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CONVERGENCE ANALYSIS OF ADI ORTHOGONAL SPLINE COLLOCATION WITHOUT PERTURBATION TERMS

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Abstract. For the heat equation on a rectangle and nonzero Dirichlet boundary conditions, we consider an ADI orthogonal spline collocation method without perturbation terms, to specify boundary values of intermediate solutions at half time levels on the vertical sides of the rectangle. We show that, at each time level, the method has optimal convergence rate in the L^2 norm in space. Numerical results for splines of orders $4, 5, 6$ confirm our theoretical convergence rates and demonstrate suboptimal convergence rates in the H^1 norm. We also demonstrate numerically that the scheme without the perturbation terms is applicable to variable coefficient problems yielding the same convergence rates obtained for the heat equation.

Key words. Convergence, alternating direction implicit method, orthogonal spline collocation, perturbation terms.

1. Introduction

The alternating direction implicit (ADI) method is a popular and useful technique for solving partial differential equations on rectangles. Such methods reduce the solution of multi-dimensional problems to the solution of a collection of independent discrete one-dimensional problems in the coordinate directions. ADI techniques have been used in recent years to solve a variety of problems in various fields such as biology, engineering, finance, physics (see, for example, [1, 8, 13, 14, 16, 19, 23, 25, 31, 32]).

ADI methods were first introduced, in the context of finite differences, by Peaceman and Rachford [20] to solve parabolic and elliptic problems with zero Dirichlet boundary conditions. When extending the ADI finite difference method to nonzero Dirichlet boundary conditions, some authors included additional terms, called 'perturbation terms', to specify intermediate solutions at half time levels on vertical sides of the rectangle (see, for example, [12, (2.8)], [27, (13), (14) on pg. 549, (35) on pg. 555], [29, (7.3.11)], [30, (4.4.20), (4.4.21)]). The inclusion of perturbation terms preserves the optimal convergence rate in the discrete H^1 norm in space. However, it has been shown in [17, 3] for the heat equation and a variable coefficient parabolic equation, respectively, that the ADI finite difference scheme without perturbation terms has optimal convergence rate in the discrete L^2 norm in space. This important finding opened the door to an application of the ADI finite difference method to the solution of parabolic equations with Dirichlet boundary conditions on nonrectangular sets. In [4], for the first time in the literature, we have formulated and analyzed an ADI finite difference method without the perturbation terms on a convex set.

Over the past several years ADI orthogonal spline collocation (OSC) has proved to be an efficient technique to solve time dependent partial differential equation problems on rectangles and rectangular polygons (see [5, 6, 13, 14, 15, 16, 22, 24, 26] and references therein). The ADI OSC scheme was analyzed in [15] for the solution of the heat equation with zero Dirichlet boundary conditions on a rectangle. The

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ADI OSC scheme with perturbation terms was analyzed in [5] for the solution of a variable coefficient parabolic equation with nonzero Dirichlet boundary conditions on a rectangle. The purpose of the present paper is to prove the optimal convergence rate in the L^2 norm of the ADI OSC scheme without perturbation terms for the solution of the heat equation

$$u_t + (L_1 + L_2)u = f(x,y,t), \quad (x,y,t) \in \Omega \times (0,T], \tag{1}$$

where $\Omega = (a,b) \times (c,d)$,

$$L_1 u = -u_{xx}, \quad L_2 u = -u_{yy}, \tag{2}$$

with the initial and nonzero Dirichlet boundary conditions given by

$$u(x,y,0) = g_1(x,y), \quad (x,y) \in \overline{\Omega}, \tag{3}$$

$$u(x,y,t) = g_2(x,y,t), \quad (x,y,t) \in \partial\Omega \times (0,T]. \tag{4}$$

While we define the ADI OSC scheme and give convergence analysis for the heat equation, we demonstrate by a numerical example that the scheme without the perturbation terms is applicable to variable coefficient parabolic problems yielding the same convergence rates as those for the heat equation. We expect the result of this paper to impact applications and convergence analysis of the ADI OSC method for parabolic equations with nonzero Dirichlet boundary conditions on non-rectangular sets [7].

In section 2 we give Preliminaries. The ADI OSC schemes with and without perturbation terms are described in section 3. Convergence analysis of the ADI scheme without perturbation terms is carried out in section 4. In section 5, errors and convergence rates of the ADI OSC schemes with and without perturbation terms are presented for splines of orders $4, 5, 6$. Concluding remarks are given in section 6.

2. Preliminaries

Let $\{x_i\}_{i=0}^{N_x}$ and $\{y_j\}_{j=0}^{N_y}$ be respectively partitions (in general nonuniform) of $[a,b]$ and $[c,d]$ such that

$$a = x_0 < x_1 < \cdots < x_{N_x-1} < x_{N_x} = b, \qquad c = y_0 < y_1 < \cdots < y_{N_y-1} < y_{N_y} = d.$$

Let $I_i^x = (x_{i-1}, x_i)$, $I_j^y = (y_{j-1}, y_j)$, $h_i^x = x_i - x_{i-1}$, $h_j^y = y_j - y_{j-1}$, and let

$$\underline{h}_x = \min_i h_i^x, \quad \overline{h}_x = \max_i h_i^x, \quad \underline{h}_y = \min_j h_j^y, \quad \overline{h}_y = \max_j h_j^y,$$

$$h = \max(\overline{h}_x, \overline{h}_y).$$

We assume that a collection of the partitions $\{x_i\}_{i=0}^{N_x} \times \{y_j\}_{j=0}^{N_y}$ of Ω is regular, that is, there exist positive constants σ_1, σ_2, and σ_3 such that for every partition in the collection, we have

$$\sigma_1 \overline{h}_x \le \underline{h}_x, \qquad \sigma_1 \overline{h}_y \le \underline{h}_y, \qquad \sigma_2 \le \frac{\overline{h}_x}{\overline{h}_y} \le \sigma_3.$$

In the following, we assume that a natural number $r \ge 3$. Let P_r denote the set of polynomials of degree $\le r$. Let $\mathcal{M}_x$, $\mathcal{M}_x^0$, $\mathcal{M}_y$, and $\mathcal{M}_y^0$ be the spaces defined by

$$\mathcal{M}_x = \{v \in C^1[a,b] : v|_{[x_{i-1},x_i]} \in P_r, i = 1, \dots, N_x\},$$
$$\mathcal{M}_x^0 = \{v \in \mathcal{M}_x : v(a) = v(b) = 0\},$$
$$\mathcal{M}_y = \{v \in C^1[c,d] : v|_{[y_{j-1},y_j]} \in P_r, j = 1, \dots, N_y\},$$
$$\mathcal{M}_y^0 = \{v \in \mathcal{M}_y : v(c) = v(d) = 0\}.$$

The dimensions of $\mathcal{M}_x$ and $\mathcal{M}_y$ are $(r-1)N_x+2$ and $(r-1)N_y+2$, respectively. Let $\mathcal{M}$ and $\mathcal{M}^0$ be the spaces defined by

$$\mathcal{M} = \mathcal{M}_x \otimes \mathcal{M}_y, \qquad \mathcal{M}^0 = \mathcal{M}_x^0 \otimes \mathcal{M}_y^0.$$

Remark 2.1. *$\mathcal{M}$ ($\mathcal{M}^0$) is the set of all functions that are finite linear combinations of products $\phi(x)\psi(y)$, where $\phi \in \mathcal{M}_x$ ($\mathcal{M}_x^0$) and $\psi \in \mathcal{M}_y$ ($\mathcal{M}_y^0$).*

Let $\{\xi_k\}_{k=1}^{r-1}$ and $\{\omega_k\}_{k=1}^{r-1}$ be respectively the nodes and weights of the $(r-1)$-point Gauss-Legendre quadrature on $(0,1)$. Note that

$$\omega_k > 0, \quad k = 1, \dots, r-1, \quad \sum_{k=1}^{r-1} \omega_k = 1. \tag{5}$$

Set $\mathcal{G}_x = \{\xi_{i,k}^x\}_{i=1,k=1}^{N_x,r-1}$, $\mathcal{G}_y = \{\xi_{j,l}^y\}_{j=1,l=1}^{N_y,r-1}$, where

$$\xi_{i,k}^x = x_{i-1} + h_i^x \xi_k, \quad \xi_{j,l}^y = y_{j-1} + h_j^y \xi_l. \tag{6}$$

Set

$$\mathcal{G} = \{(\xi^x, \xi^y) : \xi^x \in \mathcal{G}_x, \xi^y \mathcal{G}_y\}, \quad \overline{\mathcal{G}}_x = \mathcal{G}_x \cup \{a,b\}, \quad \overline{\mathcal{G}}_y = \mathcal{G}_y \cup \{c,d\}.$$

For v and w defined on $\mathcal{G}$, let $(v,w)_{\mathcal{G}}$ and $\|v\|_{\mathcal{G}}$ be given by

$$(v,w)_{\mathcal{G}} = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} (v,w)_{\mathcal{G}_{i,j}}, \quad \|v\|_{\mathcal{G}}^2 = (v,v)_{\mathcal{G}}, \tag{7}$$

where

$$(v,w)_{\mathcal{G}_{i,j}} = h_i^x h_j^y \sum_{k=1}^{r-1} \sum_{l=1}^{r-1} \omega_k \omega_l (vw)(\xi_{i,k}^x, \xi_{j,l}^y), \quad \|v\|_{\mathcal{G}_{i,j}}^2 = (v,v)_{\mathcal{G}_{i,j}}. \tag{8}$$

Corollary 5.3 of [21] implies that $v \in \mathcal{M}^0$ is uniquely defined by its values at the points of $\mathcal{G}$.

