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GLOBAL WELL-POSEDNESS FOR NAVIER-STOKES-DARCY EQUATIONS WITH THE FREE INTERFACE

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Abstract. In this paper, the Navier-Stokes-Darcy equations with the free interface are considered, which model the movement of the sea and the sand in the seafloor or the filtration of blood through the arterial wall. The global well-posedness of the solution perturbed around the constant steady state is obtained and then the almost exponential decay to the constant stationary state is gained. Finally, we present an efficient explicit discrete scheme based on finite-volume method for the free interface system and provide the numerical tests to illustrate the consistency with our analysis result.

Key words. Navier-Stokes-Darcy equations, the free interface, global well-posedness, large time behavior.

1. Introduction

In this paper, we consider the Navier-Stokes-Darcy equations with the free boundary, that is, the viscous, incompressible fluid coupled with the porous medium flow which are separated by the free interface. The equations model the movement of the sea and the sand in the seafloor or the filtration of blood through the arterial wall; refer to [29, 2] and the references therein. The mixed Stokes-Darcy model, which is the simplified model of the Navier-Stokes-Darcy equations, has a wide range of applications in science and engineering including industrial settings, especially in cases where a free flowing fluid moves over a porous medium, referring to [8, 14, 24, 26] and the reference therein.

Let us assume that the two-phase flows are confined in a domain $\Omega \subset \mathbb{R}^3$, which is separated into two free moving regions $\Omega_+(t)$ and $\Omega_-(t)$ such that $\bar{\Omega} = \bar{\Omega}_+(t) \cup \bar{\Omega}_-(t)$ and $\Omega_+(t) \cap \Omega_-(t) = \emptyset$. Here $\Omega_+(t)$ and $\Omega_-(t)$ represent the region of the upper flow and the porous matrix region, respectively, defined as

$$\Omega_+(t) = \{y \in \mathbb{T}^2 \times \mathbb{R} \mid \eta(t, y') < y_3 < 1\} \tag{1}$$

and

$$\Omega_-(t) = \{y \in \mathbb{T}^2 \times \mathbb{R} \mid -b(y') < y_3 < \eta(t, y')\}.$$

where $y = (y', y_3)$ and $y' = (y_1, y_2)$, $\mathbb{T}^2 = (2\pi L_1 \mathbb{T}) \times (2\pi L_2 \mathbb{T})$. $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the usual 1−torus and $L_1, L_2 > 0$ are fixed constants. The interface separating the domain Ω is denoted by

$$\Sigma_-(t) = \{(y', y_3) | y_3 = \eta(t, y')\}.$$

Let

$$\Sigma_+ = \{(y', y_3) | y_3 = 1\} \quad \text{and} \quad \Sigma_{-b} = \{(y', y_3) | y_3 = -b(y')\}$$

denote the fixed upper boundary of $\Omega_+(t)$ and the given lower boundary of $\Omega_-(t)$ respectively, where $b(y') \in C^\infty(\mathbb{T}^2)$ is the known function describing the location of

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the bottom Σ_{-b}. The upper fluid is described by the incompressible Navier-Stokes equations

$$\begin{cases} \rho_w \partial_t \tilde{u} + \rho_w \tilde{u} \cdot \nabla \tilde{u} + \nabla \tilde{p} = \mu \Delta \tilde{u} - g \rho_w \vec{e}_3 \\ \operatorname{div} \tilde{u} = 0 \end{cases} \quad \text{in } \Omega_+(t), \tag{2}$$

where the vector $\tilde{u}$ and the scalar function $\tilde{p}$ denote the velocity and pressure of the fluid respectively. $\rho_w > 0$ and $\mu > 0$ denote the density and viscosity of the upper fluid, $g > 0$ is the gravitational constant and $\vec{e}_3 = (0, 0, 1)^T$ is the $Z-$axis unit vector.

The lower fluid is described by the porous medium model with $\gamma > 1$

$$\partial_t \tilde{\phi} - \Delta \tilde{\phi}^\gamma = 0, \tag{3}$$

where the scalar function $\tilde{\phi}(t, \cdot) : \Omega_-(t) \to \mathbb{R}$ denotes the hydraulic head or dynamic pressure. By the Darcy's law, the velocity of the lower fluid is defined by

$$u_- := -\frac{\gamma}{\gamma - 1} \nabla \tilde{\phi}^{\gamma - 1} \quad \text{for} \quad \gamma > 1.$$

In the present paper, we consider the following boundary condition. On the upper fixed boundary Σ_+ the non-slip condition is considered

$$\tilde{u} = 0 \qquad \text{on } \Sigma_+, \tag{4}$$

and we give some fixed constant pressure $\tilde{\phi}_b > 0$ on the lower fixed boundary Σ_{-b}, i.e.

$$\tilde{\phi} = \tilde{\phi}_b \qquad \text{on } \Sigma_{-b}. \tag{5}$$

It's an open problem that what conditions on the free interface $\Sigma_-(t)$ make the problem (2) and (3) well-posed. In the present paper, the Beavers-Joesph-Saffman's interface condition on $\Sigma_-(t)$, seeing [7, 30] for the detail, is considered

$$\begin{cases} \tilde{u} \cdot \vec{n} = u_- \cdot \vec{n} \\ (\tilde{p} I - \mu \mathbb{D}(\tilde{u})) \vec{n} = \tilde{P}_\gamma \vec{n} - \rho_s g \eta \vec{n} \end{cases} \quad \text{on } \Sigma_-(t), \tag{6}$$

where

$$\tilde{P}_\gamma = \frac{\gamma}{\gamma - 1} \tilde{\phi}^{\gamma - 1} \quad \text{for } \gamma > 1,$$

I denotes the 3×3 identity matrix, $(\mathbb{D}(\tilde{u}))_{ij} = \partial_j \tilde{u}^i + \partial_i \tilde{u}^j$ describes twice of the velocity deformation tensor, the positive constant ρ_s is the density of lower fluid, satisfying

$$\rho_s > \rho_w,$$

$\vec{n}$ is the unit normal vector of $\Sigma_-(t)$ pointing to the upper fluid, given by

$$\vec{n} = \frac{(-\partial_{y_1} \eta, -\partial_{y_2} \eta, 1)}{\sqrt{1 + |\nabla_H \eta|^2}} = \frac{\vec{N}}{\sqrt{1 + |\nabla_H \eta|^2}},$$

∇_H, div_H and Δ_H denote the horizontal gradient, the horizontal divergence and the horizontal Laplace operator respectively. According to the kinematic boundary condition of the fluid, the free interface satisfies

$$\partial_t \eta + u_{-,1} \partial_{y_1} \eta + u_{-,2} \partial_{y_2} \eta = u_{-,3} \qquad \text{on } \Sigma_-(t), \tag{7}$$

where $u_{-,i} (i = 1, 2, 3)$ represents the i-th element of the velocity u_-. The initial data are given by

$$(\tilde{u}, \tilde{\phi}, \eta)|_{t=0} = (\tilde{u}_0, \tilde{\phi}_0, \eta_0), \tag{8}$$

which satisfy some certain compatibility condition presented later. We also assume the initial interface function η_0 satisfies

$$1 > \eta_0(y') > -b(y') \qquad \text{on } \mathbb{T}^2 \tag{9}$$

and the "zero-average" condition

$$\int_{\mathbb{T}^2} \eta_0 dy' = 0. \tag{10}$$

Then as the time develops, the zero-average condition of the free interface η also persists, i.e.

$$\int_{\mathbb{T}^2} \eta(t)dy' = 0 \quad \text{for } t \geq 0.$$

In fact, by (4), $(6)_1$ and (7), we get

$$\frac{d}{dt}\int_{\mathbb{T}^2} \eta dy' = \int_{\mathbb{T}^2} \partial_t \eta dy' = \int_{\Gamma_-(t)} \tilde{u}\cdot\vec{n} dS = \int_{\Omega_-(t)} \operatorname{div} \tilde{u} dy = 0,$$

where the incompressible condition in (2) is used.

By the equations (2)-(7), we can easily find the constant steady state $(\bar{u}, \bar{\phi}, \bar{p}, \tilde{\eta})$, satisfying

$$\begin{cases} \bar{u} = 0 & \text{in } \Omega_+, \\ \bar{\phi} = \tilde{\phi}_b & \text{in } \Omega_-, \\ \partial_3 \bar{p} = -\rho_w g & \text{in } \Omega_+, \\ \bar{p}(0) = \bar{P}_\gamma & \text{on } \Sigma_-, \\ \tilde{\eta} = 0 & \text{on } \Sigma_-, \end{cases} \tag{11}$$

where the equilibrium domain Ω_+, Ω_- and the equilibrium internal free surface Σ_- are defined in (12), (13) below and

$$\bar{P}_\gamma = \frac{\gamma}{\gamma-1}\tilde{\phi}_b^{\gamma-1} \quad \text{for } \gamma > 1 \ .$$

According to $(11)_3$ and $(11)_4$, we can get $\bar{p} = -\rho_w g y_3 + \bar{P}_\gamma$.

There are many works on the free boundary problems in fluid mechanics. For the single layer viscous incompressible flow, Beale [4, 5] firstly proved the local well-posedness of the solution with or without the influence of the surface tension, and then the global well-posedness of the solution is showed by Beale [5] under consideration of the surface tension and Sylvester [36] without cosieration of the surface tension. Coutand and Shkoller [10] also established the local well-posedness of the solution under consideration of the surface tension. Solonnikov [32] proved the local well-posedness of the solution in Hölder space without the effect of the surface tension and also did a series of works on the free boundary problem in fluid mechanics, referring to [33, 34, 35]. Allain [1] also got the local well-posedness of the solution by different method. Bae [3] showed the global solvability in Sobolev space by energy method. The large time behavior of the solution is showed in [6] under cosideration of the surface tension and in [27, 39, 40] for the periodic case. Recently, Guo and Tice [15, 16, 17] developed a two-tier energy method to prove the local and global well-posedness and the large time behavior of the solution.

For the two layer viscous fluid flow with free boundary, Yao and Zhu [44] investigated the global existence and uniqueness of weak solution for a viscous two-phase model. Prüss and Simonett [28] proved the local well-posedness of a free interface problem with surface tension. Hataya [18] considered the periodic free interface problem with surface tension and showed the local and global existence of the solution perturbed around the steady state Couette flow. Wang, Tice and Kim [43] considered the two viscous incompressible flow which satisfies the density of the lower fluid bigger than the upper one, and obtained the global existence of the solution and the decay to the equilibrium at almost exponential rate without the influence of surface tension or at exponential rate under the effect of surface tension. Xu and Zhang [45] also considered the global solvability of the same problem in differential space. As regards the Stokes-Darcy equations, there are many numerical results about it with the interface conditions, referring to [22, 23] and the references therein.

For simplicity, we set $\gamma = 3$ in this paper, and then the porous medium equation (3) reads as

$$\partial_t \tilde{\phi} - \Delta \tilde{\phi}^3 = 0.$$

In the present paper, we mainly consider the well-posedness and large time behavior of the solutions to Navier-Stokes-Darcy equations (2)-(7) perturbated around the constant steady state. To do these, we only need to do the a priori estimates by the two-tier energy method developed by Guo and Tice [17].

1.1. Reformulation in the flattening coordinate. The movement of the free interface $\Sigma_-(t)$ creates many mathematical difficulties. To deal with these, we will transform the free boundary problem to the problem in the new fixed domain by a special flattening coordinate transformation motivated by one introduced in [5]. The equilibrium domain and the equilibrium interface are defined as

$$\begin{aligned} \Omega_+ &:= \{(x', x_3) \in \mathbb{T}^2 \times \mathbb{R} \mid 0 < x_3 < 1\}, \\ \Omega_- &:= \{(x', x_3) \in \mathbb{T}^2 \times \mathbb{R} \mid -b(x') < x_3 < 0\} \end{aligned} \tag{12}$$

and

$$\Sigma_- := \{(x', x_3) \mid x_3 = 0\}, \tag{13}$$

where $x' = (x_1, x_2)$. Define $\Omega = \Omega_+ \cup \Omega_-$. The harmonic extension of η is defined as

$$\bar{\eta}(t, x', x_3) := \mathcal{P}\eta(t, x', x_3) \qquad \text{in } \mathbb{R}^+ \times \mathbb{T}^2 \times \mathbb{R},$$

where $\mathcal{P}\eta$ is defined in the appendix. Then the flattening transform is

$$\Xi : x \in \Omega \mapsto \Omega(t) \ni (y_1, y_2, y_3) = (x_1, x_2, x_3 + \theta\bar{\eta}), \tag{14}$$

where $\theta \in C^\infty(\mathbb{R})$ is the nonnegative smooth cut-off function, satisfying $\theta(1) = 0$, $\theta(0) = 1$ and $\theta(-\frac{b}{2}) = 0$, $\underline{b} = \min_{\mathbb{T}^2} b(x_1, x_2)$. Notice that $\Xi(t, \Sigma_+) = \Sigma_+$, $\Xi(t, \Sigma_-) = \Sigma_-(t)$ and $\Xi(t, \Sigma_{-b}) = \Sigma_{-b}$, which means the flattening transformation Ξ keeps the upper and lower boundaries unmoved and transforms the free interface $\Sigma_-(t)$ to the fixed interface Σ_-.

Via the direct computation, we obtain

$$\nabla\Xi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ A & B & J \end{pmatrix} \quad \text{and} \quad \mathscr{A} := \left(\nabla\Xi^{-1}\right)^T = \begin{pmatrix} 1 & 0 & -AK \\ 0 & 1 & -BK \\ 0 & 0 & K \end{pmatrix}, \tag{15}$$

where

$$A = \theta\partial_1\bar{\eta}, \; B = \theta\partial_2\bar{\eta}, \; J = 1 + \partial_3\theta\bar{\eta} + \theta\partial_3\bar{\eta} \;\text{ and }\; K = J^{-1}. \tag{16}$$

Notice that if the norm of the free interface function η is sufficiently small in some Sobolev space, the mapping Ξ is a C^1 diffeomorphism. Under this flattening transformation Ξ, the problem (2)-(7) can be transformed to the problem in a fixed domain problem.

$$\begin{cases} \rho_w(\partial_t u - K\partial_t \Xi_3 \partial_3 u + u\cdot\nabla_{\mathscr{A}} u) + \nabla_{\mathscr{A}}\tilde{p} = \mu\Delta_{\mathscr{A}} u - \rho_w g\vec{e}_3 & \text{in } \Omega_+,\\ \mathrm{div}_{\mathscr{A}}\, u = 0 & \text{in } \Omega_+,\\ \partial_t\tilde{\phi} - K\partial_t\Xi_3\partial_3\tilde{\phi} - \Delta_{\mathscr{A}}\tilde{\phi}^3 = 0 & \text{in } \Omega_-,\\ u = 0 & \text{on } \Sigma_+,\\ \partial_t\eta = \tilde{u}\cdot\vec{N} & \text{on } \Sigma_-,\\ u\cdot\vec{n} = -\frac{3}{2}\nabla_{\mathscr{A}}\tilde{\phi}^2\cdot\vec{n} & \text{on } \Sigma_-,\\ (\tilde{p}I - \mu\mathbb{D}_{\mathscr{A}}(u))\vec{n} = -\rho_s g\eta\vec{n} + \frac{3}{2}\tilde{\phi}^2\vec{n} & \text{on } \Sigma_-,\\ \tilde{\phi} = \tilde{\phi}_b & \text{on } \Sigma_{-b}, \end{cases}$$

where the differential operators $\nabla_{\mathscr{A}}$, $\mathrm{div}_{\mathscr{A}}$ and $\Delta_{\mathscr{A}}$ denote $(\nabla_{\mathscr{A}} f)_i := \mathscr{A}_{ij}\partial_j f$, $\mathrm{div}_{\mathscr{A}}\, X := \mathscr{A}_{ij}\partial_j X_i$ and $\Delta_{\mathscr{A}} f = \mathrm{div}_{\mathscr{A}}\,\nabla_{\mathscr{A}} f$ for any function f and any vector function X, the repeated indexes denote the sum from 1 to 3, $S_{\mathscr{A}}(p,u) := pI - \mu\mathbb{D}_{\mathscr{A}} u$ denote the stress tensor, I is the 3×3 identity matrix and $(\mathbb{D}_{\mathscr{A}} u)_{ij} = \mathscr{A}_{ik}\partial_k u_j + \mathscr{A}_{jk}\partial_k u_i$, $\vec{N} := -\partial_1\eta\vec{e}_1 - \partial_2\eta\vec{e}_2 + \vec{e}_3$.

Define $\phi = \tilde{\phi} - \tilde{\phi}_b$ and $p = \tilde{p} - \bar{p} - \partial_3\bar{p}(\theta\bar{\eta})$. Note the fact that $\bar{p}(0) = \tilde{\phi}_b$ on Σ_- and

$$\begin{aligned} &\rho_w g\delta_{i3} + \mathscr{A}_{ij}\partial_j\bar{p} + \mathscr{A}_{ij}\partial_j(\partial_3\bar{p}\theta\bar{\eta})\\ =& -\rho_w g\mathscr{A}_{ij}\partial_j\Xi_3 + \rho_w g\mathscr{A}_{i3} + \rho_w g\mathscr{A}_{ij}\partial_j(\theta\bar{\eta})\\ =& -\rho_w g\mathscr{A}_{ij}\partial_j(x_3+\theta\bar{\eta}) + \rho_w g\mathscr{A}_{i3} + \rho_w g\mathscr{A}_{ij}\partial_j(\theta\bar{\eta}) = 0. \end{aligned}$$

Then we get

$$\begin{cases} \rho_w(\partial_t u - K\partial_t\Xi_3\partial_3 u + u\cdot\nabla_{\mathscr{A}} u) + \nabla_{\mathscr{A}} p = \mu\Delta_{\mathscr{A}} u & \text{in } \Omega_+,\\ \mathrm{div}_{\mathscr{A}}\, u = 0 & \text{in } \Omega_+,\\ \partial_t\phi - K\partial_t\Xi_3\partial_3\phi - 3\tilde{\phi}_b^{\,2}\Delta_{\mathscr{A}}\phi = 3\tilde{\phi}_b\Delta_{\mathscr{A}}\phi^2 + \Delta_{\mathscr{A}}\phi^3 & \text{in } \Omega_-,\\ u = 0 & \text{on } \Sigma_+,\\ \partial_t\eta = u\cdot\vec{N} & \text{on } \Sigma_-,\\ u\cdot\vec{n} = -3\tilde{\phi}_b\nabla_{\mathscr{A}}\phi\cdot\vec{n} - \frac{3}{2}\nabla_{\mathscr{A}}\phi^2\cdot\vec{n} & \text{on } \Sigma_-,\\ (pI - \mu\mathbb{D}_{\mathscr{A}}(u))\vec{n} = -\rho g\eta\vec{n} + 3\tilde{\phi}_b\phi\vec{n} + \frac{3}{2}\phi^2\vec{n} & \text{on } \Sigma_-,\\ \phi = 0 & \text{on } \Sigma_{-b}, \end{cases} \tag{17}$$

where $\rho := \rho_s - \rho_w$.

1.2. Main results. In this paper, we mainly consider the global well-posedness and large time behavior of the strong solution to the equations (17). The compatible conditions of the initial data satisfy

$$\begin{cases} D_t^j u(0,\cdot) = 0 & \text{on } \Sigma_+,\\ \nabla_{\mathscr{A}_0}\cdot(D_t^j u(0,\cdot)) = 0 & \text{in } \Omega_+(0),\\ D_t^j(u\cdot\vec{n})(0,\cdot) = D_t^j(u_-\cdot\vec{n})(0,\cdot) & \text{on } \Sigma_-,\\ \mathbb{P}_0\left(W_3^j(0,\cdot) + \mathbb{D}_{\mathscr{A}_0} D_t^j u(0,\cdot)N_0\right) = 0 & \text{on } \Sigma_-,\\ D_t^j\phi(0,\cdot) = 0 & \text{on } \Sigma_{-b} \end{cases} \tag{18}$$

for $j = 0, 1, 2, ..., 2N-1$, where $W_3^0 := W_3 = \phi\vec{n} - \rho g\eta\vec{n}$ and

$$W_3^j = \partial_t^j W_3 + \sum_{l=0}^{j-1}\partial_t^l\mathfrak{E}^3(D_t^{j-l-1}u, \partial_t^{j-l-1}p),$$

with $\mathfrak{E}^3(v,q) = \mathbb{D}_{\mathscr{A}}(Rv)\vec{N} - (qI - \mathbb{D}_{\mathscr{A}} v)\partial_t\vec{N} + \mathbb{D}_{\partial_t\mathscr{A}} v\vec{N}$ and $D_t u = \partial_t u - Ru$ for $R = \partial_t M M^{-1}$ and $M = K\nabla\Xi$. The orthogonal projection onto the tangential space of the suface $\{(x',x_3)|x_3=\eta_0(x'),\ x'\in\mathbb{T}^2\}$ is defined as

$$\mathbb{P}_0(v)=v-(v\cdot\vec{N_0})\vec{N_0}\left|\vec{N_0}\right|^{-2}\quad\text{for } \vec{N_0}=\vec{N}(0,\cdot)=(-\partial_1\eta_0,-\partial_2\eta_0,1).$$

To describe our results clearly, the energies and dissipations of the solution to equations (17) are defined as follows. For any integer $N\geq 3$, the high-order energy is wrote as

$$\mathscr{E}_{2N}=\sum_{j=0}^{2N}\left(\left\|\partial_t^j u\right\|_{4N-2j}^2+\left\|\partial_t^j\phi\right\|_{4N-2j}^2+\left\|\partial_t^j\eta\right\|_{4N-2j}^2\right)+\sum_{j=0}^{2N-1}\left\|\partial_t^j p\right\|_{4N-2j-1}^2$$

and the corresponding dissipation as

$$\begin{aligned}\mathscr{D}_{2N}=&\sum_{j=0}^{2N}\left(\left\|\partial_t^j u\right\|_{4N-2j+1}^2+\left\|\partial_t^j\phi\right\|_{4N-2j+1}^2\right)+\sum_{j=0}^{2N-1}\left\|\partial_t^j p\right\|_{4N-2j}^2\\&+\|\eta\|_{4N-1/2}^2+\|\partial_t\eta\|_{4N-1/2}^2+\sum_{j=2}^{2N+1}\left\|\partial_t^j\eta\right\|_{4N-2j+5/2}^2.\end{aligned}$$

The low-order energy is defined as

$$\begin{aligned}\mathscr{E}_{N+2}=&\sum_{j=0}^{N+2}\left(\|\partial_t^j u\|_{2(N+2)-2j}^2+\|\partial_t^j\phi\|_{2(N+2)-2j}^2+\|\partial_t^j\eta\|_{2(N+2)-2j}^2\right)\\&+\sum_{j=0}^{N+1}\|\partial_t^j p\|_{2(N+2)-2j-1}^2\end{aligned}$$

and the low-order dissipation is

$$\begin{aligned}\mathscr{D}_{N+2}=&\sum_{j=0}^{N+2}\left(\left\|\partial_t^j u\right\|_{2(N+2)-2j+1}^2+\left\|\partial_t^j\phi\right\|_{2(N+2)-2j+1}^2\right)+\sum_{j=0}^{N+1}\left\|\partial_t^j p\right\|_{2(N+2)-2j}^2\\&+\|\eta\|_{2(N+2)-1/2}^2+\|\partial_t\eta\|_{2(N+2)-1/2}^2+\sum_{j=2}^{(N+2)+1}\left\|\partial_t^j\eta\right\|_{2(N+2)-2j+5/2}^2.\end{aligned}$$

The highest-order norm of the free interface η is defined by

$$\mathscr{F}_{2N}=\|\eta\|_{4N+1/2}^2. \tag{19}$$

Finally, the total energy function is defined as

$$\mathscr{G}_{2N}(t)=\sup_{0\leq r\leq t}\mathscr{E}_{2N}(r)+\int_0^t\mathscr{D}_{2N}(r)dr+\sup_{0\leq r\leq t}(1+r)^{4N-8}\mathscr{E}_{N+2}(r)+\sup_{0\leq r\leq t}\frac{\mathscr{F}_{2N}(r)}{(1+r)}.$$

The main results in this paper are stated as follows.

