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SUPERCONVERGENCE AND FLUX RECOVERY FOR AN ENRICHED FINITE ELEMENT METHOD

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Abstract. We introduce a flux recovery scheme for an enriched finite element method applied to an interface diffusion equation with absorption. The method is a variant of the finite element method introduced by Wang *et al.* in [20]. The recovery is done at nodes first and then extended to the whole domain by interpolation. In the case of piecewise constant diffusion coefficient, we show that the nodes of the finite elements are superconvergence points for both the primary variable p and its flux u. In particular, in the absence of the absorption term zero error is achieved at the nodes and interface point in the approximation of u and p. In the general case, pressure error at the nodes and interface point is second order. Numerical results are provided to confirm the theory.

Key words. Flux recovery technique, superconvergence, enriched finite element, immersed finite element method.

1. Introduction

We consider the interface two-point boundary value problem

$$\begin{cases} -(\beta(x)p'(x))' + w(x)p(x) = f(x), & x \in I = (a,b), \\ p(a) = p(b) = 0, \end{cases} \tag{1}$$

where $w(x) \geq 0$, and $0 < \beta \in C[a,\alpha] \cup C[\alpha,b]$ is discontinuous across the interface α with the jump conditions

$$[p]_\alpha = 0, \tag{2}$$

$$[\beta p']_\alpha = g. \tag{3}$$

Here the unknown function p may stand for the pressure or temperature in a medium with certain physical properties and the derived quantity $u := -\beta p'$ is the corresponding Darcy velocity or heat flux, which is equally important. The piecewise continuous β reflects a nonuniform material or medium property and the function $w(x)$ reflects the surroundings of the material. Problem (1) can also be viewed as the steady neutron diffusion problem [19]. However, in this paper we will refer to p as pressure. Due to its one-dimensional simple structure, many mathematical and numerical properties of related numerical methods can be explicitly worked out. For example, in this paper the flux jump g in (3) will be taken as zero. In fact, if $g \neq 0$ we can handle the nonhomogenous flux jump condition as follows[1]. Let $\tilde{p}$ be a boundary vanishing function such that $[\tilde{p}]_\alpha = 0$ and $[\beta\tilde{p}']_\alpha = g$. Among all possible $\tilde{p}$, we can choose those suitable for our numerical computation as well. For instance,

$$\tilde{p}(x) = \begin{cases} 0, & a \leq x < \alpha, \\ -\frac{g}{\beta^+(b-\alpha)}(x-\alpha)^2 + \frac{g}{\beta^+}(x-\alpha), & \alpha \leq x \leq b, \end{cases} \tag{4}$$

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where $\beta^+ = \lim_{x\to\alpha^+} \beta(x)$. A transformed new variable $p - \tilde{p}$ will then give rise to an interface problem with homogeneous jump conditions. One can generalize the above technique to higher dimensions with the help of those used in the immersed finite element shape functions construction. In general, it is very instructive to study problem (1) before moving to its higher dimensional and/or nonsteady state versions. It is in this spirit that we shall study the associated enriched finite element approximation. Recent studies of immersed finite element and volume methods on similar one-dimensional problems can be found in [4, 5].

Numerical methods for the interface problem (1) generally use meshes that are either fitted or unfitted with the interface. A method allowing unfitted meshes would be very efficient when one has to follow a moving interface in a temporal problem. For an in-depth exposition of the numerics and applications of interface problems, we refer the readers to [14] and the references therein. For our purpose here let us only mention two classes of methods: (a) the class of immersed finite element and difference methods and (b) the class of enriched finite element methods. For example, in an immersed finite element (IFE) method, the mesh is made up of interface elements where the interface intersects elements (thus immersed) and noninterface elements where the interface is absent. On a noninterface element one uses standard local shape functions, whereas on an interface element one uses piecewise standard local shape functions subject to continuity and jump conditions. Representative works on IFE methods can be found in [12, 13, 14, 15, 16, 18], among others.Recent advances in the subject of superconvergence of the IFE method are [9, 10] and the related references there in. For the enriched method, the standard finite element method is enriched with some nonstandard elements that reflect the presence of the interface. It was originally designed to handle crack problems [2, 8, 17], but for recent years efforts have been made to generalize it to fluid problems, see [20] and the references therein.

In this paper, we are interested in studying a flux recovery procedure for an enriched finite element. The procedure can produce accurate approximate flux u_h of p, once an approximate p_h has been obtained. It is important that the procedure can recover flux without having to solve any system of equations. Chou and Tang [7] initiated such methods when the mesh is fitted. Later it was generalized to the immersed interface mesh case using linear immersed finite elements (IFE) of Li *et al.* [16] and their variants for one dimensional elliptic and parabolic problems [1, 6]. In this paper we extend the methodology to enriched finite elements from the conforming P_1 elements.

The idea of the flux recovery scheme in [7] is very easy to describe in the one dimensional case. Suppose let there be given an expression of the exact flux $u(x_i)$ at some mesh point x_i in terms of a weighted integral of p, which can be obtained as follows. Let ϕ be a function with compact support K such that $I_i = [x_{i-1}, x_i] \subset K$, the interface point $\alpha \notin K$ (non-interface element), $\phi(x_{i-1}) = 0$, $\phi(x_i) = 1$. An example of such a function is the standard finite element hat function. Multiplying (1) by ϕ and integrating by parts, we see that the flux u satisfies

$$u(x_i) = -\int_{I_i} \beta p' \phi' dx - \int_{I_i} wp\phi dx + \int_{I_i} f\phi dx.$$

It is then natural to define an approximate flux u_h at x_i as

$$u_h(x_i) = -\int_{I_i} \beta p_h' \phi' dx - \int_{I_i} wp_h\phi dx + \int_{I_i} f\phi dx.$$

The error $E_i := u(x_i) - u_h(x_i)$ then satisfies

$$E_i = -\int_{I_i} \beta(p' - p_h')\phi' dx - \int_{I_i} w(p - p_h)\phi dx.$$

In the case that ϕ is linear on I_i, $w = 0$, $p = p_h$ at x_{i-1}, x_i, we immediately see that the error in flux is also zero at x_i. With a little calculation using the jump conditions (2)-(3), the same line of thought works when $\alpha \in I_i$ (interface element case). In this paper the ϕ's will be from the usual P_1 conforming hat functions and we show in Thm 4.2 that in the case of $w = 0$, the enriched finite element solution $p_h = p$ at all end nodes and as a consequence $u = u_h$ at those points as well. When $w \neq 0$, the exactness cannot be attained due to the nature of the Green's function involved (see the proof of Thm 4.2), but those points are still superconvergence points of the pressure and flux . Another feature of our scheme is that when $w = 0$ the following conservation law or discrete first fundamental theorem of calculus holds:

$$u_h(x_i) - u_h(x_{i-1}) = \int_{I_i} f(x)dx,$$

whose continuous version can be obtained for the exact flux from integrating (1). The rest of the paper is organized as follows. In Section 2, we introduce the approximate pressure space which is a modification of the one in [20]. Its approximation properties are shown in this section. We then present our flux recovery scheme in Section 3. Superconvergece properties are proven in Section 4. In particular, we show that for piecewise constant coefficient problems our method can capture the flux at nodes and at the interface points exactly for non-absorption case. Moreover, it has uniform error distribution over all nodes for general problems. Second order pressure error and first order flux error are shown at the nodes. The optimal convergence rates of the pressure and flux are shown as well. Finally, numerical examples are provided in Section 5 to confirm the theory.