Let $\{t_n\}_{n=0}^{N_t}$ be a partition of $[0,T]$ such that $t_n = n\tau$, where $\tau = T/N_t$, and let $t_{n+1/2} = (t_n + t_{n+1})/2$, $n = 0, \dots, M-1$. We introduce the notation

$$\partial_t v^n = \frac{v^{n+1} - v^n}{\tau}. \tag{9}$$

For a function s defined on $\overline{\Omega} \times [0,T]$, we use the notation

$$s^n = s(\cdot, t_n), \quad s^{n+1/2} = s(\cdot, t_{n+1/2}).$$

Assume s is a function defined on $\overline{\Omega}$. Then the Gauss interpolant $s_{\mathcal{G}} \in \mathcal{M}$ of s is defined by

$$s_{\mathcal{G}}(\xi^x, \xi^y) = s(\xi^x, \xi^y), \quad \xi^x \in \overline{\mathcal{G}}_x, \quad \xi^y \in \overline{\mathcal{G}}_y; \tag{10}$$

and for $\alpha = a, b$, the Gauss interpolant $s_{\mathcal{G}}^n(\alpha, \cdot) \in \mathcal{M}_y$ of $s^n(\alpha, y)$, $y \in [c,d]$, is defined by

$$s_{\mathcal{G}}^n(\alpha, \xi^y) = s^n(\alpha, \xi^y), \quad \xi^y \in \overline{\mathcal{G}}_y; \tag{11}$$

and for $\alpha = c, d$, the Gauss interpolant $s_{\mathcal{G}}^n(\cdot, \alpha) \in \mathcal{M}_x$ of $s^n(x, \alpha)$, $x \in [a,b]$, is defined by

$$s_{\mathcal{G}}^n(\xi^x, \alpha) = s^n(\xi^x, \alpha), \quad \xi^x \in \overline{\mathcal{G}}_x. \tag{12}$$

As in [11], for $r > 3$, let $0 < \eta_1 < \eta_2 \ldots < \eta_{r-3} < 1$, be the simple zeros of the polynomial

$$\frac{d^{r-3}}{dt^{r-3}}[t^{r-1}(t-1)^{r-1}],$$

and let $\{\eta^x_{i,m}\}_{i,m=1}^{N_x,r-3}$ and $\{\eta^y_{j,n}\}_{j,n=1}^{N_y,r-3}$ be the points given by

$$\eta^x_{i,m} = x_{i-1} + h^x_i \eta_m, \quad \eta^y_{j,n} = y_{j-1} + h^y_j \eta_n.$$

Assume s is defined on $\overline{\Omega}$. Then, the Hermite interpolant $s_{\mathcal{H}} \in \mathcal{M}$ of s is defined by

$$\begin{aligned}
&(s_{\mathcal{H}} - s)(\eta^x_{i,m}, \eta^y_{j,n}) = 0, \quad 1 \le i \le N_x, \quad 1 \le j \le N_y \\
&\frac{\partial^k (s_{\mathcal{H}} - s)}{\partial x^k}(x_i, \eta^y_{j,n}) = 0, \quad 0 \le i \le N_x, \quad 1 \le j \le N_y, \\
&\frac{\partial^l (s_{\mathcal{H}} - s)}{\partial y^l}(\eta^x_{i,m}, y_j) = 0, \quad 1 \le i \le N_x, \quad 0 \le j \le N_y, \\
&\frac{\partial^{k+l} (s_{\mathcal{H}} - s)}{\partial x^k \partial y^l}(x_i, y_j) = 0, \quad 0 \le i \le N_x, \quad 0 \le j \le N_y,
\end{aligned} \tag{13}$$

where $m, n = 1, \ldots, r-3$ and $k, l = 0, 1$. It is known [9] that $s_{\mathcal{H}}$ exists and is unique. For a sufficiently smooth function $s(x, y, t)$ and $\alpha = a, b$, the Hermite interpolant $s^k_{\mathcal{H}}(\alpha, \cdot) \in \mathcal{M}_y$ of $s^k(\alpha, y)$, $y \in [c, d]$, is defined by

$$\begin{aligned}
&(s^k_{\mathcal{H}} - s^k)(\alpha, \eta^y_{j,n}) = 0, \quad 1 \le j \le N_y, \quad n = 1, \ldots, r-3, \\
&\frac{\partial^l (s^k_{\mathcal{H}} - s^k)}{\partial y^l}(\alpha, y_j) = 0, \quad 0 \le j \le N_y, \quad l = 0, 1,
\end{aligned} \tag{14}$$

and for $\alpha = c, d$, the Hermite interpolant $s^k_{\mathcal{H}}(\cdot, \alpha) \in \mathcal{M}_x$ of $s^k(x, \alpha)$, $x \in [a, b]$, is defined by

$$\begin{aligned}
&(s^k_{\mathcal{H}} - s^k)(\eta^x_{i,m}, \alpha) = 0, \quad 1 \le i \le N_x, \quad m = 1, \ldots, r-3, \\
&\frac{\partial^l (s^k_{\mathcal{H}} - s^k)}{\partial x^l}(x_i, \alpha) = 0, \quad 0 \le i \le N_x, \quad l = 0, 1.
\end{aligned} \tag{15}$$

Throughout the paper, C denotes a generic positive constant which may depend on σ_1, σ_2, σ_3, and r but is independent of h and τ.

In the following lemma we give approximation results of $s_{\mathcal{H}}$.

Lemma 2.1. *For s defined on $\overline{\Omega}$, let $s_{\mathcal{H}} \in \mathcal{M}$ be the Hermite interpolant of s defined in (13). If $s \in H^{r+1}(\Omega)$, then*

$$\left\| \frac{\partial^l (s - s_{\mathcal{H}})}{\partial y^l} \right\|_{\mathcal{G}} \le C h^{r+1-l} \|s\|_{H^{r+1}(\Omega)}, \quad l = 0, 2. \tag{16}$$

If $s \in H^{r+2}(\Omega)$, then

$$\|\Delta(s - s_{\mathcal{H}})\|_{\mathcal{G}} \le C h^r \|s\|_{H^{r+2}(\Omega)}. \tag{17}$$

If $s \in H^{r+3}(\Omega)$, then, for $i = 1, \ldots, N_x$, $j = 1, \ldots, N_y$,

$$|(\Delta(s - s_{\mathcal{H}}), 1)_{\mathcal{G}_{i,j}}| \le C h^{r+1} (h^x_i h^y_j)^{1/2} \|s\|_{H^{r+3}(I^x_i \times I^y_j)}. \tag{18}$$

Proof. (16) follows from [2, (2.19)] with $i = 0$ and $j = 0, 2$. (17) follows from [2, (2.20)] for $i = 0, 2$. (18) follows from an inequality used in the proof of [2, (2.22)] for $i = 0, 2$; see the last unnumbered equation in the proof of Lemma 2.4 in [2] and the first half of the same proof. □

For $k = 1, 2$, we introduce the operators $A_k : \mathcal{M}^0 \to \mathcal{M}^0$ defined by

$$A_k v(\xi) = L_k v(\xi) \quad \xi \in \mathcal{G}, \tag{19}$$

where L_1, L_2 are given in (2). Properties of the operators A_k are stated in the following lemma.

Lemma 2.2. *We have*

$$(A_k v, w)_{\mathcal{G}} = (v, A_k w)_{\mathcal{G}}, \quad v, w \in \mathcal{M}^0, \quad k = 1, 2, \tag{20}$$

$$(A_k v, v)_{\mathcal{G}} > 0, \quad 0 \neq v \in \mathcal{M}^0, \quad k = 1, 2, \tag{21}$$

$$A_1 A_2 = A_2 A_1, \tag{22}$$

$$(A_1 A_2 v, v)_{\mathcal{G}} \geq 0, \quad v \in \mathcal{M}^0. \tag{23}$$

Proof. (20) and (21) follow from (7), (8), (19), (2), and [11, Lemma 3.1]. It follows from (7), (8), and [11, Lemma 3.1] that

$$(\phi_1''(x)\psi_1(y), \phi_2(x)\psi_2''(y))_{\mathcal{G}} = (\phi_1(x)\psi_1''(y), \phi_2''(x)\psi_2(y))_{\mathcal{G}},$$
$$\phi_1, \phi_2 \in \mathcal{M}_x^0,\ \psi_1, \psi_2 \in \mathcal{M}_y^0,$$

which, by Remark 2.1, implies

$$(A_1 v, A_2 w)_{\mathcal{G}} = (A_2 v, A_1 w)_{\mathcal{G}}, \quad v, w \in \mathcal{M}^0.$$

The last unnumbered equation and (20) yield (22). It is known (see [28, Theorem 4, Section 5.1]) that for any self-adjoint, non-negative operator A there exists a unique self-adjoint, non-negative square root $A^{1/2}$ which commutes with every operator commuting with A. Hence, for $v \in \mathcal{M}^0$, using (20), (21), both with $k = 1$, (22), and (21) with $k = 2$, we have

$$(A_1 A_2 v, v)_{\mathcal{G}} = (A_1^{1/2} A_1^{1/2} A_2 v, v)_{\mathcal{G}} = (A_2 A_1^{1/2} v, A_1^{1/2} v)_{\mathcal{G}} \geq 0,$$

which gives (23). □

In the analysis, we often use the ϵ-inequality

$$\alpha\beta \leq \epsilon\alpha^2 + \frac{1}{4\epsilon}\beta^2, \quad \alpha, \beta \in R, \quad \epsilon > 0, \tag{24}$$

and the inequality

$$(\alpha + \beta)^2 \leq 2(\alpha^2 + \beta^2), \quad \alpha, \beta \in R. \tag{25}$$

We also require the discrete Gronwall inequality [18] and a summation by parts in t stated in the following lemmas.

Lemma 2.3. *If $\alpha_k, \beta_k, k = 0, \ldots, N_t$ are non-negative real numbers such that $\beta_k \leq \beta_{k+1}$ and*

$$\alpha_k \leq \beta_k + \gamma\tau \sum_{n=0}^{k-1} \alpha_n, \quad k = 0, \ldots, N_t,$$

where γ is a positive constant, then

$$\alpha_n \leq e^{\gamma\tau n}\beta_n, \quad n = 0, \ldots, N_t.$$

Lemma 2.4. *Assume $w^0 = 0$. Then*

$$\tau \sum_{n=0}^{k-1} (v^n, \partial_t w^n)_{\mathcal{G}} = (v^{k-1}, w^k)_{\mathcal{G}} - \tau \sum_{n=1}^{k-1} (\partial_t v^{n-1}, w^n)_{\mathcal{G}}, \quad k = 1, \ldots, N_t.$$

Proof. Since $(v, w)_{\mathcal{G}}$ of (7) is linear in v and w, using (9), we verify that the left-hand side is equal to the right-hand side in the desired equation. □

3. ADI OSC Schemes

For $n = 0, \dots, N_t - 1$, we find $U^{n+1} \in \mathcal{M}$ such that

$$\left[\frac{U^{n+1/2} - U^n}{\tau/2} + L_1 U^{n+1/2} + L_2 U^n\right](\xi) = f^{n+1/2}(\xi), \quad \xi \in \mathcal{G}, \tag{26}$$

$$\left[\frac{U^{n+1} - U^{n+1/2}}{\tau/2} + L_1 U^{n+1/2} + L_2 U^{n+1}\right](\xi) = f^{n+1/2}(\xi), \quad \xi \in \mathcal{G}, \tag{27}$$

where

$$U^0 = g_{1,h}, \tag{28}$$

$$\begin{aligned} U^{n+1}(\alpha, y) &= g_{2,h}^{n+1}(\alpha, y), \quad y \in [c,d], \quad \alpha = a, b, \\ U^{n+1}(x, \alpha) &= g_{2,h}^{n+1}(x, \alpha), \quad x \in [a,b], \quad \alpha = c, d, \end{aligned} \tag{29}$$

and where $g_{1,h}$, $g_{2,h}^{n+1}(\alpha, \cdot)$, and $g_{2,h}^{n+1}(\cdot, \alpha)$ are the Gauss or Hermite interpolants of g_1, $g_2^{n+1}(\alpha, \cdot)$, and $g_2^{n+1}(\cdot, \alpha)$, respectively (cf. (10)–(15)). For $n = 0, \dots, N_t - 1$ and $\xi^y \in \mathcal{G}_y$, $U^{n+1/2}(\cdot, \xi^y) \in \mathcal{M}_x$ and

$$U^{n+1/2}(\alpha, \xi^y) = \left[\frac{1}{2}(U^{n+1} + U^n) + \frac{\tau}{4} L_2 (U^{n+1} - U^n)\right](\alpha, \xi^y), \quad \alpha = a, b, \tag{30}$$

if we use perturbation terms or

$$U^{n+1/2}(\alpha, \xi^y) = \frac{1}{2}(U^{n+1} + U^n)(\alpha, \xi^y), \quad \alpha = a, b, \tag{31}$$

if we do not use perturbation terms. Using (21), one can show uniqueness, and hence existence, of $U^{n+1/2}$ and U^{n+1} satisfying (26), (31) and (27), (29), respectively.