Theorem 1.1. *Assume that the initial data (u_0,ϕ_0,η_0) satisfy some compatible conditions* (18) *and η_0 satisfies the zero average condition* (10). *Let $N\geq 3$ be an integer. Then there exists a constant $0<\kappa=\kappa(N)$ such that if the initial data satisfy $\mathscr{E}_{2N}(0)+\mathscr{F}_{2N}(0)<\kappa$, the free interface problem* (17) *has a unique global solution (u,p,ϕ,η), satisfying*

$$\mathscr{G}_{2N}(t)\leq C_1\left(\mathcal{E}_{2N}(0)+\mathcal{F}_{2N}(0)\right)\quad\textit{for any}\quad t\in[0,\infty], \tag{20}$$

where $C_1>0$ is a uniform constant independent of the time t.

Remark 1.2. *Although we consider $\gamma=3$ here, the same results in Theorem 1.1 can be also established to all the cases $\gamma>1$.*

Remark 1.3. *The bound of the total energy* $\mathscr{G}_{2N}$ *in* (20) *implies that the solution to the equations* (17) *algebraically decays to the constant stationary*

$$\mathscr{E}_{N+2}(t) \leq C_2\kappa(1+t)^{-4N+8}. \tag{21}$$

Since N may be taken to be arbitrarily large, this decay result can be regarded as an "almost exponential" decay rate.

Remark 1.4. *Compared with the two-phase Navier-Stokes equations in Wang, Tice and Kim* [43]*, it is very interesting that the conditions on the free interface are different with each other. In* [43]*, the two-phase flow don't slip relatively with each other on the free interface* $\Sigma_-(t)$*, i.e.* $u_+|_{\Sigma_-(t)} = u_-|_{\Sigma_-(t)}$*. However, in our case, we only need the two-phase flow has the same normal velocity on* $\Sigma_-(t)$*, i.e.*

$$u_+\cdot\vec{n} = u_-\cdot\vec{n} \qquad\qquad on\ \Sigma_-(t),$$

which means that these two flows may slide with each other on the interface. The other difference with [43] *is that the flow in the lower domain* $\Omega_-(t)$ *is described by the scalar function* ϕ *which is governed by the porous medium equation and the velocity* u_- *is defined by the Darcy's law.*

Remark 1.5. *The main difficulty of the present paper is the existence of the free interface separated these two flows with each other. Due to the difference of the density between the upper flow and the lower one, i.e.* $\rho_w < \rho_s$*, which gives the dissperation of the free interface in the energy, we combine the energy estimates on these two flows together and obtain the a priori estimates by the two-tier energy method. Some technical estimates are used when we apply the highest order temporal derivative to the nonlinear terms of the porous medium equation. Especially, we need to be careful and make full use of the product estimate* (142) *in Lemma 7.6 exactly to control the highest order temporal derivative term* $\nabla_{\mathscr{A}}\partial_t^{2N}\phi$ *on the boundary* Σ_- *in the nonlinear terms by the dissipation besides the trace theorem.*

The rest part of paper is arranged as follows. Together with the local existence results in the appendix Theorem 7.1, we shall establish the uniform a-priori estimate in Section 2-4 under the a-priori assumption

$$\mathscr{G}_{2N}(T) \leq \delta \tag{22}$$

for some existence time $T > 0$ and small constant $\delta > 0$. In Section 2, the temporal estimate is obtained and the tangential and normal estimates are gained respectively in Section 3 and Section 4. Then the proof of Theorem 1.1 is showed in Section 5. The local existence result and useful tools are listed in the appendix. In section 6, we establish an efficient explicit discrete scheme based on finite-volume method for this free interface system in two dimension. On the other hand, we perform the numerical experiments which agree with the theoretical results in Theorem 1.1.

Notation. Throughout the paper, we use Einstein convention of summing over repeated index for vector and tensor operations. The universal constant $C > 0$ means the generic constant that depends on the parameters N and Ω, but does not depend on the initial data and the time t, and we employ the notation $a \lesssim b$ to denote $a \leqq Cb$ for the universal constant $C > 0$.

Denote $\partial^\alpha = \partial_t^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$ and $|\alpha| = 2\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$ for the multi-index $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^{1+3}$. $\nabla_H f = (\partial_1 f, \partial_2 f)$ and ∇f denote the tangential and total derivatives of f. Denote $H^k(\Omega_\pm)(k \geq 0)$ and $H^s(\Sigma_\pm)(s \in \mathbb{R})$ for the usual Sobolev space with the norm $\|\cdot\|_k$ and $\|\cdot\|_s$ respectively and $H^0(\Omega_\pm) = L^2(\Omega_\pm)$ with the norm $\|\cdot\|$. For given integers $n,\ m \geq 0$, we introduce the following notations to describe the sums of spacial derivatives

$$\|\nabla_H^{m,n} f\|^2 := \sum_{\substack{\alpha\in\mathbb{N}^2\\ m\leq|\alpha|\leq n}} \|\partial^\alpha f\|^2 \text{ and } \|\nabla^{m,n} f\|^2 := \sum_{\substack{\alpha\in\mathbb{N}^3\\ m\leq|\alpha|\leq n}} \|\partial^\alpha f\|^2$$

and the time-space derivatives

$$\left\|\bar{\nabla}_H^{m,n} f\right\|^2 := \sum_{\substack{\alpha\in\mathbb{N}^{1+2}\\ m\leq|\alpha|\leq n}} \|\partial^\alpha f\|^2 \text{ and } \left\|\bar{\nabla}^{m,n} f\right\|^2 := \sum_{\substack{\alpha\in\mathbb{N}^{1+3}\\ m\leq|\alpha|\leq n}} \|\partial^\alpha f\|^2 .$$

If $m = n \geq 0$, we have the following convention

$$\begin{aligned} \|\nabla_H^m f\|^2 = \|\nabla_H^{m,m} f\|^2, \quad \|\nabla^m f\|^2 = \|\nabla^{m,m} f\|^2, \\ \left\|\bar{\nabla}_H^m f\right\|^2 = \left\|\bar{\nabla}_H^{m,m} f\right\|^2, \quad \left\|\bar{\nabla}^m f\right\|^2 = \left\|\bar{\nabla}^{m,m} f\right\|^2. \end{aligned}$$

Denote

$$\int_{\Omega_\pm} f(x) = \int_{\Omega_\pm} f(x)dx \quad \text{and} \quad \int_{\Sigma_\pm} g(x') = \int_{\Sigma_\pm} g(x')dx'$$

for simplification.

2. Temporal energy estimates.

In this section, the temporal energy estimate will be established under the a-priori assumption (22) by the the linear geometric equations. Here we firstly give an important lemma on the flattening transform.

Lemma 2.1. *There exists a universal* $0 < \delta < 1$*, such that if the a-priori assumption* (22) *holds, then we have*

$$\|J-1\|_{L^\infty}^2 + \|A\|_{L^\infty}^2 + \|B\|_{L^\infty}^2 \leq \frac{1}{2}, \text{ and } \|K\|_{L^\infty}^2 + \|\mathcal{A}\|_{L^\infty}^2 \lesssim 1. \tag{23}$$

Proof. According to the definitions of A, B and J given in(16) and Lemmas 7.3, we may bound

$$\|J-1\|_{L^\infty}^2 + \|A\|_{L^\infty}^2 + \|B\|_{L^\infty}^2 \lesssim \|\bar{\eta}\|_3^2 \lesssim \|\eta\|_{5/2}^2 .$$

Then if δ is sufficiently small, we obtain the estimate in (23). □

Applying the differential operator ∂_t^s for $s \in \mathbb{N}$ to equation (17), we get

$$\begin{cases} \partial_t v - K\partial_t \Xi_3 \partial_3 v + u\cdot\nabla_{\mathscr{A}} v + \operatorname{div}_{\mathscr{A}} S_{\mathscr{A}}(q,v) = F^1 & \text{in } \Omega_+ \\ \operatorname{div}_{\mathscr{A}} v = F^2 & \text{in } \Omega_+ \\ \partial_t z - K\partial_t\Xi_3\partial_3 z - 3\tilde{\phi_b}^2 \Delta_{\mathscr{A}} z = F^3 & \text{in } \Omega_- \\ v = 0 & \text{on } \Sigma_+ \\ S_{\mathscr{A}}(q,v)\vec{N} = -\rho g\xi\vec{N} + 3\tilde{\phi_b} z\vec{N} + F^4 & \text{on } \Sigma_- \\ v\cdot\vec{N} = -3\tilde{\phi_b}\nabla_{\mathscr{A}} z\cdot\vec{N} + F^5 & \text{on } \Sigma_- \\ \partial_t\xi = \vec{N}\cdot v + F^6 & \text{on } \Sigma_- \\ z = 0 & \text{on } \Sigma_{-b}, \end{cases} \tag{24}$$

where $v = \partial_t^s u$, $z = \partial_t^s \phi$, $q = \partial_t^s p$ and $\xi = \partial_t^s \eta$. The nonlinear terms in (24) are listed as

$$
\begin{aligned}
F_i^1 = & \sum_{0<m<s} C_s^m \partial_t^m w \partial_3 \partial_t^{s-m} u_i + \sum_{0<m\le s} C_s^m \partial_t^m K \partial_t^{s-m} \partial_t \Xi_3 \partial_3 u_i \\
& - \sum_{0<m\le s} C_s^m \partial_t^m (u_j \mathscr{A}_{jk}) \partial_t^{s-m} \partial_k u_i \\
& - \sum_{0<m\le s} C_s^m \partial_t^m \mathscr{A}_{ik} \partial_t^{s-m} \partial_k p + \sum_{0<m<s} C_s^m \partial_t^m \mathscr{A}_{jr} \partial_t^{s-m} \partial_r (\mathscr{A}_{jk} \partial_k u_i) \\
& + \sum_{0<m\le s} C_s^m \mathscr{A}_{jr} \partial_r (\partial_t^m \mathscr{A}_{jk} \partial_k \partial_t^{s-m} u_i) + \partial_t^s \mathscr{A}_{jr} \partial_r (\mathscr{A}_{jk} \partial_k u_i),
\end{aligned} \tag{25}
$$

$$
F^2 = - \sum_{0<m<s} C_s^m \partial_t^m \mathscr{A}_{jk} \partial_k \partial_t^{s-m} u_j - \partial_t^s \mathscr{A}_{jk} \partial_k u_j, \tag{26}
$$

$$
\begin{aligned}
F_i^4 = & - \sum_{0\le m<s} C_s^m (\rho g \partial_t^m \eta \partial_t^{s-m} \vec{N}_i + \partial_t^m p \partial_t^{s-m} \vec{N}_i) \\
& + \sum_{0\le m<s} C_s^m [\partial_t^m \partial_k u_j \partial_t^{s-m} (\mathscr{A}_{ik} \vec{N}_j) + \partial_t^m \partial_k u_i \partial_t^{s-m} (\mathscr{A}_{jk} \vec{N}_j)] \\
& + \sum_{0\le m<s} 3\tilde{\phi}_b C_s^m \partial_t^m \phi \partial_t^{s-m} \vec{N}_i + \frac{3}{2} \partial_t^s (\phi^2 \vec{N}_i),
\end{aligned} \tag{27}
$$

(28)

$$
\begin{aligned}
F_i^5 = & - \sum_{0<m\le s} 3\tilde{\phi}_b C_s^m \partial_t^m \mathscr{A}_{ik} \partial_k \partial_t^{s-m} \phi \vec{N}_i - \sum_{0\le m<s} 3\tilde{\phi}_b C_s^m \partial_t^m (\mathscr{A}_{ik} \partial_k \phi) \partial_t^{s-m} \vec{N}_i \\
& - \sum_{0\le m<s} C_s^m \partial_t^m u \cdot \partial_t^{s-m} \vec{N} + 3 \partial_t^s (\phi \mathscr{A}_{ik} \partial_k \phi \vec{N}_i),
\end{aligned}
$$

$$
F^6 = \sum_{0\le m<s} C_s^m \partial_t^m u \partial_t^{s-m} \vec{N} \tag{29}
$$

and

$$
F^3 = F^{3,1} + F^{3,2}
$$

where

$$
\begin{aligned}
F^{3,1} = & \sum_{0<m<s} 3\tilde{\phi}_b^{\,2} C_s^m \partial_t^m \mathscr{A}_{jr} \partial_r \partial_t^{s-m} (\mathscr{A}_{jk} \partial_k \phi) \\
& + \sum_{0<m<s} 3\tilde{\phi}_b^{\,2} \mathscr{A}_{jr} \partial_r (\partial_t^m \mathscr{A}_{jk} \partial_k \partial_t^{s-m} \phi) + 3\tilde{\phi}_b^{\,2} \partial_t^s \mathscr{A}_{jr} \partial_r (\mathscr{A}_{jk} \partial_k \phi) \\
& + \sum_{0<m<s} C_s^m \partial_t^m (K \partial_t \Xi_3) \partial_3 \partial_t^{s-m} \phi + 3\tilde{\phi}_b^{\,2} \mathscr{A}_{jr} \partial_r (\partial_t^s \mathscr{A}_{jk} \partial_k \phi) \\
& + \sum_{0<m\le s} C_s^m \partial_t^m (\mathscr{A}_{ij} \partial_j \mathscr{A}_{ik}) \partial_t^{s-m} (3\phi^2 \partial_k \phi + 6\tilde{\phi}_b \phi \partial_k \phi) \\
& + \sum_{0<m\le s} C_s^m \partial_t^m (\mathscr{A}_{ij} \mathscr{A}_{ik}) \partial_t^{s-m} ((6 + 6\tilde{\phi}_b) \phi \partial_j \phi \partial_k \phi + 3\phi^2 \partial_j \partial_k \phi + 6\tilde{\phi}_b \phi \partial_j \partial_k \phi),
\end{aligned} \tag{30}
$$

$$
F^{3,2} = \Delta_{\mathscr{A}} \partial_t^s \phi^3 + 3\tilde{\phi}_b \Delta_{\mathscr{A}} \partial_t^s \phi^2
$$

and $w = K \partial_t \Xi_3$.

Then we have the basic energy for the system (24) as follows,

Lemma 2.2. *Assume* (u, p, ϕ, η) *is the solution to* (17) *and has suitable regularity. Then we have*

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\Omega_+}\tilde{\phi_b}J|v|^2+\frac{1}{2}\int_{\Omega_-}J|z|^2+\frac{1}{2}\int_{\Sigma_-}\tilde{\phi_b}|\xi|^2\right)$$
$$+\frac{1}{2}\int_{\Omega_+}\tilde{\phi_b}J|\mathbb{D}_{\mathscr{A}}v|^2+\int_{\Omega_-}3\tilde{\phi_b}^2J|\nabla_{\mathscr{A}}z|^2$$
$$=\int_{\Omega_+}\tilde{\phi_b}J(v\cdot F^1+qF^2)+\int_{\Omega_-}F^3Jz+\tilde{\phi_b}\int_{\Sigma_-}F^4v+F^5z+F^6\xi+\int_{\Sigma_-}\partial_t\eta\frac{|z|^2}{2}.$$

Proof. Multiplying the first equation (24) by Jv and integrating over Ω_+, we have

$$I+II=III, \tag{31}$$

where

$$I=\int_{\Omega_+}\partial_t v_i Jv_i-\partial_t\Xi_3\partial_3 v_i v_i+u_j\mathscr{A}_{jk}\partial_k v_i Jv_i,$$
$$II=\int_{\Omega_+}\mathscr{A}_{jk}\partial_k(S_{\mathscr{A}}(q,v)_{ij})Jv_i,\quad\text{and}\quad III=\int_{\Omega_+}F^1\cdot Jv.$$

Integrating by parts and noting the boundary condition $v=0$ on Σ_+ and the geometric identity $\partial_k(J\mathscr{A}_{jk})=0$ which is proved in Lemma 7.5, we get

$$\begin{aligned}I=\frac{1}{2}\frac{d}{dt}\int_{\Omega_+}J|v|^2-\frac{1}{2}\int_{\Omega_+}\partial_t J|v|^2+\frac{1}{2}\int_{\Omega_+}\partial_3\partial_t\Xi_3|v|^2+\frac{1}{2}\int_{\Sigma_-}\partial_t\Xi_3|v|^2\\-\frac{1}{2}\int_{\Omega_+}J\mathscr{A}_{jk}\partial_k u_j|v|^2-\frac{1}{2}\int_{\Sigma_-}Ju_j\mathscr{A}_{j3}|v|^2=:I_1+I_2.\end{aligned}\tag{32}$$

Since $\mathscr{A}_{jk}\partial_k u_j=0$, $\partial_t\eta=u\cdot\vec{N}$ and $\theta(0)=1$, it yields

$$\begin{aligned}I_2=&-\frac{1}{2}\int_{\Omega_+}\partial_t J|v|^2+\frac{1}{2}\int_{\Omega_+}\partial_3\partial_t\Xi_3|v|^2+\frac{1}{2}\int_{\Sigma_-}\partial_t\Xi_3|v|^2\\&-\frac{1}{2}\int_{\Omega_+}J\mathscr{A}_{jk}\partial_k u_j|v|^2-\frac{1}{2}\int_{\Sigma_-}Ju_j\mathscr{A}_{j3}|v|^2\\=&\frac{1}{2}\int_{\Omega_+}-\partial_t J|v|^2+\partial_3\partial_t\Xi_3|v|^2-\frac{1}{2}\int_{\Omega_+}J\mathscr{A}_{jk}\partial_k u_j|v|^2\\&+\frac{1}{2}\int_{\Sigma_-}\partial_t\Xi_3|v|^2-Ju_j\mathscr{A}_{j3}|v|^2.\end{aligned}$$

Note that $J\mathscr{A}_{j3}=\vec{N}_j$, so we get

$$I_2=0\quad\text{and}\quad I=\frac{1}{2}\frac{d}{dt}\int_{\Omega_+}J|v|^2.$$

Similarly, we calculate II, yielding

$$\begin{aligned}II &= -\int_{\Omega_+} S_{\mathscr{A}}(q,v)_{ij}\partial_k(Jv_i\mathscr{A}_{jk}) - \int_{\Sigma_-} v_i S_{\mathscr{A}}(q,v)_{ij}J\mathscr{A}_{j3}\\ &= -\int_{\Omega_+} S_{\mathscr{A}}(q,v)_{ij}J\mathscr{A}_{jk}\partial_k v_i - \int_{\Sigma_-} S_{\mathscr{A}}(q,v)\vec{N}\cdot v\\ &= \int_{\Omega_+} -q\mathscr{A}_{ik}\partial_k v_i J + J\frac{|\mathbb{D}_{\mathscr{A}}v|^2}{2} - \int_{\Sigma_-} S_{\mathscr{A}}(q,v)\vec{N}\cdot v\\ &= \int_{\Omega_+} -qF^2 J + J\frac{|\mathbb{D}_{\mathscr{A}}v|^2}{2} - \int_{\Sigma_-} -\xi\vec{N}\cdot v + z\vec{N}\cdot v + F^4\cdot v.\end{aligned}$$

Then we obtain

$$II = \int_{\Omega_+} -qF^2 J + J\frac{|\mathbb{D}_{\mathscr{A}}v|^2}{2} + \frac{1}{2}\partial_t\int_{\Sigma_-}|\xi|^2 + \int_{\Sigma_-} zF^6 - z\partial_t\xi - \xi F^6 - F^4 v. \tag{33}$$

Combining (32) and (33) together, we have

(34)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\int_{\Omega_+} J|v|^2 + \int_{\Sigma_-}|\xi|^2\right] + \int_{\Omega_+} J\frac{|\mathbb{D}_{\mathscr{A}}v|^2}{2} + \int_{\Sigma_-} z(F^6-\partial_t\xi) - \xi F^6 - F^4 v\\ &= \int_{\Omega_+} J(F^1\cdot v + qF^2).\end{aligned}$$

Next, multiplying the third equation of (24) by Jz and integrating over Ω_-, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\Omega_-} J|z|^2 - \frac{1}{2}\int_{\Omega_-}\partial_t J|z|^2 - \int_{\Omega_-}\partial_t\Xi_3\partial_3\frac{|z|^2}{2}\\ &- 3\tilde{\phi_b}^2\int_{\Omega_-}\mathscr{A}_{ik}\partial_k(\nabla_{\mathscr{A}}z)_i Jz = \int_{\Omega_-} F^3 Jz.\end{aligned} \tag{35}$$

Integrating by parts, it yields

$$\int_{\Omega_-}\partial_t\Xi_3\partial_3\frac{|z|^2}{2} = -\int_{\Omega_-}\partial_3\partial_t\Xi_3\frac{|z|^2}{2} + \int_{\Sigma_-}\partial_t\Xi_3\frac{|z|^2}{2} \tag{36}$$

and

$$\begin{aligned}\int_{\Omega_-}\mathscr{A}_{ik}\partial_k(\nabla_{\mathscr{A}}z)_i Jz &= \int_{\Omega_-} z\partial_k(J\mathscr{A}_{ik}(\nabla_{\mathscr{A}}z)_i)\\ &= -\int_{\Omega_-} J\mathscr{A}_{ik}(\nabla_{\mathscr{A}}z)_i\partial_k z + \int_{\Sigma_-} z(\nabla_{\mathscr{A}}z)_i J\mathscr{A}_{i3}\\ &= -\int_{\Omega_-} J|\nabla_{\mathscr{A}}z|^2 + \int_{\Sigma_-} z(\nabla_{\mathscr{A}}z)\cdot\vec{N}.\end{aligned} \tag{37}$$

The boundary condition implies

$$\begin{aligned}\int_{\Sigma_-} z(\nabla_{\mathscr{A}}z)\cdot\vec{N} &= \frac{1}{3\tilde{\phi_b}}\int_{\Sigma_-} z(F^5 - v\cdot\vec{N}) = \frac{1}{3\tilde{\phi_b}}\int_{\Sigma_-} F^5 z - z(\partial_t\xi - F^6)\\ &= \frac{1}{3\tilde{\phi_b}}\int_{\Sigma_-} F^5 z + F^6 z - z\partial_t\xi.\end{aligned} \tag{38}$$

Combining (35), (36), (37) and (38) together, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega_-} J\,|z|^2 + 3\tilde{\phi_b}^2\int_{\Omega_-} J\,|\nabla_{\mathscr{A}} z|^2 - \int_{\Sigma_-}\partial_t\eta\frac{|z|^2}{2}\\ -\tilde{\phi_b}\int_{\Sigma_-} F^5 z + F^6 z - z\partial_t\xi = \int_{\Omega_-} F^3 J z.\end{aligned}\tag{39}$$

Multiplying (34) by $\tilde{\phi_b}$ and adding the result to (39), we complete the proof. □

Next, we will estimate the nonlinear terms in the geometric form (25)-(29) by Hölder inequality, Cauchy inequality and Sobolev embedding inequalities.

