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AN EFFICIENT NONLINEAR SOLVER FOR STEADY MHD BASED ON ALGEBRAIC SPLITTING

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Abstract. We propose a new, efficient, nonlinear iteration for solving the steady incompressible MHD equations. The method consists of a careful combination of an incremental Picard iteration, Yosida splitting, and a grad-div stabilized finite element discretization. At each iteration, the Schur complement remains the same, is SPD, and can be easily and effectively preconditioned with the pressure mass matrix. Furthermore, this method decouples the block Schur complement into 2 simple Stokes Schur complement. We show that the iteration converges linearly to the discrete MHD system solution, both analytically and numerically. Several numerical tests are given which reveal very good convergence properties, and excellent results on a benchmark problem.

Key words. Steady MHD, algebraic splitting, incremental Picard Yosida method, nonlinear solver.

1. Introduction

Magnetohydrodynamics (MHD) describes the flow of electronically conducting fluids in a magnetic field, which arises in a wide range of applications such as geophysics and astrophysics [2, 3, 5, 6, 10]. We herein develop an efficient nonlinear iteration scheme to solve steady MHD in a convex domain Ω, which is given by

$$-\nu\Delta u + (u\cdot\nabla)u - s(B\cdot\nabla)B + \nabla\bar{p} = f, \tag{1}$$

$$\nabla\cdot u = 0, \tag{2}$$

$$-\nu_m\Delta B + (u\cdot\nabla)B - (B\cdot\nabla)u - \nabla\lambda = \nabla\times g, \tag{3}$$

$$\nabla\cdot B = 0, \tag{4}$$

where u is the velocity of fluid, $\bar{p}$ is a modified pressure, B is the magnetic field, λ is a variable acting as a Lagrange multiplier corresponding to the solenoidal constraint on the magnetic field, f is the body forcing, $\nabla\times g$ is the forcing on the magnetic field B, s is a coupling number, ν is the kinematic viscosity and ν_m is the magnetic diffusivity. For simplicity we consider homogeneous Dirichlet boundary conditions for both u and B and $s = 1$. With minor changes, our analysis will also hold for inhomogeneous or periodic boundaries, as well as no slip velocity together with $B\cdot n = 0$ and $(\nabla\times B)\times n = 0$ (in this case the Maxwell equation uses the curl curl form of the dissipation term).

Although MHD couples the Navier-Stokes equations (NSE) for fluids to Maxwell's equations for electromagnetics, the linear systems that arise have similar structure to those of the NSE, but in block form. Using Picard's method to solve steady MHD equations requires solving a linear saddle point system in each iteration. Difficulties arise when solving such saddle point systems, such as how to build an efficient preconditioner for iterative linear solvers for large problems [4], and how to derive a robust error estimator [11]. Several approximations are made to solving this saddle point linear system. The linear algebra problem is actually worse in the steady case, than the time dependent case, since one cannot take advantage of

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traditional splitting methods such as projection methods, or lag nonlinear terms in a temporal discretization.

We herein propose a method to solve the nonlinear system (1)-(4) based on an algebraic splitting method shown to work very well for a NSE system in [13], which will require much easier linear system solvers than standard nonlinear solvers do. It is based on a careful combination of an incremental Picard iteration, grad-div stabilization, and algebraic splitting of Yosida-Type. A derivation of the method is given below.

The standard Picard iteration scheme for (1)-(4) is given below: Guess u_0, B_0 and for $k = 1, 2, \dots$, and find u_k, p_k, B_k, λ_k satisfying

$$-\gamma\nabla(\nabla\cdot u_k) + u_{k-1}\cdot\nabla u_k - B_{k-1}\cdot\nabla B_k + \nabla p_k - \nu\Delta u_k = f, \tag{5}$$

$$\nabla\cdot u_k = 0, \tag{6}$$

$$-\gamma_m\nabla(\nabla\cdot B_k) + u_{k-1}\cdot\nabla B_k - B_{k-1}\cdot\nabla u_k - \nabla\lambda_k - \nu_m\Delta B_k = \nabla\times g, \tag{7}$$

$$\nabla\cdot B_k = 0. \tag{8}$$

Although $\nabla(\nabla\cdot u_k) = \nabla(\nabla\cdot B_k) = 0$ due to (6) and (8), when discretized with common finite element choices, such as Taylor-Hood, we only have weak enforcement of (6) and (8). Thus parameters γ, γ_m penalize the divergence error of numerical solutions. Notice these grad-div stabilization terms can make problem unstable if gammas are too large as they are singular. In practice, $\gamma, \gamma_m \sim 1$ are close to optimal parameters. Adding increments $-\nabla p_{k-1}$ and $\nabla\lambda_{k-1}$ on both sides of (5) and (7) respectively, gives an incremental Picard iteration:

$$-\gamma\nabla(\nabla\cdot u_k) + u_{k-1}\cdot\nabla u_k - B_{k-1}\cdot\nabla B_k + \nabla\delta_k^p - \nu\Delta u_k = f - \nabla p_{k-1}, \tag{9}$$

$$\nabla\cdot u_k = 0, \tag{10}$$

$$-\gamma_m\nabla(\nabla\cdot B_k) + u_{k-1}\cdot\nabla B_k - B_{k-1}\cdot\nabla u_k - \nabla\delta_k^\lambda - \nu_m\Delta B_k = \nabla\times g + \nabla\lambda_{k-1}, \tag{11}$$

$$\nabla\cdot B_k = 0. \tag{12}$$

which is equivalent to the usual Picard iteration (assuming $p_0 = \lambda_0 = 0$). After applying a finite element discretization to (9)-(12), a block linear system arises at each iteration in the form

$$\begin{pmatrix} A_1 & N_1 & C_1^T & 0 \\ N_1 & A_2 & 0 & C_1^T \\ C_1 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{B} \\ \bar{\delta^p} \\ \bar{\delta^\lambda} \end{pmatrix} = \begin{pmatrix} \tilde{F}_1 \\ \tilde{F}_2 \\ 0 \\ 0 \end{pmatrix}, \tag{13}$$

where A_1 and A_2 consist of a stiffness matrix K, divergence matrix E and contributions of convection terms, N_1 is the contribution of convection term from Maxwell's equation, and C_1 is a rectangular matrix coming from (10) or (12). The *bar* notation denotes the coefficient vectors corresponding to the associated finite element functions. This block linear system, just like for the NSE, takes the form of a saddle point system

$$\begin{pmatrix} A & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} = \begin{pmatrix} \tilde{F} \\ 0 \end{pmatrix}, \tag{14}$$

if we set $A = \begin{pmatrix} A_1 & N_1 \\ N_1 & A_2 \end{pmatrix}$, $C = \begin{pmatrix} C_1 & 0 \\ 0 & C_1 \end{pmatrix}$, $\bar{X} = \begin{pmatrix} \bar{u} \\ \bar{B} \end{pmatrix}$, $\bar{Y} = \begin{pmatrix} \bar{\delta^p} \\ \bar{\delta^\lambda} \end{pmatrix}$, $\tilde{F} = \begin{pmatrix} \tilde{F}_1 \\ \tilde{F}_2 \end{pmatrix}$. Such a system is well known to be very difficult to solve. Direct solvers are not effective for these systems, except for small problems, and typical approaches use some decomposition of the saddle point matrix. The main difficulty in the decomposition is the need to perform a linear solve with the Schur complement matrix $S := C^T A^{-1} C$, and is especially hard for large, non-symmetric systems. It is an open problem to obtain robust solvers that work in a wide variety of applications for such linear systems.