4. Convergence Analysis

In our convergence analysis of the ADI OSC scheme (26)–(29) and (31), we assume that initial and boundary conditions are approximated using the Hermite interpolant; see (28), (29). We also assume that the solution u of (1)–(4) is a sufficiently smooth function on $\overline{\Omega} \times [0, T]$. For sufficiently smooth function s on $\overline{\Omega} \times [0, T]$, we use the notation

$$\|s\|_{C([\overline{\Omega}, [0,T])} = \max_{(x,y,t) \in \overline{\Omega} \times [0,T]} |s(x, y, t|,$$

$$\|s\|_{C([0,T], H^l(\Omega))} = \max_{0 \le t \le T} \|s(\cdot, t)\|_{H^l(\Omega)},$$

assuming that both quantities exist and are finite.

4.1. Error Equations.

For $n = 0, \dots, N_t$, let $z^n \in \mathcal{M}$ be defined by

$$z^n = U^n - u^n_{\mathcal{H}}, \tag{32}$$

where, for fixed t_n, $u^n_{\mathcal{H}} = u_{\mathcal{H}}(\cdot, t_n)$ is the Hermite interpolant of u^n or equivalently $u(\cdot, t_n)$ (cf. (13)). Then it follows from (32), (3), (4), (28), (29) that

$$z^0 = 0, \tag{33}$$

and for $n = 1, \dots, N_t$,

$$z^n = 0 \quad \text{on} \quad \partial\Omega. \tag{34}$$

For $n = 0, \dots, N_t - 1$ and $\xi^y \in \mathcal{G}_y$, we introduce $z^{n+1/2}(\cdot, \xi^y) \in \mathcal{M}_x$ defined by

$$z^{n+1/2}(x, \xi^y) = U^{n+1/2}(x, \xi^y) - \widetilde{w}^{n+1/2}(x, \xi^y), \quad x \in [a, b], \tag{35}$$

where $\widetilde{w}^{n+1/2}(\cdot,\xi^y) \in \mathcal{M}_x$ is given by

$$\widetilde{w}^{n+1/2}(x,\xi^y) = \frac{1}{2}\left(u_{\mathcal{H}}^{n+1} + u_{\mathcal{H}}^{n}\right)(x,\xi^y), \quad x \in [a,b]. \tag{36}$$

Using (35), (31), (36), (32), (33), and (34), we obtain

$$z^{n+1/2}(\alpha,\xi^y) = 0, \quad \alpha = a,b, \quad \xi^y \in \mathcal{G}_y. \tag{37}$$

The truncation errors of the scheme, for $n = 0,\dots,N_t-1$, are defined by

$$T_1^n(\xi) = f^{n+1/2}(\xi) - \left[\frac{\widetilde{w}^{n+1/2} - u_{\mathcal{H}}^n}{\tau/2} + L_1\widetilde{w}^{n+1/2} + L_2 u_{\mathcal{H}}^n\right](\xi), \quad \xi \in \mathcal{G}, \tag{38}$$

and

$$T_2^n(\xi) = f^{n+1/2}(\xi) - \left[\frac{u_{\mathcal{H}}^{n+1} - \widetilde{w}^{n+1/2}}{\tau/2} + L_1\widetilde{w}^{n+1/2} + L_2 u_{\mathcal{H}}^{n+1}\right](\xi), \quad \xi \in \mathcal{G}. \tag{39}$$

Equations (38) and (39) indicate by how much $u_{\mathcal{H}}^n$, $u_{\mathcal{H}}^{n+1}$, and $\widetilde{w}^{n+1/2}$ fail to satisfy equations (26) and (27). For $n = 0,\dots,N_t-1$, we introduce T_+^n and T_-^n in $\mathcal{M}^0$ defined by

$$T_+^n(\xi) = \frac{1}{2}\left(T_1^n + T_2^n\right)(\xi), \quad T_-^n(\xi) = \frac{\tau}{4}\left(T_2^n - T_1^n\right)(\xi), \quad \xi \in \mathcal{G}. \tag{40}$$

Using (32), (35), (26), (27), (38), and (39), for $n = 0,\dots,N_t-1$, we obtain

$$\left[\frac{z^{n+1/2} - z^n}{\tau/2} + L_1 z^{n+1/2} + L_2 z^n\right](\xi) = T_1^n(\xi), \quad \xi \in \mathcal{G}, \tag{41}$$

$$\left[\frac{z^{n+1} - z^{n+1/2}}{\tau/2} + L_1 z^{n+1/2} + L_2 z^{n+1}\right](\xi) = T_2^n(\xi), \quad \xi \in \mathcal{G}, \tag{42}$$

Subtracting (42) from (41), and multiplying by $\tau/4$, we obtain

$$z^{n+1/2}(\xi) = \frac{1}{2}(z^{n+1}+z^n)(\xi) + \frac{\tau}{4}L_2(z^{n+1}-z^n)(\xi) - \frac{\tau}{4}(T_2^n - T_1^n)(\xi), \quad \xi \in \mathcal{G}. \tag{43}$$

Let $w^{n+1/2} \in \mathcal{M}^0$ be defined by

$$w^{n+1/2}(\xi) = L_2(z^{n+1} - z^n)(\xi), \quad \xi \in \mathcal{G}. \tag{44}$$

Then

$$z^{n+1/2}(x,\xi^y) = \frac{1}{2}(z^{n+1}+z^n)(x,\xi^y) + \frac{\tau}{4}w^{n+1/2}(x,\xi^y) - T_-^n(x,\xi^y),\ x \in [a,b],\ \xi^y \in \mathcal{G}_y,$$

since the left- and right-hand sides are in $\mathcal{M}_x^0$ and they are equal to one another for all $x \in \mathcal{G}_x$ by (43), (44), and (40). Substituting (43) in place of the first $z^{n+1/2}$ in (41), using (9), and substituting the last unnumbered equation in place of the second $z^{n+1/2}$ in (41), we obtain

$$\partial_t z^n(\xi) + \frac{1}{2}L_2(z^{n+1}-z^n)(\xi) - \frac{1}{2}(T_2^n - T_1^n)(\xi)$$
$$+\frac{1}{2}L_1(z^{n+1}+z^n)(\xi) + \frac{\tau}{4}L_1 w^{n+1/2}(\xi) - L_1(T_-^n)(\xi) + L_2 z^n(\xi) = T_1^n(\xi), \quad \xi \in \mathcal{G}.$$

Rearranging the above unnumbered equation and using (40), we obtain

$$\begin{aligned}\partial_t z^n(\xi) + \frac{1}{2}(L_1+L_2)(z^{n+1}+z^n)(\xi) + \frac{\tau}{4}L_1 w^{n+1/2}(\xi) \\ = T_+^n(\xi) + L_1(T_-^n)(\xi), \quad \xi \in \mathcal{G}.\end{aligned} \tag{45}$$

It follows from (44) and (19) that $w^{n+1/2} = \tau A_2 \partial_t z^n$. Hence, using (19), we rewrite (45) in the operator form as

$$\partial_t z^n + \frac{1}{2}(A_1 + A_2)(z^{n+1} + z^n) + \frac{\tau^2}{4} A_1 A_2 \partial_t z^n = T_+^n + A_1(T_-^n). \tag{46}$$

To show uniqueness of $z^{n+1} \in \mathcal{M}^0$ satisfying (46), we assume $Z \in \mathcal{M}^0$ satisfies

$$\left[\tau^{-1} Z + \frac{1}{2}(A_1 + A_2) Z + \frac{\tau}{4} A_1 A_2 Z\right](\xi) = 0, \quad \xi \in \mathcal{G}.$$

Taking the inner product $(\cdot,\cdot)_\mathcal{G}$ on both sides of the last unnumbered equation with Z and using (21) and (23), we obtain

$$0 = \tau^{-1}(Z, Z)_\mathcal{G} + \frac{1}{2}((A_1 + A_2)Z, Z)_\mathcal{G} + \frac{\tau}{4}(A_1 A_2 Z, Z)_\mathcal{G} \geq \tau^{-1}(Z, Z)_\mathcal{G}.$$

Therefore, $Z = 0$ which implies uniqueness of $z^{n+1} \in \mathcal{M}^0$.

It will be convenient in the analysis to introduce $p^{n+1}, q^{n+1} \in \mathcal{M}^0$ satisfying (cf. (46)), for $n = 0, \ldots, N_t - 1$,

$$\partial_t p^n + \frac{1}{2}(A_1 + A_2)(p^{n+1} + p^n) + \frac{\tau^2}{4} A_1 A_2 \partial_t p^n = T_+^n, \tag{47}$$

$$\partial_t q^n + \frac{1}{2}(A_1 + A_2)(q^{n+1} + q^n) + \frac{\tau^2}{4} A_1 A_2 \partial_t q^n = A_1 T_-^n, \tag{48}$$

where

$$p^0 = q^0 = 0. \tag{49}$$

It follows from (33) and (49) that $z^0 = p^0 + q^0$. Assume that $z^n = p^n + q^n, n = 0, \ldots, N_t - 1$. Then (47) and (48) imply that (46) is true with z^{n+1} replaced by $p^{n+1} + q^{n+1}$. Hence by uniqueness of z^{n+1}, we have

$$z^n = p^n + q^n, \quad n = 0, \ldots, N_t. \tag{50}$$

The representation (50) allows for a simple analysis of error bounds for (47) in comparison to that for (48).