Lemma 2.3. *Suppose that the a-priori assumption* (22) *holds. If* δ *is small enough, then we have*

$$\|\partial_t^{2N+1}J\|_0^2 + \|\partial_t^{2N+1}K\|_0^2 \lesssim \mathscr{D}_{2N}.$$

Proof. Note that $J = 1 + \partial_3\theta\bar{\eta} + \theta\partial_3\bar{\eta}$ and $K = J^{-1}$. By Lemma 7.3, we have

$$\|\partial_t^{2N+1}J\|_0^2 \lesssim \|\partial_t^{2N+1}\bar{\eta}\|_0^2 + \|\partial_t^{2N+1}\nabla\bar{\eta}\|_0^2 \lesssim \|\partial_t^{2N+1}\eta\|_{\frac{1}{2}}^2 \lesssim \mathscr{D}_{2N}.\tag{40}$$

By Lemma 2.1, we get $\frac{1}{2} \le J \le \frac{3}{2}$. (40) immediately gives

$$\|\partial_t^{2N+1}K\|_0^2 \lesssim \mathscr{D}_{2N}.$$

□

Lemma 2.4. *Let* F^1, F^2, $F^{3,1}$, F^4, F^5 *and* F^6 *be defined in* (25), (26), (30), (27), (28) *and* (29), *respectively. Suppose that the a-priori assumption* (22) *holds for some small* $\delta > 0$.

i) For $0 \le s \le 2N$, *we have*

$$\|F^1\|_0^2 + \|\partial_t(JF^2)\|_0^2 + \|F^{3,1}\|_0^2 + \|F^4\|_0^2 + \|F^5\|_0^2 + \|F^6\|_0^2 \lesssim \mathscr{E}_{2N}\mathscr{D}_{2N}.\tag{41}$$

Also, we have

$$\|F^2\|_0 \lesssim \mathscr{E}_{2N}.\tag{42}$$

ii) For $0 \le s \le N+2$, *we have*

$$\|F^1\|_0^2 + \|F^2\|_0^2 + \|\partial_t(JF^2)\|_0^2 + \|F^3\|_0^2 + \|F^4\|_0^2 + \|F^5\|_0^2 + \|F^6\|_0^2 \lesssim \mathscr{E}_{2N}\mathscr{D}_{N+2}\tag{43}$$

and

$$\|F^2\|_0^2 \lesssim \mathscr{E}_{2N}\mathscr{E}_{N+2}.$$

Proof. These estimates are directly obtained via Sobolev embedding inequalities, so we take the estimates of F^2 and F^4 as the example. We firstly estimate the term F^2, recalling the definition

$$F^2 = -\sum_{0<m<s} C_s^m \partial_t^m \mathscr{A}_{jk}\partial_k\partial_t^{s-m}u_j - \partial_t^s\mathscr{A}_{jk}\partial_k u_j.$$

Via its definition, F^2 may have the terms like $\partial_t^m(AK)\partial_t^{s-m}\partial_3 u_1$, $\partial_t^m(AK)\partial_t^{s-m}\partial_3 u_2$ and $\partial_t^m K\partial_t^{s-m}\partial_3 u_3$ for any $0 < s \le 2N$. Note that the highest order derivatives satisfy $\partial_t J \sim \partial_t\nabla\bar{\eta}$, $A \sim \nabla\bar{\eta}$ and $B \sim \nabla\bar{\eta}$. Then $\partial_t^m(AK)$ has the form of polynomial $Q(\partial_t^j\nabla\bar{\eta}, \partial_t^{m-j}\nabla\bar{\eta}, K)$, where $j = 1, 2, ...m$. By Lemma 2.1 and Lemma 7.3, it is straightforward to see that each term can be written in the form $X \cdot Y$, where X involves fewer temporal derivative than Y, and then by Sobolev embedding inequalities, Lemma 7.3 and Lemma 7.4 and combining together with the definitions of $\mathscr{E}_{2N}$ and $\mathscr{D}_{2N}$, we obtain

$$\|X\|_{L^\infty}^2 \lesssim \mathscr{E}_{2N} \text{ and } \|Y\|_0^2 \lesssim \mathscr{D}_{2N}.$$

And then $\|XY\|_0^2 \le \|X\|_{L^\infty}^2 \|Y\|_0^2 \lesssim \mathscr{E}_{2N}\mathscr{D}_{2N}$.

To the highest order $s = 2N$, $\partial_t^s \mathscr{A}_{jk}\partial_k u_j \sim \partial_t^{2N}\nabla\bar{\eta}\partial_k u_j$ implies

$$\|\partial_t^{2N}\nabla\bar{\eta}\partial_k u_j\|_0^2 \lesssim \|\partial_t^{2N}\nabla\bar{\eta}\|_0^2\|\nabla u\|_{L^\infty}^2 \lesssim \|\partial_t^{2N}\eta\|_{H^{\frac{1}{2}}}^2 \|u\|_{H^3}^2 \lesssim \mathscr{E}_{2N}\mathscr{D}_{2N}$$

which gives the desired estimate of F^2 in (41).

Similarly, we get

$$\left\|\partial_t^m \mathscr{A}_{jk}\partial_k\partial_t^{s-m}u_j\right\|_0 \lesssim \mathscr{E}_{2N} \quad \text{for } 0 < m < 2N.$$

But to estimate the term $\partial_t^{2N}\mathscr{A}_{jk}\partial_k u_j$, with the help of Lemma 7.7 and the equation $\partial_t\eta = u \cdot N$, we have

$$\begin{aligned}\left\|\partial_t^{2N}\nabla\bar{\eta}\partial_k u_j\right\|_0^2 &\lesssim \left\|\partial_t^{2N}\nabla\bar{\eta}\right\|_0^2 \|\partial_k u_j\|_{L^\infty}^2 \lesssim \left\|\partial_t^{2N}\eta\right\|_{\frac{1}{2}}^2 \|u\|_3^2 \\ &\lesssim \left\|\partial_t^{2N-1}(u\cdot\vec{N})\right\|_{\frac{1}{2}}^2 \|u\|_3^2 \lesssim \mathscr{E}_{2N}^2,\end{aligned}$$

which immediately gives (42).

To estimate F^4, it involves η, p and ϕ which has the same structure. So we only need to estimate one term $\partial_t^m p\partial_t^{s-m}\vec{N}_i$ $(0 \le m < 2N)$. If $0 \le m \le N$ and $i = 1, 2$,

$$\left\|\sum_{0\le m\le N}\partial_t^m p\partial_t^{2N-m}\vec{N}_i\right\|_0^2 \lesssim \sum_{0\le m\le N}\|\partial_t^m p\|_{L^\infty(\Sigma_-)}^2 \left\|\partial_t^{2N-m}\nabla_H\eta\right\|_0^2 \lesssim \mathscr{E}_{2N}\mathscr{D}_{2N}.$$

If $N+1 \le m \le 2N-1$ and $i = 1,\ 2$, we adopt the estimate

$$\begin{aligned}\left\|\sum_{N+1\le m\le 2N-1}\partial_t^m p\partial_t^{2N-m}\vec{N}_i\right\|_0^2 &\lesssim \sum_{N+1\le m\le 2N-1}\|\partial_t^m p\|_{H^0(\Sigma_-)}^2 \left\|\partial_t^{2N-m}\nabla_H\eta\right\|_{L^\infty}^2 \\ &\lesssim \mathscr{E}_{2N}\mathscr{D}_{2N}.\end{aligned}$$

To estimate the other terms of F^4, it yields

$$\begin{aligned}&\left\|\sum_{0\le m\le N}\partial_t^m\partial_k u_j\partial_t^{2N-m}(\mathscr{A}_{ik}\vec{N}_j)\right\|_0^2 \\ &\lesssim \sum_{0\le m\le N}\|\partial_t^m\partial_k u_j\|_{L^\infty(\Sigma_-)}^2 \left\|\partial_t^{2N-m}(\mathscr{A}_{ik}\vec{N}_j)\right\|_{H^0(\Sigma_-)}^2 \lesssim \mathscr{E}_{2N}\mathscr{D}_{2N}\end{aligned} \tag{44}$$

and

$$\begin{aligned}&\left\|\sum_{N+1\le m\le 2N-1}\partial_t^m\partial_k u_j\partial_t^{2N-m}(\mathscr{A}_{ik}\vec{N}_j)\right\|_0^2 \\ &\lesssim \sum_{N+1\le m\le 2N-1}\|\partial_t^m\partial_k u_j\|_{L^2(\Sigma_-)}^2 \left\|\partial_t^{2N-m}(\mathscr{A}_{ik}\vec{N}_j)\right\|_{L^\infty(\Sigma_-)}^2 \lesssim \mathscr{E}_{2N}\mathscr{D}_{2N}.\end{aligned} \tag{45}$$

Here we use these facts that if $j = 1, 2$ and $0 \le m \le N$

$$\begin{aligned}\left\|\partial_t^{2N-m}(\mathscr{A}_{ik}\vec{N}_j)\right\|_{H^0(\Sigma_-)}^2 &\lesssim \sum_{0\le l\le[N-\frac{m}{2}]}\left\|\partial_t^l\nabla_H\eta\right\|_{L^\infty}^2 \left\|\partial_t^{2N-m-l}\nabla_H\eta\right\|_0^2 \\ &+ \sum_{[N-\frac{m}{2}]+1\le l\le 2N-m}\left\|\partial_t^l\nabla_H\eta\right\|_0^2 \left\|\partial_t^{2N-m-l}\nabla_H\eta\right\|_{L^\infty}^2 \lesssim \mathscr{E}_{2N}\mathscr{D}_{2N},\end{aligned}$$

while $j=3$

$$\left\|\partial_t^{2N-m}\mathscr{A}_{ik}\right\|_{H^0(\Sigma_-)}^2 \lesssim \mathscr{D}_{2N}.$$

We also use that if $j=1,2$ and $N+1\le m\le 2N-1$

$$\left\|\partial_t^{2N-m}(\mathscr{A}_{ik}\vec{N}_j)\right\|_{L^\infty(\Sigma_-)}^2 \lesssim \sum_{0\le l\le 2N-m}\left\|\partial_t^l\nabla_H\eta\right\|_{L^\infty}^2\left\|\partial_t^{2N-m-l}\nabla_H\eta\right\|_{L^\infty}^2 \lesssim \mathscr{E}_{2N}\mathscr{D}_{2N}.$$

And if $j=3$, it can be easily verified that

$$\left\|\partial_t^{2N-m}\mathscr{A}_{ik}\right\|_{L^\infty(\Sigma_-)}^2 \lesssim \mathscr{D}_{2N}.$$

For the term $\partial_t^{2N}(\phi^2\vec{N}_i)$, we only need to bound the highest order term $\phi\partial_t^{2N}\phi\vec{N}_i$, yielding

$$\|\phi\partial_t^{2N}\phi\vec{N}_i\|_{H^0(\Sigma_-)}^2 \lesssim \|\phi\|_{L^\infty(\Sigma_-)}^2\|\partial_t^{2N}\phi\|_1^2\|\vec{N}_i\|_\infty \lesssim \mathscr{E}_{2N}\mathscr{D}_{2N}$$

So we complete the proof. □

To describe the temporal energy easily and clearly, we define the temporal and tangential energy and dissipation as follows. The temporal energy and dissipation are defined

$$\bar{\mathscr{E}}_n^0 = \sum_{s=0}^n\left(\|\sqrt{J}\partial_t^s u\|_0^2 + \|\partial_t^s\eta\|_0^2 + \|\sqrt{J}\partial_t^s\phi\|_0^2\right) \text{ and } \bar{\mathscr{D}}_n^0 = \sum_{s=0}^n\left(\|\mathbb{D}\partial_t^s u\|_0^2 + \|\nabla\partial_t^s\phi\|_0^2\right),$$

and the horizontal energy and dissipation are

$$\begin{aligned}\bar{\mathscr{E}}_n^+ =& \left\|\bar{\nabla}_H^{0,2n-1}(\chi u)\right\|_0^2 + \left\|\nabla_H\bar{\nabla}_H^{2n-1}(\chi u)\right\|_0^2 + \left\|\bar{\nabla}_H^{0,2n-1}(\chi\phi)\right\|_0^2\\ &+ \left\|\nabla_H\bar{\nabla}_H^{2n-1}(\chi\phi)\right\|_0^2 + \left\|\bar{\nabla}_H^{0,2n-1}\eta\right\|_0^2 + \left\|\nabla_H\bar{\nabla}_H^{2n-1}\eta\right\|_0^2\end{aligned}$$

and

$$\begin{aligned}\bar{\mathscr{D}}_n^+ =& \left\|\bar{\nabla}_H^{0,2n-1}\mathbb{D}(\chi u)\right\|_0^2 + \left\|\nabla_H\bar{\nabla}_H^{2n-1}\mathbb{D}(\chi u)\right\|_0^2\\ &+ \left\|\bar{\nabla}_H^{0,2n-1}\nabla(\chi\phi)\right\|_0^2 + \left\|\nabla_H\bar{\nabla}_H^{2n-1}\nabla(\chi\phi)\right\|_0^2.\end{aligned}$$

We also define $\bar{\mathscr{E}}_n = \bar{\mathscr{E}}_n^+ + \bar{\mathscr{E}}_n^0$ and $\bar{\mathscr{D}}_n = \bar{\mathscr{D}}_n^+ + \bar{\mathscr{D}}_n^0$ and the special energy

$$\begin{aligned}\mathscr{K} =& \|u\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty}^2 + \|\nabla^2 u\|_{L^\infty}^2 + \sum_{i=1}^2\|\nabla_H u_i\|_{H^2(\Sigma)}^2\\ &+ \|\nabla\phi\|_{L^\infty}^2 + \|\nabla^2\phi\|_{L^\infty}^2 + \sum_{i=1}^2\|\nabla_H\phi_i\|_{H^2(\Sigma)}^2.\end{aligned} \tag{46}$$

Note that $\mathscr{K} \lesssim \mathscr{E}_{N+2}$ which is directly obtained by Sobolev embedding inequalities.

Now, we first give the temporal energy estimate of $\bar{\mathscr{E}}_{2N}^0$.

Proposition 2.5. *Assume (u,p,ϕ,η) is the solution to equations* (17) *and satisfies the assumptions in Theorem 1.1. Suppose the a-priori assumption* (22) *holds for some small constant δ. Then we obtain*

$$\bar{\mathscr{E}}_{2N}^0(t) + \int_0^t \bar{\mathscr{D}}_{2N}^0 \lesssim \mathscr{E}_{2N}(0) + (\mathscr{E}_{2N}(t))^{\frac{3}{2}} + \int_0^t\sqrt{\mathscr{E}_{2N}}\mathscr{D}_{2N}. \tag{47}$$

Proof. Define $v=\partial_t^s u$, $z=\partial_t^s\phi$, $q=\partial_t^s p$ and $\xi=\partial_t^s\eta$. Applying the differential operator ∂_t^s to (17) for $s=0,1,2,...,2N$, we get the equations (24). By Lemma 2.2, integrating from 0 to T, we get

$$\begin{aligned}
&\left(\frac{1}{2}\int_{\Omega_+}\tilde{\phi}_b J\left|\partial_t^s u(t)\right|^2+\frac{1}{2}\int_{\Omega_-}J\left|\partial_t^s\phi(t)\right|^2+\frac{1}{2}\int_{\Sigma_-}\tilde{\phi}_b\left|\partial_t^s\eta(t)\right|^2\right)\\
&\qquad+\frac{1}{2}\int_0^t\int_{\Omega_+}\tilde{\phi}_b J\left|\mathbb{D}_{\mathscr{A}}\partial_t^s u\right|^2+\int_0^t\int_{\Omega_-}3\tilde{\phi}_b^{\,2}J\left|\nabla_{\mathscr{A}}\partial_t^s\phi\right|^2\\
&=\left(\frac{1}{2}\int_{\Omega_+}\tilde{\phi}_b J\left|\partial_t^s u(0)\right|^2+\frac{1}{2}\int_{\Omega_-}J\left|\partial_t^s\phi(0)\right|^2+\frac{1}{2}\int_{\Sigma_-}\tilde{\phi}_b\left|\partial_t^s\eta(0)\right|^2\right)\\
&\quad+\int_0^t\int_{\Omega_+}\tilde{\phi}_b J(\partial_t^s u\cdot F^1+\partial_t^s pF^2)+\int_0^t\int_{\Omega_-}F^3J\partial_t^s\phi+\tilde{\phi}_b\int_0^t\int_{\Sigma_-}F^4\partial_t^s u\\
&\quad+F^5\cdot\vec{N}\partial_t^s\phi+F^6\partial_t^s\eta+\frac{1}{2}\int_0^t\int_{\Sigma_-}\partial_t\eta\left|\partial_t^s\phi\right|^2.
\end{aligned}\tag{48}$$

Next, we estimate the right hand side of (48). For the term F^1, in view of (41) and Lemma 2.4, we get

$$\int_0^t\int_{\Omega_+}J\partial_t^s u\cdot F^1\lesssim\int_0^t\left\|F^1\right\|_0\left\|\partial_t^s u\right\|_0\left\|J\right\|_{L^\infty}\lesssim\int_0^t\sqrt{\mathscr{E}_{2N}}\mathscr{D}_{2N}.\tag{49}$$

But for the term $F^3=F^{3,1}+F^{3,2}$, $F^{3,1}$ is bounded form from (41) that

$$\int_0^t\int_{\Omega_-}J\partial_t^s\phi F^{3,1}\lesssim\int_0^t\left\|F^{3,1}\right\|_0\left\|\partial_t^s\phi\right\|_0\left\|J\right\|_{L^\infty}\lesssim\int_0^t\sqrt{\mathscr{E}_{2N}}\mathscr{D}_{2N}.\tag{50}$$

For the special term $F^{3,2}$, if $0\le s\le 2N-1$, it is easily verified that $\|F^{3,2}\|_0^2\lesssim\mathscr{E}_{2N}\mathscr{D}_{2N}$ and

$$\int_0^t\int_{\Omega_-}J\partial_t^s\phi F^{3,2}\lesssim\int_0^t\sqrt{\mathscr{E}_{2N}}\mathscr{D}_{2N}.$$

While $s=2N$, there are terms like $\phi^2\mathscr{A}_{ij}\mathscr{A}_{ik}\partial_j\partial_k\partial_t^{2N}\phi$ which are beyond the control of $\mathscr{D}_{2N}$. In fact, for the term $\Delta_{\mathscr{A}}(\phi^2\partial_t^{2N}\phi)$ of $\Delta_{\mathscr{A}}\partial_t^{2N}\phi^3$, it follows from integration by parts that

$$\begin{aligned}
&\int_{\Omega_-}J\Delta_{\mathscr{A}}(\phi^2\partial_t^{2N}\phi)\partial_t^{2N}\phi\\
&=-\int_{\Omega_-}J\nabla_{\mathscr{A}}(\phi^2\partial_t^{2N}\phi)\nabla_{\mathscr{A}}\partial_t^{2N}\phi+\int_{\Sigma_-}\vec{N}\cdot\nabla_{\mathscr{A}}(\phi^2\partial_t^{2N}\phi)\partial_t^{2N}\phi\\
&=-2\int_{\Omega_-}J\nabla_{\mathscr{A}}\phi\nabla_{\mathscr{A}}\partial_t^{2N}\phi\phi\partial_t^{2N}\phi-\int_{\Omega_-}J\phi^2|\nabla_{\mathscr{A}}\partial_t^{2N}\phi|^2\\
&\quad+\int_{\Sigma_-}2\phi(\partial_t^{2N}\phi)^2\vec{N}\cdot\nabla_{\mathscr{A}}\phi+\int_{\Sigma_-}\phi^2\partial_t^{2N}\phi\vec{N}\cdot\nabla_{\mathscr{A}}\partial_t^{2N}\phi\\
&=I+II+III+IV.
\end{aligned}$$

Then we get

$$\begin{aligned}
I+II&\lesssim\|J\|_\infty\|\nabla_{\mathscr{A}}\phi\|_\infty\|\phi\|_\infty\|\mathscr{A}_{ij}\|_\infty\|\partial_j\partial_t^{2N}\phi\|_0\|\partial_t^{2N}\phi\|_0\\
&\quad+\|J\|_\infty\|\phi\|_\infty^2\|\mathscr{A}_{ij}\|_\infty^2\|\partial_j\partial_t^{2N}\phi\|_0^2\lesssim\sqrt{\mathscr{E}_{2N}}\mathscr{D}_{2N}
\end{aligned}$$

and

$$\begin{aligned}
III+IV &\lesssim \|\phi\|_{L^\infty(\Sigma_-)}\|\vec{N}_i\|_{L^\infty(\Sigma_-)}\|\mathcal{A}_{ij}\|_{L^\infty(\Sigma_-)}\|\partial_j\phi\|_{L^\infty(\Sigma_-)}\|\partial_t^{2N}\phi\|^2_{H^0(\Sigma_-)} \\
&\quad + \|\phi\|^2_{L^\infty(\Sigma_-)}\|\vec{N}\cdot\nabla_{\mathscr{A}}\partial_t^{2N}\phi\|_{H^{-\frac12}(\Sigma_-)}\|\partial_t^{2N}\phi\|_{H^{\frac12}(\Sigma_-)} \\
&\lesssim \sqrt{\mathscr{E}_{2N}}\mathscr{D}_{2N} + \|\phi\|^2_{L^\infty(\Sigma_-)}\|\mathscr{A}_{ij}\partial_j\partial_t^{2N}\phi\|_{H^{-\frac12}(\Sigma_-)}\|\vec{N}_i\|_2\|\partial_t^{2N}\phi\|_{H^{\frac12}(\Sigma_-)} \\
&\lesssim \sqrt{\mathscr{E}_{2N}}\mathscr{D}_{2N} + \|\phi\|^2_{L^\infty(\Sigma_-)}\|\mathscr{A}_{ij}\|_{H^2(\Sigma_-)}\|\partial_j\partial_t^{2N}\phi\|_{H^{-\frac12}(\Sigma_-)}\|\vec{N}_i\|_2\|\partial_t^{2N}\phi\|_{H^{\frac12}(\Sigma_-)} \\
&\lesssim \sqrt{\mathscr{E}_{2N}}\mathscr{D}_{2N},
\end{aligned}$$

where we use (142) and the trace theory. Together with (50), we get

$$\int_0^t\int_{\Omega_-} J\partial_t^s\phi F^3 \lesssim \int_0^t \sqrt{\mathscr{E}_{2N}}\mathscr{D}_{2N}. \tag{51}$$

For the terms F^4, F^5 and F^6, we obtain

$$\begin{aligned}
&\int_0^t\int_{\Sigma_-} F^4\partial_t^s u + F^5\cdot\vec{N}\partial_t^s\phi + F^6\partial_t^s\eta \\
&\lesssim \int_0^t \left\|F^4\right\|_0\|\partial_t^s u\|_{H^0(\Sigma_-)} + \left\|F^6\right\|_0\|\partial_t^s\eta\|_0 + \left\|F^5\cdot\vec{N}\right\|_0\|\partial_t^s\phi\|_{H^0(\Sigma_-)} \\
&\lesssim \int_0^t \sqrt{\mathscr{E}_{2N}\mathscr{D}_{2N}}\sqrt{\mathscr{D}_{2N}} + (1+\|\eta\|^2_{5/2})\left\|F^5\right\|_0\|\partial_t^s\phi\|_{H^{\frac12}(\Omega_-)} \\
&\lesssim \int_0^t \sqrt{\mathscr{E}_{2N}}\mathscr{D}_{2N}.
\end{aligned} \tag{52}$$

It is easy to get that

$$\int_0^t\int_{\Sigma_-}\partial_t\eta\,|\partial_t^s\phi|^2 \lesssim \int_0^t \|\partial_t\eta\|_{L^\infty}\|\partial_t^s\phi\|^2_{H^0(\Sigma_-)} \lesssim \int_0^t\sqrt{\mathscr{E}_{2N}}\mathscr{D}_{2N}. \tag{53}$$