Our approach herein is to avoid such solvers through approximation methods, but with then approximation error going to 0 as the nonlinear iteration converges. That is, instead of solving (14), we perform an inexact LU decomposition of the first block matrix, and obtain

$$\begin{aligned}\begin{pmatrix} A & C^T \\ C & 0 \end{pmatrix} &\approx \begin{pmatrix} A & 0 \\ C & -C\tilde{A}^{-1}C^T \end{pmatrix} \begin{pmatrix} I & A^{-1}C^T \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} A & C^T \\ C & C(A^{-1} - \tilde{A}^{-1})C^T \end{pmatrix},\end{aligned} \tag{15}$$

where $\tilde{A} = \begin{pmatrix} \nu K + \gamma E & 0 \\ 0 & \nu_m K + \gamma E \end{pmatrix}$ is an approximation of A that is symmetric, positive and definite (SPD), and thus so is the approximate Schur complement $\tilde{S} := C\tilde{A}^{-1}C^T$. Moreover, $\tilde{S}$ stays the same at each iteration (while the usual Picard iteration needs to update at each iteration) and *decouples into two Stokes Schur complements*. With this benefit, we only have to build preconditioners once, and can use robust and efficient iterative solvers. One important feature of IPY method is the incremental MHD formulation. Without it, one may still apply Yosida method to the stabilized MHD formula, but it will only converge to solutions that do not preserve divergence-free property.

This paper is arranged as follows. In section 2, we give mathematical preliminaries and notation used throughout. Section 3 analyzes the proposed incremental Picard-Yosida method (IPY), and section 4 presents two numerical tests that illustrate the effectiveness of the method.

2. Mathematical preliminaries

We consider a convex domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) that is open, connected, and either with Lipschitz boundary $\partial\Omega$ or with Ω being a polygon. The $L^2(\Omega)$ norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot, \cdot)$, and $L_0^2(\Omega)$ denotes the zero mean subspace of $L^2(\Omega)$. Throughout this paper, it is understood by context whether a particular space is scalar or vector valued, and so we do not distinguish notation.

The natural function spaces for our problem are

$$X := \left(H_0^1(\Omega)\right)^d, \qquad Q := L_0^2(\Omega).$$

In the space X, the Poincare inequality is known to hold: There exists $\lambda > 0$, dependent only on the size of Ω, such that for every $v \in X$,

$$\|v\| \leq \lambda \|\nabla v\|.$$

The dual space of X will be denoted by X^*, with norm $\|\cdot\|_{-1}$.

Let τ_h be a conforming, shape-regular, and simplicial triangulation of Ω with h_T denoting the maximum element diameter. We denote with P_k the space of degree k globally continuous piecewise polynomials with respect to τ_h, and P_k^{disc} the space of degree k piecewise polynomials that can be discontinuous across elements.

Throughout the paper, we consider only discrete spaces $(X_h,\ Q_h) \subset (X, Q)$ that satisfy the LBB condition: there exists a constant β, independent of h, satisfying

$$\inf_{q\in Q_h}\sup_{v\in X_h}\frac{(\nabla\cdot v, q)}{\|q\|\|\nabla v\|}\geq\beta>0. \tag{16}$$

Common examples of such elements are (P_2, P_1) Taylor-Hood (TH) elements, and (P_k, P^{disc}_{k-1}) Scott-Vogelius (SV) elements on meshes with particular structure [15, 1, 9], and see [7, 8].

Define the discretely divergence free velocity space by

$$V_h := \{v \in X_h,\ (\nabla\cdot v, q) = 0,\ \forall q \in Q_h\},$$

and the nonlinear form

$$b^*(u, v, w) = \frac{1}{2}(u\cdot\nabla v, w) - \frac{1}{2}(u\cdot\nabla w, v).$$

Lemma 2.1. *There exists a finite constant $M > 0$ depending only on Ω such that*

$$|b^*(u, v, w)| \leq M\|\nabla u\|\|\nabla v\|\|\nabla w\|, \tag{17}$$

for every $u, v, w \in X$. Moreover, we have the following property:

$$b^*(u, v, w) = -b^*(u, w, v), \quad \textit{for all } u, v, w \in X. \tag{18}$$

We omit the proof of this lemma, but note it can be easily verified by using Holder's inequality, Sobolev inequalities and the Poincare inequality [12].

2.1. Discrete steady MHD. The discrete steady MHD system takes the form: find $(u, \bar p, B, \lambda) \in (X_h, Q_h, X_h, Q_h)$ satisfying for all $(v, q, w, r) \in (X_h, Q_h, X_h, Q_h)$,

$$\gamma(\nabla\cdot u,\nabla\cdot v)+\nu(\nabla u,\nabla v)+b^*(u,u,v)-b^*(B,B,v)-(\bar p,\nabla\cdot v)=(f,v), \tag{19}$$

$$(\nabla\cdot u, q) = 0, \tag{20}$$

$$\gamma_m(\nabla\cdot B,\nabla\cdot w)+\nu_m(\nabla B,\nabla w)+b^*(u,B,w)-b^*(B,u,w) +(\lambda,\nabla\cdot w)=(\nabla\times g,w), \tag{21}$$

$$(\nabla\cdot B, r) = 0. \tag{22}$$

Lemma 2.2. *Solutions to* (19)-(22) *exist and satisfy*

$$\|\nabla u\|^2 + \|\nabla B\|^2 \leq \mathcal{K}^{-2}\left(\|f\|^2_{-1} + |\nabla\times g\|^2_{-1}\right). \tag{23}$$

If

$$\alpha := M\mathcal{K}^{-2}\sqrt{\|f\|^2_{-1} + \|\nabla\times g\|^2_{-1}} < \frac{1}{2},$$

where $\mathcal{K} := \min\{\nu, \nu_m\}$, then they are also unique.

Proof. For simplicity, we set $\gamma = \gamma_m = 0$. One can easily extend the proof to the case of $\gamma, \gamma_m > 0$. There are 3 steps to finish this proof. First we show that the solutions are bounded by data if they exist. Then we give existence and uniqueness.

We begin by choosing $v = u, w = B$ in (19)-(22), which vanishes pressure and lambda terms, and leaves

$$\nu\|\nabla u\|^2 = b^*(B, B, u) + (f, u),$$
$$\nu_m\|\nabla B\|^2 = b^*(B, u, B) + (\nabla\times g, B).$$

Adding these equations cancels the nonlinear terms, and then applying Young's inequality produces

$$\nu\|\nabla u\|^2 + \nu_m\|\nabla B\|^2 \leq \nu^{-1}\|f\|^2_{-1} + \nu_m^{-1}\|\nabla\times g\|^2_{-1}.$$

Multiplying $\mathcal{K}^{-1}$ by both sides gets

$$\|\nabla u\|^2 + \|\nabla B\|^2 \le \mathcal{K}^{-2}\left(\|f\|_{-1}^2 + \|\nabla \times g\|_{-1}^2\right). \tag{24}$$

Next, we show the solution's existence using the Leray-Schauder fixed point theorem. Define $T : X_h^* \times X_h^* \to V_h \times V_h$ to be the solution operator of the Stokes problem, where X_h^* is the dual space of X_h. Specifically, $T(\hat{f}, \hat{g}) = (u, B)$ where $(u, B) \in V_h \times V_h$ solves

$$\begin{aligned} -\nu(\nabla u, \nabla v) &= (\hat{f}, v), \\ -\nu_m(\nabla B, \nabla w) &= (\hat{g}, w), \end{aligned}$$

for all $v, w \in V_h$. The Lax-Milgram theorem ensures T exists and is bounded:

$$\|T\| := \sup_{\hat{f}, \hat{g} \in X^*} \frac{\|T(\hat{f}, \hat{g})\|_X}{\|(\hat{f}, \hat{g})\|_*} = \sup_{\hat{f}, \hat{g} \in X_h^*} \frac{\|\nabla u\|}{\|\hat{f}\|_*} \cdot \frac{\|\nabla B\|}{\|\hat{g}\|_*} \le \nu^{-1}\nu_m^{-1} \le \mathcal{K}^{-2}.$$

Define the nonlinear operator $N : X_h \times X_h \to X_h^* \times X_H^*$ such that

$$N\begin{pmatrix} u \\ B \end{pmatrix} = \begin{pmatrix} f - u \cdot \nabla u + B \cdot \nabla B \\ \nabla \times g - u \cdot \nabla B + B \cdot \nabla u \end{pmatrix}, \tag{25}$$

which is continuous and bounded. It is enough to show that the first component in (25) is continuous and bounded.

$$\begin{aligned} \|f - u \cdot \nabla u + B \cdot \nabla B\|_{-1} &\le \|f\|_{-1} + M\left(\|\nabla u\|^2 + \|\nabla B\|^2\right) \\ &\le \|f\|_{-1} + M\mathcal{K}^{-2}\left(\|f\|_{-1}^2 + \|\nabla \times g\|_{-1}^2\right) \\ &< \infty, \end{aligned}$$

thanks to (24). Hence the operator N is bounded.