4.2. Error Bounds. In order to bound p^n and q^n of (47), (48), and (49), we introduce

$$\eta(\cdot, t) = u(\cdot, t) - u_\mathcal{H}(\cdot, t), \quad t \in [0, T], \tag{51}$$

where, for fixed t, $u_\mathcal{H}(\cdot, t)$ is the Hermite interpolant of $u(\cdot, t)$ (cf. (13)). The next two lemmas are concerned with T_+^n and T_-^n appearing on the right hand sides of (47) and (48), respectively.

Lemma 4.1. *For T_+^n, $n = 0, \ldots, N_t - 1$, defined in (40), we have*

$$T_+^n(\xi) = S^n(\xi) + \frac{1}{2}(L_1 + L_2)(\eta^{n+1} + \eta^n)(\xi), \quad \xi \in \mathcal{G}, \tag{52}$$

where

$$\|S^n\|_\mathcal{G} \leq C(\tau^2 + h^{r+1}). \tag{53}$$

Proof. It follows from (40), (38), (39), (9), (1), (36), and (51) that, for $n = 0, \dots, N_t - 1$,

$$T_+^n(\xi) = f^{n+1/2}(\xi) - \left[\partial_t u_{\mathcal{H}}^n + L_1 \widetilde{w}^{n+1/2} + \tfrac{1}{2} L_2 (u_{\mathcal{H}}^{n+1} + u_{\mathcal{H}}^n)\right](\xi)$$

$$= u_t^{n+1/2}(\xi) - \partial_t u_{\mathcal{H}}^n(\xi) + (L_1 + L_2) u^{n+1/2}(\xi) - \tfrac{1}{2}(L_1 + L_2)(u_{\mathcal{H}}^{n+1} + u_{\mathcal{H}}^n)(\xi)$$

$$= S^n(\xi) + \tfrac{1}{2}(L_1 + L_2)(\eta^{n+1} + \eta^n)(\xi), \quad \xi \in \mathcal{G},$$

where

(54)
$$S^n(\xi) = u_t^{n+1/2}(\xi) - \partial_t u^n(\xi) + \partial_t \eta^n(\xi) + (L_1 + L_2)\Big[u^{n+1/2}(\xi) - \frac{u^{n+1} + u^n}{2}(\xi)\Big], \ \xi \in \mathcal{G}.$$

This proves (52). Using (7), (8), (9), Taylor's theorem, and (5), we obtain

$$\left\| u_t^{n+1/2}(\cdot) - \partial_t u^n(\cdot) \right\|_{\mathcal{G}}^2 \le C\tau^4 \|u_{ttt}\|_{C(\overline{\Omega} \times [0,T])}^2 . \tag{55}$$

In a similar way, using (2), we obtain

$$\left\| L_1 \left(u^{n+1/2}(\cdot) - \frac{u^{n+1} + u^n}{2}(\cdot) \right) \right\|_{\mathcal{G}}^2 \le C\tau^4 \|u_{xxtt}\|_{C(\overline{\Omega} \times [0,T])}^2 , \tag{56}$$

$$\left\| L_2 \left(u^{n+1/2}(\cdot) - \frac{u^{n+1} + u^n}{2}(\cdot) \right) \right\|_{\mathcal{G}}^2 \le C\tau^4 \|u_{yytt}\|_{C(\overline{\Omega} \times [0,T])}^2 . \tag{57}$$

Using (7), (8), the Cauchy Schwarz inequality, (51), $(u_{\mathcal{H}})_t = (u_t)_{\mathcal{H}}$, (16) with $l = 0$ and s replaced by $u_t(\cdot, t)$, we obtain

$$\|\partial_t \eta^n\|_{\mathcal{G}}^2 = \left\| \tau^{-1} \int_{t_n}^{t_{n+1}} \eta_t(\cdot, t) dt \right\|_{\mathcal{G}}^2 \le \tau^{-1} \int_{t_n}^{t_{n+1}} \|\eta_t(\cdot, t)\|_{\mathcal{G}}^2 \, dt \tag{58}$$

$$= \tau^{-1} \int_{t_n}^{t_{n+1}} \|[u_t - (u_t)_{\mathcal{H}}](\cdot, t)\|_{\mathcal{G}}^2 \, dt \le C h^{2r+2} \|u_t\|_{C([0,T], H^{r+1}(\Omega))}^2 .$$

Using (54), the triangle inequality, and (55)–(58), we have

$$\|S^n\|_{\mathcal{G}} \le C\tau^2 \Big[\|u_{ttt}\|_{C(\overline{\Omega} \times [0,T])} + \|u_{xxtt}\|_{C(\overline{\Omega} \times [0,T])} + \|u_{yytt}\|_{C(\overline{\Omega} \times [0,T])} \Big] \tag{59}$$

$$+ C h^{r+1} \|u_t\|_{C([0,T], H^{r+1}(\Omega))} ,$$

which yields (53). $\square$

Remark 4.1. *If $f(\xi, t_{n+1/2})$ in (26) and (27) of the ADI OSC scheme is replaced by $[f(\xi, t_{n+1}) + f(\xi, t_n)]/2$, then (54) is replaced by*

$$S^n(\xi) = \frac{1}{2}(u_t^{n+1} + u_t^n)(\xi) - \partial_t u^n(\xi) + \partial_t \eta^n(\xi), \quad \xi \in \mathcal{G}.$$

and (59) is replaced by

$$\|S^n\|_{\mathcal{G}} \le C\tau^2 \|u_{ttt}\|_{C(\overline{\Omega} \times [0,T])} + C h^{r+1} \|u_t\|_{C([0,T], H^{r+1}(\Omega))} .$$

Lemma 4.2. *For T_-^n, $n = 0, \dots, N_t - 1$, defined in (40), we have*

$$\|T_-^n\|_{\mathcal{G}} \le C\tau^2, \ \ n = 0, \dots, N_t - 1, \quad \|\partial_t T_-^{n-1}\|_{\mathcal{G}} \le C\tau^2, \ \ n = 1, \dots, N_t - 1. \tag{60}$$

Proof. Using (40), (38), (39), (36), (2), and (51), for $n = 0, \dots, N_t - 1$, we obtain

$$(61) \qquad T_-^n(\xi) = \left[\widetilde{w}^{n+1/2} - \frac{u_{\mathcal{H}}^{n+1} + u_{\mathcal{H}}^n}{2} - \frac{\tau}{4} L_2(u_{\mathcal{H}}^{n+1} - u_{\mathcal{H}}^n)\right](\xi)$$

$$= \frac{\tau}{4}\left[(u_{\mathcal{H}}^{n+1})_{yy} - (u_{\mathcal{H}}^n)_{yy}\right](\xi) = \frac{\tau}{4}\left[(u_{yy}^{n+1} - u_{yy}^n) - (\eta_{yy}^{n+1} - \eta_{yy}^n)\right](\xi), \quad \xi \in \mathcal{G}.$$

Using (7), (8), Taylor's theorem, and (5), we obtain

$$(62) \qquad \left\|u_{yy}^{n+1} - u_{yy}^n\right\|_{\mathcal{G}}^2 \le C\tau^2 \left\|u_{yyt}\right\|_{C(\overline{\Omega}\times[0,T])}^2.$$

Using (7), (8), the Cauchy Schwarz inequality, (51), $u_{yyt} = u_{tyy}$, $(u_{\mathcal{H}})_{yyt} = [(u_t)_{\mathcal{H}}]_{yy}$, (16) with $l = 2$ and s replaced by $u_t(\cdot, t)$, we obtain

$$(63) \qquad \left\|\eta_{yy}^{n+1} - \eta_{yy}^n\right\|_{\mathcal{G}}^2 = \left\|\int_{t_n}^{t_{n+1}} \eta_{yyt}(\cdot, t)dt\right\|_{\mathcal{G}}^2 \le \tau \int_{t_n}^{t_{n+1}} \left\|\eta_{yyt}(\cdot, t)dt\right\|_{\mathcal{G}}^2$$

$$= \tau \int_{t_n}^{t_{n+1}} \left\|\left[u_t - (u_t)_{\mathcal{H}}\right]_{yy}(\cdot, t)dt\right\|_{\mathcal{G}}^2 \le C\tau^2 h^{2r-2} \left\|u_t\right\|_{C([0,T],H^{r+1}(\Omega))}^2.$$

The first bound in (60) follows from (61), the triangle inequality, (62), and (63).

Using (9) and (61), for $n = 1, \dots, N_t - 1$, we obtain

$$(64) \qquad \partial_t T_-^{n-1}(\xi) = \frac{1}{\tau}(T_-^n - T_-^{n-1})(\xi)$$

$$= \frac{\tau^2}{4}\left[\frac{u_{yy}^{n+1} - 2u_{yy}^n + u_{yy}^{n-1}}{\tau^2}\right](\xi) + \frac{\tau^2}{4}\left[\frac{\eta_{yy}^{n+1} - 2\eta_{yy}^n + \eta_{yy}^{n-1}}{\tau^2}\right](\xi), \quad \xi \in \mathcal{G}.$$

Using (7), (8), Taylor's theorem, and (5), we obtain

$$(65) \qquad \left\|\frac{u_{yy}^{n+1} - 2u_{yy}^n + u_{yy}^{n-1}}{\tau^2}\right\|_{\mathcal{G}}^2 \le C \left\|u_{yytt}\right\|_{C(\overline{\Omega}\times[0,T])}^2.$$

Using (7), (8), Cauchy Schwarz inequality, (51), $u_{yytt} = u_{ttyy}$, $(u_{\mathcal{H}})_{yytt} = [(u_{tt})_{\mathcal{H}}]_{yy}$, (16) with $l = 2$ and s replaced by $u_{tt}(\cdot, t)$, we obtain

$$(66) \qquad \left\|\frac{\eta_{yy}^{n+1} - 2\eta_{yy}^n + \eta_{yy}^{n-1}}{\tau^2}\right\|_{\mathcal{G}}^2 = \left\|\tau^{-2}\int_{t_{n-1}}^{t_{n+1}} (\tau - |t - n\tau|)\eta_{yytt}(\cdot, t)dt\right\|_{\mathcal{G}}^2$$

$$\le \tau^{-1}\int_{t_{n-1}}^{t_{n+1}} \left\|\eta_{yytt}(\cdot, t)\right\|_{\mathcal{G}}^2 dt = \tau^{-1}\int_{t_{n-1}}^{t_{n+1}} \left\|\left[u_{tt} - (u_{tt})_{\mathcal{H}}\right]_{yy}(\cdot, t)\right\|_{\mathcal{G}}^2 dt$$

$$\le Ch^{2r-2} \left\|u_{tt}\right\|_{C([0,T],H^{r+1}(\Omega))}^2.$$

The second bound in (60) follows from (64), the triangle inequality, (65), and (66). □