For the term F^2, if $s<2N$, we can directly derive by Sobolev embedding inequalities

$$\begin{aligned}
\int_0^t\int_{\Omega_+} J\partial_t^s pF^2 &\lesssim \int_0^t \|J\|_{L^\infty}\|\partial_t^s p\|_0\left\|F^2\right\|_0 \\
&\lesssim \int_0^t \sqrt{\mathscr{D}_{2N}}\sqrt{\mathscr{E}_{2N}\mathscr{D}_{2N}} \lesssim \int_0^t\sqrt{\mathscr{E}_{2N}}\mathscr{D}_{2N}.
\end{aligned}$$

When $s=2N$, we can't directly control the pressure term $\partial_t^{2N}p$ by $\mathscr{D}_{2N}$, so we integrate by parts with respect to the time t

$$\begin{aligned}
\int_0^t\int_{\Omega_+} J\partial_t^s pF^2 =: -\int_0^t\int_{\Omega_+}\partial_t^{2N-1}p\partial_t(JF^2) + \int_{\Omega_+}(\partial_t^{2N-1}pJF^2)(t) \\
-\int_{\Omega_+}(\partial_t^{2N-1}pJF^2)(0).
\end{aligned}$$

Then we get

$$\begin{aligned}
-\int_0^t\int_{\Omega_+}\partial_t^{2N-1}p\partial_t(JF^2) \lesssim \int_0^t\left\|\partial_t^{2N-1}p\right\|_0\left\|\partial_t(JF^2)\right\|_0 \lesssim \int_0^t\sqrt{\mathscr{D}_{2N}}\sqrt{\mathscr{E}_{2N}\mathscr{D}_{2N}} \\
\lesssim \int_0^t\sqrt{\mathscr{E}_{2N}}\mathscr{D}_{2N}
\end{aligned}$$

and in view of (42)

$$\begin{aligned}&\int_{\Omega_+}(\partial_t^{2N-1}pJF^2)(t)-\int_{\Omega_+}(\partial_t^{2N-1}pJF^2)(0)\\&\lesssim\|J\|_{L^\infty}\left\|\partial_t^{2N-1}p(t)\right\|_0\left\|F^2(t)\right\|_0+\|J(0)\|_{L^\infty}\left\|\partial_t^{2N-1}p(0)\right\|_0\left\|F^2(0)\right\|_0\\&\lesssim\mathscr{E}_{2N}(0)+(\mathscr{E}_{2N}(t))^{\frac{3}{2}},\end{aligned}$$

so, combining these together, it yields

$$\int_0^t\int_{\Omega_+}J\partial_t^s pF^2\lesssim\mathscr{E}_{2N}(0)+(\mathscr{E}_{2N}(t))^{\frac{3}{2}}+\int_0^t\sqrt{\mathscr{E}_{2N}}\mathscr{D}_{2N}. \tag{54}$$

Finally, adding (49), (51), (52), (53) and (54) together, we have

$$\begin{aligned}&\left(\frac{1}{2}\int_{\Omega_+}J\left|\partial_t^s u(t)\right|^2+\frac{1}{2}\int_{\Omega_-}J\left|\partial_t^s\phi(t)\right|^2+\frac{1}{2}\int_{\Sigma_-}\left|\partial_t^s\eta(t)\right|^2\right)\\&\qquad+\frac{1}{2}\int_0^t\int_{\Omega_+}J\left|\mathbb{D}_{\mathscr{A}}\partial_t^s u\right|^2+\int_0^t\int_{\Omega_-}J\left|\nabla_{\mathscr{A}}\partial_t^s\phi\right|^2\\&\lesssim\mathscr{E}_{2N}(0)+(\mathscr{E}_{2N}(t))^{\frac{3}{2}}+\int_0^t\sqrt{\mathscr{E}_{2N}}\mathscr{D}_{2N}.\end{aligned} \tag{55}$$

To get the temporal energy estimate by (55), we need to rewrite the terms in the left hand of (55). Note that

$$J\left|\mathbb{D}_{\mathscr{A}}\partial_t^s u\right|^2=\left|\mathbb{D}\partial_t^s u\right|^2+(J-1)\left|\mathbb{D}\partial_t^s u\right|^2+J(\mathbb{D}_{\mathscr{A}}\partial_t^s u+\mathbb{D}\partial_t^s u):(\mathbb{D}_{\mathscr{A}}\partial_t^s u-\mathbb{D}\partial_t^s u) \tag{56}$$

and

$$(\mathbb{D}_{\mathscr{A}}\partial_t^s u\pm\mathbb{D}\partial_t^s u)_{ij}=(\mathscr{A}_{ik}\pm\delta_{ik})\partial_k\partial_t^s u_j+(\mathscr{A}_{jk}\pm\delta_{jk})\partial_k\partial_t^s u_i,$$

which yields

$$\left|\mathbb{D}_{\mathscr{A}}\partial_t^s u+\mathbb{D}\partial_t^s u\right|\lesssim(1+\sqrt{\mathscr{E}_{2N}})\left|\nabla\partial_t^s u\right|\quad\text{and}\quad\left|\mathbb{D}_{\mathscr{A}}\partial_t^s u-\mathbb{D}\partial_t^s u\right|\lesssim\sqrt{\mathscr{E}_{2N}}\left|\nabla\partial_t^s u\right|.$$

Then we get

$$\begin{aligned}\int_0^t\int_{\Omega_+}J\left|(\mathbb{D}_{\mathscr{A}}\partial_t^s u+\mathbb{D}\partial_t^s u):(\mathbb{D}_{\mathscr{A}}\partial_t^s u-\mathbb{D}\partial_t^s u)\right|&\lesssim\int_0^t\int_{\Omega_+}(\sqrt{\mathscr{E}_{2N}}+\mathscr{E}_{2N})\left|\nabla\partial_t^s u\right|^2\\&\lesssim\int_0^t(\sqrt{\mathscr{E}_{2N}}+\mathscr{E}_{2N})\int_{\Omega_+}\left|\nabla\partial_t^s u\right|^2\lesssim\int_0^t\sqrt{\mathscr{E}_{2N}}\mathscr{D}_{2N}\end{aligned}$$

and

$$\int_0^t\int_{\Omega_+}|J-1|\left|\mathbb{D}\partial_t^s u\right|^2\lesssim\int_0^t\sqrt{\mathscr{E}_{2N}}\mathscr{D}_{2N}.$$

Similarly, we have

$$J\left|\nabla_{\mathscr{A}}\partial_t^s\phi\right|^2=\left|\nabla\partial_t^s\phi\right|^2+(J-1)\left|\nabla\partial_t^s\phi\right|^2+J(\nabla_{\mathscr{A}}\partial_t^s\phi+\nabla\partial_t^s\phi)\cdot(\nabla_{\mathscr{A}}\partial_t^s\phi-\nabla\partial_t^s\phi)$$

and

$$(\nabla_{\mathscr{A}}\partial_t^s\phi\pm\nabla\partial_t^s\phi)_i=(\mathscr{A}_{ij}\pm\delta_{ij})\partial_j\partial_t^s\phi,$$

which yield

$$\begin{aligned}\int_0^t\int_{\Omega_-}\left|J(\nabla_{\mathscr{A}}\partial_t^s\phi+\nabla\partial_t^s\phi)\cdot(\nabla_{\mathscr{A}}\partial_t^s\phi-\nabla\partial_t^s\phi)\right|&\lesssim\int_0^t\int_{\Omega_-}(\sqrt{\mathscr{E}_{2N}}+\mathscr{E}_{2N})\left|\nabla\partial_t^s\phi\right|^2\\&\lesssim\int_0^t(\sqrt{\mathscr{E}_{2N}}+\mathscr{E}_{2N})\int_{\Omega_-}\left|\nabla\partial_t^s\phi\right|^2\lesssim\int_0^t\sqrt{\mathscr{E}_{2N}}\mathscr{D}_{2N}\end{aligned}$$

and

$$\int_0^t \int_{\Omega_-} |J-1| \, |\nabla \partial_t^s \phi|^2 \lesssim \int_0^t \sqrt{\mathscr{E}_{2N}} \mathscr{D}_{2N}. \tag{57}$$

Combining the above inequalities together, we get

$$\begin{aligned} &\left(\int_{\Omega_+} J \, |\partial_t^s u(t)|^2 + \int_{\Omega_-} J \, |\partial_t^s \phi(t)|^2 + \int_{\Sigma_-} |\partial_t^s \eta(t)|^2 \right) \\ &\qquad + \int_0^t \int_{\Omega_+} |\mathbb{D} \partial_t^s u|^2 + \int_0^t \int_{\Omega_-} |\nabla \partial_t^s \phi|^2 \\ &\lesssim \mathscr{E}_{2N}(0) + (\mathscr{E}_{2N}(t))^{\frac{3}{2}} + \int_0^t \sqrt{\mathscr{E}_{2N}} \mathscr{D}_{2N} \end{aligned}$$

for $0 \le s \le 2N$. Summing the above inequality over $0 \le s \le 2N$, we obtain (47) and then complete the proof. □

Next, we present the corresponding estimate to the $N+2$ level.

Proposition 2.6. *Suppose that the assumptions in Proposition 2.5 hold, then we have*

$$\frac{d}{dt} \left(\bar{\mathscr{E}}_{N+2}^0 - 2 \int_\Omega J \partial_t^{N+1} p F^2 \right) + \bar{\mathscr{D}}_{N+2}^0 \lesssim \sqrt{\mathscr{E}_{2N}} \mathscr{D}_{N+2}. \tag{58}$$

Proof. Applying $\partial_t^{\alpha_0}$ $(0 \le \alpha_0 \le N+2)$ to (17), it yields that $v = \partial_t^{\alpha_0} u$, $z = \partial_t^{\alpha_0} \phi$, $q = \partial_t^{\alpha_0} p$ and $\xi = \partial_t^{\alpha_0} \eta$ solve the (24). By Lemma 2.2, we have

$$\begin{aligned} &\frac{d}{dt} \left(\frac{1}{2} \int_{\Omega_+} \tilde{\phi}_b J \, |\partial_t^{\alpha_0} u(t)|^2 + \frac{1}{2} \int_{\Omega_-} J \, |\partial_t^{\alpha_0} \phi(t)|^2 + \frac{1}{2} \int_{\Sigma_-} \tilde{\phi}_b \, |\partial_t^{\alpha_0} \eta(t)|^2 \right) \\ &\qquad + \frac{1}{2} \int_{\Omega_+} \tilde{\phi}_b J \, |\mathbb{D}_{\mathscr{A}} \partial_t^{\alpha_0} u|^2 + \int_{\Omega_-} 3 \tilde{\phi}_b^{\,2} J \, |\nabla_{\mathscr{A}} \partial_t^{\alpha_0} \phi|^2 \\ &= \int_{\Omega_+} \tilde{\phi}_b J (\partial_t^{\alpha_0} u \cdot F^1 + \partial_t^{\alpha_0} p F^2) + \int_{\Omega_-} F^3 J \partial_t^{\alpha_0} \phi \\ &\qquad + \tilde{\phi}_b \int_{\Sigma_-} F^4 \partial_t^{\alpha_0} u + F^5 \cdot \vec{N} \partial_t^{\alpha_0} \phi + F^6 \partial_t^{\alpha_0} \eta + \frac{1}{2} \int_{\Sigma_-} \partial_t \eta \, |\partial_t^{\alpha_0} \phi|^2 . \end{aligned} \tag{59}$$

Then we only need to estimate the right hand of (59). Firstly, by (43), the term F^1 is controlled

$$\int_{\Omega_+} J \partial_t^{\alpha_0} u \cdot F^1 \lesssim \|J\|_{L^\infty} \left\| F^1 \right\|_0 \|\partial_t^{\alpha_0} u\| \lesssim \sqrt{\mathscr{E}_{2N} \mathscr{D}_{N+2}} \sqrt{\mathscr{D}_{N+2}} = \sqrt{\mathscr{E}_{2N}} \mathscr{D}_{N+2}. \tag{60}$$

The term F^3 is estimated

$$\int_{\Omega_-} F^3 J \partial_t^{\alpha_0} \phi \lesssim \|J\|_{L^\infty} \left\| F^3 \right\|_0 \|\partial_t^{\alpha_0} \phi\| \lesssim \sqrt{\mathscr{E}_{2N} \mathscr{D}_{N+2}} \sqrt{\mathscr{D}_{N+2}} = \sqrt{\mathscr{E}_{2N}} \mathscr{D}_{N+2}.$$

For the terms F^4, F^5 and F^6, in view of the trace theory and (43), we obtain that

$$\begin{aligned}&\int_{\Sigma_-} F^4\partial_t^{\alpha_0}u + F^5\cdot\vec{N}\partial_t^{\alpha_0}\phi + F^6\partial_t^{\alpha_0}\eta\\ &\lesssim \left\|F^4\right\|_0 \|\partial_t^{\alpha_0}u\|_{H^0(\Sigma_-)} + \left\|F^5\cdot\vec{N}\right\|_0 \|\partial_t^{\alpha_0}\phi\|_0 + \left\|F^6\right\|_0 \|\partial_t^{\alpha_0}\eta\|\\ &\lesssim \sqrt{\mathscr{E}_{2N}\mathscr{D}_{N+2}}\sqrt{\mathscr{D}_{N+2}} + (1+\|\eta\|_{5/2}^2)\left\|F^5\right\|_0 \|\partial_t^{\alpha_0}\phi\|_{H^{\frac{1}{2}}(\Omega_-)}\\ &\lesssim \sqrt{\mathscr{E}_{2N}}\mathscr{D}_{N+2}.\end{aligned}$$

For the last term in (59), it yields

$$\int_{\Sigma_-}\partial_t\eta\left|\partial_t^s\phi\right|^2 \lesssim \|\partial_t\eta\|_{L^\infty}\|\partial_t^s\phi\|_{H^0(\Sigma_-)}^2 \lesssim \sqrt{\mathscr{E}_{2N}}\mathscr{D}_{N+2}.$$

For the term F^2, if $0\le\alpha_0<N+2$, we can directly control this term by the right hand side of (58), i.e.

$$\int_{\Omega_+} J\partial_t^{\alpha_0}pF^2 \lesssim \|J\|_{L^\infty}\|\partial_t^{\alpha_0}p\|_0\left\|F^2\right\|_0 \lesssim \sqrt{\mathscr{E}_{2N}}\mathscr{D}_{N+2}. \tag{61}$$

If $\alpha_0 = N+2$, it yields

$$\int_{\Omega_+} J\partial_t^{N+2}pF^2 = -\int_{\Omega_+}\partial_t^{N+1}p\partial_t(JF^2) + \frac{d}{dt}\int_{\Omega_+} J\partial_t^{N+1}pF^2.$$

Thanks to (43), we have

$$\begin{aligned}-\int_{\Omega_+}\partial_t^{N+1}p\partial_t(JF^2) &\lesssim \left\|\partial_t^{N+1}p\right\|_0\left\|\partial_t(JF^2)\right\|_0\\ &\lesssim \sqrt{\mathscr{D}_{N+2}}\sqrt{\mathscr{E}_{2N}\mathscr{D}_{N+2}} = \sqrt{\mathscr{E}_{2N}}\mathscr{D}_{N+2}.\end{aligned}$$

Adding these two together, we obtain

$$\int_{\Omega_+} J\partial_t^{N+2}pF^2 \lesssim \frac{d}{dt}\int_{\Omega_+} J\partial_t^{N+1}pF^2 + \sqrt{\mathscr{E}_{2N}}\mathscr{D}_{N+2}. \tag{62}$$

Combining (60)-(61) and (62) together, we deduce

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\int_{\Omega_+} J\left|\partial_t^{\alpha_0}u(t)\right|^2 + \frac{1}{2}\int_{\Omega_-} J\left|\partial_t^{\alpha_0}\phi(t)\right|^2 + \frac{1}{2}\int_{\Sigma_-}\left|\partial_t^{\alpha_0}\eta(t)\right|^2\right.\\ &\left.-\,\delta_{\alpha_0,N+2}\int_{\Omega_+} J\partial_t^{N+1}pF^2\right) + \frac{1}{2}\int_{\Omega_+} J\left|\mathbb{D}_{\mathscr{A}}\partial_t^{\alpha_0}u\right|^2 + \int_{\Omega_-} J\left|\nabla_{\mathscr{A}}\partial_t^{\alpha_0}\phi\right|^2\\ &\lesssim \sqrt{\mathscr{E}_{2N}}\mathscr{D}_{N+2}.\end{aligned}$$

We can argue as in (56)-(57) in Proposition 2.5 to get that

$$\frac{1}{2}\int_{\Omega_+}\left|\mathbb{D}\partial_t^{\alpha_0}u\right| \lesssim \frac{1}{2}\int_{\Omega_+} J\left|\mathbb{D}_{\mathscr{A}}\partial_t^{\alpha_0}u\right|^2 + \sqrt{\mathscr{E}_{2N}}\mathscr{D}_{N+2}$$

and

$$\int_{\Omega_-}\left|\nabla\partial_t^{\alpha_0}\phi\right| \lesssim \int_{\Omega_-} J\left|\nabla_{\mathscr{A}}\partial_t^{\alpha_0}\phi\right|^2 + \sqrt{\mathscr{E}_{2N}}\mathscr{D}_{N+2}.$$

Finally, we sum over α_0 from 0 to $N+2$ to complete the proof. □

3. Horizontal energy estimate.

In this section, the horizontal energy estimate is gained via the linear perturbed form as follows

$$(63)\quad \begin{cases} \partial_t u + \nabla p - \Delta u = \Phi^1 & \text{in } \Omega_+, \\ \operatorname{div} u = \Phi^2 & \text{in } \Omega_+, \\ \partial_t \phi - 3\tilde{\phi_b}^2 \Delta\phi = \Phi^3 & \text{in } \Omega_-, \\ u = 0 & \text{on } \Sigma_+, \\ \partial_t \eta - u_3 = \Phi^5 & \text{on } \Sigma_-, \\ (pI - \mathbb{D}u)\vec{e}_3 = -\rho g \eta \vec{e}_3 + 3\tilde{\phi_b}\phi\vec{e}_3 + \Phi^7 & \text{on } \Sigma_-, \\ u \cdot \vec{e}_3 = -3\tilde{\phi_b}\nabla\phi \cdot \vec{e}_3 + \Phi^8 & \text{on } \Sigma_-, \\ \phi = 0 & \text{on } \Sigma_{-b}, \end{cases}$$

where the nonlinear terms in the right side are listed as

$$\begin{aligned} \Phi^1_i = {} & (\delta_{ij} - \mathscr{A}_{ij})\partial_j p + \mathscr{A}_{jl}\partial_l(\mathscr{A}_{jk}\partial_k u_i) - \partial_{11}u_i - \partial_{22}u_i - \partial_{33}u_i \\ & + \rho K \partial_t \Xi_3 \partial_3 u_i - \rho u_j \mathscr{A}_{jk}\partial_k u_i \\ = {} & (\delta_{ij} - \mathscr{A}_{ij})\partial_j p + \mathscr{A}_{jl}\partial_l \mathscr{A}_{jk}\partial_k u_i + (A^2+B^2)K^2\partial_{33}u_i + (K^2-1)\partial_{33}u_i \\ & + \rho K \partial_t \Xi_3 \partial_3 u_i - \rho u_j \mathscr{A}_{jk}\partial_k u_i + \sum_{l\neq k} \mathscr{A}_{jl}\mathscr{A}_{jk}\partial_l\partial_k u_i, \end{aligned}$$

$$\Phi^2 = \operatorname{div} u - \operatorname{div}_{\mathscr{A}} u = (1-K)\partial_3 u_3 + BK\partial_3 u_2 + AK\partial_3 u_1,$$

$$\begin{aligned} \Phi^3 = {} & -3\tilde{\phi_b}^2\Delta\phi + 3\tilde{\phi_b}^2\Delta_{\mathscr{A}}\phi + K\partial_t\Xi_3\partial_3\phi + 3\tilde{\phi_b}\Delta_{\mathscr{A}}\phi^2 + \Delta_{\mathscr{A}}\phi^3 \\ = {} & -3\tilde{\phi_b}^2\partial_1^2\phi - 3\tilde{\phi_b}^2\partial_2^2\phi - 3\tilde{\phi_b}^2\partial_3^2\phi + 3\tilde{\phi_b}^2\mathscr{A}_{ik}\partial_k\mathscr{A}_{ij}\partial_j\phi + 3\tilde{\phi_b}^2\mathscr{A}_{ik}\mathscr{A}_{ij}\partial_k\partial_j\phi \\ & + K\partial_t\Xi_3\partial_3\phi + 3\tilde{\phi_b}\Delta_{\mathscr{A}}\phi^2 + \Delta_{\mathscr{A}}\phi^3 \\ = {} & 3\tilde{\phi_b}^2[-\partial_1(AK)\partial_3\phi + AK\partial_3(AK)\partial_3\phi - \partial_2(BK)\partial_3\phi + BK\partial_3(BK)\partial_3\phi \\ & + K\partial_3 K\partial_3\phi + (K^2-1)\partial_3^2\phi - 2AK\partial_1\partial_3\phi - 2BK\partial_2\partial_3\phi + (A^2+B^2)K^2\partial_3^2\phi] \\ & + K\partial_t\Xi_3\partial_3\phi + 3\tilde{\phi_b}\Delta_{\mathscr{A}}\phi^2 + \Delta_{\mathscr{A}}\phi^3, \end{aligned}$$

$$\Phi^5 = -u_1\partial_1\eta - u_2\partial_2\eta,$$

$$\begin{aligned} \Phi^7 = {} & -\partial_1\eta \begin{pmatrix} \rho g\eta - p + \phi + 2\partial_1 u_1 - 2AK\partial_3 u_1 + \frac{3}{2}\phi^2 \\ \partial_2 u_1 + \partial_1 u_2 - BK\partial_3 u_1 - AK\partial_3 u_2 \\ \partial_1 u_3 + K\partial_3 u_1 - AK\partial_3 u_3 \end{pmatrix} \\ & -\partial_2\eta \begin{pmatrix} \partial_2 u_1 + \partial_1 u_2 - BK\partial_3 u_1 - AK\partial_3 u_2 \\ \rho g\eta - p + \phi + 2\partial_2 u_2 - 2BK\partial_3 u_2 + \frac{3}{2}\phi^2 \\ \partial_2 u_3 - BK\partial_3 u_3 + K\partial_3 u_2 \end{pmatrix} + \begin{pmatrix} (K-1)\partial_3 u_1 - AK\partial_3 u_3 \\ (K-1)\partial_3 u_2 - BK\partial_3 u_3 \\ 2(K-1)\partial_3 u_3 + \frac{3}{2}\phi^2 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \Phi^8 & = u\cdot\vec{e}_3 - u\cdot\vec{N} + 3\tilde{\phi_b}\nabla\phi\cdot\vec{e}_3 - 3\tilde{\phi_b}\nabla_{\mathscr{A}}\phi\cdot\vec{N} - 3\phi\vec{N}\cdot\nabla_{\mathscr{A}}\phi \\ & = \partial_1\eta(u_1 + \partial_1\phi - AK\partial_3\phi) + \partial_2\eta(u_2 + \partial_2\phi - BK\partial_3\phi) \\ & \quad + (1-K)\partial_3\phi - 3\phi\vec{N}\cdot\nabla_{\mathscr{A}}\phi. \end{aligned}$$

Choose the smooth cut-off function $\chi(x_3) \in C_0^\infty(\mathbb{R})$ defined as

$$(64)\quad \chi(x_3) = \begin{cases} 1 & \text{if} \quad -\frac{4b}{5} < x_3 < \frac{4}{5}, \\ 0 & \text{if} \quad x_3 \geq \frac{5}{6} \quad \text{or} \quad x_3 \leq -\frac{5b}{6}, \end{cases}$$

where $\underline{b} = \min_{\mathbb{T}^2} b$. Then multiplying (63) by χ, it yields

$$\begin{cases} \partial_t(\chi u) + \nabla(\chi p) - \Delta(\chi u) = \chi\Phi^1 + H^1 & \text{in } \Omega_+, \\ \operatorname{div} \chi u = \chi\Phi^2 + H^2 & \text{in } \Omega_+, \\ \partial_t(\chi\phi) - 3\tilde{\phi_b}^2\Delta(\chi\phi) = \chi\Phi^3 + H^3 & \text{in } \Omega_-, \\ \chi u = 0 & \text{on } \Sigma_+, \\ \partial_t\eta - \chi u_3 = \chi\Phi^5 & \text{on } \Sigma_-, \\ ((\chi p)I - \mathbb{D}(\chi u))\vec{e}_3 = -\rho g\eta\vec{e}_3 + 3\tilde{\phi_b}(\chi\phi)\vec{e}_3 + \Phi^7 & \text{on } \Sigma_-, \\ (\chi u)\cdot\vec{e}_3 = -3\tilde{\phi_b}\nabla(\chi\phi)\cdot\vec{e}_3 + \Phi^8 & \text{on } \Sigma_-, \\ \chi\phi = 0 & \text{on } \Sigma_{-b}, \end{cases} \tag{65}$$

where the new linear terms in the right side of (65) are defined as

$$H^1 = \partial_3\chi(p\vec{e}_3 - 2\partial_3 u) - \partial_3^2\chi u, \quad H^2 = \partial_3\chi u_3 \ \text{ and } \ H^3 = -\partial_3^2\chi\phi - 2\partial_3\chi\partial_3\phi. \tag{66}$$

Applying the differential operator $\partial^\alpha := \partial_t^{\alpha_0}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}$ for $\alpha \in \mathbb{N}^{1+2}$ to equations (65), we get

$$\begin{cases} \partial_t v + \nabla q - \Delta v = G^1 & \text{in } \Omega_+, \\ \operatorname{div} v = G^2 & \text{in } \Omega_+, \\ \partial_t z - 3\tilde{\phi_b}^2\Delta z = G^3 & \text{in } \Omega_-, \\ v = 0 & \text{on } \Sigma_+, \\ \partial_t\xi - v_3 = G^5 & \text{on } \Sigma_-, \\ (qI - \mathbb{D}v)\vec{e}_3 = -\rho g\xi\vec{e}_3 + 3\tilde{\phi_b}z\vec{e}_3 + G^7 & \text{on } \Sigma_-, \\ v\cdot\vec{e}_3 = -3\tilde{\phi_b}\nabla z\cdot\vec{e}_3 + G^8 & \text{on } \Sigma_-, \\ z = 0 & \text{on } \Sigma_{-b}, \end{cases} \tag{67}$$

where $v = \chi\partial^\alpha u$, $z = \chi\partial^\alpha\phi$, $q = \chi\partial^\alpha p$, $\xi = \partial^\alpha\eta$ and the nonlinear terms in the right side of (67) are

$$\begin{aligned} G^1 &= \chi\partial^\alpha\Phi^1 + \partial^\alpha H^1, \qquad G^2 = \chi\partial^\alpha\Phi^2 + \partial^\alpha H^2, \quad G^3 = \chi\partial^\alpha\Phi^3 + \partial^\alpha H^3, \\ G^5 &= \chi\partial^\alpha\Phi^5, \qquad\qquad\qquad\qquad G^7 = \partial^\alpha\Phi^7, \quad G^8 = \partial^\alpha\Phi^8. \end{aligned}$$

The basic energy equality is gained as follows.