2. Approximation properties of the enrichment finite element space

The weak formulation of the problem (1)–(3) is: Given $f \in L^2(I)$, find $p \in H_0^1(I)$ such that

$$a(p, q) = (f, q) \quad \forall q \in H_0^1(I), \tag{5}$$

where

$$a(p, q) = \int_a^b \beta(x)p'(x)q'(x)dx + \int_a^b w(x)p(x)q(x)dx, \qquad (f, q) = \int_a^b f(x)q(x)dx.$$

We now introduce an approximation space for its solution. Let $a = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_n = b$ be a partition of I and the interface point $\alpha \in (x_k, x_{k+1})$ for some k. In a two dimensional interface problem, the curved interface is usually approximated by a line segment and the area between them is consequently $O(h^2)$. To reflect this feature for one dimensional case, we introduce an $\alpha - \epsilon$ point to mimic the discretized interface. The quantity ϵ will be called discrete interface deviation. Accordingly, we construct a piecewise linear function associated with the interface element $[x_k, x_{k+1}]$ such that

$$\psi(x_k) = 0 = \psi(x_{k+1}), \qquad [\psi]_{\alpha-\epsilon} = 0, \qquad [\psi']_{\alpha-\epsilon} = 1.$$

More specifically,

$$
(6) \qquad \psi(x) = \begin{cases} 0, & x \in [a, x_k], \\ \dfrac{(x_{k+1} - (\alpha - \epsilon))(x_k - x)}{x_{k+1} - x_k}, & x \in [x_k, \alpha - \epsilon], \\ \dfrac{((\alpha - \epsilon) - x_k)(x - x_{k+1})}{x_{k+1} - x_k}, & x \in [\alpha - \epsilon, x_{k+1}], \\ 0, & x \in [x_{k+1}, b]. \end{cases}
$$

Note that in the above construction, we assumed that ϵ is such that $x_k \le \alpha - \epsilon \le x_{k+1}$. We choose ϵ after the unique interface element containing α has been located. It will be seen later that a minimum assumption of $\epsilon = \mathcal{O}(h^s)$ for some positive integer $s \ge 2$ is required for optimal error estimates. Of course, the case of $\epsilon = 0$ is most common and natural for normal one dimensional consideration.

For $\bar{I} = \cup_0^{n-1} I_i, I_i = [x_i, x_{i+1}]$, let S_h be the conforming linear finite element space

$$
(7) \qquad \begin{aligned} S_h &= \{p_h \in C(\bar{I}) : p_h|_{I_i} \in P_1, i = 0, \ldots, n-1, p(a) = p(b) = 0\} \\ &= \mathrm{Span}\{\phi_i, i = 1, 2, \ldots, n-1\}, \end{aligned}
$$

where ϕ_i are the usual global hat functions with vanishing boundary condition. We denote the usual P_1-interpolation operator by $\pi_h : C(\bar{I}) \to S_h$,

$$
\pi_h g = \sum_{i=1}^{n-1} g(x_i)\phi_i.
$$

Following an idea in Wang *et al.*[20], we define an enriched finite element space

$$
(8) \qquad \begin{aligned} \overline{S}_h &= S_h \oplus \psi S_h = \{p_h + q_h\psi : \, p_h, q_h \in S_h\} \\ &= \mathrm{Span}\{\phi_1, \phi_2, \ldots, \phi_{n-1}, \phi_k\psi, \phi_{k+1}\psi\} \end{aligned}
$$

and its associated enriched finite element method for problem (1): Find $p_h \in \overline{S}_h$ such that

$$
(9) \qquad a(p_h, q_h) = (f, q_h) \quad \forall q_h \in \overline{S}_h.
$$

Let $\tilde{H}^2(I) = H_0^1(I) \cap H^2(I^-) \cap H^2(I^+)$, where $I^- = (a, \alpha), I^+ = (\alpha, b)$. It is essential for the enriched space to have good approximation properties for the functions in $\tilde{H}^2(I)$ that satisfy the jump conditions (2)-(3) in which the exact solution p lies. We also need to introduce the following sets: $I_h^- = (a, \alpha - \epsilon), I_h^+ = (\alpha - \epsilon, b)$. For $p \in \tilde{H}^2(I)$, let $p_i, i = 1, 2$ be the extensions of p restricted to I^- and I^+ to $H^2(I)$, respectively. Thus $p_2' - p_1'$ is in $H^1(I) \subset C(\bar{I})$ due to the Sobolev inequality. This implies that the usual P_1–interpolation $\pi_h(p_2' - p_1') \in S_h$ is well defined (we do not need to use the Clement interpolator as in Eq. (42) of [20] for two dimensions). We define the interpolation operator $I_h : \tilde{H}^2(I) \to \overline{S}_h$

$$
(10) \qquad \begin{aligned} I_h p =& \pi_h p + \pi_h(p_2' - p_1')\psi, \\ =& \frac{p_1(x_k)(x_{k+1} - x) + p_2(x_{k+1})(x - x_k)}{x_{k+1} - x_k} + \pi_h(p_2' - p_1')(x)\psi(x) \end{aligned}
$$

on $[x_k, x_{k+1}]$.

Let $\chi_i, i = 1, 2$ be the characteristic functions of I^- and I^+, respectively. Similarly, let $\chi_{h,i}, i = 1, 2$ be the characteristic functions of I_h^- and I_h^+, respectively.

Following Eqs (24) and (25) of [20], we define

$$V_h = \{v = v_{h,1}\chi_1 + v_{h,2}\chi_2; v_{h,i} \in S_h, i = 1,2\},$$
$$V_h^* = \{v = v_{h,1}\chi_{h,1} + v_{h,2}\chi_{h,2}; v_{h,i} \in S_h, i = 1,2\}.$$

Note that the functions in the above spaces may be discontinuous at α and $\alpha - \epsilon$. Define the auxiliary interpolations $\bar{I}_h : \tilde{H}^2(I) \to V_h$ and $\bar{I}_h^* : \tilde{H}^2(I) \to V_h^*$

$$\bar{I}_h p = \pi_h p_1 \chi_1 + \pi_h p_2 \chi_2,$$
$$\bar{I}_h^* p = \pi_h p_1 \chi_{h,1} + \pi_h p_2 \chi_{h,2},$$

so that $\bar{I}_h p(x_i) = p(x_i) = \bar{I}_h^* p(x_i)$ for $i = 0, 1, \ldots, n$ with $\bar{I}_h p(\alpha^+) = p(\alpha)$.

Below we use conventional Sobolev norm notation. For example, $|u|_{1,J}$ denotes the usual H^1-seminorm for $u \in H^1(J)$, and $||u||^2_{2,I^- \cup I^+} = ||u||^2_{2,I^-} + ||u||^2_{2,I^+}$ for $u \in \tilde{H}^2(I)$, and so on. Sometimes, we use $||u||^2_{2,I}$ for $||u||^2_{2,I^- \cup I^+}$ for simplicity. To derive a bound for the term $|p - I_h p|_{1,I^- \cup I^+}$ we split the error as follows:

$$|p - I_h p|_{1,I^- \cup I^+} \le |p - \bar{I}_h p|_{1,I^- \cup I^+} + |\bar{I}_h p - \bar{I}_h^* p|_{1,I^- \cup I^+} + |\bar{I}_h^* p - I_h p|_{1,I^- \cup I^+}. \tag{11}$$

From the classical approximation theory

$$|p - \bar{I}_h p|_{1,I^- \cup I^+} \le Ch\|p\|_{2,I^- \cup I^+}. \tag{12}$$

The next two lemmas give estimates for the remaining two terms on the right side of (11).

Lemma 2.1. *Let ϵ be the discrete interface deviation. For any $p \in \tilde{H}^2(I)$ we have*

$$|\bar{I}_h p - \bar{I}_h^* p|_{1,I^- \cup I^+} \le C(h + \epsilon^{1/2})\|p\|_{2,I^- \cup I^+}.$$

Proof. Note that

$$\int_{\alpha-\epsilon}^{\alpha} \frac{d}{du}(u - (\alpha - \epsilon))\left((p_2' - p_1')(u)\right)^2 du$$
$$= \int_{\alpha-\epsilon}^{\alpha} \left((p_2' - p_1')(u)\right)^2 du + \int_{\alpha-\epsilon}^{\alpha} 2(u - (\alpha - \epsilon))(p_2' - p_1')(u)(p_2 - p_1)''(u)du,$$

which implies

$$\epsilon\left((p_2' - p_1')(\alpha)\right)^2 = |p_2 - p_1|^2_{1,(\alpha-\epsilon,\alpha)} + 2\int_{\alpha-\epsilon}^{\alpha}(u - (\alpha-\epsilon))(p_2 - p_1)'(u)(p_2 - p_1)''(u)du$$

and hence

$$\begin{aligned} |p_2 - p_1|^2_{1,(\alpha-\epsilon,\alpha)} = &-2\int_{\alpha-\epsilon}^{\alpha}(u - (\alpha - \epsilon))(p_2 - p_1)'(u)(p_2 - p_1)''(u)du... \\ &+ \epsilon\left((p_2' - p_1')(\alpha)\right)^2 \\ \le &2\epsilon\left\|(p_2 - p_1)'\right\|_{0,(\alpha-\epsilon,\alpha)}\left\|(p_2 - p_1)''\right\|_{0,(\alpha-\epsilon,\alpha)} + \epsilon\left((p_2 - p_1)'(\alpha)\right)^2 \\ \le &\epsilon\left|p_2 - p_1\right|^2_{1,(\alpha-\epsilon,\alpha)} + \epsilon\left|p_2 - p_1\right|^2_{2,(\alpha-\epsilon,\alpha)} + \epsilon\left((p_2 - p_1)'(\alpha)\right)^2. \end{aligned}$$