For continuity, assume that we have a convergent sequence $(u_n, B_n) \to (u, B)$ in H^1. Then, the first component of $N\begin{pmatrix} u_n \\ B_n \end{pmatrix} - N\begin{pmatrix} u \\ B \end{pmatrix}$ is

$$\begin{aligned} &\| - u_n \cdot \nabla u_n + B_n \cdot \nabla B_n - (u \cdot \nabla u + B \cdot \nabla B)\| \\ =&\| - (u_n - u) \cdot \nabla u_n - u \cdot \nabla(u_n - u) + B \cdot \nabla(B_n - B) + (B_n - B) \cdot \nabla B_n\| \\ \le& M\|\nabla(u_n - u)\|\|\nabla u_n\| + M\|\nabla u\|\|\nabla(u_n - u)\| \\ &+ M\|\nabla B\|\|\nabla(B_n - B)\| + M\|\nabla(B_n - B)\|\|\nabla B_n\| \\ \to& 0, \quad as \quad n \to \infty. \end{aligned}$$

Similarly, one can show that the second component goes to zero as n increases. Therefore, $T \circ N$ is compact. Note that the solution of (19)-(22) is a fixed point of

$$T\left(N\begin{pmatrix} u \\ B \end{pmatrix}\right) = \begin{pmatrix} u \\ B \end{pmatrix}.$$

By the Leray-Schauder fixed point theorem, thanks to the priori solution bound, there exists a solution to (19) -(22) .

Now we show uniqueness. Let $(u_1, p_1, B_1, \lambda_1)$ and $(u_2, p_2, B_2, \lambda_2)$ be the solutions of (19)-(22). Denote $e_u = u_1 - u_2$, $e_B = B_1 - B_2$, $e_p = p_1 - p_2$, $e_\lambda = \lambda_1 - \lambda_2$.

Subtracting the discretized MHD system with these two solutions, we have for $v, w, \in X_h$

$$\begin{aligned}\nu(\nabla e_u, \nabla v) - (e_p, \nabla \cdot v)\\ +b^*(u_1, e_u, v) + b^*(e_u, u_2, v) - b^*(B_1, e_B, v) - b^*(e_B, B_2, v) =&0,\\ (\nabla \cdot e_u, q) =&0,\end{aligned} \tag{26}$$

$$\begin{aligned}\nu_m(\nabla e_B, \nabla w) + (e_\lambda, \nabla \cdot w)\\ +b^*(u_1, e_B, w) + b^*(e_u, B_2, w) - b^*(B_1, e_u, w) - b^*(e_B, u_2, w) =&0,\\ (\nabla \cdot e_B, r) =&0.\end{aligned} \tag{27}$$

Setting $v = e_u, w = e_B$ vanishes the second and third terms in (26), (27), leaving

$$\begin{aligned}\nu\|\nabla e_u\|^2 + b^*(e_u, u_2, e_u) - b^*(B_1, e_B, e_u) - b^*(e_B, B_2, e_u) =&0,\\ \nu_m\|\nabla e_B\|^2 + b^*(e_u, B_2, e_B) - b^*(B_1, e_u, e_B) - b^*(e_B, u_2, e_B) =&0.\end{aligned}$$

Adding these equations vanishes the fourth terms in each equation and produces

$$\begin{aligned}\nu\|\nabla e_u\|^2 + \nu_m\|\nabla e_B\|^2 \le& M\|\nabla e_u\|^2\|\nabla u_2\| + M\|\nabla e_B\|\|\nabla e_u\|\|\nabla B_2\|\\ &+ M\|\nabla e_u\|\|\nabla e_B\|\|\nabla B_2\| + M\|\nabla e_B\|^2\|\nabla u_2\|,\end{aligned} \tag{28}$$

which reduces to

$$\nu(1-\alpha)\|\nabla e_u\|^2 + \nu_m(1-\alpha)\|\nabla e_B\|^2 \le 2M\|\nabla e_B\|\|\nabla e_u\|\|\nabla B_2\|,$$

thanks to the definition of α.

Applying Young's inequality and (24) now yields

$$\begin{aligned}&\nu(1-\alpha)\|\nabla e_u\|^2 + \nu_m(1-\alpha)\|\nabla e_B\|^2\\ \le&\nu^{-1}(1-\alpha)^{-1}M^2\left(\|\nabla e_u\|^2 + \|\nabla e_B\|^2\right)\|\nabla B_2\|^2\\ \le&(1-\alpha)^{-1}\mathcal{K}\alpha^2\left(\|\nabla e_u\|^2 + \|\nabla e_B\|^2\right),\end{aligned}$$

which is equivalent to

$$\frac{\mathcal{K}(1-2\alpha)}{1-\alpha}\left(\|\nabla e_u\|^2 + \|\nabla e_B\|^2\right) \le 0. \tag{29}$$

Thus $e_u = e_B = 0$ for $0 < \alpha < \frac{1}{2}$. By the inf-sup condition and (26), (27), one can easily get $e_p = e_\lambda = 0$ as well. Therefore we have proved the uniqueness of (19) - (22), provided $\alpha < \frac{1}{2}$. ☐

3. The incremental Picard-Yosida iteration

In this section, we present the (usual) Picard iteration and our new incremental Picard-Yosida iteration, for steady MHD. We analytically show these methods converge to the solution of the discretized coupled nonlinear system (19)-(22).

3.1. Usual Picard iteration. We now present the Picard iteration for MHD.

Algorithm 3.1. *The usual Picard iteration for steady MHD takes the form*

Step 1: Guess $u_0, B_0 \in X_h$.
Step k: Find $(u_k, p_k, B_k, \lambda_k) \in (X_h, Q_h, X_h, Q_h)$ *satisfying for all* $(v, q, w, r) \in$

(X_h, Q_h, X_h, Q_h),

$$b^*(u_{k-1}, u_k, v) - b^*(B_{k-1}, B_k, v) - (p_k, \nabla\cdot v) + \nu(\nabla u_k, \nabla v) + \gamma(\nabla\cdot u_k, \nabla\cdot v) = (f, v), \tag{30}$$

$$(\nabla\cdot u_k, q) = 0, \tag{31}$$

$$b^*(u_{k-1}, B_k, w) - b^*(B_{k-1}, u_k, w) + (\lambda_k, \nabla\cdot w) + \nu_m(\nabla B_k, \nabla w) + \gamma_m(\nabla\cdot B_k, \nabla\cdot w) = (\nabla\times g, w), \tag{32}$$

$$(\nabla\cdot B_k, r) = 0. \tag{33}$$

The corresponding block matrix system for (30)-(33) is given by

$$\begin{pmatrix} A & C^T \\ C & 0 \end{pmatrix}\begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}.$$

With LU decomposition, we obtain

$$\begin{pmatrix} A & 0 \\ C & -CA^{-1}C^T \end{pmatrix}\begin{pmatrix} I & A^{-1}C^T \\ 0 & I \end{pmatrix}\begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

and this reduces to the following 3 step linear solve process::

Step 1: Find $\bar{Z}$ satisfying $A\bar{Z} = F$,
Step 2: Find $\bar{Y}$ satisfying $C\bar{Z} - CA^{-1}C^T\bar{Y} = 0$,
Step 3: Find $\bar{X}$ satisfying $A\bar{X} + C^T\bar{Y} = F$.