Lemma 3.1. *Assume that v, z, q and ξ are the solutions to the above equations and have suitable regularities, then we have*

(68)
$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\int_{\Omega_+}|v|^2 + \frac{1}{2}\int_{\Omega_-}3\,|z|^2 + \frac{1}{2}\int_{\Sigma_-}\rho g\,|\xi|^2\right) + \frac{1}{2}\int_{\Omega_+}|\mathbb{D}v|^2 + \int_{\Omega_-}9\tilde{\phi_b}^2\,|\nabla z|^2 \\ &= \int_{\Omega_+} v\cdot(G^1 - \nabla G^2) + \int_{\Omega_+} qG^2 + \int_{\Omega_-} 3G^3 z + \int_{\Sigma_-}\rho g\xi G^5 + G^7\cdot v + 3\tilde{\phi_b}G^8 z. \end{aligned}$$

Proof. We omit the calculus here, since it is similar with Lemma 2.2. □

To obtain the tangential energy estimate, we firstly bound the nonlinear terms $\Phi^1 - \Phi^8$ by Sobolev embedding inequalities.

Lemma 3.2. *Suppose the a-priori assumption (22) holds for some small constant $\delta > 0$. Then there exists a positive constant $\theta(0 < \theta \le 1)$, such that the following* *inequalities hold*

$$\left\|\bar{\nabla}^{0,4N-2}\Phi^1\right\|_0^2+\left\|\bar{\nabla}^{0,4N-2}\Phi^2\right\|_1^2+\left\|\bar{\nabla}^{0,4N-2}\Phi^3\right\|_0^2+\left\|\bar{\nabla}_H^{0,4N-2}\Phi^5\right\|_{\frac{1}{2}}^2 + \left\|\bar{\nabla}_H^{0,4N-2}\Phi^7\right\|_{\frac{1}{2}}^2+\left\|\bar{\nabla}_H^{0,4N-2}\Phi^8\right\|_{\frac{1}{2}}^2 \lesssim \mathscr{E}_{2N}^{1+\theta}, \tag{69}$$

$$\begin{aligned}&\left\|\bar{\nabla}^{0,4N-2}\Phi^1\right\|_0^2+\left\|\bar{\nabla}^{0,4N-2}\Phi^2\right\|_1^2+\left\|\bar{\nabla}^{0,4N-2}\Phi^3\right\|_0^2+\left\|\bar{\nabla}_H^{0,4N-2}\Phi^5\right\|_{\frac{1}{2}}^2\\ &+\left\|\bar{\nabla}_H^{0,4N-2}\Phi^7\right\|_{\frac{1}{2}}^2+\left\|\bar{\nabla}_H^{0,4N-2}\Phi^8\right\|_{\frac{1}{2}}^2+\left\|\bar{\nabla}^{4N-3}\partial_t\Phi^1\right\|_0^2+\left\|\bar{\nabla}^{4N-3}\partial_t\Phi^2\right\|_1^2+\left\|\bar{\nabla}^{4N-3}\partial_t\Phi^3\right\|_0^2\\ &+\left\|\bar{\nabla}_H^{4N-3}\partial_t\Phi^5\right\|_{\frac{1}{2}}+\left\|\bar{\nabla}_H^{4N-3}\partial_t\Phi^7\right\|_{\frac{1}{2}}+\left\|\bar{\nabla}_H^{4N-3}\partial_t\Phi^8\right\|_{\frac{1}{2}}\lesssim \mathscr{E}_{2N}^{\theta}\mathscr{D}_{2N},\end{aligned} \tag{70}$$

$$\begin{aligned}&\left\|\nabla^{4N-1}\Phi^1\right\|_0^2+\left\|\nabla^{4N-1}\Phi^2\right\|_1^2+\left\|\nabla^{4N-1}\Phi^3\right\|_0^2+\left\|\nabla_H^{4N-1}\Phi^5\right\|_{\frac{1}{2}}^2+\left\|\nabla_H^{4N-1}\Phi^7\right\|_{\frac{1}{2}}^2\\ &+\left\|\nabla_H^{4N-1}\Phi^8\right\|_{\frac{1}{2}}^2\lesssim \mathscr{E}_{2N}^{\theta}\mathscr{D}_{2N}+\mathscr{K}\mathscr{F}_{2N},\end{aligned} \tag{71}$$

where the definition of $\mathscr{K}$ is given in (46).

Proof. For the sake of brevity, we only present a sketch for the proof of (69) and (70). Since all the terms of Φ^i are the product which are comprised by at least two terms, we estimate these according to Sobolev embedding inequalities, trace theory and Lemma 7.6, Lemma 7.3 and Lemma 7.4. We take the term Φ^3 for example. Recalling the definition of Φ^3, it yields

$$\left\|\bar{\nabla}^{0,4N-2}\Phi^3\right\|_0^2=\sum_{\substack{\alpha\in\mathbb{N}^{1+3}\\0\le|\alpha|\le 4N-2}}\left\|\partial^\alpha\Phi^3\right\|_0^2$$

and

$$\begin{aligned}\Phi^3=&-\partial_1(AK)\partial_3\phi+AK\partial_3(AK)\partial_3\phi-\partial_2(BK)\partial_3\phi+BK\partial_3(BK)\partial_3\phi\\ &+K\partial_3K\partial_3\phi+(K^2-1)\partial_3^2\phi-2AK\partial_1\partial_3\phi-2BK\partial_2\partial_3\phi+(A^2+B^2)K^2\partial_3^2\phi\\ &+K\partial_t\Xi_3\partial_3\phi+3\tilde{\phi}_b\Delta_{\mathscr{A}}\phi^2+\Delta_{\mathscr{A}}\phi^3.\end{aligned}$$

Applying ∂^α to Φ^3, for the highest order term, we have terms of the form in $\bar{\nabla}^{0,4N-2}\Phi^3$

$$\begin{gathered}\partial^\gamma\bar{\eta}\mathscr{P}(A,B,K)\partial_3\phi,\quad \partial^\nu\bar{\eta}\mathscr{P}_1(A,B,K)\partial^\alpha\partial_3\phi,\\ (K^2-1)\partial^\alpha\nabla^2\phi,\quad AK\partial^\alpha\nabla^2\phi,\quad BK\partial^\alpha\nabla\phi,\end{gathered} \tag{72}$$

$$\begin{gathered}\partial^\alpha\nabla\bar{\eta}\mathscr{P}_2(A,B,K)\partial_3^2\phi,\quad A^2K^2\partial^\alpha\nabla^2\phi,\quad B^2K^2\partial^\alpha\nabla^2\phi,\\ K^2\partial^\beta\bar{\eta}\partial_t\bar{\eta}\partial_3\phi,\quad K\partial_t\partial^\alpha\bar{\eta}\partial_3\phi,\quad K\partial_t\bar{\eta}\partial_3\partial^\alpha\phi\end{gathered} \tag{73}$$

where $|\gamma|=4N$, $|\nu|=2$, $|\alpha|=4N-2$, $|\beta|=4N-1$ with $\gamma,\nu,\alpha,\beta\in\mathbb{N}^3$ and $\mathscr{P}(A,B,K),\mathscr{P}_1(A,B,K)$,
$\mathscr{P}_2(A,B,K)$ are polynomials. It is easy to verify that each term can be controlled by $\mathscr{E}_{2N}^{1+\theta}$, we take $\partial^\gamma\bar{\eta}\mathscr{P}(A,B,K)\partial_3\phi$ as example. Sobolev embedding inequalities give $\|\mathscr{P}(A,B,K)\|_\infty\lesssim 1+\mathscr{E}_{2N}\lesssim 1$, by Lemma 7.3, we have

$$\begin{aligned}\|\partial^\gamma\bar{\eta}\mathscr{P}(A,B,K)\partial_3\phi\|_0^2&\lesssim\left\|\nabla^{4N}\bar{\eta}\right\|_0^2\|\mathscr{P}(A,B,K)\|_\infty^2\|\partial_3\phi\|_\infty^2\\ &\lesssim\left\|\nabla_H^{4N-\frac{1}{2}}\eta\right\|_0^2\|\phi\|_3^2\lesssim\mathscr{E}_{2N}^{1+\theta}\end{aligned}$$

and the other terms of $\partial^\alpha \Phi^3$ can be directly estimated by $\mathscr{E}_{2N}^{1+\theta}$ by Sobolev embedding inequalities. (69) and (70) can be gained by the same idea.
Next, we just present some special terms in (71) which will be controlled by $\mathscr{E}_{2N}^{\theta}\mathscr{D}_{2N} + \mathscr{K}\mathscr{F}_{2N}$. To estimate $\nabla^{4N-1}\Phi^1$, we only bound the highest derivatives, i.e.

$$\partial^\beta \bar{\eta} \mathscr{Q}(A, B, J, K) \partial^\gamma u$$

with $\mathscr{Q}(A, B, J, K)$ a polynomial, where $\beta, \gamma \in \mathbb{N}^3$ and $|\beta| = 4N + 1$ and $|\gamma| = 1$. Via Lemma 7.3, it yields

$$\left\|\nabla^{4N+1}\bar{\eta}\right\|_0^2 \lesssim \|\eta\|_{4N+\frac{1}{2}}^2 = \mathscr{F}_{2N}$$

and according to Sobolev embedding, it gets

$$\|\mathscr{Q}(A, B, J, K)\|_{L^\infty} \lesssim 1 + \mathscr{E}_{2N}^{\theta}.$$

Then we have

$$\begin{aligned} \left\|\partial^\beta \bar{\eta} \mathscr{Q}(A, B, J, K) \partial^\gamma u\right\|_0^2 &\lesssim \left\|\partial^\beta \bar{\eta}\right\|_0^2 \|\mathscr{Q}(A, B, J, K)\|_{L^\infty}^2 \|\partial^\gamma u\|_{L^\infty}^2 \\ &\lesssim \left\|\nabla^{4N+1}\bar{\eta}\right\|_0^2 \|\nabla u\|_{L^\infty}^2 \lesssim \mathscr{K}\mathscr{F}_{2N}. \end{aligned}$$

To control $\nabla^{4N-1}\Phi^2$, there are terms of the form

$$\partial^\beta \bar{\eta} \mathscr{Q}_1(A, B, K) \partial^\gamma u$$

with $\mathscr{Q}_1(A, B, K)$ a polynomial, where $\beta, \gamma \in \mathbb{N}^3$ and $|\beta| = 4N$ and $|\gamma| = 1$. By Sobolev embedding, we have

$$\|\mathscr{Q}_1(A, B, K)\|_{C^1}^2 \lesssim 1 + \mathscr{E}_{2N}^{\theta} \lesssim 1$$

and then

$$\begin{aligned} \left\|\partial^\beta \bar{\eta} \mathscr{Q}_1(A, B, K) \partial^\gamma u\right\|_1^2 &\lesssim \|\mathscr{Q}_1(A, B, K)\|_{C^1}^2 \left\|\partial^\beta \bar{\eta} \partial^\gamma u\right\|_1^2 \\ &\lesssim \left\|\partial^\beta \bar{\eta} \partial^\gamma u\right\|_0^2 + \left\|\nabla \partial^\beta \bar{\eta} \partial^\gamma u\right\|_0^2 + \left\|\partial^\beta \bar{\eta} \nabla \partial^\gamma u\right\|_0^2 \\ &\lesssim \left\|\nabla^{4N}\bar{\eta}\right\|_0^2 (\|\nabla u\|_{L^\infty}^2 + \left\|\nabla^2 u\right\|_{L^\infty}^2) + \left\|\nabla^{4N+1}\bar{\eta}\right\|_0^2 \|\nabla u\|_{L^\infty}^2 \\ &\lesssim \mathscr{E}_{2N}\mathscr{D}_{2N} + \mathscr{K}\mathscr{F}_{2N}, \end{aligned}$$

where Sobolev embedding inequalities and Lemma 7.3 are used in the last inequality.
To estimate $\nabla^{4N-1}\Phi^3$, except $\partial^\gamma \bar{\eta} \mathscr{P}(A, B, K) \partial_3 \phi$ with $|\gamma| = 4N + 1$, other terms can be bounded by $\mathscr{E}_{2N}^{\theta}\mathscr{D}_{2N}$. We give a simple estimate for $\partial^\gamma \bar{\eta} \mathscr{P}(A, B, K) \partial_3 \phi$ as follows,

$$\begin{aligned} \|\partial^\gamma \bar{\eta} \mathscr{P}(A, B, K) \partial_3 \phi\|_0^2 &\lesssim \left\|\nabla^{4N+1}\bar{\eta}\right\|_0^2 \|\mathscr{P}(A, B, K)\|_\infty^2 \|\partial_3 \phi\|_\infty^2 \\ &\lesssim \|\eta\|_{4N+\frac{1}{2}}^2 \|\nabla \phi\|_\infty^2 \lesssim \mathscr{K}\mathscr{F}_{2N} \end{aligned}$$

To estimate $\nabla_H^{4N-1}\Phi^5$, we only need to control $u_i \partial^\gamma \eta$ $(i = 1, 2)$ with $\gamma \in \mathbb{N}^2$ and $|\gamma| = 4N$. In view of Lemma 7.7, it yields

$$\|u_i \partial^\gamma \eta\|_{\frac{1}{2}}^2 \lesssim \|u_i\|_{C^1}^2 \|\partial^\gamma \eta\|_{\frac{1}{2}}^2 \lesssim \mathscr{F}_{2N}\mathscr{K}. \tag{74}$$

To control $\nabla_H^{4N-1}\Phi^7$, it only needs to estimate $\partial^\beta \eta \mathscr{Q}_2(A, B, K) \partial^\gamma u$ with $|\beta| = 4N$ and $|\gamma| = 1$, where $\mathscr{Q}_2(A, B, K)$ a polynomial, $\beta \in \mathbb{N}^2$ and $\gamma \in \mathbb{N}^3$. Via Sobolev embedding inequalities, it yields

$$\|\mathscr{Q}_2(A, B, K)\|_{C^1}^2 \lesssim 1 + \mathscr{E}_{2N}^{\theta},$$

and then

$$\begin{aligned}\left\|\partial^\beta\eta\mathscr{Q}_2(A,B,K)\partial^\gamma u\right\|_{\frac12}^2 &\lesssim \left\|\partial^\beta\eta\right\|_{\frac12}^2 \|\mathscr{Q}_2(A,B,K)\partial^\gamma u\|_{C^1}^2\\ &\lesssim \|\eta\|_{4N+\frac12}^2 \|\mathscr{Q}_2(A,B,K)\|_{C^1}^2 \|\partial^\gamma u\|_{C^1(\Sigma_-)}^2 \lesssim \mathscr{F}_{2N}\mathscr{K}.\end{aligned}$$

To estimate $\nabla_H^{4N-1}\Phi^8$, we only need to control $u\cdot\nabla_H^{4N}\eta$ and $\partial^\beta\eta\mathscr{Q}_3(A,B,K)\partial^\gamma\phi$ with $|\beta|=4N$ and $|\gamma|=1$, where $\mathscr{Q}_3(A,B,K)$ a polynomial, $\beta\in\mathbb{N}^2$ and $\gamma\in\mathbb{N}^3$. Note that

$$\|\mathscr{Q}_3(A,B,K)\|_{C^1}^2 \lesssim 1+\mathscr{E}_{2N}^\theta,$$

and it yields

$$\begin{aligned}\left\|\partial^\beta\eta\mathscr{Q}_3(A,B,K)\partial^\gamma\phi\right\|_{\frac12}^2 &\lesssim \left\|\partial^\beta\eta\right\|_{\frac12}^2 \|\mathscr{Q}_3(A,B,K)\partial^\gamma\phi\|_{C^1}^2\\ &\lesssim \left\|\nabla_H^{4N}\eta\right\|_{\frac12}^2 \|\mathscr{Q}_3(A,B,K)\|_{C^1}^2 \|\nabla\phi\|_{C^1}^2\\ &\lesssim \mathscr{F}_{2N}\mathscr{K}.\end{aligned}$$

Combining these inequalities together, we complete the proof of (71). □

Similarly, the $N+2$-order estimates of the nonlinear terms Φ^i are obtained and the proof is omitted here.

Lemma 3.3. *There exists a positive constant θ, such that the following inequalities hold.*

(75)
$$\begin{aligned}&\left\|\bar\nabla^{0,2(N+2)-2}\Phi^1\right\|_0^2+\left\|\bar\nabla^{0,2(N+2)-2}\Phi^2\right\|_1^2+\left\|\bar\nabla^{0,2(N+2)-2}\Phi^3\right\|_0^2+\left\|\bar\nabla_H^{0,2(N+2)-2}\Phi^5\right\|_{\frac12}^2\\ &\qquad+\left\|\bar\nabla_H^{0,2(N+2)-2}\Phi^7\right\|_{\frac12}^2+\left\|\bar\nabla_H^{0,2(N+2)-2}\Phi^8\right\|_{\frac12}^2\lesssim\mathscr{E}_{2N}^\theta\mathscr{E}_{N+2},\end{aligned}$$

(76)
$$\begin{aligned}&\left\|\bar\nabla^{0,2(N+2)-1}\Phi^1\right\|_0^2+\left\|\bar\nabla^{0,2(N+2)-1}\Phi^2\right\|_1^2+\left\|\bar\nabla^{0,2(N+2)-1}\Phi^3\right\|_0^2+\left\|\bar\nabla_H^{0,2(N+2)-1}\Phi^5\right\|_{\frac12}^2\\ &\qquad+\left\|\bar\nabla_H^{0,2(N+2)-1}\Phi^7\right\|_{\frac12}^2+\left\|\bar\nabla_H^{0,2(N+2)-1}\Phi^8\right\|_{\frac12}^2\lesssim\mathscr{E}_{2N}^\theta\mathscr{D}_{N+2}.\end{aligned}$$

With these estimates in hand, the horizontal energy estimates of the solutions to equations (17) are gained as follows.