Consequently,

$$|p_2 - p_1|^2_{1,(\alpha-\epsilon,\alpha)} \le \epsilon\left\|p_2 - p_1\right\|^2_{2,(\alpha-\epsilon,\alpha)} + \epsilon\left((p_2 - p_1)'(\alpha)\right)^2. \tag{13}$$

Moreover, using the trace inequality on (a, α) we get

$$\left((p_2 - p_1)'(\alpha)\right)^2 \le C\|p_2 - p_1\|^2_{2,I^-}. \tag{14}$$

Applying (14) to (13), we find that

$$|p_2 - p_1|^2_{1,(\alpha-\epsilon,\alpha)} \le C\epsilon\left\|p_2 - p_1\right\|^2_{2,(\alpha-\epsilon,\alpha)} + C\epsilon\|p_2 - p_1\|^2_{2,I^-}. \tag{15}$$

By the fact that $\bar{I}_h p - \bar{I}_h^* p = \pi_h p_1 - \pi_h p_2$ on $(\alpha-\epsilon,\alpha)$ and (15), we conclude

$$\begin{aligned} |\bar{I}_h p - \bar{I}_h^* p|_{1,(\alpha-\epsilon,\alpha)} &\le |\pi_h p_2 - p_2|_{1,(\alpha-\epsilon,\alpha)} + |\pi_h p_1 - p_1|_{1,(\alpha-\epsilon,\alpha)} + |p_2 - p_1|_{1,(\alpha-\epsilon,\alpha)} \\ &\le C\epsilon\left(\|p_1\|_{2,I^-\cup I^+} + \|p_2\|_{2,I^-\cup I^+}\right) + C\epsilon^{1/2}\|p_2 - p_1\|_{2,I^-} \\ &\le Ch\left(\|p_1\|_{2,I^-\cup I^+} + \|p_2\|_{2,I^-\cup I^+}\right) + C\epsilon^{1/2}\|p_2 - p_1\|_{2,I^-} \\ &\le Ch\|p\|_{2,I^-\cup I^+} + C\epsilon^{1/2}\|p\|_{2,I^-\cup I^+} \end{aligned}$$

where we have used the boundedness of the extension operators. This complete the proof. ☐

Lemma 2.2. *Let $p \in \tilde{H}^2(I)$ and let ϵ be the discrete interface deviation. Then*

$$|\bar{I}_h^* p - I_h p|_{1,I^-\cup I^+} \le C(h + h^{-1}\epsilon)\|p\|_{2,I^-\cup I^+}$$

Proof. Using definition (10) of $I_h p$ we have on the interval $[x_k, \alpha-\epsilon]$

$$\begin{aligned} \left(\bar{I}_h^* p - I_h p\right)' &= \left(\bar{I}_h^* p - \pi_h p\right)' - \left(\pi_h(p_2' - p_1')(x)\psi(x)\right)' \\ &= \left(\left(\bar{I}_h^* p - \pi_h p\right)' + \frac{(x_{k+1}-\alpha)(p_2'(\alpha) - p_1'(\alpha))}{x_{k+1}-x_k}\right) \\ &\quad - \left(\frac{(x_{k+1}-\alpha)(p_2'(\alpha) - p_1'(\alpha))}{x_{k+1}-x_k} + \left(\pi_h(p_2' - p_1')(x)\psi(x)\right)'\right). \end{aligned} \tag{16}$$

Noting that $\bar{I}_h^* p(x_k) = \pi_h p_1(x_k)$, we have for $x \in [x_k, \alpha-\epsilon]$

$$\left(\bar{I}_h^* p - \pi_h p\right)'(x) = \frac{p_1(x_{k+1}) - p_2(x_{k+1})}{x_{k+1}-x_k} = \frac{p_1(x_{k+1}) - p_1(\alpha) + p_2(\alpha) - p_2(x_{k+1})}{x_{k+1}-x_k}.$$

Thus

$$\begin{aligned} &\left|\left(\bar{I}_h^* p - \pi_h p\right)'(x) + \frac{(x_{k+1}-\alpha)(p_2'(\alpha) - p_1'(\alpha))}{x_{k+1}-x_k}\right| \\ &\qquad = \left|\frac{1}{x_{k+1}-x_k}\left(\int_\alpha^{x_{k+1}}\int_\alpha^x p_1''(y)dydx + \int_\alpha^{x_{k+1}}\int_x^\alpha p_2''(y)dydx\right)\right| \\ &\qquad \le \frac{1}{x_{k+1}-x_k}\left(\int_\alpha^{x_{k+1}}\int_x^\alpha |p_1''(y)|dydx + \int_\alpha^{x_{k+1}}\int_x^\alpha |p_2''(y)|dydx\right) \\ &\qquad \le \frac{x_{k+1}-\alpha}{x_{k+1}-x_k} h^{1/2}\|p''\|_{0,I^-\cup I^+} \\ &\qquad \le Ch^{1/2}\|p''\|_{0,I^-\cup I^+}. \end{aligned} \tag{17}$$

Clearly,

$$\begin{aligned} &\left(\pi_h(p_2' - p_1')(x)\psi(x)\right)' = \frac{x_{k+1} - (\alpha-\epsilon)}{(x_{k+1}-x_k)^2}\big[((p_2'-p_1')(x_{k+1}) - (p_2'-p_1')(x_k))(x_k - x) \\ &\quad - ((p_2'-p_1')(x_k)(x_{k+1}-x) + (p_2'-p_1')(x_{k+1})(x - x_k))\big] \\ &= \frac{x_{k+1} - (\alpha-\epsilon)}{(x_{k+1}-x_k)^2}\times \\ &\big[2(p_2'-p_1')(x_{k+1})(x_k - x) - (p_2'-p_1')(x_k)(x_k - x) - (p_2'-p_1')(x_k)(x_{k+1}-x)\big]. \end{aligned}$$

That is,

$$
\begin{aligned}
&(\pi_h(p_2' - p_1')(x)\psi(x))' + \frac{(x_{k+1}-\alpha)(p_2'(\alpha) - p_1'(\alpha))}{x_{k+1}-x_k} \\
=&\frac{(x_{k+1}-\alpha)(p_2'-p_1')(\alpha)}{x_{k+1}-x_k} + \frac{x_{k+1}-(\alpha-\epsilon)}{(x_{k+1}-x_k)^2}\times \\
&\Big[2(p_2'-p_1')(x_{k+1})(x_k - x) - (p_2'-p_1')(x_k)(x_k-x) - (p_2'-p_1')(x_k)(x_{k+1}-x)\Big] \\
=&\frac{((\alpha-\epsilon)-\alpha)(p_2'-p_1')(\alpha)}{x_{k+1}-x_k} + \frac{x_{k+1}-(\alpha-\epsilon)}{(x_{k+1}-x_k)^2}\Big[(x_{k+1}-x_k)(p_2'-p_1')(\alpha) \\
&+ 2(p_2'-p_1')(x_{k+1})(x_k - x) - (p_2'-p_1')(x_k)(x_k-x) - (p_2'-p_1')(x_k)(x_{k+1}-x)\Big] \\
=&\frac{((\alpha-\epsilon)-\alpha)(p_2'-p_1')(\alpha)}{x_{k+1}-x_k} + \frac{x_{k+1}-(\alpha-\epsilon)}{(x_{k+1}-x_k)^2}\Big[(x_{k+1}-x)\int_{x_k}^{\alpha}(p_2'-p_1')'(y)dy \\
&+ (x-x_k)\int_{x_{k+1}}^{\alpha}(p_2'-p_1')'(y)dy + (x_k - x)\int_{x_k}^{x_{k+1}}(p_2'-p_1')'(y)dy\Big] \\
=&J_1 + J_2,
\end{aligned}
$$

where

$$
|J_1| = \left|\frac{((\alpha-\epsilon)-\alpha)(p_2'-p_1')(\alpha)}{x_{k+1}-x_k}\right| \le Ch^{-1}\epsilon\|p\|_{2,I^-\cup I^+}. \tag{18}
$$