The equivalent finite element systems are:

Step 1: Find $(\hat{z}_k, \hat{\mu}_k)$ satisfies

$$\begin{aligned}(\gamma D + \nu K + N(u_{k-1}))\,\hat{z}_k - N(B_{k-1})\hat{\mu}_k &= \hat{f},\\ (\gamma_m D + \nu_m K + N(u_{k-1}))\,\hat{\mu}_k - N(B_{k-1})\hat{z}_k &= \hat{g},\end{aligned}$$

Step 2: Find $(\hat{\chi}, \hat{p}_k, \hat{\psi}, \hat{\lambda}_k)$ satisfying for

$$\begin{aligned}(\gamma D + \nu K + N((u_{k-1}))\hat{\chi} - N(B_{k-1})\hat{\psi} + C_1^T\hat{p}_k &= 0,\\ (\gamma D + \nu K + N(u_{k-1})\hat{\psi} - N(B_{k-1})\hat{\chi} + C_1^T\hat{\lambda}_k &= 0,\\ C_1\hat{\chi} &= -C_1\hat{z}_k,\\ C_1\hat{\psi} &= -C_1\hat{\nu}_k.\end{aligned}$$

Step 3: Find $(\hat{u}_k, \hat{B}_k)$ satisfies

$$\begin{aligned}(\gamma D + \nu K + N(u_{k-1}))\,\hat{u}_k - N(B_{k-1})\hat{B}_k &= \hat{f},\\ (\gamma_m D + \nu_m K + N(u_{k-1}))\,\hat{B}_k - N(B_{k-1})\hat{u}_k &= \hat{g},\end{aligned}$$

where D represent the grad-div matrix, K is the stiffness matrix, and $N(\cdot)$ arises from the nonlinear convection terms. In step 2, the difficulty is to solve the fully coupled Schur complement, which is essentially system (30)-(33) with different RHS.

Lemma 3.2. *The usual Picard iteration converges linearly (with contraction number at most α) to the unique solution of* (19)-(22)*, provided the small data condition*

$$\alpha := M\mathcal{K}^{-2}(\|f\|_{-1}^2 + \|\nabla\times g\|_{-1}^2)^{-1/2} < \frac{1}{2}$$

is satisfied, where $\mathcal{K} := \min\{\nu, \nu_m\}$.

Proof. We begin by showing the stability of Algorithm 3.1. Letting $v = u_k$ in (30) of Step k vanishes the first term and pressure term, and provides

$$\gamma\|\nabla \cdot u_k\|^2 + \frac{\nu}{2}\|\nabla u_k\|^2 \leq b^*(B_{k-1}, B_k, u_k) + \frac{\nu^{-1}}{2}\|f\|_{-1}^2,$$

thanks to Young's inequality. Similarly, we have the following bound by setting $w = B_k$ in (32),

$$\gamma_m\|\nabla \cdot u_k\|^2 + \frac{\nu_m}{2}\|\nabla u_k\|^2 \leq b^*(B_{k-1}, u_k, B_k) + \frac{\nu_m^{-1}}{2}\|\nabla \times g\|_{-1}^2.$$

Adding the two inequalities above and applying (18), produces

$$\|\nabla u_k\|^2 + \|\nabla B_k\|^2 \leq \mathcal{K}^{-2}\left(\|f\|_{-1}^2 + \|\nabla \times g\|_{-1}^2\right), \tag{34}$$

thanks to the definition of $\mathcal{K}$.

Next, we show the usual Picard iteration converges. Denote $\delta_k := u_k - u_{k-1}$, $\delta_k^B := B_k - B_{k-1}$, $\delta_k^p := p_k - p_{k-1}$, $\delta_k^\lambda := \lambda_k - \lambda_{k-1}$.

Subtracting two successive iterations of algorithm 3.1 at Steps k and $k-1$ yields

(35)
$$\begin{aligned}\gamma(\nabla \cdot \delta_k, \nabla \cdot v) + \nu(\nabla \delta_k, \nabla v) =&(\delta_k^p, \nabla \cdot v) - b^*(\delta_{k-1}, u_{k-1}, v) - b^*(u_{k-1}, \delta_k, v)\\ &+ b^*(B_{k-1}, \delta_k^B, v) + b^*(\delta_{k-1}^B, B_{k-1}, v),\end{aligned}$$

$$(\nabla \cdot \delta_k, q) = 0, \tag{36}$$

(37)
$$\begin{aligned}\gamma_m(\nabla \cdot \delta_k^B, \nabla \cdot w) + \nu_m(\nabla \delta_k^B, \nabla w) =& -(\delta_k^\lambda, \nabla \cdot w) - b^*(\delta_{k-1}, B_{k-1}, w)\\ &- b^*(u_{k-1}, \delta_k^B, w) + b^*(B_{k-1}, \delta_k, w)\\ &+ b^*(\delta_{k-1}^B, u_{k-1}, w),\end{aligned}$$

$$(\nabla \cdot \delta_k^B, r) = 0. \tag{38}$$

Letting $v = \delta_k$ in (35) and $w = \delta_k^B$ in (37) vanishes the pressure term, Lagrange term, and the fifth term in both (35),(37). Then adding them vanishes the sixth terms on both equations, and gives the following bound after dropping the two grad-div stabilization terms,

$$\begin{aligned}&\nu\|\nabla \delta_k\|^2 + \nu_m\|\nabla \delta_k^B\|^2\\ \leq& 2M^2(\nu^{-1}\|f\|_{-1}^2 + \nu_m^{-1}\|\nabla \times g\|_{-1}^2)(\|\nabla \delta_{k-1}\|^2 + \|\nabla \delta_{k-1}^B\|^2),\end{aligned}$$

and thus

$$\|\nabla \delta_k\|^2 + \|\nabla \delta_k^B\|^2 \leq 2\alpha^2(\|\nabla \delta_{k-1}\|^2 + \|\nabla \delta_{k-1}^B\|^2),$$

thanks to the Young's inequality, (17), (18), (34), and the definition of α.

Since $\alpha < \frac{1}{2}$ by assumption, the sequence $\{\|\nabla \delta_k\|^2 + \|\nabla \delta_k^B\|^2\}_k^\infty$ is contractive and therefore the usual Picard iteration is convergent to some $(\tilde{u}, \tilde{p}, \tilde{B}, \tilde{\lambda}) \in (X_h, Q_h, X_h, Q_h)$. Because of the inf-sup condition, we have that $(\tilde{u}, \tilde{p}, \tilde{B}, \tilde{\lambda})$ satisfies (19) and (21). Moreover, $u_k, B_k \in V_h$ implies their limits $\tilde{u}, \tilde{B}$ are also inside V_h and thus $(\tilde{u}, \tilde{p}, \tilde{B}, \tilde{\lambda})$ must be the unique solution of (19)-(22). □

3.2. The incremental Picard-Yosida iteration. The usual Picard algorithm is classical and easy to implement, however, as discussed above, it suffers problems in that it requires solving large nonsymmetric Schur complement linear system at each iteration.

The main difficulties of the Picard iteration come from solving the Step 2 (Schur complement) linear system, since these Schur complements are large and non-symmetric. Furthermore, it is updated at each iteration, so we potentially need to build new preconditioners at every iteration. Due to these difficulties, we now propose an alternative method. This procedure is formalized in the following algorithm. We prove below that it converges linearly to the steady MHD solution (19)-(22).

Algorithm 3.3. *The incremental Picard-Yosida iteration for the steady MHD is defined by:*

Step 1: Guess $(u_0, p_0, B_0, \lambda_0) \in (X_h, Q_h, X_h, Q_h)$.