Proposition 3.4. *Assume (u,p,ϕ,η) is the solution to the equations* (17) *and the assumptions in Theorem 1.1 and the a-priori assumption* (22) *hold for some small positive constant δ. Then for any $\varepsilon\in(0,1)$, there exists a constant $C(\varepsilon)>0$, such that*

(77) $$\bar{\mathscr{E}}_{2N}^+(t)+\int_0^t\bar{\mathscr{D}}_{2N}^+\lesssim\bar{\mathscr{E}}_{2N}^+(0)+\int_0^t\mathscr{E}_{2N}^\theta\mathscr{D}_{2N}+\sqrt{\mathscr{D}_{2N}\mathscr{K}\mathscr{F}_{2N}}+\varepsilon\mathscr{D}_{2N}+C(\varepsilon)\bar{\mathscr{D}}_{2N}^0.$$

Proof. Suppose $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{N}^{1+2}$ such that $\alpha_0 \le 2N-1$ and $|\alpha| \le 4N$. Lemma 3.1 gives

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\int_{\Omega_+} |\chi\partial^\alpha u|^2 + \frac{1}{2}\int_{\Omega_-} 3\,|\chi\partial^\alpha \phi|^2 + \frac{1}{2}\int_{\Sigma_-} \rho g\,|\partial^\alpha \eta|^2\right)\\
&\quad + \frac{1}{2}\int_{\Omega_+} |\mathbb{D}(\chi\partial^\alpha u)|^2 + \int_{\Omega_-} 9\tilde{\phi_b}^2\,|\nabla(\chi\partial^\alpha\phi)|^2\\
&= \int_{\Omega_+} (\chi\partial^\alpha u)\cdot(\chi\partial^\alpha\Phi^1 + \partial^\alpha H^1 - \nabla(\chi\partial^\alpha\Phi^2 + \partial^\alpha H^2))\\
&\quad + \int_{\Omega_+} \chi\partial^\alpha p(\chi\partial^\alpha\Phi^2 + \partial^\alpha H^2) + \int_{\Omega_-} 3(\chi\partial^\alpha\Phi^3 + \partial^\alpha H^3)\chi\partial^\alpha\phi\\
&\quad + \int_{\Sigma_-} \rho g\partial^\alpha\eta\partial^\alpha\Phi^5 + \partial^\alpha\Phi^7\cdot(\chi\partial^\alpha u) + 3\tilde{\phi_b}\partial^\alpha\Phi^8\chi\partial^\alpha\phi,
\end{aligned} \tag{78}$$

where H^1, H^2 and H^3 are given by (66). Then we only need to estimate the right hand side of (78). The definitions of H^i $(i = 1, 2, 3)$ imply

$$\partial^\alpha H^1 = \partial_3\chi(\partial^\alpha p\vec{e}_3 - 2\partial_3\partial^\alpha u) - \partial_3^2\chi\partial^\alpha u, \quad \partial^\alpha H^2 = \partial_3\chi\partial^\alpha u_3$$

and

$$\partial^\alpha H^3 = -\partial_3^2\chi\partial^\alpha\phi - 2\partial_3\chi\partial_3\partial^\alpha\phi,$$

yielding

$$\begin{aligned}
&\int_{\Omega_+} \chi\partial^\alpha u\partial^\alpha H^1 - \chi\partial^\alpha u\cdot\nabla(\partial^\alpha H^2) + \chi\partial^\alpha p\partial^\alpha H^2 + \int_{\Omega_-} \chi\partial^\alpha\phi\partial^\alpha H^3\\
&\lesssim \|\partial^\alpha u\|_0\left(\|\partial^\alpha p\|_0 + \|\partial^\alpha u\|_1\right) + \|\partial^\alpha\phi\|_0\,\|\partial^\alpha\phi\|_1\\
&\lesssim \|\partial_t^{\alpha_0} u\|_{4N-2\alpha_0}\sqrt{\mathscr{D}_{2N}} + \|\partial_t^{\alpha_0}\phi\|_{4N-2\alpha_0}\sqrt{\mathscr{D}_{2N}}.
\end{aligned}$$

Via Sobolev interpolation theory, it gives

$$\|\partial_t^{\alpha_0} u\|_{4N-2\alpha_0} \lesssim \|\partial_t^{\alpha_0} u\|_0^{\theta_1}\,\|\partial_t^{\alpha_0} u\|_{4N-2\alpha_0+1}^{1-\theta_1}$$

and

$$\|\partial_t^{\alpha_0}\phi\|_{4N-2\alpha_0} \lesssim \|\partial_t^{\alpha_0}\phi\|_0^{\theta_1}\,\|\partial_t^{\alpha_0}\phi\|_{4N-2\alpha_0+1}^{1-\theta_1}$$

for $\theta_1 = \frac{1}{4N-2\alpha_0+1}$. Combining these with Lemma 7.8 and Lemma 7.9 together, we obtain that for any $\varepsilon > 0$

$$\begin{aligned}
&\|\partial_t^{\alpha_0} u\|_{4N-2\alpha_0}\sqrt{\mathscr{D}_{2N}} + \|\partial_t^{\alpha_0}\phi\|_{4N-2\alpha_0}\sqrt{\mathscr{D}_{2N}}\\
&\lesssim \|\partial^{\alpha_0} u\|_0^{\theta_1}\,\mathscr{D}_{2N}^{\frac{1-\theta_1}{2}}\sqrt{\mathscr{D}_{2N}} + \|\partial^{\alpha_0}\phi\|_0^{\theta_1}\,\mathscr{D}_{2N}^{\frac{1-\theta_1}{2}}\sqrt{\mathscr{D}_{2N}}\\
&\lesssim \|\nabla\partial^{\alpha_0} u\|_0^{\theta_1}\,\mathscr{D}_{2N}^{\frac{1-\theta_1}{2}}\sqrt{\mathscr{D}_{2N}} + \|\nabla\partial^{\alpha_0}\phi\|_0^{\theta_1}\,\mathscr{D}_{2N}^{\frac{1-\theta_1}{2}}\sqrt{\mathscr{D}_{2N}}\\
&\lesssim (\bar{\mathscr{D}}_{2N}^0)^{\frac{\theta_1}{2}}\mathscr{D}_{2N}^{1-\frac{\theta_1}{2}} \lesssim \varepsilon(1-\frac{\theta_1}{2})\mathscr{D}_{2N} + \frac{\theta_1}{2}\varepsilon^{1-\frac{2}{\theta_1}}\bar{\mathscr{D}}_{2N}^0.
\end{aligned}$$

Next ,we turn to estimate the nonlinear terms Φ^i. If $|\alpha| \le 4N-1$, in view of (70) and (71) in Lemma 3.2, we get

$$\begin{aligned}
&\int_{\Omega_+} (\chi\partial^\alpha u)\cdot\chi\partial^\alpha\Phi^1 - \chi\partial^\alpha u\cdot\nabla(\chi\partial^\alpha\Phi^2) + \chi\partial^\alpha p\chi\partial^\alpha\Phi^2 + \int_{\Omega_-} (\chi\partial^\alpha\Phi^3)\chi\partial^\alpha\phi\\
&\lesssim \|\partial^\alpha u\|_0\left\|\partial^\alpha\Phi^1\right\|_0 + \|\partial^\alpha u\|_0\left\|\partial^\alpha\Phi^2\right\|_1 + \|\partial^\alpha p\|_0\left\|\partial^\alpha\Phi^2\right\|_0 + \left\|\partial^\alpha\Phi^3\right\|_0\|\partial^\alpha\phi\|_0\\
&\lesssim \sqrt{\mathscr{D}_{2N}}\sqrt{\mathscr{E}_{2N}^\theta\mathscr{D}_{2N} + \mathscr{K}\mathscr{F}_{2N}}.
\end{aligned}$$

When $|\alpha| = 4N$, integrating by parts with respect to the horizontal direction, we get

$$\begin{aligned}\int_{\Omega_+} (\chi\partial^\alpha u)\cdot\chi\partial^\alpha\Phi^1 &= -\int_{\Omega_+} \chi\partial^{\alpha+\beta}u\chi\partial^\gamma\Phi^1\\ &\lesssim \left\|\partial^{\alpha+\beta}u\right\|_0\left\|\partial^\gamma\Phi^1\right\|_0 \lesssim \sqrt{\mathscr{D}_{2N}}\sqrt{\mathscr{E}_{2N}^\theta\mathscr{D}_{2N}+\mathscr{K}\mathscr{F}_{2N}},\end{aligned}$$

$$\begin{aligned}\int_{\Omega_+} \chi\partial^\alpha u\cdot\nabla(\chi\partial^\alpha\Phi^2) &= -\int_{\Omega_+} \chi\partial^{\alpha+\beta}u\cdot\nabla(\chi\partial^\gamma\Phi^2)\\ &\lesssim \left\|\partial^{\alpha+\beta}u\right\|_0\left\|\partial^\gamma\Phi^2\right\|_1 \lesssim \sqrt{\mathscr{D}_{2N}}\sqrt{\mathscr{E}_{2N}^\theta\mathscr{D}_{2N}+\mathscr{K}\mathscr{F}_{2N}}\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega_+} \chi\partial^\alpha p\chi\partial^\alpha\Phi^2 &\lesssim \left\|\partial^\alpha p\right\|_0\left\|\partial^\alpha\Phi^2\right\|_0 \lesssim \left\|\partial^\alpha p\right\|_0\left\|\bar\nabla^{4N-1}\Phi^2\right\|_1\\ &\lesssim \sqrt{\mathscr{D}_{2N}}\sqrt{\mathscr{E}_{2N}^\theta\mathscr{D}_{2N}+\mathscr{K}\mathscr{F}_{2N}},\end{aligned}$$

where $\alpha = \beta+\gamma$ and ∂^β only contains the horizontal direction derivative, satisfying $|\gamma| = 4N-1$ and $|\beta| = 1$. To control the terms Φ^3, it gives

$$\begin{aligned}\int_{\Omega_-} \chi\partial^\alpha\Phi^3\chi\partial^\alpha\phi &= -\int_{\Omega_-} \chi\partial^\gamma\Phi^3\chi\partial^{\alpha+\beta}\phi \lesssim \left\|\partial^\gamma\Phi^3\right\|_0\left\|\partial^{\alpha+\beta}\phi\right\|_0\\ &\lesssim \sqrt{\mathscr{D}_{2N}}\sqrt{\mathscr{E}_{2N}^\theta\mathscr{D}_{2N}+\mathscr{K}\mathscr{F}_{2N}}.\end{aligned}$$

Via the trace theory and Lemma 3.2, we obtain

$$\begin{aligned}&\int_{\Sigma_-} \Phi^7\cdot(\chi\partial^\alpha u)+\Phi^8\chi\partial^\alpha\phi = -\int\int_{\Sigma_-} \partial^{\alpha-\beta}\Phi^7\chi\partial^{\alpha+\beta}u+\partial^{\alpha-\beta}\Phi^8\chi\partial^{\alpha+\beta}\phi\\ &\lesssim \left\|\partial^\alpha u\right\|_{H^{\frac12}(\Sigma_-)}\left\|\bar\nabla_H^{0,4N-1}\Phi^7\right\|_{H^{\frac12}(\Sigma_-)} + \left\|\partial^\alpha\phi\right\|_{H^{\frac12}(\Sigma_-)}\left\|\bar\nabla_H^{0,4N-1}\Phi^8\right\|_{H^{\frac12}(\Sigma_-)}\\ &\lesssim \sqrt{\mathscr{D}_{2N}}\sqrt{\mathscr{E}_{2N}^\theta\mathscr{D}_{2N}+\mathscr{K}\mathscr{F}_{2N}}.\end{aligned}$$

To estimate the term Φ^5, we split into two cases: $\alpha_0 \geq 1$ and $\alpha_0 = 0$. If $\alpha_0 \geq 1$, we get

$$\begin{aligned}\int_{\Sigma_-} \rho g\partial^\alpha\eta\partial^\alpha\Phi^5 &= -\int_{\Sigma_-} \rho g\partial^{\alpha+\beta}\eta\partial^{\alpha-\beta}\Phi^5 \lesssim \left\|\partial^{\alpha+\beta}\eta\right\|_{H^{-\frac12}(\Sigma_-)}\left\|\partial^{\alpha-\beta}\Phi^5\right\|_{H^{\frac12}(\Sigma_-)}\\ &\lesssim \left\|\partial^\alpha\eta\right\|_{H^{\frac12}(\Sigma_-)}\left\|\bar\nabla_H^{0,4N-1}\Phi^5\right\|_{\frac12}\\ &\lesssim \sqrt{\mathscr{D}_{2N}}\sqrt{\mathscr{E}_{2N}^\theta\mathscr{D}_{2N}+\mathscr{K}\mathscr{F}_{2N}}.\end{aligned}$$

If $\alpha_0 = 0$, it means ∂^α only involves horizontal spatial derivatives, so we may use Lemma 5.1 in [17] to bound

$$\int_{\Sigma_-} \rho g\partial^\alpha\eta\partial^\alpha\Phi^5 \lesssim \sqrt{\mathscr{D}_{2N}}\sqrt{\mathscr{E}_{2N}^\theta\mathscr{D}_{2N}+\mathscr{K}\mathscr{F}_{2N}}.$$

Combining the above estimates together, we conclude

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\int_{\Omega_+}|\chi\partial^\alpha u|^2+\frac{1}{2}\int_{\Omega_-}|\chi\partial^\alpha\phi|^2+\frac{1}{2}\int_{\Sigma_-}|\partial^\alpha\eta|^2\right)\\
&\qquad+\frac{1}{2}\int_{\Omega_+}|\mathbb{D}(\chi\partial^\alpha u)|^2+\int_{\Omega_-}|\nabla(\chi\partial^\alpha\phi)|^2\\
\lesssim&\sqrt{\mathscr{D}_{2N}}\sqrt{\mathscr{E}_{2N}^\theta\mathscr{D}_{2N}+\mathscr{K}\mathscr{F}_{2N}}+\varepsilon(1-\frac{\theta_1}{2})\mathscr{D}_{2N}+\frac{\theta_1}{2}\varepsilon^{1-\frac{2}{\theta_1}}\bar{\mathscr{D}}_{2N}^0\\
\lesssim&\mathscr{E}_{2N}^{\frac{\theta}{2}}\mathscr{D}_{2N}+\sqrt{\mathscr{D}_{2N}\mathscr{K}\mathscr{F}_{2N}}+\varepsilon\mathscr{D}_{2N}+\varepsilon^{1-\frac{2}{\theta_1}}\bar{\mathscr{D}}_{2N}^0.
\end{aligned}
\tag{79}
$$

Then we complete the proof via integrating over the time from 0 to t and sum with respect to α. □

Next, we show the $(N+2)$th-order energy estimates of the solution.

Proposition 3.5. *Assume (u,p,ϕ,η) is the solution to the equations* (17) *and the assumptions in Theorem 1.1 and the a-priori assumption* (22) *hold for some small positive constant δ. Then for any $\varepsilon\in(0,1)$, there exists a constant $C(\varepsilon)>0$, such that*

$$
\frac{d}{dt}\bar{\mathscr{E}}_{N+2}^{+}+\bar{\mathscr{D}}_{N+2}^{+}\lesssim\mathscr{E}_{2N}^{\theta}\mathscr{D}_{N+2}+\varepsilon\mathscr{D}_{N+2}+C(\varepsilon)\bar{\mathscr{D}}_{N+2}^0. \tag{80}
$$

Proof. By Lemma 3.1, we still have the same energy identity (78) in Proposition 3.4 for $\alpha_0\le N+1$ and $|\alpha|\le 2(N+2)$. So we need to estimate the right side of (78). To bound the linear term, we have

$$
\begin{aligned}
&\int_{\Omega_+}\chi\partial^\alpha u\partial^\alpha H^1-\chi\partial^\alpha u\cdot\nabla(\partial^\alpha H^2)+\chi\partial^\alpha p\partial^\alpha H^2+\int_{\Omega_-}\chi\partial^\alpha\phi\partial^\alpha H^3\\
\lesssim&\|\partial^{\alpha_0}u\|_{2(N+2)-2\alpha_0}\sqrt{\mathscr{D}_{N+2}}+\|\partial_t^{\alpha_0}\phi\|_{2(N+2)-2\alpha_0}\sqrt{\mathscr{D}_{N+2}}.
\end{aligned}
\tag{81}
$$

Via the interpolation theory, it gives

$$
\|\partial^{\alpha_0}u\|_{2(N+2)-2\alpha_0}\lesssim\|\partial^{\alpha_0}u\|_0^{\theta_1}\|\partial^{\alpha_0}u\|_{2(N+2)-2\alpha_0+1}^{1-\theta_1}
$$

and

$$
\|\partial_t^{\alpha_0}\phi\|_{2(N+2)-2\alpha_0}\lesssim\|\partial_t^{\alpha_0}\phi\|_0^{\theta_1}\|\partial_t^{\alpha_0}\phi\|_{2(N+2)-2\alpha_0+1}^{1-\theta_1}
$$

for $\theta_1=\frac{1}{2N-2\alpha_0+5}$. Combining these two inequalities with (81) together, it gets

$$
\begin{aligned}
&\int_{\Omega_+}\chi\partial^\alpha u\partial^\alpha H^1-\chi\partial^\alpha u\cdot\nabla(\partial^\alpha H^2)+\chi\partial^\alpha p\partial^\alpha H^2+\int_{\Omega_-}\chi\partial^\alpha\phi\partial^\alpha H^3\\
\lesssim&\varepsilon(1-\frac{\theta_1}{2})\mathscr{D}_{N+2}+\frac{\theta_1}{2}\varepsilon^{1-\frac{2}{\theta_1}}\bar{\mathscr{D}}_{N+2}^0\\
\lesssim&\varepsilon\mathscr{D}_{N+2}+\varepsilon^{-2N+2\alpha_0-4}\bar{\mathscr{D}}_{N+2}^0.
\end{aligned}
$$

To estimate the nonlinear terms Φ^i, by Lemma 3.3, we get

$$
\begin{aligned}
&\int_{\Omega_+}(\chi\partial^\alpha u)\cdot\chi\partial^\alpha\Phi^1-\chi\partial^\alpha u\cdot\nabla(\chi\partial^\alpha\Phi^2)+\chi\partial^\alpha p\chi\partial^\alpha\Phi^2+\int_{\Omega_-}(\chi\partial^\alpha\Phi^3)\chi\partial^\alpha\phi\\
\lesssim&\sqrt{\mathscr{D}_{N+2}}\sqrt{\mathscr{E}_{2N}^\theta\mathscr{D}_{N+2}},
\end{aligned}
$$

where $\alpha=(\alpha_0,\alpha_1,\alpha_2)$ satisfies $|\alpha|\le 2(N+2)-1$. If $|\alpha|=2(N+2)$, choosing β and γ such that $\partial^\alpha=\partial^\beta\partial^\gamma$ and $|\beta|=1$ and integrating by parts with respect to the horizontal direction, we obtain

$$\int_{\Omega_+} (\chi\partial^\alpha u)\cdot\chi\partial^\alpha\Phi^1 = -\int_{\Omega_+} \chi\partial^{\alpha+\beta}u\chi\partial^\gamma\Phi^1 \lesssim \sqrt{\mathscr{D}_{N+2}}\sqrt{\mathscr{E}_{2N}^\theta\mathscr{D}_{N+2}}$$

and

$$\int_{\Omega_+} \chi\partial^\alpha u\cdot\nabla(\chi\partial^\alpha\Phi^2) = -\int_{\Omega_+} \chi\partial^{\alpha+\beta}u\cdot\nabla(\chi\partial^\gamma\Phi^2) \lesssim \sqrt{\mathscr{D}_{N+2}}\sqrt{\mathscr{E}_{2N}^\theta\mathscr{D}_{N+2}}.$$

Similarly, we get

$$\begin{aligned}\int_{\Omega_+} \chi\partial^\alpha p\chi\partial^\alpha\Phi^2 &\lesssim \|\partial^\alpha p\|_0\left\|\partial^\alpha\Phi^2\right\|_0 \lesssim \|\partial^\alpha p\|_0\left\|\bar\nabla^{2(N+2)-1}\Phi^2\right\|_1\\ &\lesssim \sqrt{\mathscr{D}_{N+2}}\sqrt{\mathscr{E}_{2N}^\theta\mathscr{D}_{N+2}}\end{aligned}$$

and

$$\int_{\Omega_-} \chi\partial^\alpha\Phi^3\chi\partial^\alpha\phi = -\int_{\Omega_-} \chi\partial^\gamma\Phi^3\chi\partial^{\alpha+\beta}\phi \lesssim \sqrt{\mathscr{D}_{N+2}}\sqrt{\mathscr{E}_{2N}^\theta\mathscr{D}_{N+2}}.$$

Via the trace theory and Lemma 3.3, it gets

$$\begin{aligned}\int_{\Sigma_-} \Phi^7\cdot(\chi\partial^\alpha u) + \Phi^8\chi\partial^\alpha\phi &= -\int_{\Sigma_-} \partial^{\alpha-\beta}\Phi^7\chi\partial^{\alpha+\beta}u + \partial^{\alpha-\beta}\Phi^8\chi\partial^{\alpha+\beta}\phi\\ &\lesssim \sqrt{\mathscr{D}_{N+2}}\sqrt{\mathscr{E}_{2N}^\theta\mathscr{D}_{N+2}}.\end{aligned}$$

To bound the nonlinear term Φ^5, we split into two cases: $\alpha_0 \geq 1$ and $\alpha_0 = 0$. If $\alpha_0 \geq 1$, it yields

$$\begin{aligned}\int_{\Sigma_-} \rho g\partial^\alpha\eta\partial^\alpha\Phi^5 &= -\int_{\Sigma_-} \rho g\partial^{\alpha+\beta}\eta\partial^{\alpha-\beta}\Phi^5 \lesssim \left\|\partial^{\alpha+\beta}\eta\right\|_{H^{-\frac12}(\Sigma_-)}\left\|\partial^{\alpha-\beta}\Phi^5\right\|_{H^{\frac12}(\Sigma_-)}\\ &\lesssim \sqrt{\mathscr{D}_{N+2}}\sqrt{\mathscr{E}_{2N}^\theta\mathscr{D}_{N+2}}.\end{aligned}$$

If $\alpha_0 = 0$, it implies ∂^α only has the horizontal derivatives and $\alpha_1+\alpha_2 = 2(N+2)$. By the Leibniz rule, we get that

$$\begin{aligned}\partial^\alpha\Phi^5 &= -\partial^\alpha(\nabla_H\eta\cdot u_H)\\ &= -\sum_{\substack{0\leq\beta\leq\alpha\\ |\beta|\leq 2(N+2)-2}} C_\alpha^\beta\nabla_H\partial^{\alpha-\beta}\eta\cdot\partial^\beta u_H - \sum_{\substack{0\leq\beta\leq\alpha\\ |\beta|\geq 2(N+2)-1}} C_\alpha^\beta\nabla_H\partial^{\alpha-\beta}\eta\cdot\partial^\beta u_H\\ &=: I + II,\end{aligned}$$

where $u_H = (u_1, u_2)$ and C_α^β is the constant depending only on α and β. Then by Lemma 7.6, it yields

$$\begin{aligned}\int_{\Sigma_-} \rho g\partial^\alpha\eta I &\lesssim \|\partial^\alpha\eta\|_{H^{-\frac12}(\Sigma_-)}\left\|\nabla_H\partial^{\alpha-\beta}\eta\right\|_{\frac12}\left\|\partial^\beta u\right\|_{H^2(\Sigma_-)}\\ &\lesssim \sqrt{\mathscr{D}_{N+2}}\sqrt{\mathscr{E}_{2N}}\sqrt{\mathscr{D}_{N+2}}\end{aligned}$$

and

$$\begin{aligned}\int_{\Sigma_-} \rho g\partial^\alpha\eta II &\lesssim \|\partial^\alpha\eta\|_{H^{-\frac12}(\Sigma_-)}\left\|\nabla_H\partial^{\alpha-\beta}\eta\cdot\partial^\beta u\right\|_{H^{\frac12}(\Sigma_-)}\\ &\lesssim \|\partial^\alpha\eta\|_{H^{-\frac12}(\Sigma_-)}\left\|\nabla_H\partial^{\alpha-\beta}\eta\right\|_2\left\|\partial^\beta u\right\|_{H^{\frac12}(\Sigma_-)}\\ &\lesssim \sqrt{\mathscr{D}_{N+2}}\sqrt{\mathscr{E}_{2N}}\sqrt{\mathscr{D}_{N+2}}.\end{aligned}$$

Combining these estimates together and summing with respect to α, we obtain the estimate (80). □

4. Normal energy estimates.

In this section, we will give the normal energy estimate, i.e. $\mathscr{E}_n$ and $\mathscr{D}_n$ are comparable to $\bar{\mathscr{E}}_n$ and $\bar{\mathscr{D}}_n$ for both at the $(N+2)$th and $(2N)$th level, respectively.