Lemma 4.3. *Let* $v \in \mathcal{M}^0$ *and*

$$(67) \qquad \overline{v}_{i,j} = \left(h_i^x h_j^y\right)^{-1}(v, 1)_{\mathcal{G}_{i,j}}, \quad i = 1, \dots, N_x, \quad j = 1, \dots, N_y.$$

Then

$$(68) \qquad \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} \left\|v - \overline{v}_{i,j}\right\|_{\mathcal{G}_{i,j}}^2 \le Ch^2(-\Delta v, v)_{\mathcal{G}}.$$

Proof. It follows from (67) and (8) that

$$\overline{v}_{i,j} = \sum_{\mu=1}^{r-1}\sum_{\nu=1}^{r-1} \omega_\mu \omega_\nu v(\xi_{i,\mu}, \xi_{j,\nu}). \tag{69}$$

Using (8), (5), (69), (25), and the Cauchy Schwarz inequality, we have

(70)

$$\|v - \overline{v}_{i,j}\|^2_{\mathcal{G}_{i,j}} = h_i^x h_j^y \sum_{k=1}^{r-1}\sum_{l=1}^{r-1} \omega_k \omega_l \left(\sum_{\mu=1}^{r-1}\sum_{\nu=1}^{r-1} \omega_\mu \omega_\nu \left[v(\xi^x_{i,k}, \xi^y_{j,l}) - v(\xi^x_{i,\mu}, \xi^y_{j,\nu}) \right] \right)^2$$

$$= h_i^x h_j^y \sum_{k=1}^{r-1}\sum_{l=1}^{r-1} \omega_k \omega_l \left(\sum_{\mu=1}^{r-1}\sum_{\nu=1}^{r-1} \omega_\mu \omega_\nu \Big[v(\xi^x_{i,k}, \xi^y_{j,l}) - v(\xi^x_{i,\mu}, \xi^y_{j,l}) + v(\xi^x_{i,\mu}, \xi^y_{j,l}) - v(\xi^x_{i,\mu}, \xi^y_{j,\nu}) \Big] \right)^2$$

$$= h_i^x h_j^y \sum_{k=1}^{r-1}\sum_{l=1}^{r-1} \omega_k \omega_l \left(\sum_{\mu=1}^{r-1}\sum_{\nu=1}^{r-1} \omega_\mu \omega_\nu \left[\int_{\xi^x_{i,\mu}}^{\xi^x_{i,k}} v_x(s, \xi^y_{j,l}) ds + \int_{\xi^y_{j,\nu}}^{\xi^y_{j,l}} v_y(\xi^x_{i,\mu}, s) ds \right] \right)^2$$

$$\le 2 h_i^x h_j^y \sum_{k=1}^{r-1}\sum_{l=1}^{r-1} \omega_k \omega_l \sum_{\mu=1}^{r-1}\sum_{\nu=1}^{r-1} \omega_\mu \omega_\nu \left[\left| \int_{\xi^x_{i,\mu}}^{\xi^x_{i,k}} v_x(s, \xi^y_{j,l}) ds \right|^2 + \left| \int_{\xi^y_{j,\nu}}^{\xi^y_{j,l}} v_y(\xi^x_{i,\mu}, s) ds \right|^2 \right]$$

$$\le C h_i^x h_j^y \left[\sum_{l=1}^{r-1} \omega_l \left(\int_{I_i^x} |v_x(s, \xi^y_{j,l})| ds \right)^2 + \sum_{k=1}^{r-1} \omega_k \left(\int_{I_j^y} |v_y(\xi^x_{i,k}, s)| ds \right)^2 \right]$$

$$\le C h_i^x h_j^y \left[h_i^x \sum_{l=1}^{r-1} \omega_l \, \|v_x(\cdot, \xi^y_{j,l})\|^2_{L^2(I_i^x)} + h_j^y \sum_{k=1}^{r-1} \omega_k \, \|v_y(\xi^x_{i,k}, \cdot)\|^2_{L^2(I_j^y)} \right]$$

$$\le C h^2 \left[h_j^y \sum_{l=1}^{r-1} \omega_l \, \|v_x(\cdot, \xi^y_{j,l})\|^2_{L^2(I_i^x)} + h_i^x \sum_{k=1}^{r-1} \omega_k \, \|v_y(\xi^x_{i,k}, \cdot)\|^2_{L^2(I_j^y)} \right].$$

Using (70), [11, Lemma 3.3], (7), and (8), we obtain

$$\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} \|v - \overline{v}_{i,j}\|^2_{\mathcal{G}_{i,j}}$$

$$\le C h^2 \left[\sum_{j=1}^{N_y} h_j^y \sum_{l=1}^{r-1} \omega_l \, \|v_x(\cdot, \xi^y_{j,l})\|^2_{L^2(a,b)} + \sum_{i=1}^{N_x} h_i^x \sum_{k=1}^{r-1} \omega_k \, \|v_y(\xi^x_{i,k}, \cdot)\|^2_{L^2(c,d)} \right]$$

$$\le C h^2 \Bigg[\sum_{j=1}^{N_y} h_j^y \sum_{l=1}^{r-1} \omega_l \, \sum_{i=1}^{N_x} h_i^x \sum_{k=1}^{r-1} \omega_k \, (-v_{xx} v)(\xi^x_{i,k}, \xi^y_{j,l}) + \sum_{i=1}^{N_x} h_i^x \sum_{k=1}^{r-1} \omega_k \, \sum_{j=1}^{N_y} h_j^y \sum_{l=1}^{r-1} \omega_l \, (-v_{yy} v)(\xi^x_{i,k}, \xi^y_{j,l}) \Bigg]$$

$$= C h^2 \left[(-v_{xx}, v)_{\mathcal{G}} + (-v_{yy}, v)_{\mathcal{G}} \right] = C h^2 (-\Delta v, v)_{\mathcal{G}},$$

which proves (68). $\square$

We obtain an error bound for (47), (49) in the following theorem.

Theorem 4.1. *For $p^n \in \mathcal{M}^0$, $n = 0, \ldots, N_t$, satisfying (47) and (49), we have*

$$(-\Delta p^n, p^n)_{\mathcal{G}} \le C(\tau^4 + h^{2r+2}). \tag{71}$$

Proof. Taking the inner product $(\cdot,\cdot)_{\mathcal{G}}$ on both sides of (47) with $2\tau\partial_t p^n$ and using (7), (9), and (23), we obtain, for $n = 0, \ldots, N_t - 1$,

$$2\tau\|\partial_t p^n\|^2_{\mathcal{G}} + ((A_1 + A_2)(p^{n+1} + p^n), p^{n+1} - p^n)_{\mathcal{G}} \le 2\tau(T^n_+, \partial_t p^n)_{\mathcal{G}}. \tag{72}$$

Using (52), the Cauchy Schwarz inequality, (24), (53), and (25), we have

$$\begin{aligned} 2\tau(T^n_+, \partial_t p^n)_{\mathcal{G}} &\le 2\tau\|S^n\|_{\mathcal{G}}\|\partial_t p^n\|_{\mathcal{G}} + \tau((L_1 + L_2)(\eta^{n+1} + \eta^n), \partial_t p^n)_{\mathcal{G}} \\ &\le \tau\|\partial_t p^n\|^2_{\mathcal{G}} + C\tau(\tau^4 + h^{2r+2}) + \tau((L_1 + L_2)(\eta^{n+1} + \eta^n), \partial_t p^n)_{\mathcal{G}}. \end{aligned} \tag{73}$$

Using (7), (2), and linearity in w of $(v, w)_{\mathcal{G}_{i,j}}$ of (8), we have

$$\tau\left((L_1 + L_2)(\eta^{n+1} + \eta^n), \partial_t p^n\right)_{\mathcal{G}} = I^{(n)} + II^{(n)}, \tag{74}$$

where

$$I^{(n)} = \tau\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}(-\Delta(\eta^{n+1} + \eta^n), \partial_t(p^n - \overline{p}^n_{i,j}))_{\mathcal{G}_{i,j}}, \tag{75}$$

$$II^{(n)} = \tau\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}(-\Delta(\eta^{n+1} + \eta^n), \partial_t\overline{p}^n_{i,j})_{\mathcal{G}_{i,j}}, \tag{76}$$

$$\overline{p}^n_{i,j} = \left(h^x_i h^y_j\right)^{-1}(p^n, 1)_{\mathcal{G}_{i,j}}. \tag{77}$$

First, we bound $II^{(n)}$. Since $(v, w)_{\mathcal{G}_{i,j}}$ of (8) is linear in v, it follows from (77) and (9) that

$$\partial_t\overline{p}^n_{i,j} = (h^x_i h^y_j)^{-1}(\partial_t p^n, 1)_{\mathcal{G}_{i,j}}.$$

Using the last unnumbered equation, the Cauchy Schwarz inequality, (8), (5), the triangle inequality, (51), (18) with s replaced by u^{n+1} and u^n, and using (24), we have

$$\begin{aligned} (-\Delta(\eta^{n+1} + \eta^n), \partial_t\overline{p}^n_{i,j})_{\mathcal{G}_{i,j}} &= (h^x_i h^y_j)^{-1}(\partial_t p^n, 1)_{\mathcal{G}_{i,j}}(-\Delta(\eta^{n+1} + \eta^n), 1)_{\mathcal{G}_{i,j}} \\ &\le \|\partial_t p^n\|_{\mathcal{G}_{i,j}}\,(h^x_i h^y_j)^{-1/2}\,\left\{|(\Delta\left[u^{n+1} - (u^{n+1})_{\mathcal{H}}\right], 1)_{\mathcal{G}_{i,j}}|\right. \\ &\qquad\qquad \left. + |(\Delta\left[u^n - (u^n)_{\mathcal{H}}\right], 1)_{\mathcal{G}_{i,j}}|\right\} \\ &\le \|\partial_t p^n\|_{\mathcal{G}_{i,j}} C h^{r+1}\left[\|u^{n+1}\|_{H^{r+3}(I^x_i\times I^y_j)} + \|u^n\|_{H^{r+3}(I^x_i\times I^y_j)}\right] \\ &\le \|\partial_t p^n\|^2_{\mathcal{G}_{i,j}} + Ch^{2r+2}\left[\|u^{n+1}\|^2_{H^{r+3}(I^x_i\times I^y_j)} + \|u^n\|^2_{H^{r+3}(I^x_i\times I^y_j)}\right]. \end{aligned} \tag{78}$$