Step k consists of the following 3 steps:

k.1 Find $z_k, \mu_k \in X_h$ *satisfying for all* $v, w \in X_h$,

$$\begin{aligned}\gamma(\nabla\cdot z_k, \nabla\cdot v) + b^*(u_{k-1}, z_k, v) - b^*(B_{k-1}, \mu_k, v) + \nu(\nabla z_k, \nabla v)\\ =(f, v) + (p_{k-1}, \nabla\cdot v),\\ \gamma_m(\nabla\cdot \mu_k, \nabla\cdot w) + b^*(u_{k-1}, \mu_k, w) - b^*(B_{k-1}, z_k, w) + \nu_m(\nabla\mu_k, \nabla w)\\ =(\nabla\times g, w) - (\lambda_{k-1}, \nabla\cdot w).\end{aligned}$$

k.2 Find $(\chi_k, \delta_k^p, \psi, \delta_\lambda^k) \in (X_h, Q_h, X_h, Q_h)$ *satisfying for all* $(v, q, w, r) \in (X_h, Q_h, X_h, Q_h)$,

$$\begin{aligned}\gamma(\nabla\cdot\chi_k, \nabla\cdot v) - (\delta_k^p, \nabla\cdot v) + \nu(\nabla\chi_k, \nabla v) &=0,\\ (\nabla\cdot\chi_k, q) &= -(\nabla\cdot z_k, q),\\ \gamma_m(\nabla\cdot\psi_k, \nabla\cdot w) + (\delta_k^\lambda, \nabla\cdot w) + \nu(\nabla\psi_k, \nabla w) &=0,\\ (\nabla\cdot\psi_k, r) &= -(\nabla\cdot\mu_k, r)\end{aligned}$$

k.3 Set $p_k := p_{k-1} + \delta_k^p$, $\lambda_k := \lambda_{k-1} + \delta_k^\lambda$ *and then find* $u_k, B_k \in X_h$ *satisfying for all* $v, w \in X_h$,

$$\begin{aligned}\gamma(\nabla\cdot u_k, \nabla\cdot v) + b^*(u_{k-1}, u_k, v) - b^*(B_{k-1}, B_k, v) + \nu(\nabla u_k, \nabla v)\\ =(f, v) + (p_k, \nabla\cdot v),\\ \gamma_m(\nabla\cdot B_k, \nabla\cdot w) + b^*(u_{k-1}, B_k, w) - b^*(B_{k-1}, u_k, w) + \nu_m(\nabla B_k, \nabla w)\\ =(\nabla\times g, w) - (\lambda_k, \nabla\cdot w).\end{aligned}$$

This yields the algebraic formulation

$$\begin{aligned}(\gamma D + \nu K + N(u_{k-1}))\,\hat z_k - N(B_{k-1})\hat\mu_k &=\hat f + C^T\hat p_{k-1},\\ (\gamma_m D + \nu_m K + N(u_{k-1}))\,\hat\mu_k - N(B_{k-1})\hat z_k &=\hat g - C^T\hat\lambda_{k-1},\\ C(\nu K + \gamma D)^{-1}C^T\hat\delta_k^p &= -C\hat z_k,\\ C(\nu_m K + \gamma_m D)^{-1}C^T\hat\delta_k^\lambda &= -C\hat\mu_k,\\ \hat p_k &=\hat\delta_k^p + \hat p_{k-1},\\ \hat\lambda_k &=\hat\delta_k^\lambda + \hat\lambda_{k-1},\\ (\gamma D + \nu K + N(u_{k-1}))\,\hat u_k - N(B_{k-1})\hat B_k &=\hat f + C^T\hat p_k,\\ (\gamma_m D + \nu_m K + N(u_{k-1}))\,\hat B_k - N(B_{k-1})\hat u_k &=\hat g - C^T\hat\lambda_k,\end{aligned}$$

where the Schur complement decouples into 2 SPD Stokes Schur complements, which are the same at each iteration, and are easily and robustly solvable with CG as inner and outer solvers, and standard preconditioners.

Theorem 3.4. *Assume the initial guess satisfies* $\mathcal{K}\|\nabla(u-u_0)\|^2+\mathcal{K}\|\nabla(B-B_0)\|^2+\gamma^{-1}\|\bar{p}-p_0\|^2+\gamma_m^{-1}\|\lambda-\lambda_0\| \le \|\nabla u\|^2+\|\nabla B\|^2$ *and* $\gamma \ge \nu, \gamma_m \ge \nu_m$, *where* $(u,\bar{p},B,\lambda)$ *is the solution of* (19)*-*(22)*, and* (u_0,p_0,B_0,λ_0) *to be the initial guess. Let* $(u_k,p_k,B_k,\lambda_k) \in (X_h,Q_h,X_h,Q_h)$ *be the Step k solution of Algorithm 3.3. Then it converges linearly (with contraction ratio at most* α*) to* $(u,\bar{p},B,\lambda)$*, provided the data satisfies*

$$\alpha < \min\left\{\frac{1}{2},\ (24(5+\mathcal{K}^{-1}))^{-1}\beta^2,\ (36\beta^{-2}+2+\sqrt{2})^{-1}\right\}.$$

Remark 3.5. *Proving convergence of this iteration requires a stronger condition on* α *than for usual Picard. This arises due to the use of the inf-sup condition in the proof, which seems necessary due to the step 2 approximation. However, our numerical tests and analysis below show that this strong requirement of* α *does not affect the contraction ratio.*

Proof. We denote $e_k^u := u-u_k,\ e_k^z = u-z_k,\ e_k^B := B-B_k,\ e_k^\mu := B-\mu_k,\ e_k^p := \bar{p}-p_k,\ e_k^\lambda := \lambda-\lambda_k$.

During the proof, we assume that

$$\mathcal{K}\|\nabla(u-u_{k-1})\|^2+\mathcal{K}\|\nabla(B-B_{k-1})\|^2+\gamma^{-1}\|\bar{p}-p_{k-1}\|^2+\gamma_m^{-1}\|\lambda-\lambda_{k-1}\| \le \|\nabla u\|^2+\|\nabla B\|^2 \tag{39}$$

holds and then show that the sequence $\{\mathcal{K}\|\nabla(u-u_{k-1})\|^2+\mathcal{K}\|\nabla(B-B_{k-1})\|^2+\gamma^{-1}\|\bar{p}-p_{k-1}\|^2+\gamma_m^{-1}\|\lambda-\lambda_{k-1}\|\}$ is decreasing and this implies the condition at the next iteration.

Using the assumption (39), the triangle inequality and (23), we give the following estimation:

$$\begin{aligned}
&\|\nabla u_{k-1}\|^2+\|\nabla B_{k-1}\|^2\\
\le&2\|\nabla e_{k-1}^u\|^2+2\|\nabla e_{k-1}^B\|^2+2\|\nabla u\|^2+2\|\nabla B\|^2\\
\le&2(1+\mathcal{K}^{-1})\left(\|\nabla u\|^2+\|\nabla B\|^2\right)\\
\le&2(1+\mathcal{K}^{-1})\mathcal{K}^{-2}(\|f\|_{-1}^2+\|\nabla\times g\|_{-1}^2).
\end{aligned} \tag{40}$$

Subtracting the Step $k.1$ from (19), (21) of the discrete steady MHD system, we get that for all $v,w \in X_h$,

$$\begin{aligned}
&\gamma(\nabla\cdot e_k^z,\nabla\cdot v)+\nu(\nabla e_k^z,\nabla v)=(e_{k-1}^p,\nabla\cdot v)\\
&\qquad -b^*(u_{k-1},e_k^z,v)-b^*(e_{k-1}^u,u,v)+b^*(e_{k-1}^B,B,v)+b^*(B_{k-1},e_k^\mu,v),\\
&\gamma_m(\nabla\cdot e_k^\mu,\nabla\cdot w)+\nu_m(\nabla e_k^\mu,\nabla w)=-(e_{k-1}^\lambda,\nabla\cdot w)\\
&\qquad -b^*(u_{k-1},e_k^\mu,w)-b^*(e_{k-1}^u,B,w)+b^*(e_{k-1}^B,u,w)+b^*(B_{k-1},e_k^z,w).
\end{aligned}$$