Proposition 4.1. *Assume (u, p, ϕ, η) is the solution to equations (17) and the assumptions in Theorem 1.1 and the a-priori assumption (22) hold for some small positive constant δ. Then there exists a positive constant $\theta > 0$, such that*

$$\mathscr{E}_{2N} \lesssim \bar{\mathscr{E}}_{2N} + \mathscr{E}_{2N}^{1+\theta} \tag{82}$$

and

$$\mathscr{E}_{N+2} \lesssim \bar{\mathscr{E}}_{N+2} + \mathscr{E}_{2N}^{\theta}\mathscr{E}_{N+2}. \tag{83}$$

Proof. Define the norm of the nonlinear terms as

$$\begin{aligned}\mathscr{Z}_n = \sum_{j=1}^{n-1} &\left\|\partial_t^j \Phi^1\right\|_{2n-2j-2}^2 + \left\|\partial_t^j \Phi^2\right\|_{2n-2j-1}^2 + \left\|\partial_t^j \Phi^3\right\|_{2n-2j-2}^2 \\ &+ \left\|\partial_t^j \Phi^7\right\|_{2n-2j-\frac{3}{2}}^2 + \left\|\partial_t^j \Phi^8\right\|_{2n-2j-\frac{3}{2}}^2.\end{aligned} \tag{84}$$

By the a-priori assumption (22), it gives $\|J-1\|_{L^\infty} \le 1/2$, and then we have

$$\frac{1}{2}\sum_{j=0}^{n}\left\|\partial_t^j u\right\|_0^2 \le \sum_{j=0}^{n}\left\|\sqrt{J}\partial_t^j u\right\|_0^2 \le \frac{3}{2}\sum_{j=0}^{n}\left\|\partial_t^j u\right\|_0^2$$

and

$$\frac{1}{2}\sum_{j=0}^{n}\left\|\partial_t^j \phi\right\|_0^2 \le \sum_{j=0}^{n}\left\|\sqrt{J}\partial_t^j \phi\right\|_0^2 \le \frac{3}{2}\sum_{j=0}^{n}\left\|\partial_t^j \phi\right\|_0^2.$$

The definitions of $\bar{\mathscr{E}}_n^0$ and $\bar{\mathscr{E}}_n^+$ give that

$$\|\partial_t^n u\|_0^2 + \|\partial_t^n \phi\|_0^2 + \sum_{j=0}^{n}\|\partial_t \eta\|_{2n-2j}^2 \lesssim \bar{\mathscr{E}}_n. \tag{85}$$

By the tace theory, we get that if $0 \le j \le n-1$,

$$\left\|\partial_t^j \phi\right\|_{H^{2n-2j-\frac{3}{2}}(\Sigma_-)}^2 \lesssim \left\|\bar{\nabla}_H^{0,2n-1}\phi\right\|_{H^{-\frac{1}{2}}(\Sigma_-)}^2 \lesssim \left\|\bar{\nabla}_H^{0,2n-1}(\chi\phi)\right\|_0^2$$

and

$$\left\|\partial_t^j u\right\|_{H^{2n-2j-\frac{3}{2}}(\Sigma_-)}^2 \lesssim \left\|\bar{\nabla}_H^{0,2n-1}u\right\|_{H^{-\frac{1}{2}}(\Sigma_-)}^2 \lesssim \left\|\bar{\nabla}_H^{0,2n-1}(\chi u)\right\|_0^2.$$

Applying ∂_t^j $(0 \le j \le n-1)$ to the equations (63), it gives

$$\begin{cases}\nabla\partial_t^j p - \Delta\partial_t^j u = \partial_t^j\Phi^1 - \partial_t^{j+1}u & \text{in } \Omega_+,\\ \operatorname{div}\partial_t^j u = \partial_t^j \Phi^2 & \text{in } \Omega_+,\\ \partial_t^j u = 0 & \text{on } \Sigma_+,\\ (\partial_t^j pI - \mathbb{D}\partial_t^j u)\vec{e}_3 = -\rho g\partial_t^j\eta\vec{e}_3 + \partial_t^j\phi\vec{e}_3 + \partial_t^j\Phi^7 & \text{on } \Sigma_-\end{cases} \tag{86}$$

and

$$\begin{cases}-\Delta\partial_t^j\phi = \partial_t^j\Phi^3 - \partial_t^{j+1}\phi & \text{in } \Omega_-,\\ \partial_t^j u\cdot\vec{e}_3 = -\nabla\partial_t^j\phi\cdot\vec{e}_3 + \partial_t^j\Phi^8 & \text{on } \Sigma_-,\\ \partial_t^j\phi = 0 & \text{on } \Sigma_{-b}.\end{cases} \tag{87}$$

Via the Stokes estimates, combining with (84) and (85) together, we get

$$\begin{aligned}(88)\quad &\left\|\partial_t^j u\right\|_{2n-2j}^2+\left\|\partial_t^j p\right\|_{2n-2j-1}^2\lesssim\left\|\partial_t^{j+1} u\right\|_{2n-2j-2}^2+\left\|\partial_t^j \Phi^1\right\|_{2n-2j-2}^2\\ &+\left\|\partial_t^j \Phi^2\right\|_{2n-2j-1}^2+\left\|\partial_t^j \Phi^7\right\|_{2n-2j-\frac{3}{2}}^2+\left\|\partial_t^j \eta\right\|_{2n-2j-\frac{3}{2}}^2+\left\|\partial_t^j \phi\right\|_{H^{2n-2j-\frac{3}{2}}(\Sigma_-)}^2.\end{aligned}$$

And via the elliptic estimates, it yields

$$\begin{aligned}(89)\quad \left\|\partial_t^j \phi\right\|_{2n-2j}^2\lesssim &\left\|\partial_t^j \Phi^3\right\|_{2n-2j-2}^2+\left\|\partial_t^{j+1} \phi\right\|_{2n-2j-2}^2+\left\|\partial_t^j \Phi^8\right\|_{H^{2n-2j-\frac{3}{2}}(\Sigma_-)}^2\\ &+\left\|\partial_t^j u\cdot\vec{e}_3\right\|_{H^{2n-2j-\frac{3}{2}}(\Sigma_-)}^2.\end{aligned}$$

Adding (88) and (89) together, we obtain

(90)

$$\begin{aligned}&\left\|\partial_t^j u\right\|_{2n-2j}^2+\left\|\partial_t^j p\right\|_{2n-2j-1}^2+\left\|\partial_t^j \phi\right\|_{2n-2j}^2\\ \lesssim&\left\|\partial_t^{j+1} u\right\|_{2n-2j-2}^2+\left\|\partial_t^{j+1} \phi\right\|_{2n-2j-2}^2+\left\|\partial_t^j \Phi^1\right\|_{2n-2j-2}^2\\ &+\left\|\partial_t^j \Phi^2\right\|_{2n-2j-1}^2+\left\|\partial_t^j \Phi^3\right\|_{2n-2j-2}^2+\left\|\partial_t^j \Phi^7\right\|_{2n-2j-\frac{3}{2}}^2+\left\|\partial_t^j \eta\right\|_{2n-2j-\frac{3}{2}}^2+\bar{\mathscr{E}}_n^+\\ \lesssim&\left\|\partial_t^{j+1} u\right\|_{2n-2j-2}^2+\left\|\partial_t^{j+1} \phi\right\|_{2n-2j-2}^2+\mathscr{Z}_n+\bar{\mathscr{E}}_n.\end{aligned}$$

If $j=n-1$, (90) gives

$$\left\|\partial_t^{n-1} u\right\|_2^2+\left\|\partial_t^{n-1} p\right\|_1^2+\left\|\partial_t^{n-1} \phi\right\|_2^2\lesssim\|\partial_t^n u\|_0^2+\|\partial_t^n \phi\|_0^2+\mathscr{Z}_n+\bar{\mathscr{E}}_n\lesssim\mathscr{Z}_n+\bar{\mathscr{E}}_n,$$

which immediately gives

$$(91)\quad \sum_{0\le j\le n-1}\left\|\partial_t^j u\right\|_{2n-2j}^2+\left\|\partial_t^j p\right\|_{2n-2j-1}^2+\left\|\partial_t^j \phi\right\|_{2n-2j}^2\lesssim\mathscr{Z}_n+\bar{\mathscr{E}}_n.$$

Then adding (85) and (91) together, we get that

$$\mathscr{E}_n\lesssim\mathscr{Z}_n+\bar{\mathscr{E}}_n.$$

Combining the above inequality with the estimates of the nonlinear terms $\mathscr{Z}_{2N}$ (69) and (75) together, we complete the proof. □

Next, we give the comparable results on the dissipation.

Proposition 4.2. *Assume* (u,p,ϕ,η) *is the solution to equations* (17) *and the assumptions in Theorem 1.1 and the a-priori assumption* (22) *hold for some small positive constant* δ*. Then there exists a positive constant* $\theta>0$*, such that*

$$(92)\quad \mathscr{D}_{2N}\lesssim\bar{\mathscr{D}}_{2N}+\mathscr{K}\mathscr{F}_{2N}+\mathscr{E}_{2N}^\theta\mathscr{D}_{2N}$$

and

$$(93)\quad \mathscr{D}_{N+2}\lesssim\bar{\mathscr{D}}_{N+2}+\mathscr{E}_{2N}^\theta\mathscr{D}_{N+2}.$$

Proof. Denote that

$$\begin{aligned}\mathscr{X}_n=&\left\|\bar\nabla^{0,2n-1}\Phi^1\right\|_0^2+\left\|\bar\nabla^{0,2n-1}\Phi^2\right\|_1^2+\left\|\bar\nabla^{0,2n-1}\Phi^3\right\|_0^2+\left\|\bar\nabla_H^{0,2n-1}\Phi^5\right\|_{\frac{1}{2}}^2\\ &+\left\|\bar\nabla_H^{2n-2}\partial_t\Phi^5\right\|_{\frac{1}{2}}^2+\left\|\bar\nabla_H^{0,2n-1}\Phi^7\right\|_{\frac{1}{2}}^2.\end{aligned}$$

By the definition of $\bar{\mathscr{D}}_n^+$ and $\bar{\mathscr{D}}_n^0$ and the Korn inequality, we get that

(94)
$$\begin{aligned}&\left\|\bar{\nabla}_H^{0,2n-1}u\right\|^2_{H^1(\Omega_{1,+})}+\left\|\nabla_H\bar{\nabla}_H^{2n-1}u\right\|^2_{H^1(\Omega_{1,+})}+\left\|\bar{\nabla}_H^{0,2n-1}\phi\right\|^2_{H^1(\Omega_{1,-})}\\&+\left\|\nabla_H\bar{\nabla}_H^{2n-1}\phi\right\|^2_{H^1(\Omega_{1,-})}\\ \lesssim&\left\|\bar{\nabla}_H^{0,2n-1}(\chi u)\right\|^2_1+\left\|\nabla_H\bar{\nabla}_H^{2n-1}(\chi u)\right\|^2_1+\left\|\bar{\nabla}_H^{0,2n-1}(\chi\phi)\right\|^2_1+\left\|\nabla_H\bar{\nabla}_H^{2n-1}(\chi\phi)\right\|^2_1\\ \lesssim&\bar{\mathscr{D}}_n^+\end{aligned}$$

and

$$\sum_{s=0}^{n}\left\|\partial_t^j u\right\|^2_{H^1(\Omega_{1,+})}+\left\|\partial_t^j\phi\right\|^2_{H^1(\Omega_{1,-})}\lesssim\sum_{s=0}^{n}\left\|\partial_t^j u\right\|^2_1+\left\|\partial_t^j\phi\right\|^2_1\lesssim\bar{\mathscr{D}}_n^0, \tag{95}$$

where $\Omega_{1,+}=\Omega_+\cap\Omega_1$, $\Omega_{1,-}=\Omega_-\cap\Omega_1$ and

$$\Omega_1=\left\{x\in\mathbb{R}^3|(x_1,x_2)\in\mathbb{T}^2,x_3\in\left(-\frac{4}{5}\underline{b},\frac{4}{5}\right)\right\}.$$

Note that $\chi\equiv 1$ on Ω_1. Adding (94) and (95) together, it yields

$$\left\|\bar{\nabla}_H^{0,2n}u\right\|^2_{H^1(\Omega_{1,+})}+\left\|\bar{\nabla}_H^{0,2n}\phi\right\|^2_{H^1(\Omega_{1,-})}\lesssim\bar{\mathscr{D}}_n. \tag{96}$$

Via (96), the higher regularity of u on the horizontal boundary Σ_- is gained

$$\begin{aligned}\left\|\partial_t^j u\right\|^2_{H^{2n-2j+\frac12}(\Sigma_-)}&\lesssim\left\|\partial_t^j u\right\|^2_{H^{\frac12}(\Sigma_-)}+\left\|\nabla_H^{2n-2j}\partial_t^j u\right\|^2_{H^{\frac12}(\Sigma_-)}\\&\lesssim\left\|\partial_t^j u\right\|^2_{H^1(\Omega_{1,+})}+\left\|\bar{\nabla}_H^{0,2n}u\right\|^2_{H^1(\Omega_{1,+})}\lesssim\bar{\mathscr{D}}_n.\end{aligned}$$

Similarly, we obtain the estimate on ϕ

$$\left\|\partial_t^j\phi\right\|^2_{H^{2n-2j+\frac12}(\Sigma_-)}\lesssim\left\|\partial_t^j\phi\right\|^2_{H^1(\Omega_{1,+})}+\left\|\bar{\nabla}_H^{0,2n}\phi\right\|^2_{H^1(\Omega_{1,+})}\lesssim\bar{\mathscr{D}}_n.$$

Applying ∂_t^j $(0\le j\le n-1)$ to the equations (63), it yields

$$\begin{cases}-\Delta\partial_t^j u+\nabla\partial_t^j p=\partial_t^j\Phi^1-\partial_t^{j+1}u & \text{in }\Omega_+,\\ \operatorname{div}\partial_t^j u=\partial_t^j\Phi^2 & \text{in }\Omega_+,\\ \partial_t^j u=0 & \text{on }\Sigma_+,\\ \partial_t^j u=\partial_t^j u & \text{on }\Sigma_-\end{cases}$$

and

$$\begin{cases}-\Delta\partial_t^j\phi=\partial_t^j\Phi^3-\partial_t^{j+1}\phi & \text{in }\Omega_-,\\ \partial_t^j\phi=\partial_t^j\phi & \text{on }\Sigma_-,\\ \partial_t^j\phi=0 & \text{on }\Sigma_{-b}.\end{cases}$$

Via the Stokes estimate and the standard elliptic estimates, it gives

$$\begin{aligned}&\left\|\partial_t^j u\right\|^2_{2n-2j+1}+\left\|\nabla\partial_t^j p\right\|^2_{2n-2j-1}\\ \lesssim&\left\|\partial_t^{j+1}u\right\|^2_{2n-2j-1}+\left\|\partial_t^j\Phi^1\right\|^2_{2n-2j-1}+\left\|\partial_t^j\Phi^2\right\|^2_{2n-2j}+\left\|\partial_t^j u\right\|^2_{H^{2n-2j+\frac12}(\Sigma_-)}\\ \lesssim&\left\|\partial_t^{j+1}u\right\|^2_{2n-2j-1}+\left\|\partial_t^j\Phi^1\right\|^2_{2n-2j-1}+\left\|\partial_t^j\Phi^2\right\|^2_{2n-2j}+\bar{\mathscr{D}}_n\end{aligned} \tag{97}$$

and

$$\begin{aligned}\left\|\partial_t^j\phi\right\|_{2n-2j+1}^2 &\lesssim \left\|\partial_t^j\Phi^3\right\|_{2n-2j-1}^2+\left\|\partial_t^{j+1}\phi\right\|_{2n-2j-1}^2+\left\|\partial_t^j\phi\right\|_{H^{2n-2j+\frac{1}{2}}(\Sigma_-)}^2\\ &\lesssim \left\|\partial_t^j\Phi^3\right\|_{2n-2j-1}^2+\left\|\partial_t^{j+1}\phi\right\|_{2n-2j-1}^2+\bar{\mathscr{D}}_n.\end{aligned}\tag{98}$$

Summing (97) and (98) together, we get

$$\begin{aligned}&\left\|\partial_t^j u\right\|_{2n-2j+1}^2+\left\|\nabla\partial_t^j p\right\|_{2n-2j-1}^2+\left\|\partial_t^j\phi\right\|_{2n-2j+1}^2\\ \lesssim &\left\|\partial_t^{j+1}u\right\|_{2n-2j-1}^2+\left\|\partial_t^{j+1}\phi\right\|_{2n-2j-1}^2+\mathscr{X}_n+\bar{\mathscr{D}}_n.\end{aligned}\tag{99}$$

Via (99), we claim that

$$\sum_{j=0}^{n}\left\|\partial_t^j u\right\|_{2n-2j+1}^2+\left\|\partial_t^j\phi\right\|_{2n-2j+1}^2+\sum_{j=0}^{n-1}\left\|\nabla\partial_t^j p\right\|_{2n-2j-1}^2\lesssim\mathscr{X}_n+\bar{\mathscr{D}}_n.\tag{100}$$

To prove this claim, combining (99) with (95) together, we get that if $j=n-1$,

$$\left\|\partial_t^{n-1}u\right\|_3^2+\left\|\nabla\partial_t^{n-1}p\right\|_1^2+\left\|\partial_t^{n-1}\phi\right\|_3^2\lesssim\left\|\partial_t^n u\right\|_1^2+\left\|\partial_t^n\phi\right\|_1^2+\mathscr{X}_n+\bar{\mathscr{D}}_n\lesssim\mathscr{X}_n+\bar{\mathscr{D}}_n.$$

By induction, we assume that the following inequalities hold

$$\left\|\partial_t^{n-l}u\right\|_{2l+1}^2+\left\|\nabla\partial_t^{n-l}p\right\|_{2l-1}^2+\left\|\partial_t^{n-l}\phi\right\|_{2l+1}^2\lesssim\mathscr{X}_n+\bar{\mathscr{D}}_n\quad\text{for } i\le l\le n-1.\tag{101}$$

Then if $j=n-(i+1)$, combining (99) and (101) together, we have

$$\begin{aligned}&\left\|\partial_t^{n-(l+1)}u\right\|_{2(l+1)+1}^2+\left\|\nabla\partial_t^{n-(l+1)}p\right\|_{2(l+1)-1}^2+\left\|\partial_t^{n-(l+1)}\phi\right\|_{2(l+1)+1}^2\\ \lesssim&\left\|\partial_t^{n-l}u\right\|_{2l+1}^2+\left\|\partial_t^{n-l}\phi\right\|_{2l+1}^2+\mathscr{X}_n+\bar{\mathscr{D}}_n\\ \lesssim&\mathscr{X}_n+\bar{\mathscr{D}}_n,\end{aligned}$$

which implies that (101) holds for all $0<l\le n$. Summing with respect to l from 0 to n, we obtain the claim (100).

To complete the proof, we need to estimate the other terms in $\mathscr{D}_{2n}$. To get the estimate of η, we will fully take advantage of the boundary condition

$$\rho g\eta=\phi+\Phi_3^7+2\partial_3 u_3-p.\tag{102}$$

Since the estimate $\|p\|_0^2$ isn't gained, (100) gives the estimate $\|\nabla p\|^2$. Then applying ∇_H to the equation (102), it yields

$$\begin{aligned}&\|\nabla_H\eta\|_{2n-\frac{3}{2}}^2\\ \lesssim&\|\nabla_H\phi\|_{H^{2n-\frac{3}{2}}(\Sigma_-)}^2+\left\|\nabla_H\Phi^7\right\|_{2n-\frac{3}{2}}^2+\|\nabla_H\partial_3 u_3\|_{H^{2n-\frac{3}{2}}(\Sigma_-)}^2+\|\nabla_H p\|_{H^{2n-\frac{3}{2}}(\Sigma_-)}^2\\ \lesssim&\|\phi\|_{2n}^2+\left\|\nabla_H\Phi^7\right\|_{2n-\frac{3}{2}}^2+\|u\|_{2n+1}^2+\|\nabla p\|_{2n-1}^2.\end{aligned}\tag{103}$$

Note that $\int_{\Sigma_-}\eta=0$, and via the Poincäre inequality and (100), (103) gives

$$\begin{aligned}\|\eta\|_{2n-\frac{1}{2}}^2&\lesssim\|\eta\|_0^2+\|\nabla_H\eta\|_{2n-\frac{3}{2}}^2\lesssim\|\nabla_H\eta\|_{2n-\frac{3}{2}}^2\\ &\lesssim\|\phi\|_{2n}^2+\left\|\nabla_H\Phi^7\right\|_{2n-\frac{3}{2}}^2+\|u\|_{2n+1}^2+\|\nabla p\|_{2n-1}^2\\ &\lesssim\mathscr{X}_n+\bar{\mathscr{D}}_n.\end{aligned}\tag{104}$$

To deal with the term $\|\partial_t\eta\|^2_{2n-\frac{1}{2}}$, we apply the operator ∂_t to the boundary condition $\partial_t\eta = u_3 + \Phi^5$, yielding

$$
\begin{aligned}
\|\partial_t\eta\|^2_{2n-\frac{1}{2}} &\lesssim \|u_3\|^2_{H^{2n-\frac{1}{2}}(\Sigma_-)} + \left\|\Phi^5\right\|^2_{2n-\frac{1}{2}} \lesssim \|u\|^2_{2n} + \left\|\Phi^5\right\|^2_{2n-\frac{1}{2}} \\
&\lesssim \bar{\mathscr{D}}_n + \left\|\Phi^5\right\|^2_{\frac{1}{2}} + \left\|\nabla_H^{2n-1}\Phi^5\right\|^2_{\frac{1}{2}} \\
&\lesssim \bar{\mathscr{D}}_n + \mathscr{X}_n.
\end{aligned} \tag{105}
$$

Applying the temporal operator ∂_t^j $(2 \le j \le n+1)$ to the equation $(63)_5$, we get

$$
\begin{aligned}
\left\|\partial_t^j\eta\right\|^2_{2n-2j+\frac{5}{2}} &\lesssim \left\|\partial_t^{j-1}u_3\right\|^2_{H^{2n-2j+\frac{5}{2}}(\Sigma_-)} + \left\|\partial_t^{j-1}\Phi^5\right\|^2_{2n-2j+\frac{5}{2}} \\
&\lesssim \left\|\partial_t^{j-1}u_3\right\|^2_{2n-2(j-1)+1} + \left\|\partial_t^{j-1}\Phi^5\right\|^2_{2n-2(j-1)+\frac{1}{2}} \\
&\lesssim \mathscr{X}_n + \bar{\mathscr{D}}_n.
\end{aligned} \tag{106}
$$

Combining (104), (105) and (106) together, we get

$$
\|\eta\|^2_{2n-\frac{1}{2}} + \|\partial_t\eta\|^2_{2n-\frac{1}{2}} + \sum_{j=2}^{n+1}\left\|\partial_t^j\eta\right\|^2_{2n-2j+\frac{5}{2}} \lesssim \mathscr{X}_n + \bar{\mathscr{D}}_n. \tag{107}
$$

Applying ∂_t^j $(0 \le j \le n-1)$ to the equation (102), via (100) and (107), we have

$$
\begin{aligned}
\left\|\partial_t^j p\right\|^2_{H^0(\Sigma_-)} &\lesssim \left\|\partial_t^j\eta\right\|^2_0 + \left\|\partial_t^j\phi\right\|^2_{H^0(\Sigma_-)} + \left\|\partial_3\partial_t^j u_3\right\|^2_{H^0(\Sigma_-)} + \left\|\partial_t^j\Phi^7\right\|^2_0 \\
&\lesssim \left\|\partial_t^j\eta\right\|^2_0 + \left\|\partial_t^j\phi\right\|^2_1 + \left\|\partial_t^j u\right\|^2_2 + \left\|\partial_t^j\Phi^7\right\|^2_0 \\
&\lesssim \mathscr{X}_n + \bar{\mathscr{D}}_n.
\end{aligned} \tag{108}
$$

Combining (108), Lemma 7.8 and (100) together, we get

$$
\left\|\partial_t^j p\right\|^2_1 \lesssim \left\|\partial_t^j p\right\|^2_{L^2(\Sigma_-)} + \left\|\partial_3\partial_t^j p\right\|^2_{L^2(\Omega_+)} \lesssim \mathscr{X}_n + \bar{\mathscr{D}}_n.
$$

Then we conclude that

$$
\mathscr{D}_n \lesssim \mathscr{X}_n + \bar{\mathscr{D}}_n.
$$

Then Lemma 3.2 and Lemma 3.3 respectively give the estimate of $\mathscr{X}_{2N}$ to the case $n = 2N$ and $n = N+2$, which completes the proof. □

To gain the a priori estimate, we need to control the L^2 norm of the highest order derivative of the free surface. Next lemma shows the estimate of the transport equation. The proof we omitted here can be found in [17].