Here we used the one dimensional Sobolev imbedding (14) and boundedness of the extension operators. Furthermore,

$$
\begin{aligned}
|J_2| =&\frac{x_{k+1}-(\alpha-\epsilon)}{(x_{k+1}-x_k)^2}\Bigg|(x_{k+1}-x)\int_{x_k}^{\alpha}(p_2'-p_1')'(y)dy \\
&+ (x_k - x)\int_{\alpha}^{x_{k+1}}(p_2'-p_1')'(y)dy + (x_k-x)\int_{x_k}^{x_{k+1}}(p_2'-p_1')'(y)dy\Bigg| \\
\le&\frac{x_{k+1}-(\alpha-\epsilon)}{(x_{k+1}-x_k)^2}\Bigg[|x_{k+1}-x|\int_{x_k}^{\alpha}|p_2''-p_1''|(y)dy + |x_k - x|\times \\
&\left(\int_{\alpha}^{x_{k+1}}|(p_2''-p_1'')(y)|dy + \int_{x_k}^{\alpha}|(p_2''-p_1'')(y)|dy + \int_{\alpha}^{x_{k+1}}|(p_2''-p_1'')(y)|dy\right)\Bigg] \\
\le&C\frac{x_{k+1}-(\alpha-\epsilon)}{(x_{k+1}-x_k)^2}\times \\
&\Big[|x_{k+1}-x|(\alpha-x_k)^{1/2} + |x_k - x|\left(2(x_{k+1}-\alpha)^{1/2} + (\alpha - x_k)^{1/2}\right)\Big]\|p''\|_{0,I^-\cup I^+}.
\end{aligned} \tag{19}
$$

Now applying (18) and (19) to (16)

$$
\begin{aligned}
&|\left(\bar{I}_h^* p - I_h p\right)'| \le Ch^{-1}\epsilon\|p''\|_{0,I^-\cup I^+} + C\frac{x_{k+1}-(\alpha-\epsilon)}{(x_{k+1}-x_k)^2}\times \\
&\Big[|x_{k+1}-x|(\alpha-x_k)^{1/2} + |x_k - x|\left(2(x_{k+1}-\alpha)^{1/2} + (\alpha-x_k)^{1/2}\right)\Big]\|p''\|_{0,I^-\cup I^+}.
\end{aligned}
$$

Obviously,

$$
\|\bar{I}_h^* p - I_h p\|_{1,[x_k,\alpha-\epsilon]} \le C(h + h^{-1}\epsilon)\|p\|_{2,I^-\cup I^+}.
$$

Similarly, we can calculate error estimates for the intervals $[\alpha-\epsilon,\alpha]$ and $[\alpha, x_{k+1}]$. □

Applying (12), Lemma 2.1, and Lemma 2.2 to (11), we obtain the following theorem.

Theorem 2.3. *Let the interface deviation $\epsilon = \mathcal{O}(h^2)$. Then for any $p \in \tilde{H}^2(I)$, there exists a constant $C > 0$ independent of h such that*

$$|p - I_h p|_{1,I} \leq Ch\|p\|_{2,I^- \cup I^+}. \tag{20}$$

3. Construction of the approximate flux

We denote the exact flux by $u = -\beta p'$ and the approximate flux by u_h. The construction of u_h follows the same line as in Chou [6]: first we develop a formula of the exact flux u in terms of pressure p and then use the same formula with replacements of u by u_h and p by p_h. It is proper to point that at this stage that u_h below is not defined as $-\beta p_h'$.

To shorten the presentation of the equations, we collect the two terms in (1) as

$$F(x) := f(x) - w(x)p(x), \text{ and its discrete version: } F_h := f(x) - w(x)p_h(x). \tag{21}$$

To derive a formula of u on a noninterface element, we multiply (1) by ϕ_i and integrate by parts over $[x_{i-1}, x_i], i \neq k+1, 1 \leq i \leq n$ to obtain

$$u(x_i^-) = -\beta(x_i)p'(x_i) = -\int_{x_{i-1}}^{x_i} \beta p' \phi_i' \, dx + \int_{x_{i-1}}^{x_i} F\phi_i \, dx$$

and do the same for $[x_i, x_{i+1}]$, $i \neq k$, $0 \leq i \leq n-1$ to get

$$u(x_i^+) = -\beta(x_i)p'(x_i) = \int_{x_i}^{x_{i+1}} \beta p' \phi_i' \, dx - \int_{x_i}^{x_{i+1}} F\phi_i \, dx.$$

Furthermore, the above two relations also hold when $i = k+1, k$. For instance, to get the expression for $u(x_k^+)$, we integrate (1) against ϕ_k over $[x_k, x_{k+1}]$ to get

$$-\int_{x_k}^{\alpha} (\beta p')'\phi_k \, dx - \int_{\alpha}^{x_{k+1}} (\beta p')'\phi_k dx = \int_{x_k}^{x_{k+1}} F\phi_k dx.$$

By integration by parts on the two integrals on the left side and by the continuity of the flux at α, the left side becomes

$$\int_{x_k}^{\alpha} \beta p' \phi_k' dx - (\beta p')(\alpha^-)\phi_k(\alpha) - u(x_k^+) + \int_{\alpha}^{x_{k+1}} \beta p' \phi_k' dx + (\beta p')(\alpha^+)\phi_k(\alpha)$$

or

$$\int_{x_k}^{x_{k+1}} \beta p' \phi_k' dx - u(x_k^+).$$

The remaining case can be derived similarly.

Hence

$$u(x_i^-) = -\beta(x_i)p'(x_i) = -\int_{x_{i-1}}^{x_i} \beta p' \phi_i' \, dx + \int_{x_{i-1}}^{x_i} F\phi_i \, dx, \quad 1 \leq i \leq n, \tag{22}$$

and

$$u(x_i^+) = -\beta(x_i)p'(x_i) = \int_{x_i}^{x_{i+1}} \beta p' \phi_i' \, dx - \int_{x_i}^{x_{i+1}} F\phi_i \, dx, \quad 0 \leq i \leq n-1. \tag{23}$$

Thus, if p_h is a good approximation for p, it is, in view of (22)-(23), natural to define $u_h(x_i^-)$ and $u_h(x_i^+)$ as

$$u_h(x_i^-) = -\int_{x_{i-1}}^{x_i} \beta(x)p_h'\phi_i' dx + \int_{x_{i-1}}^{x_i} F_h\phi_i dx, \quad 1 \leq i \leq n,$$

$$u_h(x_i^+) = \int_{x_i}^{x_{i+1}} \beta(x)p_h'\phi_i' dx - \int_{x_i}^{x_{i+1}} F_h\phi_i dx, \quad 0 \leq i \leq n-1.$$

However, $u_h(x_i^-) = u_h(x_i^+)$, which can be seen by replacing q_h with ϕ_i in (9). Consequently, $u_h(x_i)$ is well defined, i.e.,

Formulas for the approximate flux at nodes:

$$u_h(x_i) = -\int_{x_{i-1}}^{x_i} \beta(x) p_h' \phi_i' dx + \int_{x_{i-1}}^{x_i} (f - wp_h)\phi_i dx, \quad 1 \le i \le n \tag{24}$$

$$= \int_{x_i}^{x_{i+1}} \beta(x) p_h' \phi_i' dx - \int_{x_i}^{x_{i+1}} (f - wp_h)\phi_i dx, \quad 0 \le i \le n-1. \tag{25}$$