Using (76), (78), and (7), we obtain

$$II^{(n)} \le \tau\|\partial_t p^n\|^2_{\mathcal{G}} + \tau Ch^{2r+2}\left[\|u^{n+1}\|^2_{H^{r+3}(\Omega)} + \|u^n\|^2_{H^{r+3}(\Omega)}\right]. \tag{79}$$

Combining (72), (73), (74), and (79), we obtain

$$((A_1 + A_2)(p^{n+1} + p^n), p^{n+1} - p^n)_{\mathcal{G}} \le I^{(n)} + C\tau(\tau^4 + h^{2r+2}).$$

For given $k = 1, \ldots, N_t$, we sum the last unnumbered equation from $n = 0$ to $k-1$, use (20), $p^0 = 0$ of (49), (19), and (2), to obtain

$$(80) \qquad (-\Delta p^k, p^k)_{\mathcal{G}} \le \sum_{n=0}^{k-1} I^{(n)} + C(\tau^4 + h^{2r+2}), \quad k = 1, \ldots, N_t.$$

Next, using (75), $p^0 = 0$ of (49) which implies $\overline{p}^0_{i,j} = 0$ by (77), and Lemma 2.4 with the subscript $\mathcal{G}$ replaced by $\mathcal{G}_{i,j}$, we obtain

$$(81) \qquad \sum_{n=0}^{k-1} I^{(n)} = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \tau \sum_{n=0}^{k-1} (-\Delta(\eta^{n+1} + \eta^n), \partial_t(p^n - \overline{p}^n_{i,j}))_{\mathcal{G}_{i,j}}$$

$$= \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} (-\Delta(\eta^k + \eta^{k-1}), p^k - \overline{p}^k_{i,j})_{\mathcal{G}_{i,j}} + \tau \sum_{n=1}^{k-1} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} (\partial_t \Delta(\eta^n + \eta^{n-1}), p^n - \overline{p}^n_{i,j})_{\mathcal{G}_{i,j}}.$$

To bound the first term on the right-hand side in (81), we use twice the Cauchy Schwarz inequality, (7), (8), the triangle inequality, (51), Lemma 4.3 with $v, \overline{v}_{i,j}$ replaced respectively by $p^k, \overline{p}^k_{i,j}$, (17) with s replaced by u^k and u^{k+1}, and (24), to obtain

(82)

$$\sum_{i=1}^{N_x} \sum_{j=1}^{N_y} (-\Delta(\eta^k + \eta^{k-1}, p^k - \overline{p}^k_{i,j})_{\mathcal{G}_{i,j}} \le \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \|\Delta(\eta^k + \eta^{k-1})\|_{\mathcal{G}_{i,j}} \, \|p^k - \overline{p}^k_{i,j}\|_{\mathcal{G}_{i,j}}$$

$$\le \|\Delta(\eta^k + \eta^{k-1})\|_{\mathcal{G}} \left(\sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \|p^k - \overline{p}^k_{i,j}\|^2_{\mathcal{G}_{i,j}} \right)^{1/2}$$

$$\le \left(\|\Delta[u^k - (u^k)_{\mathcal{H}}]\|_{\mathcal{G}} + \|\Delta[u^{k-1} - (u^{k-1})_{\mathcal{H}}]\|_{\mathcal{G}} \right) Ch(-\Delta p^k, p^k)^{1/2}_{\mathcal{G}}$$

$$\le (-\Delta p^k, p^k)^{1/2}_{\mathcal{G}} C h^{r+1} \|u\|_{C([0,T],H^{r+2}(\Omega))}$$

$$\le \frac{1}{2}(-\Delta p^k, p^k)_{\mathcal{G}} + C h^{2r+2} \|u\|^2_{C([0,T],H^{r+2}(\Omega))}.$$

To bound the second term on the right hand side of (81), using (9), (7), (8), the Cauchy Schwarz inequality, (51), $(\Delta u)_t = \Delta(u_t)$, $[\Delta(u_{\mathcal{H}})]_t = \Delta[(u_t)_{\mathcal{H}}]$, and (17) with s replaced by $u_t(\cdot, t)$, we obtain

$$\|\partial_t \Delta \eta^n\|^2_{\mathcal{G}} = \left\| \tau^{-1} \int_{t_n}^{t_{n+1}} (\Delta\eta)_t(\cdot, t) dt \right\|^2_{\mathcal{G}} \le \tau^{-1} \int_{t_n}^{t_{n+1}} \|(\Delta\eta)_t(\cdot, t)\|^2_{\mathcal{G}} dt$$

$$= \tau^{-1} \int_{t_n}^{t_{n+1}} \left\| \Delta[u_t - (u_t)_{\mathcal{H}}](\cdot, t) \right\|^2_{\mathcal{G}} dt \le C h^{2r} \|u_t\|^2_{C([0,T],H^{r+2}(\Omega))}.$$

Then following derivations in (82) and using the last unnumbered equation, we have

$$(83) \qquad \tau \sum_{n=1}^{k-1} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} (\partial_t \Delta(\eta^n + \eta^{n-1}), p^n - \overline{p}^n_{i,j})_{\mathcal{G}_{i,j}}$$

$$\le \tau \sum_{n=1}^{k-1} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \|\partial_t \Delta(\eta^n + \eta^{n-1})\|_{\mathcal{G}_{i,j}} \|p^n - \overline{p}^n_{i,j}\|_{\mathcal{G}_{i,j}}$$

$$
\begin{aligned}
&\leq \tau \sum_{n=1}^{k-1} \|\partial_t \Delta(\eta^n + \eta^{n-1})\|_{\mathcal{G}} \left(\sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \|p^n - \overline{p}_{i,j}^n\|_{\mathcal{G}_{i,j}} \right)^{1/2} \\
&\leq \tau \sum_{n=1}^{k-1} \left(\|\partial_t \Delta \eta^n\|_{\mathcal{G}} + \|\partial_t \Delta \eta^{n-1}\|_{\mathcal{G}} \right) Ch(-\Delta p^n, p^n)_{\mathcal{G}}^{1/2} \\
&\leq \tau \sum_{n=1}^{k-1} (-\Delta p^n, p^n)_{\mathcal{G}}^{1/2} Ch^{r+1} \|u_t\|_{C([0,T],H^{r+2}(\Omega))} \\
&\leq \frac{\tau}{2} \sum_{n=1}^{k-1} (-\Delta p^n, p^n)_{\mathcal{G}} + Ch^{2r+2} \|u_t\|^2_{C([0,T],H^{r+2}(\Omega))}.
\end{aligned}
$$

Substituting (82) and (83) into (81), we obtain

$$
\sum_{n=1}^{k-1} I^{(n)} \leq \frac{1}{2}(-\Delta p^k, p^k)_{\mathcal{G}} + \frac{\tau}{2} \sum_{n=1}^{k-1} (-\Delta p^n, p^n)_{\mathcal{G}} + Ch^{2r+2}, \quad k = 1, \dots, N_t.
$$

Substituting the last unnumbered equation into (80), multiplying by 2, and using $p^0 = 0$ of (49), we obtain

$$
(-\Delta p^k, p^k)_{\mathcal{G}} \leq \tau \sum_{n=1}^{k-1} (-\Delta p^n, p^n)_{\mathcal{G}} + Ch^{2r+2}, \quad k = 0, \dots, N_t.
$$

Lemma 2.3, applied to the last unnumbered equation with $\alpha_k = (-\Delta p^k, p^k)_{\mathcal{G}}$, $\beta_k = Ch^{2r+2}$ and $\gamma = 1$, completes the proof. □

Next we obtain an error bound for (48), (49).

Theorem 4.2. *For $q^n \in \mathcal{M}^0$, $n = 0, \dots, N_t$, satisfying (48), (49), we have*

$$
\|q^n\|_{\mathcal{G}} \leq C\tau^2. \tag{84}
$$

Proof. (21) with $k = 1$ implies existence of A_1^{-1}. Multiplying (48) by A_1^{-1} and taking the inner product $(\cdot,\cdot)_{\mathcal{G}}$ on both sides with $2\tau\partial_t p^n$, we obtain

$$
\begin{aligned}
2\tau(A_1^{-1}\partial_t q^n, \partial_t q^n)_{\mathcal{G}} + \tau(q^{n+1} + q^n, \partial_t q^n)_{\mathcal{G}} + \tau(A_1^{-1}A_2(q^{n+1} + q^n), \partial_t q^n)_{\mathcal{G}} \\
+ \frac{\tau^3}{2}(A_2\partial_t q^n, \partial_t q^n)_{\mathcal{G}} = 2\tau(T_-^n, \partial_t q^n)_{\mathcal{G}}, \quad n = 0, \dots, N_t - 1.
\end{aligned} \tag{85}
$$

It follows from (20) and (22) that

$$
(A_1^{-1}A_2 v, w)_{\mathcal{G}} = (v, A_2 A_1^{-1} w)_{\mathcal{G}} = (v, A_1^{-1} A_2 w)_{\mathcal{G}}, \quad v, w \in \mathcal{M}^0,
$$

which, along with (9), implies

$$
\tau(A_1^{-1}A_2(q^{n+1} + q^n), \partial_t q^n)_{\mathcal{G}} = (A_1^{-1}A_2 q^{n+1}, q^{n+1})_{\mathcal{G}} - (A_1^{-1}A_2 q^n, q^n)_{\mathcal{G}}.
$$

Using (9), we also have

$$
\tau((q^{n+1} + q^n), \partial_t q^n)_{\mathcal{G}} = (q^{n+1}, q^{n+1})_{\mathcal{G}} - (q^n, q^n)_{\mathcal{G}}.
$$

It follows from (21) that the first and last terms on the left hand side of (85) are non-negative. Dropping these terms and using the last two unnumbered equations in (85), we obtain

$$
\begin{aligned}
(q^{n+1}, q^{n+1})_{\mathcal{G}} - (q^n, q^n)_{\mathcal{G}} + (A_1^{-1}A_2 q^{n+1}, q^{n+1})_{\mathcal{G}} - (A_1^{-1}A_2 q^n, q^n)_{\mathcal{G}} \\
\leq 2\tau(T_-^n, \partial_t q^n)_{\mathcal{G}}, \quad n = 0, \dots, N_t - 1.
\end{aligned} \tag{86}
$$

It follows from (20), (22), and (23) that

$$
(A_1^{-1}A_2 v, v)_{\mathcal{G}} \geq 0, \quad v \in \mathcal{M}^0.
$$

For $k = 1, \dots, N_t$, summing (86) from $n = 0$ to $k-1$, using the last unnumbered equation, and $q^0 = 0$ of (49), we obtain

$$\|q^k\|_{\mathcal{G}}^2 \le 2\tau \sum_{n=0}^{k-1} (T_-^n, \partial_t q^n)_{\mathcal{G}}. \tag{87}$$

