{v} - \Pi_h \mathbf{v})}\|^2 &= \sum_{T \in \mathcal{T}_h} \int_T \overline{\nabla \cdot (\mathbf{v} - \Pi_h \mathbf{v})}\ \overline{\nabla \cdot (\mathbf{v} - \Pi_h \mathbf{v})}\, dT\\ &= \sum_{T \in \mathcal{T}_h} \overline{\nabla \cdot (\mathbf{v} - \Pi_h \mathbf{v})} \int_T \nabla \cdot (\mathbf{v} - \Pi_h \mathbf{v})\, dT = 0.\end{aligned}$$

Based on this, Theorems 1 and 2 can be derived in a similar way for this renovated Crouzeix-Raviart element pair.

*(IV) The MINI pair* $((P_1 + B_3)^2, P_1)$ *is stable for Stokes problems but cannot be reused with Scheme (15) for linear elasticity.* Different than $\mathrm{BR}_1$, the MINI element enriches the $P_1^2$ space by cubic bubble functions ($B_3 = \mathrm{Span}(\lambda_1\lambda_2\lambda_3)$) for element interiors [6, 19]. The matching space for pressure approximation consists of continuous piecewise linear polynomials.

It is not a surprise to see the abnormal numerical results in Table 7, which are obtained from using Scheme (15) with the MINI space to Example 1. Especially, for a large $\lambda$ value ($\lambda = 1.6667 * 10^8$), there is no convergence in the energy norm or $L^2$-norm.

TABLE 7. Abnormal numerical results obtained from applying Scheme (15) with the MINI elements to Example 1 (with Dirichlet boundary conditions) on uniform triangular meshes.

| $h$ | $\|\mathbf{u}-\mathbf{u}_h\|_h$ | $\|\mathbf{u}-\mathbf{u}_h\|$ | $\|\sigma-\sigma_h\|$ | $\|\nabla\cdot\mathbf{u}-\nabla\cdot\mathbf{u}_h\|$ |
|---|---|---|---|---|
| $\nu = 0.499$ or $\lambda = 1.6644 * 10^2$ | | | | |
| 1/8 | 3.8607e+00 | 8.8190e-01 | 8.9388e+01 | 3.7877e-01 |
| 1/16 | 3.4943e+00 | 7.2828e-01 | 5.3657e+01 | 2.2725e-01 |
| 1/32 | 2.7441e+00 | 4.5979e-01 | 4.0901e+01 | 1.7328e-01 |
| 1/64 | 1.8139e+00 | 2.1059e-01 | 3.0005e+01 | 1.2717e-01 |
| 1/128 | 1.0376e+00 | 7.2192e-02 | 1.8613e+01 | 7.8906e-02 |
| $\nu = 0.499999999$ or $\lambda = 1.6667 * 10^8$ | | | | |
| 1/8 | 4.0193e+00 | 9.5535e-01 | 8.7270e+07 | 3.7026e-01 |
| 1/16 | 4.0267e+00 | 9.6023e-01 | 4.4149e+07 | 1.8731e-01 |
| 1/32 | 4.0286e+00 | 9.6149e-01 | 2.2139e+07 | 9.3929e-02 |
| 1/64 | 4.0291e+00 | 9.6180e-01 | 1.1078e+07 | 4.6999e-02 |
| 1/128 | 4.0292e+00 | 9.6186e-01 | 5.5399e+06 | 2.3504e-02 |

Theoretically, we can also see why Scheme (15) cannot be used with the MINI space for linear elasticity. For a triangle $T$, a local projection operator $\Pi_T : H^1(T)^2 \to (P_1(T)+B_3)^2$ is defined as

$$\Pi_T\mathbf{v} = \widetilde{\Pi}_T\mathbf{v} + \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \lambda_1\lambda_2\lambda_3,$$

where $\widetilde{\Pi}_T : H^1(T)^2 \to P_1(T)^2$ is the interpolation operator, $B_3 = \mathrm{Span}\{\lambda_1\lambda_2\lambda_3\}$, and

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \frac{1}{\int_T \lambda_1\lambda_2\lambda_3 dT} \int_T (\mathbf{v} - \widetilde{\Pi}_h\mathbf{v})\, dT.$$

For a triangular mesh $\mathcal{T}_h$, we define

$$V_h = \{\mathbf{v}_h \in C^0(\Omega)^2 : \mathbf{v}_h|_T \in (P_1(T)+B_3)^2,\ \forall T \in \mathcal{T}_h\}.$$

The global projection operator $\Pi_h : H_0^1(\Omega)^2 \to V_h$ is given by

$$(\Pi_h\mathbf{v})_T = \Pi_T(\mathbf{v}|_T), \qquad \forall T \in \mathcal{T}_h.$$

The MINI element is designed with the following property

$$\int_T (\mathbf{v} - \Pi_h\mathbf{v})dT = \mathbf{0}, \quad \forall T \in \mathcal{T}_h, \quad \forall \mathbf{v} \in H_0^1(\Omega)^2.$$

What we need for the analysis in this paper to apply is rather

$$\int_T \nabla\cdot(\mathbf{v} - \Pi_h\mathbf{v})dT = 0.$$

Therefore, the MINI element is not to be used with Scheme (15) for linear elasticity, which is based on the primal formulation for displacement.

However, we want to point out that the MINI element works for elasticity in the mixed method investigated in [27].

## 7. Concluding Remarks

This paper presents CG-type finite element solvers for linear elasticity on triangular and tetrahedral meshes based on renovated Bernardi-Raugel elements. These methods provide essential enrichment to the classical linear Lagrangian elements to render them *locking-free*. These methods have 2nd order convergence in displacement and 1st order convergence in stress and dilation (divergence of displacement), when the exact solution has full regularity. Three frequently tested examples (in 2-dim and 3-dim) are examined to demonstrate the accuracy and robustness of these new solvers.

There are many other higher order stable element pairs for Stokes flow, e.g., Taylor-Hood $(P_2^2, P_1)$ for triangles. It is interesting to see whether and how these element pairs can be reused for linear elasticity. This is currently under our investigation and will be reported in our future work.

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