Adding these equations after setting $v=e_k^z,\ w=e_k^\mu$, which vanishes the fourth and last terms in both equations, produces

$$\begin{aligned}
&\gamma\|\nabla\cdot e_k^z\|^2+\nu\|\nabla e_k^z\|^2+\gamma_m\|\nabla\cdot e_k^\mu\|^2+\nu_m\|\nabla e_k^\mu\|^2=(e_{k-1}^p,\nabla\cdot e_k^z)\\
&-b^*(e_{k-1}^u,u,e_k^z)+b^*(e_{k-1}^B,B,e_k^z)-(e_{k-1}^\lambda,\nabla\cdot e_k^\mu)-b^*(e_{k-1}^u,B,e_k^\mu)+b^*(e_{k-1}^B,u,e_k^\mu).
\end{aligned}$$

Applying Cauchy-Schwarz and Young's inequalities now yields

$$\begin{aligned}&\frac{\gamma}{2}\|\nabla\cdot e_k^z\|^2+\frac{\nu}{2}\|\nabla e_k^z\|^2+\frac{\gamma_m}{2}\|\nabla\cdot e_k^\mu\|^2+\frac{\nu_m}{2}\|\nabla e_k^\mu\|^2\\ \le&\frac{1}{2}\gamma^{-1}\|e_{k-1}^p\|^2+\frac{1}{2}\gamma_m^{-1}\|e_{k-1}^\lambda\|^2\\ &+\nu^{-1}M^2\left(\|\nabla e_{k-1}^u\|^2+\|\nabla e_{k-1}^B\|^2\right)\left(\|\nabla u\|^2+\|\nabla B\|^2\right).\end{aligned}$$

Multiplying both sides by 2 and using (23) gives

$$\begin{aligned}\gamma\|\nabla\cdot e_k^z\|^2+\nu\|\nabla e_k^z\|^2+\gamma_m\|\nabla\cdot e_k^\mu\|^2+\nu_m\|\nabla e_k^\mu\|^2&\\ \le\gamma^{-1}\|e_{k-1}^p\|^2+\gamma_m^{-1}\|e_{k-1}^\lambda\|^2+2\mathcal{K}\alpha^2\left(\|\nabla e_{k-1}^u\|^2+\|\nabla e_{k-1}^B\|^2\right).&\end{aligned}\tag{41}$$

Dropping the grad-div stabilization terms, we obtain

$$\|\nabla e_k^z\|^2+\|\nabla e_k^\mu\|^2\le\mathcal{K}^{-1}\left(\gamma^{-1}\|e_{k-1}^p\|^2+\gamma_m^{-1}\|e_{k-1}^\lambda\|^2\right)+2\alpha^2(\|\nabla e_{k-1}^u\|^2+\|\nabla e_{k-1}^B\|^2).\tag{42}$$

Similarly, one can get the following bound from step $k.3$:

$$\begin{aligned}\gamma\|\nabla\cdot e_k^u\|^2+\nu\|\nabla e_k^u\|^2+\gamma_m\|\nabla\cdot e_k^B\|^2+\nu_m\|\nabla e_k^B\|^2&\\ \le\gamma^{-1}\|e_k^p\|^2+\gamma_m^{-1}\|e_k^\lambda\|^2+2\mathcal{K}\alpha^2(\|\nabla e_{k-1}^u\|^2+\|\nabla e_{k-1}^B\|^2).&\end{aligned}\tag{43}$$

Next, we bound $\|e_k^p\|$ and $\|e_k^\lambda\|$. Adding Step $k.1$ and Step $k.2$, and then subtracting the respective equations from (19) and (21) yields

$$\begin{aligned}&\gamma(\nabla\cdot(u-z_k-\chi_k),\nabla\cdot v)+\nu(\nabla(u-z_k-\chi_k),\nabla v)\\ =&(e_k^p,\nabla\cdot v)-b^*(u_{k-1},e_k^z,v)-b^*(e_{k-1}^u,u,v)\\ &+b^*(e_{k-1}^B,B,v)+b^*(B_{k-1},e_k^\mu,v),\end{aligned}\tag{44}$$

$$\begin{aligned}&\gamma_m(\nabla\cdot(B-\mu_k-\psi_k),\nabla\cdot w)+\nu_m(\nabla(B-\mu_k-\psi_k),\nabla w)\\ =&-(e_k^\lambda,\nabla\cdot w)-b^*(u_{k-1},e_k^\mu,w)-b^*(e_{k-1}^u,B,w)\\ &+b^*(e_{k-1}^B,u,w)+b^*(B_{k-1},e_k^z,w).\end{aligned}\tag{45}$$

Notice that $u-z_k-\chi_k, B-\mu_k-\psi_k\in V_h$. Adding these equations and setting $v=\xi_u:=u-z_k-\chi_k$ and $w=\xi_B:=B-\mu_k-\psi_k$ vanishes the third terms in both equations, and leaves

$$\begin{aligned}&\gamma\|\nabla\cdot\xi_u\|^2+\nu\|\nabla\xi_u\|^2+\gamma_m\|\nabla\cdot\xi_B)\|^2+\nu_m\|\nabla\xi_B\|^2\\ =&b^*(u_{k-1},e_k^z,\xi_u)-b^*(e_{k-1}^u,u,\xi_u)+b^*(e_{k-1}^B,B,\xi_u)+b^*(B_{k-1},e_k^\mu,\xi_u)\\ &-b^*(u_{k-1},e_k^\mu,\xi_B)-b^*(e_{k-1}^u,B,\xi_B)+b^*(e_{k-1}^B,u,\xi_B)+b^*(B_{k-1},e_k^z,\xi_B).\end{aligned}$$

Utilizing (17), Cauchy-Schwarz and Young's inequalities, we get

$$\begin{aligned}&\gamma\|\nabla\cdot\xi_u\|^2+\nu\|\nabla\xi_u\|^2+\gamma_m\|\nabla\cdot\xi_B\|^2+\nu_m\|\nabla\xi_B\|^2\\ \le&2\mathcal{K}^{-1}M^2(\|\nabla u\|^2+\|\nabla B\|^2)\left(4(\|\nabla e_k^z\|^2+\|\nabla e_k^\mu\|^2)+\|\nabla e_{k-1}^u\|^2+\|\nabla e_{k-1}^B\|^2\right)\\ \le&2\mathcal{K}\alpha^2\left(4(\|\nabla e_k^z\|^2+\|\nabla e_k^\mu\|^2)+\|\nabla e_{k-1}^u\|^2+\|\nabla e_{k-1}^B\|^2\right).\end{aligned}\tag{46}$$

Returning back to the pressure terms, we apply the inf-sup condition to (44) and (45), and obtain

$$\begin{aligned}\beta\|e_k^p\|\le&\gamma\|\nabla\cdot\xi_u\|+\nu\|\nabla\xi_u\|+M\|\nabla u_{k-1}\|\|\nabla e_k^z\|+M\|\nabla e_{k-1}^u\|\|\nabla u\|\\ &+M\|\nabla e_{k-1}^B\|\|\nabla B\|+M\|\nabla B_{k-1}\|\|\nabla e_k^\mu\|,\\ \beta\|e_k^\lambda\|\le&\gamma_m\|\nabla\cdot\xi_B\|+\nu_m\|\nabla\xi_B\|+M\|\nabla u_{k-1}\|\|\nabla e_k^\mu\|+M\|\nabla e_{k-1}^u\|\|\nabla B\|\\ &+M\|\nabla e_{k-1}^B\|\|\nabla u\|+M\|\nabla B_{k-1}\|\|\nabla e_k^z\|.\end{aligned}$$