Using $q^0 = 0$ of (49), Lemma 2.4, the Cauchy Schwarz inequality, (24), and (60), we have

$$\begin{aligned} 2\tau \sum_{n=0}^{k-1} (T_-^n, \partial_t q^n)_{\mathcal{G}} &= 2(T_-^{k-1}, q^k)_{\mathcal{G}} - 2\tau \sum_{n=1}^{k-1} (\partial_t T_-^{n-1}, q^n)_{\mathcal{G}} \\ &\le \frac{1}{2}\|q^k\|_{\mathcal{G}}^2 + 2\|T_-^{k-1}\|_{\mathcal{G}}^2 + \tau \sum_{n=1}^{k-1} \|\partial_t T_-^{n-1}\|_{\mathcal{G}}^2 + \tau \sum_{n=1}^{k-1} \|q^n\|_{\mathcal{G}}^2 \\ &\le \frac{1}{2}\|q^k\|_{\mathcal{G}}^2 + \tau \sum_{n=1}^{k-1} \|q^n\|_{\mathcal{G}}^2 + C\tau^4. \end{aligned} \tag{88}$$

Combining (87), (88), and using $q^0 = 0$ of (49), we obtain

$$\|q^k\|_{\mathcal{G}}^2 \le 2\tau \sum_{n=1}^{k-1} \|q^n\|_{\mathcal{G}}^2 + C\tau^4, \quad k = 0, \dots, N_t.$$

Lemma 2.3 applied to the last unnumbered equation with $\alpha_k = \|q^k\|_{\mathcal{G}}^2$, $\beta_k = C\tau^4$, and $\gamma = 2$, completes the proof. □

Theorem 4.3. *Assume $U_h^n, n = 0, \dots, N_t$, satisfy (26)–(29) and (31). Then,*

$$\max_{0 \le n \le N_t} \|u^n - U^n\|_{L^2(\Omega)} \le C(\tau^2 + h^{r+1}).$$

Proof. For $n = 0, \dots, N_t$ and p^n of (47) and (49), using the Poincaré inequality, [11, Lemma 3.1], (7), (8), (21) with $k = 2$, (19), (2), and (71), we have

$$\begin{aligned} \sum_{j=1}^{N_y} h_j^y \sum_{l=1}^{r-1} \omega_l \int_a^b [p^n]^2(x, \xi_{j,l}^y) dx &\le C \sum_{j=1}^{N_y} h_j^y \sum_{l=1}^{r-1} \omega_l \int_a^b [p_x^n]^2(x, \xi_{j,l}^y) dx \\ &\le C \sum_{j=1}^{N_y} h_j^y \sum_{l=1}^{r-1} \omega_l \sum_{i=1}^{N_x} h_i^y \sum_{k=1}^{r-1} \omega_l (-p_{xx}^n p^n)(\xi_{i,k}^x, \xi_{j,l}^y) = C(-p_{xx}^n, p^n)_{\mathcal{G}} \\ &\le C(-p_{xx}^n, p^n)_{\mathcal{G}} + (A_2 p^n, p^n)_{\mathcal{G}} \le C(-\Delta p^n, p^n)_{\mathcal{G}} \le C(\tau^4 + h^{2r+2}). \end{aligned}$$

For $n = 0, \dots, N_t$ and q^n of (48) and (49), using the second inequality in [21, (5.25)], (7), (8), and (84), we have

$$\begin{aligned} \sum_{j=1}^{N_y} h_j^y \sum_{l=1}^{r-1} \omega_l \int_a^b [q^n]^2(x, \xi_{j,l}^y) dx \le& C \sum_{j=1}^{N_y} h_j^y \sum_{l=1}^{r-1} \omega_l \sum_{i=1}^{N_x} h_i^x \sum_{k=1}^{r-1} \omega_k [q^n]^2(\xi_{i,k}, \xi_{j,l}^y) \\ =& C\|q^n\|_{\mathcal{G}}^2 \le C\tau^4. \end{aligned}$$

For $n = 0, \dots, N_t$ and z^n of (32), using the second inequality in [21, (5.25)], (50), (25), and the last two unnumbered equations, we have

$$\|z^n\|_{L^2(\Omega)}^2 = \int_a^b \int_c^d [z^n]^2(x, y) dy dx \le C \int_a^b \sum_{j=1}^{N_y} h_j^y \sum_{l=1}^{r-1} \omega_l [z^n]^2(x, \xi_{j,l}^y) dx$$

$$\leq C\sum_{j=1}^{N_y} h_j^y \sum_{l=1}^{r-1}\omega_l \int_a^b [p^n]^2(x,\xi_{j,l}^y)dx + C\sum_{j=1}^{N_y} h_j^y \sum_{l=1}^{r-1}\omega_l \int_a^b [q^n]^2(x,\xi_{j,l}^y)dx$$

$$\leq C(\tau^4 + h^{2r+2}).$$

For $n = 0, \ldots, N_t$, it follows from the triangle inequality, (32), (16) with $l = 0$ and s replaced by u^n, the last unnumbered equation, and the inequality $\sqrt{\alpha+\beta} \leq \sqrt{\alpha} + \sqrt{\beta}$, $\alpha \geq 0$, $\beta \geq 0$, that

$$\|u^n - U^n\|_{L^2(\Omega)} \leq \|u^n - u_{\mathcal{H}}^n\|_{L^2(\Omega)} + \|z^n\|_{L^2(\Omega)} \leq C(\tau^2 + h^{r+1}). \qquad \square$$

5. Numerical Results

With $\Omega = (-1,1)\times(-1,1)$ and $T = 1$, we choose f, g_1, g_2 so that the exact solution of (1)–(4) is

$$u(x,y,t) = e^{\frac{1}{1+t}} e^{xy}.$$

The ADI OSC scheme (26)–(29) has been implemented as described in [5, Section 5] for $r \geq 3$ using MATLAB. For approximation of the boundary condition on vertical sides, we have tested both approaches, (30) which uses perturbation terms and (31) which does not use perturbation terms.

The L^2 and H^1 norms of the error $e = u(\cdot,T) - U^{N_t}$ are approximated using p-point Gauss-Legendre quadratures in x and y with suitable p so that quadrature errors are negligible. Let $\{\widetilde{\xi}_k\}_{k=1}^p$ and $\{\widetilde{\omega}_k\}_{k=1}^p$ be respectively the nodes and weights of the p-point Gauss-Legendre quadrature on $(0,1)$. We set $\widetilde{\mathcal{G}}_x = \{\widetilde{\xi}_{i,k}^x\}_{i=1,k=1}^{N_x,\ p}$ and $\widetilde{\mathcal{G}}_y = \{\widetilde{\xi}_{j,l}^y\}_{j=1,l=1}^{N_y,\ p}$, where $\widetilde{\xi}_{i,k}^x$ and $\widetilde{\xi}_{j,l}^y$ are given by the right-hand sides in (6) with ξ_k and ξ_l replaced by $\widetilde{\xi}_k$ and $\widetilde{\xi}_l$, respectively. Then, the L^2 and H^1 norm errors are approximated by the formulae

$$\|e\|_{L^2(\Omega)}^2 \approx \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} h_i^x h_j^y \sum_{k=1}^{p}\sum_{l=1}^{p} \widetilde{\omega}_k^x \widetilde{\omega}_l^y e^2(\widetilde{\xi}_{i,k}^x, \widetilde{\xi}_{j,l}^y) \tag{89}$$

and

$$\|e\|_{H^1(\Omega)}^2 \approx \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} h_i^x h_j^y \sum_{k=1}^{p}\sum_{l=1}^{p} \widetilde{\omega}_k^x \widetilde{\omega}_l^y \left[e^2 + e_x^2 + e_y^2\right](\widetilde{\xi}_{i,k}^x, \widetilde{\xi}_{j,l}^y),$$

respectively. We explain how to evaluate $U^{N_t}(\widetilde{\xi}^x, \widetilde{\xi}^y)$, $U_y^{N_t}(\widetilde{\xi}^x, \widetilde{\xi}^y)$, $U_x^{N_t}(\widetilde{\xi}^x, \widetilde{\xi}^y)$, $\widetilde{\xi}^x \in \widetilde{\mathcal{G}}_x$, $\widetilde{\xi}^y \in \widetilde{\mathcal{G}}_y$.

(a) For each $\xi^x \in \mathcal{G}_x$ we know the coefficients $u_{\xi^x,j}$ in

$$U^{N_t}(\xi^x, y) = \sum_{j=1}^{N_y(r-1)+2} u_{\xi^x,j} B_j^y(y), \quad y \in [c,d], \tag{90}$$

where the B splines B_j^y form a basis for $\mathcal{M}_y$. We use (90) to compute $U^{N_t}(\xi^x, \widetilde{\xi}^y)$, $U_y^{N_t}(\xi^x, \widetilde{\xi}^y)$, $\xi^x \in \mathcal{G}_y$, $\widetilde{\xi}^y \in \widetilde{\mathcal{G}}_y$.

(b) For each $\widetilde{\xi}^y \in \widetilde{\mathcal{G}}_y$, we have

$$U^{N_t}(x, \widetilde{\xi}^y) = \sum_{i=1}^{N_x(r-1)+2} \alpha_{i,\widetilde{\xi}^y} B_i^x(x), \quad x \in [a,b], \tag{91}$$

TABLE 1. L^2 norm errors; initial, boundary conditions via Hermite interpolant.

	N	With (30) Error	With (30) Rate	With (31) Error	With (31) Rate
r=3	4	5.7–04		3.9–04	
	9	2.2–05	4.002	1.5–05	4.008
	16	2.2–06	4.000	1.5–06	4.013
	25	3.7–07	4.000	2.5–07	4.011
r=4	4	1.5–04		9.6–05	
	9	2.5–06	5.001	1.6–06	5.031
	16	1.4–07	5.000	9.1–08	5.018
	25	1.5–08	5.000	9.7–09	5.011
r=5	4	3.6–05		2.4–05	
	9	2.8–07	6.000	1.8–07	6.029
	16	8.9–09	6.000	5.6–09	6.015
	25	6.1–10	6.000	3.9–10	6.010

where the B splines B_i^x form a basis for $\mathcal{M}_x$. We compute the coefficients $\alpha_{i,\widetilde{\xi}^y}$ in (91) by solving the interpolation problem

$$\sum_{i=1}^{N_x(r-1)+2} \alpha_{i,\widetilde{\xi}^y} B_i^x(\xi^x) = U^{N_t}(\xi^x, \widetilde{\xi}^y), \quad \xi^x \in \overline{\mathcal{G}}_x,$$

where $U^{N_t}(\xi^x, \widetilde{\xi}^y)$, $\xi^x \in \mathcal{G}_x$, $\widetilde{\xi}^y \in \widetilde{\mathcal{G}}_x$, are known from part (a) and $U^{N_t}(\xi^x, \widetilde{\xi}^y)$, $\xi^x = a, b$, are known from the first equation in (29). Then, we use (91) to compute $U^{N_t}(\widetilde{\xi}^x, \widetilde{\xi}^y)$, $U_x^{N_t}(\widetilde{\xi}^x, \widetilde{\xi}^y)$, $\widetilde{\xi}^x \in \widetilde{\mathcal{G}}^x$, $\widetilde{\xi}^y \in \widetilde{\mathcal{G}}_y$.