To define u_h at the interface point α, we proceed as follows. Integrating (1) over $[x_k, \alpha]$ and $[\alpha, x_{k+1}]$, we have

$$u(\alpha) = u(x_k) + \int_{x_k}^{\alpha} F(x)dx, \tag{26}$$

$$u(\alpha) = u(x_{k+1}) + \int_{x_{k+1}}^{\alpha} F(x)dx. \tag{27}$$

Thus it is natural to define

$$u_h(\alpha^-) = u_h(x_k) + \int_{x_k}^{\alpha} F_h(x)dx, \tag{28}$$

$$u_h(\alpha^+) = u_h(x_{k+1}) + \int_{x_{k+1}}^{\alpha} F_h(x)dx. \tag{29}$$

However, using (24) and the fact $\phi_k + \phi_{k+1} = 1$, we can derive easily that $u_h(\alpha^-) = u_h(\alpha^+)$ and so

Formulas for the approximate flux at interface point:

$$u_h(\alpha) = u_h(x_k) + \int_{x_k}^{\alpha} (f(x) - w(x)p_h(x))dx \tag{30}$$

$$= u_h(x_{k+1}) - \int_{\alpha}^{x_{k+1}} (f(x) - w(x)p_h(x))dx. \tag{31}$$

Global definition of u_h

Finally, we define $u_h(x)$ as the continuous piecewise liner function that interpolates at $a = x_0 < x_1 < ... < x_k < \alpha < x_{k+1} < ... < x_n = b$. That is,

$$u_h(x) = \sum_0^k u_h(x_i)\phi_i(x) + u_h(\alpha)\phi_\alpha(x) + \sum_{k+1}^n u_h(x_i)\phi_i(x). \tag{32}$$

4. Convergence

Theorem 4.1. *Let p be the exact pressure and p_h be the approximate pressure of the equations (5) and (9), respectively. Then*

$$\|p - p_h\|_{0,I} + h\|p - p_h\|_{1,I} \le Ch^2\|p\|_{2,I^- \cup I^+},$$

provided that the interface deviation $\epsilon = \mathcal{O}(h^2)$.

Proof. Subtracting (5) from (9), we have

$$a(p - p_h, q_h) = 0 \quad \forall q_h \in \overline{S}_h.$$

Then using the boundedness and coercivity properties of the bilinear form $a(\cdot,\cdot)$, we get

$$\begin{aligned}\beta_* |p - p_h|_{1,I}^2 &\le a(p - p_h, p - p_h) = a(p - p_h, p - q_h)\\ &\le \beta^* |p - p_h|_{1,I} |p - q_h|_{1,I},\end{aligned}$$

where $\beta^* = \sup_{x\in[a,b]} \beta(x)$ and $\beta_* = \inf_{x\in[a,b]} \beta(x)$. Thus by Cea's Lemma

$$\begin{aligned} |p - p_h|_{1,I} \leq & \frac{\beta^*}{\beta_*} \inf |p - q_h|_{1,I} \\ \leq & \frac{\beta^*}{\beta_*} |p - I_h p_h|_{1,I} \\ \leq & Ch\|p\|_{2,I^- \cup I^+}. \end{aligned}$$

Then the usual duality argument leads to

$$\|p - p_h\|_{0,I} \leq Ch^2 \|p\|_{2,I^- \cup I^+}.$$

□

4.1. Superconvergence and pointwise pressure and flux approximation. From now on we assume the interface deviation $\epsilon = \mathcal{O}(h^2)$.

Theorem 4.2. *Under the assumption that the function w in* (1) *is zero, the following statements hold.*

(i) Superconvergence of pressure at nodes and interface point. *Let $0 < \beta \in C[a, \alpha) \cup C(\alpha, b]$ be piecewise constant and let the discrete interface deviation $\epsilon = 0$. Then*

$$p_h(x) = p(x) \quad \forall x = x_i, i = 0, \ldots, n, \text{ and } \alpha$$

where the approximate pressure p_h is defined in (9) and the exact pressure p in (5)*.*

(ii) Superconvergence of flux at nodes and interface point. *Let $0 < \beta \in C[a, \alpha) \cup C(\alpha, b]$ be piecewise constant and let the discrete interface deviation $\epsilon = 0$. Then*

$$u_h(x) = u(x) \quad \forall x = x_i, i = 0, \ldots, n, \text{ and } \alpha,$$

where the approximate flux u_h is defined in (24) *and u is the exact flux $-\beta p'$.*

(iii) Uniform error at nodes and interface point. *Let $0 < \beta \in C[a, \alpha) \cup C(\alpha, b]$, and let the discrete interface deviation $\epsilon = 0$. Then, the errors at the nodes and the interface point are identical, i.e.,*

$$E(x) := u(x) - u_h(x) = C \qquad \forall x = x_i, i = 0, \ldots, n \text{ and } \alpha, \tag{33}$$

where C is a constant.

(iv) First order flux error at nodes and interface point. *Furthermore, if $\beta \in C^1(a, \alpha) \cap C^1(\alpha, b)$, then the constant error in (33) satisfies the following property: there exists a positive constant $\tilde{C}$ such that*

$$|u(x) - u_h(x)| \leq \tilde{C}h \tag{34}$$

for all $x = x_i, i = 0, \ldots, n$ and α.

Proof. We prove (i) first.

Let $G(x, \xi), \xi \neq \alpha$ be the Green's function satisfying

$$a(G, v) = <\delta(x - \xi), v>, \quad v \in H_0^1(a, b).$$

By working out the closed form of G satisfying the classical formulation

$$-(\beta G')' = \delta(x - \xi), \quad [G]_\alpha = 0, \quad [\beta G']_\alpha = 0, \quad G(a, \xi) = G(b, \xi) = 0,$$

we see that G can be expressed in terms of $\int_d^x \frac{1}{\beta(t)}dt$ for different d. For instance, the Green's function for $(a,b) = (0,1)$ and $\xi < \alpha$ takes the form

$$G(x,\xi) = \begin{cases} A\int_0^x \frac{1}{\beta(t)}dt, & 0 < x \le \xi, \\ (A-1)\int_\xi^x \frac{1}{\beta(t)}dt + A\int_0^\xi \frac{1}{\beta(t)}dt, & \xi \le x \le \alpha, \\ (1-A)\int_x^1 \frac{1}{\beta(t)}dt, & \alpha \le x \le 1, \end{cases}$$

where

$$A = \frac{\int_\xi^1 \frac{1}{\beta(t)}dt}{\int_0^1 \frac{1}{\beta(t)}dt}.$$

Note that G takes this simple form since $w = 0$ and note also that $G(x,\alpha) = \lim_{\xi\to\alpha} G(x,\xi)$. Since the coefficient β is piecewise constant, it is easy to see that Green's functions $G = G(\cdot, x_i), 0 \le i \le n$ and $G(\cdot,\alpha)$ are continuous piecewise linear with respect to the partition $P_\alpha : a = x_0 < x_1 < \ldots < x_k < \alpha < x_{k+1} < \ldots < x_n = b$. Let $G_h = \pi_h G \in S_h$ be the usual conforming linear interpolant of G. Then it is easy to check with $\epsilon = 0$ that

$$G - G_h = \phi\psi \in \bar{S}_h, \tag{35}$$

where $\phi \in S_h$ is the "trapezoidal" function defined by

$$\phi(x) = \begin{cases} 0 & \text{if } x \in [x_0, x_k] \cup [x_{k+1}, x_n] \\ \frac{G(\alpha)-G_h(\alpha)}{\psi(\alpha)} & \text{if } x \in [x_k, x_{k+1}] \end{cases}.$$

Now letting $G = G(x, x_i)$ and using Galerkin orthogonality property, we see that for $0 \le i \le n$

$$e(x_i) = a(G,e) = a(G - G_h, e) + a(G_h, e) = 0.$$

The case $e(\alpha)$ can be handled the same way. This proves (i).

Using (22) and (24) with $w = 0$, we have for $i = 1, \ldots, n$,

$$\begin{aligned} u_h(x_i) - u(x_i) &= \int_{x_{i-1}}^{x_i} \beta(p' - p_h')\phi_i'\, dx \\ &= \frac{\beta}{x_i - x_{i-1}}\int_{x_{i-1}}^{x_i} (p' - p_h')dx = \frac{\beta}{x_i - x_{i-1}}(p - p_h)|_{x_{i-1}}^{x_i} = 0. \end{aligned}$$

Similarly for $i = 0$. As a consequence of the nodal exactness, (30) becomes

$$u_h(\alpha) = u(x_k) + \int_{x_k}^{\alpha} f(x)dx = u(\alpha). \tag{36}$$

This completes the proof of (ii).