Squaring on both sides and multiplying by $\gamma^{-1}, \gamma_m^{-1}$ respectively, produces

$$\begin{aligned}\gamma^{-1}\beta^2\|e_k^p\|^2 \le & 6\left(\gamma\|\nabla\cdot\xi_u\|^2+\nu^2\|\nabla\xi_u\|^2\right)\\ &+6\nu^{-1}M^2\left(\|\nabla u_{k-1}\|^2\|\nabla e_k^z\|^2+\|\nabla e_{k-1}^u\|^2\|\nabla u\|^2\right.\\ &\left.+\|\nabla e_{k-1}^B\|^2\|\nabla B\|^2+\|\nabla B_{k-1}\|^2\|\nabla e_k^\mu\|^2\right),\\ \gamma_m^{-1}\beta^2\|e_k^\lambda\|^2 \le & 6\left(\gamma_m\|\nabla\cdot\xi_B\|^2+\nu_m\|\nabla\xi_B\|^2\right)\\ &+6\nu_m^{-1}M^2\left(\|\nabla u_{k-1}\|^2\|\nabla e_k^\mu\|^2+\|\nabla e_{k-1}^u\|^2\|\nabla B\|^2\right.\\ &\left.+\|\nabla e_{k-1}^B\|^2\|\nabla u\|^2+\|\nabla B_{k-1}\|^2\|\nabla e_k^z\|^2\right),\end{aligned}$$

thanks to the assumption $\nu \le \gamma, \nu_m \le \gamma_m$. Adding these estimates yields

$$\begin{aligned}&\gamma^{-1}\|e_k^p\|^2+\gamma_m^{-1}\|e_k^\lambda\|^2\\ \le&18\beta^{-2}\mathcal{K}\alpha^2(\|\nabla e_{k-1}^u\|^2+\|\nabla e_{k-1}^B\|^2)\\ &+12\beta^{-2}\mathcal{K}\alpha^2(5+\mathcal{K}^{-1})(\|\nabla e_k^z\|^2+\|\nabla e_k^\mu\|^2),\end{aligned}\tag{47}$$

thanks to (46) and (40). Applying (42) to the above inequality gives

$$\begin{aligned}&\gamma^{-1}\|e_k^p\|^2+\gamma_m^{-1}\|e_k^\lambda\|^2\\ \le&12\beta^{-2}(5+\mathcal{K}^{-1})\alpha^2(\gamma^{-1}\|e_{k-1}^p\|^2+\gamma_m^{-1}\|e_{k-1}^\lambda\|^2)\\ &+\left(18\beta^{-2}\mathcal{K}\alpha^2+24\beta^{-2}\mathcal{K}\alpha^4(5+\mathcal{K}^{-1})\right)(\|\nabla e_{k-1}^u\|^2+\|\nabla e_{k-1}^B\|^2),\end{aligned}\tag{48}$$

and this reduces (43) to

$$\begin{aligned}&\mathcal{K}\|\nabla e_k^u\|^2+\mathcal{K}\|\nabla e_k^B\|^2\\ \le&12\beta^{-2}(5+\mathcal{K}^{-1})\alpha^2(\gamma^{-1}\|e_{k-1}^p\|^2+\gamma_m^{-1}\|e_{k-1}^\lambda\|^2)\\ &+\left(18\beta^{-2}\mathcal{K}\alpha^2+24\beta^{-2}\mathcal{K}\alpha^4(5+\mathcal{K}^{-1})+2\mathcal{K}\alpha^2\right)(\|\nabla e_{k-1}^u\|^2+\|\nabla e_{k-1}^B\|^2).\end{aligned}\tag{49}$$

Now adding (48) and (49) provides

$$\begin{aligned}&\gamma^{-1}\|e_k^p\|^2+\gamma_m^{-1}\|e_k^\lambda\|^2+\mathcal{K}\|\nabla e_k^u\|^2+\mathcal{K}\|\nabla e_k^B\|^2\\ \le&24\beta^{-2}(5+\mathcal{K}^{-1})\alpha^2(\gamma^{-1}\|e_{k-1}^p\|^2+\gamma_m^{-1}\|e_{k-1}^\lambda\|^2)\\ &+\left(18\beta^{-2}+24\beta^{-2}\alpha^2(5+\mathcal{K}^{-1})+1\right)2\alpha^2(\mathcal{K}\|\nabla e_{k-1}^u\|^2+\mathcal{K}\|\nabla e_{k-1}^B\|^2).\end{aligned}$$

We use the small data assumption on α to finally obtain

$$\begin{aligned}&\gamma^{-1}\|e_k^p\|^2+\gamma_m^{-1}\|e_k^\lambda\|^2+\mathcal{K}\|\nabla e_k^u\|^2+\mathcal{K}\|\nabla e_k^B\|^2\\ \le&\alpha\left(\gamma^{-1}\|e_{k-1}^p\|^2+\gamma_m^{-1}\|e_{k-1}^\lambda\|^2\right)\\ &+2\left(\alpha+18\beta^{-2}+1\right)\alpha^2(\mathcal{K}\|\nabla e_{k-1}^u\|^2+\mathcal{K}\|\nabla e_{k-1}^B\|^2)\\ \le&\alpha\left(\gamma^{-1}\|e_{k-1}^p\|^2+\gamma_m^{-1}\|e_{k-1}^\lambda\|^2+\mathcal{K}\|\nabla e_{k-1}^u\|^2+\mathcal{K}\|\nabla e_{k-1}^B\|^2\right).\end{aligned}$$

We have thus proven that $\{\gamma^{-1}\|e_k^p\|^2+\gamma_m^{-1}\|e_k^\lambda\|^2+\mathcal{K}\|\nabla e_k^u\|^2+\mathcal{K}\|\nabla e_k^B\|^2\}$ is a contractive sequence in k, and therefore converges. Since the solution of the problem (19)-(22) is unique and bounded by the data, we have the limit of the incremental Picard-Yosida iteration converges to the solution of (19)-(22). □

4. Numerical tests for incremental Picard-Yosida iterations

In this section, we present results for two experiments: Hartmann flow, and flow past a backward facing step, to show that the incremental Picard-Yosida method is both an accurate and efficient method.

4.1. Hartmann flow. An important application of MHD is Hartmann flow, which is the steady-state flow of an incompressible, electronically conducting fluid between two parallel perfect insulating planes/walls. Here we consider a square domain of size 2 centered at origin with no-slip boundary conditions for velocity $u = 0$, an uniform magnetic field $< 0, 1 >^T$ on walls, prescribe parabolic velocity inflow/outflow profile same as the exact velocity solution u defined below. The known Hartmann solutions are:

$$u = \begin{pmatrix} \frac{1}{Ha} * \frac{1}{\tanh(Ha)} * \frac{1-\cosh(Ha*y)}{\cosh(Ha)} \\ 0 \end{pmatrix},$$

$$B = \begin{pmatrix} \frac{-y}{Ha} + \frac{1}{Ha} * \frac{1}{\tanh(Ha)} * \frac{\sinh(Ha*y)}{\cosh(Ha)} \\ 1 \end{pmatrix},$$

where the Hartmann number is defined as $Ha := \sqrt{Re * Re_m}$. We run the simulation using Taylor-Hood elements (P_2, P_1) with fixed $\gamma = \gamma_m = 1$ and varying Ha and h, solve it using either the usual Picard method, or IPY iteration, and aim at achieving three goals. First, we show that IPY is a convergent nonlinear solver, and its solution is the same as usual Picard iteration. Second, we compare the IPY solution with the known solution from literature [14] and find excellent agreement. Third, we compare IPY and usual Picard algorithms for efficiency, and observe that although IPY requires more iterations to reach the same convergence tolerance, the total algorithm time is much less. Overall, we find IPY is an efficient and accurate algorithm for solving this test problem.