(c) For each $\widetilde{\xi}^y \in \widetilde{\mathcal{G}}_y$, we have

$$U_y^{N_t}(x, \widetilde{\xi}^y) = \sum_{i=1}^{N_x(r-1)+2} \beta_{i,\widetilde{\xi}^y} B_i^x(x), \quad x \in [a, b]. \tag{92}$$

We compute the coefficients $\beta_{i,\widetilde{\xi}^y}$ in (92) by solving the interpolation problem

$$\sum_{i=1}^{N_x(r-1)+2} \alpha_{i,\widetilde{\xi}^y} B_i^x(\xi^x) = U_y^{N_t}(\xi^x, \widetilde{\xi}^y), \quad \xi^x \in \overline{\mathcal{G}}_x,$$

where $U_y^{N_t}(\xi^x, \widetilde{\xi}^y)$, $\xi^x \in \mathcal{G}_x$, $\widetilde{\xi}^y \in \mathcal{G}_y$, are known from part (a) and $U_y^{N_t}(\xi^x, \widetilde{\xi}^y) = (g_{2,h})_y(\xi^x, \widetilde{\xi}^y)$, $\xi^x = a, b$, (see the first equation in (29)). Then, we use (92) to compute $U_y^{N_t}(\widetilde{\xi}^x, \widetilde{\xi}^y)$, $\widetilde{\xi}^x \in \widetilde{\mathcal{G}}^x$, $\widetilde{\xi}^y \in \widetilde{\mathcal{G}}^y$.

For $r = 3, 4, 5$, the L^2 norm errors and convergence rates of the ADI OSC scheme are presented in Table 1. We use Hermite interpolation to approximate the initial and boundary conditions; see (28), (29). Based on our analysis of the scheme, we expect the maximum norm error in time to be $O(\tau^2)$ and the L^2 norm error in space to be $O(h^{r+1})$. Hence, for $N = 4, 9, 16, 25$, we have chosen $N_x = N_y = 2N$ ($h = h_x = h_y = 2/(2N) = 1/N$) and $N_t = (\sqrt{N})^{(r+1)}$ so that $\tau^2 = h^{(r+1)}$. When computing the errors, we choose $p = r + 2$ so that the quadrature error in (89) is negligible since we expect this error to be $O(h^{2p-1})$.

For $r = 3, 4, 5$, the convergence rates using the L^2 norm errors in Table 1 are approximately 4, 5, 6, respectively, which is optimal and confirms the findings of our

TABLE 2. H^1 norm errors; initial, boundary conditions via Hermite interpolant.

	N	With (30) Error	With (30) Rate	With (31) Error	With (31) Rate
r=3	4	6.0–03		1.9–02	
	9	5.2–04	3.008	2.8–03	2.329
	16	9.3–05	2.999	6.3–04	2.604
	25	2.4–05	2.999	2.1–04	2.436
r=4	4	1.5–03		6.6–03	
	9	5.8–05	4.003	4.2–04	3.403
	16	5.8–06	4.000	5.6–05	3.490
	25	9.7–07	4.000	1.2–05	3.486
r=5	4	3.7–04		2.2–03	
	9	6.4–06	5.001	6.0–05	4.436
	16	3.6–07	5.000	4.5–06	4.486
	25	3.9–08	5.000	6.1–07	4.490

TABLE 3. L^2 norm errors; initial, boundary conditions via Gauss interpolant.

	N	With (30) Error	With (30) Rate	With (31) Error	With (31) Rate
r=3	4	5.8–04		3.8–04	
	9	2.3–05	4.002	1.5–05	4.007
	16	2.3–06	4.000	1.5–06	4.012
	25	3.8–07	4.000	2.5–07	4.010
r=4	4	1.5–04		9.6–05	
	9	2.5–06	5.001	1.6–06	5.031
	16	1.4–07	5.000	9.1–08	5.018
	25	1.5–08	5.000	9.7–09	5.011
r=5	4	3.6–05		2.4–05	
	9	2.8–07	6.000	1.8–07	6.029
	16	8.9–09	6.000	5.7–09	6.015
	25	6.1–10	6.000	3.9–10	6.010

theoretical analysis. While the convergence rates are the same for two approaches (30) and (31), the L^2 norm errors for (31) appear to be smaller.

For $r = 3, 4, 5$, the H^1 norm errors and convergence rates of the ADI OSC scheme are shown in Table 2. We use Hermite interpolation to approximate the initial and boundary conditions. Since we expect the maximum norm error in time to be $O(\tau^2)$ and the H^1 norm error in space to be $O(h^r)$, for $N = 4, 9, 16, 25$, we have chosen $N_x = N_y = 2N$ ($h = h_x = h_y = 2/(2N) = 1/N$), $N_t = (\sqrt{N})^r$ so that $\tau^2 = h^r$. We again choose $p = r + 2$.

For $r = 3, 4, 5$, the optimal rates using the H^1 norm errors in Table 2 are approximately $3, 4, 5$, respectively, when the approach (30) is used. This is consistent with the result proved in [5] for $r = 3$. We do not obtain the optimal H^1 rates for the approach (31); for h sufficiently small, the respective rates for $r = 3, 4, 5$ are approximately $2.5, 3.5, 4.5$. Thus, when perturbation terms are not used, we

TABLE 4. H^1 norm errors; initial, boundary conditions via Gauss interpolant.

		With (30)		With (31)	
	N	Error	Rate	Error	Rate
r=3	4	6.0–03		1.8–02	
	9	5.2–04	3.011	2.8–03	2.326
	16	9.3–05	3.001	6.3–04	2.603
	25	2.4–05	3.000	2.1–04	2.435
r=4	4	1.5–03		6.6–03	
	9	5.8–05	4.003	4.2–04	3.403
	16	5.8–06	4.000	5.6–05	3.490
	25	9.7–07	4.000	1.2–05	3.486
r=5	4	3.7–04		2.2–03	
	9	6.4–06	5.001	6.0–05	4.436
	16	3.6–07	5.000	4.5–06	4.486
	25	3.9–08	5.000	6.1–07	4.490

TABLE 5. L^2 and H^1 norm errors; initial, boundary conditions via Hermite interpolant.

		L^2 norm		H^1 norm	
	N	Error	Rate	Error	Rate
r=3	4	6.4–03		3.5–02	
	9	2.5–05	4.004	4.9–03	2.446
	16	2.5–06	4.010	1.2–03	2.464
	25	4.1–07	4.010	3.9–04	2.462
r=4	4	1.6–04		1.3–02	
	9	2.7–06	5.037	7.7–04	3.470
	16	1.5–07	5.021	1.0–04	3.465
	25	1.6–08	5.014	2.2–05	3.474
r=5	4	4.1–05		4.2–03	
	9	3.0–07	6.034	1.1–04	4.472
	16	9.5–09	6.018	8.6–06	4.474
	25	6.5–10	6.012	1.2–06	4.482

only obtain suboptimal convergence rates in the H^1 norm, that is, 0.5 less than the optimal rate.

In Tables 3 and 4 we present results similar to those in Tables 1 and 2 respectively. All parameters are kept the same but instead of using Hermite interpolation, we use Gauss interpolation to approximate the initial and boundary conditions.

Comparing Tables 1 and 2 with Tables 3 and 4 respectively, we observe that the corresponding L^2 and H^1 norm errors and rates are comparable. Numerically, it is easier to approximate the initial and boundary conditions using Gauss rather than Hermite interpolation.

Finally, we demonstrate that the ADI OSC scheme without the perturbation terms is applicable to variable coefficient problems. If (2) is replaced by

$$L_1 u = -a_1(x,y,t)u_{xx}, \quad L_2 u = -a_2(x,y,t)u_{yy},$$

TABLE 6. L^2 and H^1 norm errors; initial, boundary conditions via Gauss interpolant.

		L^2 norm		H^1 norm	
	N	Error	Rate	Error	Rate
r=3	4	6.3–04		3.5–02	
	9	2.5–05	4.003	4.9–03	2.444
	16	2.5–06	4.009	1.2–03	2.463
	25	4.1–07	4.010	3.9–04	2.461
r=4	4	1.6–04		1.3–02	
	9	2.7–06	5.037	7.7–04	3.470
	16	1.5–07	5.021	1.0–04	3.465
	25	1.6–08	5.014	2.2–05	3.474
r=5	4	4.1–05		4.2–03	
	9	3.0–07	6.034	1.1–04	4.472
	16	9.5–09	6.018	8.6–06	4.474
	25	6.5–10	6.012	1.2–06	4.482

then in (26), (27), $L_i, i = 1, 2$, is replaced by $L_i^{n+1/2}, i = 1, 2$, see, for example, [5, (3.1)]. We choose

$$a_1(x, y, t) = (1/4)(1 + x^2 + y^2 + t^2), \quad a_2(x, y, t) = \sin(x + y) + (1/3)(t + 4)$$

with all other parameters the same as in the case of the heat equation.

The ADI OSC scheme has been implemented with boundary conditions on vertical sides approximated without the perturbation terms (cf. (31)). Tables 5 and 6 give the L^2 and H^1 norm errors and the corresponding convergence rates with initial and boundary conditions approximated using Hermite and Gauss interpolants, respectively.

Comparing results of Table 5 with the last two columns of Tables 1 and 2 and comparing results of Table 6 with the last two columns of Tables 3 and 4, we see that there is good agreement. This indicates that the ADI OSC scheme without the perturbation terms for the variable coefficient problem also yields the optimal L^2 norm rate and suboptimal H^1 norm rate.

6. Concluding Remarks

We have shown that the ADI OSC scheme for the heat equation on a rectangle without perturbation terms on vertical sides for nonzero Dirichlet boundary conditions has optimal convergence rate in the L^2 norm. Numerical results confirm the same. This new finding is important for applications of the ADI OSC scheme to non-linear problems, problems with other types of boundary conditions and to problems on non-rectangular regions, in which case it is impossible to use perturbation terms. Numerical results also suggest that if perturbation terms are not used, then a subotimal convergence rate is obtained for the H^1 norm.

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