As for (iii), we will prove the assertion for $i = 1, \ldots, n-1$, as the endpoint cases can be handled similarly. By (22) and (24),

$$E(x_i) = -\int_{x_{i-1}}^{x_i} \beta(p' - p_h')\phi_i' dx - \int_{x_{i-1}}^{x_i} w(p - p_h)\phi_i\, dx,$$

and similarly

$$E(x_{i+1}) = -\int_{x_i}^{x_{i+1}} \beta(p' - p_h')\phi_{i+1}' dx + \int_{x_i}^{x_{i+1}} w(p - p_h)\phi_{i+1}\, dx$$
$$= \int_{x_i}^{x_{i+1}} \beta(p' - p_h')\phi_i' dx - \int_{x_i}^{x_{i+1}} w(p - p_h)\phi_i\, dx,$$

where we have used the fact that $\phi_i' = -\phi_{i+1}'$ over the interval $[x_i, x_{i+1}]$. Thus

$$E(x_i) - E(x_{i+1}) = -\int_{x_{i-1}}^{x_{i+1}} \beta(p' - p_h')\phi_i' dx - \int_{x_{i-1}}^{x_{i+1}} w(p - p_h)\phi_i\, dx$$
$$= -\,a(p - p_h, \phi_i) = 0. \tag{37}$$

As for the interface point, we use (30) and subtract from it the corresponding equation for the exact flux u to get $E(\alpha) = E(x_j)$. This completes the proof of (iii) actually for $w \geq 0$.

We now prove assertion (iv). Let $h = \max_{0\leq i\leq n} h_i, h_i = |x_{i+1} - x_i|$ is attained by element $[x_l, x_{l+1}]$ for some $l, 0 \leq l \leq n-1$. Let $\bar{\beta}$ be a fixed value $\beta(\eta), \eta \in (x_{l+1}, x_l)$. Setting $i = l$ in (23) and (25), we have

$$u(x_l) - u_h(x_l) = \int_{x_l}^{x_{l+1}} \beta(p' - p_h')\phi_l' dx$$
$$= \int_{x_l}^{x_{l+1}} (\beta - \bar{\beta})(p' - p_h')\phi_l' dx + \int_{x_l}^{x_{l+1}} \bar{\beta}(p' - p_h')\phi_l' dx$$
$$= -\,h^{-1} \int_{x_l}^{x_{l+1}} (\beta - \bar{\beta})(p' - p_h') dx - h^{-1} \int_{x_l}^{x_{l+1}} \bar{\beta}(p' - p_h') dx$$
$$= -\,h^{-1} \int_{x_l}^{x_{l+1}} (\beta - \bar{\beta})(p' - p_h') dx$$
$$-\,\bar{\beta}h^{-1}(p(x_{l+1}) - p_h(x_{l+1})) + \bar{\beta}h^{-1}(p(x_l) - p_h(x_l))$$
$$= I_1 + I_2 + I_3.$$

Here

$$|I_1| = |h^{-1} \int_{x_l}^{x_{l+1}} (\beta - \bar{\beta})(p_h' - p') dx| \leq h^{-1}(||\beta'||_{\infty,(a,\alpha)} h)(C_1 h ||p||_{2,I}) \leq Ch,$$

where we have used Thm 4.1 to estimate $|p' - p_h'|_{1,(a,x_l)}$.

$$|I_3| = |\bar{\beta}h^{-1}(p_h(x_l) - p(x_l))| \leq (||\beta||_{\infty,I} h^{-1})(C_2 h^2 ||\beta||_{\infty,I} ||p||_{2,I}) \leq Ch,$$

where we have used (38) in Thm 4.3 below to estimate $|p(x_l) - p_h(x_l)|$. The term I_2 can be similarly estimated. Finally, we remark that $[x_l, x_{l+1}]$ was chosen so that we can avoid imposing an unnecessary(due to the $h_i^{-1}, i \neq l$ terms) quasi-uniform mesh condition to prove (iv). □

It remains to prove the second order pressure error at the nodes used in the last proof.

Theorem 4.3. Second order pressure error at nodes. *Let $\beta \in C^1(a, \alpha) \cap C^1(\alpha, b)$ and $0 \leq w \in C[a, b]$. Then there exists a constant $C > 0$ such that*

$$|p(\xi) - p_h(\xi)| \leq Ch^2 ||\beta||_{\infty,I} ||p||_{2,I}, \quad \xi = x_i, 1 \leq i \leq n-1, \alpha. \tag{38}$$

where C depends on certain norms of the Green's function at ξ.

Proof. Let $G(x,\xi)$ be the Green's function satisfying

$$a(G,v) =< \delta(x-\xi), v>, \quad v \in H_0^1(a,b).$$

Then from [19], we know that for $\xi = x_i, \alpha,\ g = G(\cdot,\xi) \in H^2(\Omega)$, for $\Omega = I_i = (x_i, x_{i+1}), i \neq k$ and $\Omega = (x_k,\alpha), (\alpha, x_{k+1})$. By the local approximation estimates in Section 2, we see that there exists $\hat{I}_h g \in \bar{S}_h$, an interpolant of g, such that

$$|g - \hat{I}_h g|_{1,\Omega} \leq Ch||g''||_{0,\Omega} \tag{39}$$

for all the Ω's listed above. Now

$$e(x_i) = a(g,e) = a(g - \hat{I}_h g, e) = (\beta(g-\hat{I}_h g)', e'),$$

implies by (39) that

$$\begin{aligned} |e(x_i)| \leq & ||\beta||_{\infty,I}||g' - (\hat{I}_h g)'||_{0,I}||e'||_{0,I} \\ \leq & ||\beta||_{\infty,I}(C_1 h||g||_{2,*})(C_2 h||p||_{2,I}) \\ \leq & C_3 h^2 ||g||_{2,*}||p||_{2,I}, \end{aligned}$$

where by Thm. 4.1

$$||e'||_{0,I} \leq C_2 h||p||_{2,I}$$

and where $||g||_{2,*}^2 := \sum ||g||_{2,\Omega}^2$, the summation being over all Ω's listed above. □

Theorem 4.4. *Under the assumption that the function $w \in C[a,b]$ in (1) is non-negative, the following statements hold.*

(i) Uniform error at nodes and interface point.
Let $0 < \beta \in C([a,\alpha) \cup C(\alpha,b]$. Then, the errors at the nodes and the interface point are identical, i.e.,

$$E(x) := u(x) - u_h(x) = C \qquad \forall x = x_i, i = 0,\dots,n \text{ and } \alpha, \tag{40}$$

where C is a constant.

(ii) First order flux error at nodes and interface point.
Let $0 < \beta \in C^1(a,\alpha) \cap C^1(\alpha,b)$, then there exists a positive constant $\tilde{C}$ such that

$$|u(x) - u_h(x)| \leq \tilde{C}h \tag{41}$$

for all $x = x_i, i = 0,\dots,n$ and α.

Proof. Assertion (i) was already shown in the proof of (iii) of Thm. 4.2; see Eq. (37).

We now prove assertion (ii). Let $h = \max_{0\leq i\leq n}|x_{i+1} - x_i| = |x_{l+1} - x_l|$ for some l. Let $\bar{\beta}$ be a fixed value $\beta(\eta), \eta \in (x_{l+1}, x_l)$. Setting $i = l$ in (23) and (25), we have

$$\begin{aligned} u(x_l) - u_h(x_l) &= \int_{x_l}^{x_{l+1}} \beta(p' - p_h')\phi_l' dx + \int_{x_l}^{x_{l+1}} w(p - p_h)\phi_l dx \\ &= J_1 + J_2. \end{aligned}$$

The term J_1 has already been estimated in the proof of (iv) of Thm. 4.2, and the J_2 term is obviously $\mathcal{O}(h)$. This complete the proof. □

4.2. L_2-Convergence of the approximate flux.

Lemma 4.5. *Let $f \in C(J), J = [s,t]$ and let $\tilde{\pi}f \in P_1$ be the linear polynomial over J such that*

$$\tilde{\pi}f(s) = f(s) + \delta,$$
$$\tilde{\pi}f(t) = f(t) + \gamma.$$

That is, $\tilde{\pi}f$ interpolates using perturbed $f(s)$ and $f(t)$. Then

$$||f - \tilde{\pi}f||^2_{0,J} \leq \frac{1}{8}|J|^4||f''||^2_{0,J} + 2|J|\max\{|\delta|^2, |\gamma|^2\}, \tag{42}$$

where $|J| = t - s$.