One main purpose of this experiment is to compare the ability of the nonlinear solver of Picard and IPY method. So we state solvers used for each method. Recall the only difference between these two methods is the Schur complement solve in Step $k.2$, thus we use sparse direct solver for Step $k.1$, $k.3$ of both methods, but different iterative solver for Step $k.2$. Specifically, we use preconditioned BICGSTAB iterative solver for Schur complement solve with block diagonal pressure mass matrix preconditioner, and preconditioned CG for the SPD Schur complement with pressure mass matrix preconditioner.

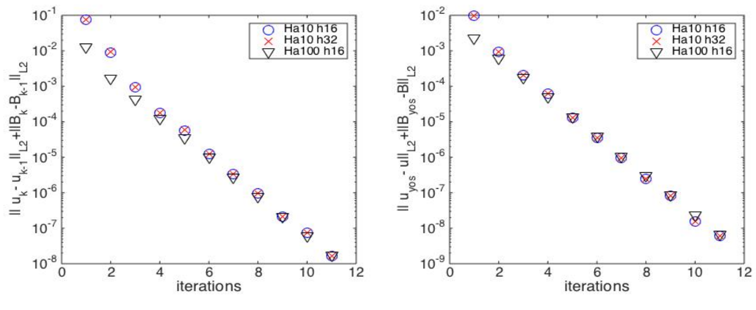

FIGURE 1. Plots are IPY difference in two successive iterations (left) and error to the solution of Picard iteration (right) under varies mesh size and Ha number.

For a convergence criteria, we use the L^2 norm of the difference in successive iterates to be below 10^{-6}. We use the author's Matlab code on uniform meshes of mesh size $h = 1/16, 1/32$, with $Ha = 1, 10, 50, 100$. In Figure 1, we show the IPY difference in successive iterations $\|u_k - u_{k-1}\| + \|B_k - B_{k-1}\|$ and error

$\|u_k - u_{dir}\| + \|B_k - B_{dir}\|$ where (u_k, B_k) is the Step k solution of Algorithm 3.3 and (u_{dir}, B_{dir}) is the solution of Algorithm 3.1. These two figures show that with a good initial guess, the IPY method converges linearly. We observe parameters, mesh size and Ha, have no effect on contraction ratio, this is because in the result of Theorem 3.4, h and Ha play no role in the contraction ratio.

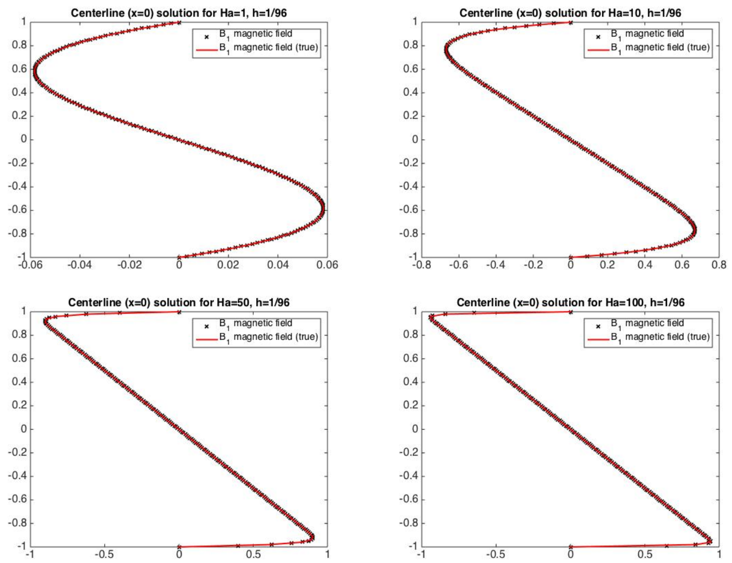

FIGURE 2. Magnetic field of Hartmann flow on centerline with different $Ha = 1, 10, 50, 100$.

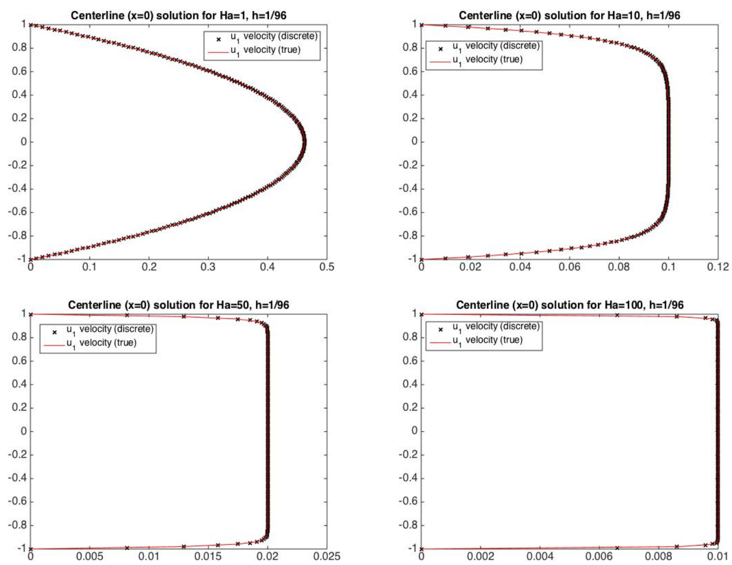

FIGURE 3. Velocity of Hartmann flow on centerline with different $Ha = 1, 10, 50, 100$.

We also compare solutions from Algorithm 3.3 with the known Hartmann solution. Testing on a mesh that provides 499,590 degrees of freedom (dof) with mesh size $h = 1/96$, we vary $Ha = \{1, 10, 50, 100\}$ and plot velocity field and magnetic field at the centerline (dropping from the top center). Comparing the solutions solved by IPY and the known solution from the literature, we observe excellent agreement in Figures 2 and 3.

Lastly, we compare the efficiency of Algorithm 3.3 with Algorithm 3.1 with fixed parameter $Ha = 100$. Note the only difference for these two methods is the Schur complement solve in Step $k.2$, and we use the preconditioned BICGSTAB solver for Step 2 of Picard iteration, while preconditioned CG iterative solver is used for Step 2 of IPY method. We precondition both with the pressure mass matrix, and use direct solvers (backslash) for the inner solves. We run the Hartmann problem with fixed $Ha = 100$ and various mesh size $1/h$. Table 1 records the number of iterations, and average linear solve timefor Schur complement solver. We observe the average solve time and number of iterations for IPY are less than the ones for Picard iteration. This improvements magnify especially for small mesh size. Thus IPY is suitable for problems with large dof comparing to Picard method.

TABLE 1. This table shows the linear solve time, and number of outer iterations for the Schur complement solve for both Picard iteration and IPY with parameter $Ha = 100$.

$1/h$	IncPicardYosida		Picard	
	# of its	avg. time	# of its	avg. time
16	7	.064166	8	0.21325
32	7	.22973	9.5	0.70962
64	7	1.02948	9	3.9455
96	7	2.4593	8.5	9.0392
128	7	4.5640	16.5	30.4966
196	8	15.5408	10.5	46.749

5. Conclusion

The proposed incremental Picard-Yosida method is a simple, efficient, and easily implementable solver for steady MHD. It completely avoids the main issues, i.e. the complicated Schur complements, and the linear solves that it does require are easy to handle. Comparing to the usual Picard iteration, we only need to build and precondition the approximated Schur complement once because it is the same at each iteration. Moreover, the approximated Schur complement decouples into 2 SPD Stokes Schur complements, we can use CG for both inner and outer solves, and effectively precondition the inner and outer solves with standard methods. We proved herein that the IPY method converges to the solution of discrete coupled steady MHD system linearly, under a data condition similar to that required for uniqueness. We also presented two numerical tests (Hartmann flow and flow past a backward facing step) that verified IPY is both an accurate and efficient method.

In the future, we plan to extend IPY to Newton-type iterations for steady MHD, and extend IPY methodology to other steady multi-physics problems.

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