Proof. It suffices to prove (42) for $J = [0,1]$, as the general case can be obtained by standard scaling argument. Define $\tilde{e}(x) = \tilde{\pi}f(x) - f(x)$. Let $T \in P_1$ be the linear polynomial over $[0,1]$ with $T(0) = \delta, T(1) = \gamma$. Since $\tilde{\pi}f \in P_1$, we see that $e = \tilde{e} - T$ is the solution of the boundary value problem

$$e''(x) = f(x), \quad e(0 = e(1) = 0,$$

which has a Green's function representation [19]

$$e(x) = \int_0^1 g(x,\xi)f''(\xi)d\xi,$$

where

$$g(x,\xi) = \begin{cases} (\xi - 1)x & \text{if } 0 \leq x \leq \xi, \\ \xi(x-1) & \text{if } \xi \leq x \leq 1. \end{cases}$$

Thus using the Cauchy-Schwarz inequality we see that

$$\begin{aligned} |e(x)| &\leq \int_0^1 |g(x,\xi)||f''(\xi)|d\xi \\ &\leq x(1-x)\int_0^1 |f''(\xi)|d\xi \\ &\leq \frac{1}{4}\int_0^1 |f''(\xi)| \cdot 1\, d\xi \\ &\leq \frac{1}{4}||f''||_{0,J}. \end{aligned}$$

Noting that

$$|\tilde{e}(x)|^2 \leq 2|e(x)|^2 + 2|T(x)|^2 \leq \frac{1}{8}||f''||^2_{0,J} + 2\max\{|\delta|^2, |\gamma|^2\}$$

and integrating give the result. □

Theorem 4.6. *Let u be the exact flux and let u_h be the approximate flux as defined by* (32)*. Then*

$$\|u - u_h\|_{0,I} = \mathcal{O}(h).$$

Proof. Applying Lemma 4.5 with $f = u$ and $\tilde{\pi}f = u_h$, and using (34) and definition (32), we can derive the claim easily.

□

TABLE 1. Maximum error at the nodes and the interface point of approximate pressure for $\epsilon = 0$.

Problem 1	h=1/32	h=1/64	h=1/128	h = 1/256	m	order
pErr@Nodes	3.1225e-17	6.245e-17	2.25514e-16	8.95117e-16	2	≈ exact
pErr@Nodes	8.67362e-18	1.21431e-17	2.77556e-17	2.94903e-17	5	≈ exact
pErr@Nodes	1.83881e-16	8.67362e-18	1.04083e-17	4.77049e-17	10	≈ exact
pErr@alp	2.71051e-19	1.35525e-18	3.79471e-19	1.21973e-17	2	≈ exact
pErr@alp	5.42101e-20	1.09775e-18	3.79471e-19	4.78404e-18	5	≈ exact
pErr@alp	2.51399e-18	2.10064e-19	2.23617e-19	1.0571e-18	10	≈ exact

TABLE 2. Maximum error at the nodes and the interface point of approximate flux $\epsilon = 0$.

Problem 1	h=1/32	h=1/64	h=1/128	h = 1/256	m	order
uErrEndNodes	2.91434e-16	8.88178e-16	1.27676e-15	6.25888e-15	2	≈ exact
uErrEndNodes	8.32667e-17	3.95517e-16	4.44089e-16	6.45317e-16	5	≈ exact
uErrEndNodes	1.04083e-15	8.32667e-17	1.75207e-16	5.46438e-16	10	≈ exact
uErr@alp	6.93889e-17	2.35922e-16	1.80411e-16	4.02456e-16	2	≈ exact
uErr@alp	2.77556e-17	2.42861e-16	1.38778e-16	2.17187e-15	5	≈ exact
uErr@alp	8.06646e-16	2.42861e-17	3.81639e-17	1.30104e-16	10	≈ exact

5. Numerical examples

Problem 1. Consider

$$-(\beta p')' = f(x) = x^m, \quad p(0) = p(1) = 0,$$

where m is a nonnegative integer. The interface point is located at α and

$$\beta(x) = \begin{cases} \beta^- & x \in [0, \alpha), \\ \beta^+ & x \in (\alpha, 1]. \end{cases}$$

The exact solution is

(43)
$$p(x) = \begin{cases} \dfrac{-1}{(m+1)(m+2)\beta^-} x^{m+2} + \dfrac{t^-}{\beta^-} x & x \le \alpha, \\ \dfrac{-1}{(m+1)(m+2)\beta^+} x^{m+2} + \dfrac{t^+}{\beta^+} x - \dfrac{t^+}{\beta^+} - \dfrac{-1}{(m+1)(m+2)\beta^+} & x \ge \alpha, \end{cases}$$

where

$$\begin{aligned} t^+ &= t^- \\ &= \left(\frac{\alpha - 1}{\beta^+} - \frac{\alpha}{\beta^-} \right) \\ &\quad \times \left(\frac{-\alpha^{m+2}}{(m+1)(m+2)\beta^-} + \frac{\alpha^{m+2}}{(m+1)(m+2)\beta^+} - \frac{1}{(m+1)(m+2)\beta^+} \right). \end{aligned}$$

The flux

$$u(x) = -\beta p'(x) = \frac{1}{m+1} x^{m+1} - t^- \tag{44}$$

Table 3. Maximum error at the nodes and the interface point of approximate pressure for $\epsilon \neq 0$.

Problem 1	h=1/32	h=1/64	h=1/128	h = 1/256	m	order
pErr@Nodes	3.15009e-08	8.06272e-09	2.03912e-09	5.0978e-10	2	≈ 2
pErr@Nodes	9.93075e-09	2.5418e-09	6.42839e-10	1.6071e-10	5	≈ 2
pErr@Nodes	3.17658e-09	8.13052e-10	2.05627e-10	5.14067e-11	10	≈ 2
pErr@alp	1.52793e-10	3.81982e-11	9.54955e-12	2.38737e-12	2	≈ 2
pErr@alp	4.81684e-11	1.20421e-11	3.01053e-12	7.52633e-13	5	≈ 2
pErr@alp	1.54078e-11	3.85194e-12	9.62987e-13	2.40747e-13	10	≈ 2

Table 4. Maximum error at the nodes and the interface point of approximate flux $\epsilon \neq 0$.

Problem 1	h=1/32	h=1/64	h=1/128	h = 1/256	m	order
uErrEndNodes	4.80013e-08	1.20003e-08	3.00008e-09	7.50025e-10	2	≈ 2
uErrEndNodes	1.51326e-08	3.78315e-09	9.45787e-10	2.36447e-10	5	≈ 2
uErrEndNodes	4.8405e-09	1.21012e-09	3.02531e-10	7.56332e-11	10	≈ 2
uErr@alp	4.80013e-08	1.20003e-08	3.00008e-09	7.50019e-10	2	≈ 2
uErr@alp	1.51326e-08	3.78315e-09	9.45787e-10	2.36447e-10	5	≈ 2
uErr@alp	4.8405e-09	1.21012e-09	3.02531e-10	7.5633e-11	10	≈ 2

is smooth over $[0,1]$. For the numerical runs, we set $\beta^- = 100$, $\beta^+ = 1$, $f(x) = x^m$, $\alpha = 1/\pi$ and calculate the maximum pressure and flux error at nodes

$$pErrEndNodes = \max_{1 \leq i \leq n-1} |p(x_i) - p_h(x_i)|,$$
$$uErrEndNodes = \max_{1 \leq i \leq n-1} |u(x_i) - u_h(x_i)|,$$

respectively. At the interface point α errors are given by

$$pErr@alp = |p(\alpha) - p_h(\alpha)|, \tag{45}$$
$$uErr@alp = |u(\alpha) - u_h(\alpha)|. \tag{46}$$

In Tables 1 and 4 below we list error at the nodes and the interface points for different mesh sizes and m values for pressure and flux, respectively. In Tables 1-2 we display results when discrete interface deviation $\epsilon = 0$, and in Tables 3-4, results when $\epsilon \neq 0$. In Tables 1-2, the pressure and the flux at the nodes and at the interface point are exact, as predicted by assertions (i) and (ii) of Thm 4.2. In Tables 3-4, convergence rate of the pressure at the nodes and at the interface are of second order, as predicted by assertions of Thm. 4.3. The convergence rate of the flux is one order higher than predicted by assertion (iv) of Thm 4.2.

Problem 2. Consider

$$-(\beta p')' + qp = f(x), \quad p(0) = p(1) = 0,$$

where m and α are defined in a same way as in Problem 1. We used the same exact solution $p(x)$ and $u(x)$ defined in (43) and (44). For the numerical simulation we set $q = 1$, $f(x) = x^m + p(x)$, and β and α values are same as in Problem 1. Numerical results in Tables 5-6 confirm the convergence rates for pressure and flux at nodes are as predicted in Thm 4.3 and Thm 4.4, respectively.

Problem 3. Consider

$$-(\beta p')' = f(x) = 2x, \quad p(0) = p(1) = 0,$$

TABLE 5. Maximum error at the nodes and the interface point of approximate pressure for $\epsilon \neq 0$.

Problem 2	h=1/32	h=1/64	h=1/128	1/256	m	order
pErr@Nodes	2.12816e-06	5.32012e-07	1.33033e-07	3.3259e-08	2	≈ 2
pErr@Nodes	9.41154e-07	2.35043e-07	5.87781e-08	1.46934e-08	5	≈ 2
pErr@Nodes	3.86372e-07	9.65054e-08	2.40991e-08	6.02308e-09	10	≈ 2
pErr@alp	2.66184e-08	6.67455e-09	1.67527e-09	4.18978e-10	2	≈ 2
pErr@alp	8.35828e-09	2.08711e-09	5.21795e-10	1.30441e-10	5	≈ 2
pErr@alp	2.66642e-09	6.63353e-10	1.65634e-10	4.13957e-11	10	≈ 2

TABLE 6. Maximum error at the nodes and the interface point of approximate flux for $\epsilon \neq 0$.

Problem 2	h=1/32	h=1/64	h=1/128	1/256	m	order
uErrEndNodes	4.24662e-05	2.12656e-05	1.06348e-05	5.31814e-06	2	≈ 1
uErrEndNodes	1.87214e-05	9.368e-06	4.68535e-06	2.34273e-06	5	≈ 1
uErrEndNodes	7.62747e-06	3.81564e-06	1.90887e-06	9.54486e-07	10	≈ 1
uErr@alp	4.24662e-05	2.12656e-05	1.06348e-05	5.31814e-06	2	≈ 1
uErr@alp	1.87214e-05	9.368e-06	4.68535e-06	2.34273e-06	5	≈ 1
uErr@alp	7.62747e-06	3.81563e-06	1.90488e-06	9.54471e-07	10	≈ 1

TABLE 7. Maximum error at the nodes and the interface point of approximate pressure for $\epsilon \neq 0$.

Problem 3	h=1/32	h=1/64	h=1/128	1/256	m	order
pErr@Nodes	1.20395e-04	3.50209e-05	9.70887e-06	2.44886e-06	2	≈ 2
pErr@alp	9.88708e-05	2.63583e-05	6.89375e-06	1.72811e-06	2	≈ 2
uErrEndNodes	3.33275e-04	8.86077e-05	2.31149e-05	5.80029e-06	5	≈ 1
uErr@alp	3.33468e-04	8.86558e-05	2.31269e-05	5.8033e-06	10	≈ 2

where m is a nonnegative integer. The interface point is located at α and

$$\beta(x) = \begin{cases} x^2 + 1 & x \in [0, \alpha), \\ x^2 & x \in (\alpha, 1]. \end{cases}$$

The exact solution is

$$p(x) = \begin{cases} -x + (1-d)\tan^{-1} x & x \in [0, \alpha), \\ -x + \dfrac{d}{x} + (1-d) & x \in (\alpha, 1]. \end{cases}$$

where

$$d = \frac{\alpha \tan^{-1}\alpha - \alpha}{1 - \alpha + \alpha \tan^{-1}\alpha}.$$

The flux

$$u(x) = -\beta p' = x^2 + d$$

is continuous. Numerical results in Table 7 confirm the convergence rates for pressure and flux at nodes are as predicted in Thm 4.3 and Thm 4.4, respectively.

References

[1] C. Attanayake and S. H. Chou, An immersed interface method for Pennes bioheat transfer equation, Discrete and Continuous Dynamical Systems Series B, 2015(20), 323–337.

[2] T. Belytschko and T. Black, Elastic crack growth in finite elements with minimal remeshing, Int. J. Numer. Methods Eng. 1999(45), 601–620.

[3] E. Burman, J. Guzman, M. Sanchez-Uribe, and M. Sarkis, Robust flux error estimation of an unfitted Nitsche method for high-contrast interface problems, arXiv:1602.00603 [math.NA], (2016).

[4] W. Cao, X. Zhang, and Z. Zhang, Superconvergence of immersed finite element methods for interface problems, Advances in Computational Mathematics, 2017(43), 795-821.

[5] W. Cao, X. Zhang, Z. Zhang and Q. Zou, Superconvergence of immersed finite volume methods for one-dimensional interface problem, J. Scientific Computing, 2017(73), 543-565.

[6] S. H. Chou, An immersed finite element method with interface flux capturing recovery, Discrete and Continuous Dynamical Systems Series B, 2012(17), 2343–2357.

[7] S. H. Chou and S. Tang, Conservative $P1$ conforming and nonconforming Galerkin FEMs: effective flux evaluation via a nonmixed method approach, SIAM J. Numer. Anal. 2000(38), 660–680.

[8] T.-P. Fries and T. Belytschko, The extended/generalized finite element method: an overview of the method and its applications, Int. J. Numer. Methods Eng. 2010(84), 253–304.

[9] H. Guo, X. Yang, Z. Zhang, Superconvergence of partially penalized immersed finite element methods, IMA Journal of Numerical Analysis, 2018(38) 2123–2144.

[10] H. Guo, X. Yang, Gradient recovery for elliptic interface problem:II. Immersed finite element methods, Journal of Computational Physics, 2017(338), 606–619.

[11] A. Hansbo and P. Hansbo, An unfitted finite element method, based on Nitsche's method, for elliptic interface problems, Comput. Methods Appl. Mech. Eng., (2002)(191), 5537–5552.

[12] X. He, Bilinear immersed finite elements for interface problems, Ph.D. thesis, Virginia Tech., Blacksberg, VA, 2009.

[13] Z. Li, The immersed interface method using a finite element formulation, Applied Numerical Mathemtics, 1998(27), 253–267.

[14] Z. Li and K. Ito, The immersed interface method: Numerical solutions of PDEs involving interfaces and irregular domains, SIAM, Philadelphia, 2006.

[15] Z. Li, T. Lin, Y. Lin, and R. C. Rogers, An immersed finite element space and its approximation capability, Numer. Methods. Partial Differential Equations, 2004(20), 338–367.

[16] Z. Li, T. Lin, and X. Wu, New Cartesian grid methods for interface problems using the finite element formulation, Numer. Math., 2003(96), 61–98.

[17] N. Moes, J. Dolbow, and T. Belytschko, A finite element method for crack growth without remeshing, Int. J. Numer. Methods Eng., 1999(46), 131–150.

[18] T. Lin, Y. Lin, R. Rogers, and M. L. Ryan, A rectangular immersed finite element space for interface problems, Advances in Computation: Theory and Practice, 2001(7), 107–114.

[19] I. Stakgold, Green's functions and boundary value problems, 2nd ed., Wiley-Interscience, New York, 1998.

[20] H. Wang, J. Chen, P. Sun, and F. Qin, A conforming enriched finite element method for elliptic interface problems, Applied Numerical Mathematics, 2018(127), 1–17.

[21] H. Wu and Y. Xiao, An unfitted hp-interface penalty finite element method for elliptic interface problems, J. of Computational Mathematics, 2019(37), 316–339.

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