Lemma 4.3. *Assume (u, p, ϕ, η) is the solution to equations* (17) *and the assumptions in Theorem 1.1 hold. Then there exists a constant $C > 0$, such that*

$$
\begin{aligned}
\sup_{0\le r\le t}\mathscr{F}_{2N}(r) &\lesssim \exp\left(C\int_0^t\sqrt{\mathscr{K}(r)}dr\right) \\
&\times\left[\mathscr{F}_{2N}(0) + t\int_0^t(1+\mathscr{E}_{2N}(r))\mathscr{D}_{2N}(r)dr + \left(\int_0^t\sqrt{\mathscr{K}(r)\mathscr{F}_{2N}(r)}dr\right)^2\right].
\end{aligned} \tag{109}
$$

Via Lemma 4.3, we immediately get that

Lemma 4.4. *Assume (u, p, ϕ, η) is the solution to equations (17) and the assumptions in Theorem 1.1 and the a-priori assumption (22) hold for some small positive constant δ. Then we have*

$$\sup_{0\le r\le t} \mathscr{F}_{2N}(r) \lesssim \mathscr{F}_{2N}(0) + t\int_0^t \mathscr{D}_{2N} \tag{110}$$

for all $0 \le t \le T$.

Proof. The assumption $\mathscr{G}_{2N}(T) < \delta$ gives

$$\mathscr{K}(r) \lesssim \mathscr{E}_{N+2}(r),$$

and combining Lemma 4.3 together, we complete the proof. □

To close the a priori estimate, we also need

Lemma 4.5. *Assume (u, p, ϕ, η) is the solution to equations (17) and the assumptions in Theorem 1.1 and the a-priori assumption (22) hold for some small positive constant δ. Then we have*

$$\int_0^t \mathscr{K}(r)\mathscr{F}_{2N}(r)dr \lesssim \delta\mathscr{F}_{2N}(0) + \delta\int_0^t \mathscr{D}_{2N}(r)dr \tag{111}$$

and

$$\int_0^t \sqrt{\mathscr{K}(r)\mathscr{F}_{2N}(r)\mathscr{D}_{2N}(r)}dr \lesssim \sqrt{\delta}\mathscr{F}_{2N}(0) + \sqrt{\delta}\int_0^t \mathscr{D}_{2N}(r)dr. \tag{112}$$

Proof. Lemma 4.4 gives that

$$\begin{aligned}
&\int_0^t \mathscr{K}(r)\mathscr{F}_{2N}(r)dr \lesssim \int_0^t \mathscr{K}(r)\left(\mathscr{F}_{2N}(0) + r\int_0^r \mathscr{D}_{2N}(s)ds\right)dr \\
\lesssim&\left(\mathscr{F}_{2N}(0) + \int_0^t \mathscr{D}_{2N}(s)ds\right)\cdot\left[\int_0^t (1+r)\mathscr{K}(r)dr\right] \\
\lesssim&\left(\mathscr{F}_{2N}(0) + \int_0^t \mathscr{D}_{2N}(s)ds\right)\cdot\int_0^t (1+r)^{9-4N}\cdot(1+r)^{4N-8}\cdot\mathscr{E}_{N+2}(r)dr \\
\lesssim&\delta\left(\mathscr{F}_{2N}(0) + \int_0^t \mathscr{D}_{2N}(s)ds\right),
\end{aligned}$$

which gives the inequality (111). (112) is gained directly by Hölder inequality and (111) and then we complete the proof. □

Next, we will give the bound of the higher order energy.

Proposition 4.6. *Assume (u, p, ϕ, η) is the solution to equations (17) and the assumptions in Theorem 1.1 hold. Then there exists a positive small constant δ, such that if the a-priori assumption (22) holds, the following inequality is established*

$$\sup_{0\le r\le T} \mathscr{E}_{2N}(r) + \int_0^T \mathscr{D}_{2N}(r)dr + \sup_{0\le r\le T}\frac{\mathscr{F}_{2N}(r)}{1+r} \lesssim \mathscr{F}_{2N}(0) + \mathscr{E}_{2N}(0). \tag{113}$$

Proof. The a-priori assumption (22) implies $\mathscr{E}_{2N}(t) < \delta$ for $0 \le t \le T$. In view of (82) and (92), it yields

$$\bar{\mathscr{E}}_{2N} \lesssim \mathscr{E}_{2N} \lesssim \bar{\mathscr{E}}_{2N} \tag{114}$$

and

$$\bar{\mathscr{D}}_{2N} \lesssim \mathscr{D}_{2N} \lesssim \bar{\mathscr{D}}_{2N} + \mathscr{K}\mathscr{F}_{2N}. \tag{115}$$

Multiplying (47) by $1+M$ and adding (77) together, it gets

$$\bar{\mathscr{E}}^0_{2N}(t)+\bar{\mathscr{E}}^+_{2N}(t)+\int_0^t \left[\bar{\mathscr{D}}^+_{2N}+(1+M)\bar{\mathscr{D}}^0_{2N}\right] \lesssim \bar{\mathscr{E}}^+_{2N}(0)+(1+M)\mathscr{E}_{2N}(0)+(1+M)\mathscr{E}^{\frac{3}{2}}_{2N}$$
$$+\int_0^t \left[(1+M)\sqrt{\mathscr{E}_{2N}}\mathscr{D}_{2N}+\mathscr{E}^{\theta}_{2N}\mathscr{D}_{2N}+\sqrt{\mathscr{D}_{2N}\mathscr{K}\mathscr{F}_{2N}}+\varepsilon\mathscr{D}_{2N}+C(\varepsilon)\bar{\mathscr{D}}^0_{2N}\right],$$

where the constant M will be choosen later. Combining this ineqality with (114) and (115) together, it yields

$$\mathscr{E}_{2N}(t)+\int_0^t \left[\mathscr{D}_{2N}+M\bar{\mathscr{D}}^0_{2N}\right] \lesssim (2+M)\mathscr{E}_{2N}(0)+(1+M)\mathscr{E}^{\frac{3}{2}}_{2N}$$
$$+\int_0^t \left[(1+M)\mathscr{E}^{\kappa}_{2N}\mathscr{D}_{2N}+\sqrt{\mathscr{D}_{2N}\mathscr{K}\mathscr{F}_{2N}}+\mathscr{K}\mathscr{F}_{2N}+\varepsilon\mathscr{D}_{2N}+C(\varepsilon)\bar{\mathscr{D}}^0_{2N}\right],$$

where $\kappa = min\{\frac{1}{2},\theta\}$. Via (111) and (112), it gets

$$\begin{aligned}
&\mathscr{E}_{2N}(t)+\int_0^t \left[\mathscr{D}_{2N}+M\bar{\mathscr{D}}^0_{2N}\right]\\
\le& C_1(2+M)\mathscr{E}_{2N}(0)+C_1(1+M)\mathscr{E}^{\frac{3}{2}}_{2N}+C_2(1+M)\int_0^t \mathscr{E}^{\kappa}_{2N}\mathscr{D}_{2N}\\
&+C_2\varepsilon\int_0^t \mathscr{D}_{2N}+C_2C(\varepsilon)\int_0^t \bar{\mathscr{D}}^0_{2N}+C_3\sqrt{\delta}\mathscr{F}_{2N}(0)+C_3\sqrt{\delta}\int_0^t \mathscr{D}_{2N}(r)dr,
\end{aligned}$$

where C_1, C_2 and C_3 are the positive constants independent on t, ε, δ and M and $C(\varepsilon)=\varepsilon^{-8N+4j-1}$ for $0\le j\le 2N-1$. Choosing ε small enough such that $C_2\varepsilon<\frac{1}{6}$, M large enough such that $M>\frac{2C_2}{\varepsilon^{8N+1}}$ and δ small enough such that $C_1(1+M)\delta^{\frac{1}{2}}<\frac{1}{2}$, $C_2(1+M)\delta^{\kappa}<\frac{1}{6}$ and $C_3\delta^{\frac{1}{2}}<\frac{1}{6}$, we get

$$\mathscr{E}_{2N}(t)+\int_0^t \mathscr{D}_{2N} \lesssim \mathscr{E}_{2N}(0)+\mathscr{F}_{2N}(0), \tag{116}$$

which controls the former two part of the left of (113). According to (110), it gets

$$\mathscr{F}_{2N}(r)\lesssim \mathscr{F}_{2N}(0)+r\int_0^r \mathscr{D}_{2N}$$

for all $0\le r\le T$. And then

$$\frac{\mathscr{F}_{2N}(r)}{1+r}\lesssim \frac{\mathscr{F}_{2N}(0)}{1+r}+\frac{r}{1+r}\int_0^r \mathscr{D}_{2N}\lesssim \mathscr{F}_{2N}(0)+\mathscr{E}_{2N}(0),$$

which gives the proof of this proposition. □

To gain the decay of the $N+2$-th level energy, the following interpolation inequality is needed.

Lemma 4.7. *There exists a constant $\delta\in(0,1)$, such that if $\mathscr{G}_{2N}(T)<\delta$, then the following inequality holds*

$$\mathscr{E}_{N+2}\lesssim (\mathscr{D}_{N+2})^{\frac{4N-8}{4N-7}}(\mathscr{E}_{2N})^{\frac{1}{4N-7}}.$$

Proof. This lemma is proved by the definitions of energy and dissipation and the standard interpolation theory, so the proof is omitted here. □

Then the decay of the $N+2$-th level energy is showed as follows.

Proposition 4.8. *Assume (u,p,ϕ,η) is the solution to equations (17) and the assumptions in Theorem 1.1 hold. Then there exists a positive constant δ, such that if the a-priori assumption (22) holds, we get*

$$\sup_{0\le r\le t}(1+r)^{4N-8}\mathscr{E}_{N+2}\lesssim \mathscr{F}_{2N}(0)+\mathscr{E}_{2N}(0). \tag{117}$$

Proof. Via (83) and (93), it gets

$$\bar{\mathscr{E}}_{N+2}\lesssim \mathscr{E}_{N+2}\lesssim \bar{\mathscr{E}}_{N+2} \tag{118}$$

and

$$\bar{\mathscr{D}}_{N+2}\lesssim \mathscr{D}_{N+2}\lesssim \bar{\mathscr{D}}_{N+2}. \tag{119}$$

Multiplying (58) by $1+M$ and adding the derived results with (80) together, it yields

(120)
$$\frac{d}{dt}\left(\bar{\mathscr{E}}^{+}_{N+2}+\bar{\mathscr{E}}^{0}_{N+2}+M\bar{\mathscr{E}}^{0}_{N+2}-2(1+M)\int_\Omega J\partial_t^{N+1}pF^2\right)+\bar{\mathscr{D}}_{N+2}+M\bar{\mathscr{D}}^{0}_{N+2}$$
$$\lesssim \mathscr{E}^{\theta}_{2N}\mathscr{D}_{N+2}+\epsilon\mathscr{D}_{N+2}+C(\epsilon)\bar{\mathscr{D}}^{0}_{N+2}+(1+M)\sqrt{\mathscr{E}_{2N}}\mathscr{D}_{N+2},$$

where the positive constant M is determined later and $C(\epsilon)=\epsilon^{-2N-4}$. Combining (120) and (119) together, we get

$$\frac{d}{dt}\left(\bar{\mathscr{E}}^{+}_{N+2}+\bar{\mathscr{E}}^{0}_{N+2}+M\bar{\mathscr{E}}^{0}_{N+2}-2(1+M)\int_\Omega J\partial_t^{N+1}pF^2\right)+\frac{1}{C_1}\mathscr{D}_{N+2}+M\bar{\mathscr{D}}^{0}_{N+2}$$
$$\le C_2\big(\mathscr{E}^{\theta}_{2N}\mathscr{D}_{N+2}+\epsilon\mathscr{D}_{N+2}+C(\epsilon)\bar{\mathscr{D}}^{0}_{N+2}+(1+M)\sqrt{\mathscr{E}_{2N}}\mathscr{D}_{N+2}\big),$$

where C_1 and C_2 are the universal constants. Then we take ϵ small enough such that $C_2\epsilon<\frac{1}{4C_1}$, $M=C_2C(\epsilon)$ and δ small enough such that $C_2\delta^\kappa<\frac{1}{8C_1}$ and $C_2(1+M)\delta^\kappa<\frac{1}{8C_1}$ for $\kappa=\min\{\theta,\frac12\}$, yielding

$$\frac{d}{dt}\left(\bar{\mathscr{E}}^{+}_{N+2}+\bar{\mathscr{E}}^{0}_{N+2}+M\bar{\mathscr{E}}^{0}_{N+2}-2(1+M)\int_\Omega J\partial_t^{N+1}pF^2\right)+\frac{1}{2C_1}\mathscr{D}_{N+2}\le 0.$$

Note that

$$\begin{aligned}\left|2(1+M)\int_\Omega J\partial_t^{N+1}pF^2\right| &\le 2(1+M)\left\|J\right\|_{L^\infty}\left\|\partial_t^{N+1}p\right\|_0\left\|F^2\right\|_0\\ &\le C_4(1+M)\mathscr{E}_{2N}^{\frac{\theta}{2}}\bar{\mathscr{E}}_{N+2}\\ &\le \frac34\bar{\mathscr{E}}_{N+2},\end{aligned}$$

where we take δ small enough such that $C_4(1+M)\delta^{\frac{\theta}{2}}\le\frac34$. Denote

$$m(t)=\bar{\mathscr{E}}_{N+2}+M\bar{\mathscr{E}}^{0}_{N+2}-2(1+M)\int_\Omega J\partial_t^{N+1}pF^2,$$

and then we have

$$0\le\frac14\bar{\mathscr{E}}_{N+2}+M\bar{\mathscr{E}}^{0}_{N+2}\le m(t)\le\frac74\bar{\mathscr{E}}_{N+2}+M\bar{\mathscr{E}}^{0}_{N+2}\le C_5\mathscr{E}_{N+2} \tag{121}$$

for some universal constant $C_5>0$. Via Lemma 4.7, it yields

$$C_5\mathscr{E}_{N+2}\le C_6(\mathscr{D}_{N+2})^{\frac{4N-8}{4N-7}}(\mathscr{E}_{2N})^{\frac{1}{4N-7}}$$

and then according to (113), we get

$$0\le m(t)\le C_6(\mathscr{D}_{N+2})^{\frac{4N-8}{4N-7}}(\mathscr{E}_{2N})^{\frac{1}{4N-7}}\le C_7(\mathscr{D}_{N+2})^{\frac{4N-8}{4N-7}}W_0^{\frac{1}{4N-7}},$$

where $C_7 \geq C_5 \geq 1$ and $W_0 = \mathscr{F}_{2N}(0) + \mathscr{E}_{2N}(0)$. So we immediately obtain

$$\frac{d}{dt}m(t) + C_8 m^{1+q}(t) \leq 0, \tag{122}$$

where $q = \frac{1}{4N-8}$ and $C_8 = \frac{1}{2C_1 C_7^{\frac{4N-7}{4N-8}} W_0^q}$. Solving (122), we get

$$m(t) \leq \frac{1}{\left(qC_8 t + \frac{1}{m^q(0)}\right)^{\frac{1}{q}}}.$$

Denote $f(\lambda) = \frac{1+\lambda}{M\lambda+A}$, which is the monotonically increasing function with respect to λ in $\lambda \geq 0$. So it yields $f(\lambda) \leq \frac{1}{M}$. (121) gives

$$m(0) \leq C_5 \mathscr{E}_{N+2}(0) \leq C_5 \mathscr{E}_{2N}(0) \leq C_5 W_0.$$

Since $C_1 \geq 1$ and $C_7 \geq C_5 \geq 1$, we get

$$qC_8 m^q(0) = \frac{q}{2C_1 C_7^{1+q}}\left(\frac{m(0)}{W_0}\right)^q \leq \frac{q}{2C_1 C_7}\left(\frac{C_5}{C_7}\right)^q \leq 1,$$

which yields

$$(1+t)^{\frac{1}{q}} m(t) \leq \frac{(1+t)^{\frac{1}{q}}}{\left(qC_8 t + \frac{1}{m^q(0)}\right)^{\frac{1}{q}}} \leq \frac{1}{(qC_8)^{\frac{1}{q}}} = \left(\frac{2C_1 C_7^{1+q}}{q}\right)^{\frac{1}{q}} W_0 \tag{123}$$

for all $t \geq 0$. Combining (118), (121) and (123) together, we get

$$(1+t)^{\frac{1}{q}} \mathscr{E}_{N+2}(t) \lesssim (1+t)^{\frac{1}{q}} \bar{\mathscr{E}}_{N+2}(t) \lesssim (1+t)^{\frac{1}{q}} m(t) \lesssim \mathscr{F}_{2N}(0) + \mathscr{E}_{2N}(0),$$

which gives the proof of Proposition 4.8. □

Adding (113) and (117) together, we conclude the a priori estimate as follows.

Proposition 4.9. *Assume (u, p, ϕ, η) is the solution to equations (17) and the assumptions in Theorem 1.1 hold. Then there exists a positive constant δ, such that if the a-priori assumption (22) holds, we get*

$$\mathscr{G}_{2N}(t) \leq C_0(\mathscr{F}_{2N}(0) + \mathscr{E}_{2N}(0)) \qquad \text{for all } 0 \leq t \leq T,$$

where C_0 is a positive constant independent of t.

5. Proof of Theorem 1.1.

In this section, we will prove Theorem 1.1 by the a priori estimate Proposition 4.9 and the local existence results Theorem 7.1 in the appendix. Firstly, we will rewrite the local existence result Theorem 7.1 as another expression which can be directly used to prove Theorem 1.1.

Proposition 5.1. *Suppose the initial data $(u_0, \phi_0, \eta_0) \in H^{4N} \times H^{4N} \times H^{4N+\frac{1}{2}}$ and satisfy the compatible conditions (18). Then for $\forall \epsilon > 0$ small enough, there exist constants $\delta_0 = \delta_0(\epsilon) > 0$ and $T_0 = C(\epsilon)\min\{1, \frac{1}{\|\eta_0\|^2_{4N+\frac{1}{2}}}\} > 0$ where $C(\epsilon)$ is a positive constant only depending on ϵ, such that if $0 < T \leq T_0$, $\|u_0\|^2_{4N} + \|\phi_0\|^2_{4N} + \|\eta_0\|^2_{4N} \leq \delta_0$, the equations (17) have a unique solution (u, ϕ, p, η) on the interval $[0, T]$, satisfying*

$$\sup_{0 \leq r \leq T} \mathscr{E}_{2N}(r) + \int_0^T \mathscr{D}_{2N}(r)dr + \left\|\partial_t^{2N+1} u\right\|^2_{L^2_T(\mathscr{H}^1_{(1)})^*} + \left\|\partial_t^{2N+1}\phi\right\|^2_{L^2_T(\mathscr{H}^1_{(2)})^*} \leq \epsilon \tag{124}$$

and

$$\sup_{0\le r\le T}\mathscr{F}_{2N}(r)\le C_9\mathscr{F}_{2N}(0)+\epsilon. \tag{125}$$

Here C_9 is a universal constant.

Proof. Via Theorem 7.1, choosing δ_0 small enough such that $C(\|u_0\|_{4N}^2+\|\phi_0\|_{4N}^2+\|\eta_0\|_{4N}^2+T\|\eta_0\|_{4N+\frac{1}{2}}^2)<\epsilon$, we immediately get the desired result. □

To show Theorem 1.1, we also need

Lemma 5.2. *Assume (u,p,ϕ,η) is the solution to equations (17) and the assumptions in Theorem 1.1 hold. Let $N\ge 3$ and suppose $0<T_1<T_2$. Then we have the following estimate*

$$\begin{aligned}\mathscr{G}_{2N}(T_2)\le{}&\mathscr{G}_{2N}(T_1)+\sup_{T_1\le r\le T_2}\mathscr{E}_{2N}(r)+\int_{T_1}^{T_2}\mathscr{D}_{2N}(r)dr\\&+\frac{1}{1+T_1}\sup_{T_1\le r\le T_2}\mathscr{F}_{2N}(r)+C_{10}(T_2-T_1)^2(1+T_2)^{4N-8}\sup_{T_1\le r\le T_2}\mathscr{E}_{2N}(r),\end{aligned}\tag{126}$$

where C_{10} is a positive constant.

Proof. By the definition of $\mathscr{G}_{2N}(t)$, it is easy to find

$$\begin{aligned}\mathscr{G}_{2N}(T_2)\le{}&\mathscr{G}_{2N}(T_1)+\sup_{T_1\le r\le T_2}\mathscr{E}_{2N}(r)+\int_{T_1}^{T_2}\mathscr{D}_{2N}(r)dr\\&+\frac{1}{1+T_1}\sup_{T_1\le r\le T_2}\mathscr{F}_{2N}(r)+\sup_{T_1\le r\le T_2}\big((1+r)^{4N-8}\mathscr{E}_{N+2}(r)\big).\end{aligned}$$

Since $N\ge 3$, we get that

$$\begin{aligned}&\sum_{0\le j\le N+2}\Bigg(\sup_{T_1\le r\le T_2}\left\|\partial_t^{j+1}u(r)\right\|_{2(N+2)-2j}^2+\sup_{T_1\le r\le T_2}\left\|\partial_t^{j+1}\phi(r)\right\|_{2(N+2)-2j}^2\\&\qquad+\sup_{T_1\le r\le T_2}\left\|\partial_t^{j+1}\eta(r)\right\|_{2(N+2)-2j}^2\Bigg)\\&+\sum_{0\le j\le N+2}\Bigg(\sup_{T_1\le r\le T_2}\left\|\partial_t^{j}u(r)\right\|_{2(N+2)-2j}^2+\sup_{T_1\le r\le T_2}\left\|\partial_t^{j}\phi(r)\right\|_{2(N+2)-2j}^2\\&\qquad+\sup_{T_1\le r\le T_2}\left\|\partial_t^{j}\eta(r)\right\|_{2(N+2)-2j}^2\Bigg)\\&\lesssim\sup_{T_1\le r\le T_2}\mathscr{E}_{2N}(r)\end{aligned}\tag{127}$$

and

$$\begin{aligned}&\sum_{0\le j\le N+1}\sup_{T_1\le t\le T_2}\left\|\partial_t^{j+1}p\right\|_{2(N+2)-2j-1}^2+\sum_{0\le j\le N+1}\sup_{T_1\le t\le T_2}\left\|\partial_t^{j}p\right\|_{2(N+2)-2j-1}^2\\&\lesssim\sup_{T_1\le t\le T_2}\mathscr{E}_{2N}(t).\end{aligned}$$

Since

$$\begin{aligned}(1+t)^{\frac{4N-8}{2}}\partial_t^j\phi(t)={}&(1+T_1)^{\frac{4N-8}{2}}\partial_t^j\phi(T_1)+\int_{T_1}^t\frac{4N-8}{2}(1+s)^{\frac{4N-10}{2}}\partial_t^j\phi(s)ds\\&+\int_{T_1}^t(1+s)^{\frac{4N-8}{2}}\partial_t^{j+1}\phi(s)ds\end{aligned}$$

for any $0 \le j \le N+2$ and $t \in [T_1, T_2]$, we get that

$$\begin{aligned}
&\left\|(1+t)^{\frac{4N-8}{2}}\partial_t^j\phi(t)\right\|_{2N+4-2j}\\
\lesssim&\left\|(1+T_1)^{\frac{4N-8}{2}}\partial_t^j\phi(T_1)\right\|_{2N+4-2j}+\int_{T_1}^{T_2}(1+s)^{\frac{4N-8}{2}}\left\|\partial_t^{j+1}\phi(s)\right\|_{2N+4-2j}ds\\
&+\int_{T_1}^{T_2}\frac{4N-8}{2}(1+s)^{\frac{4N-10}{2}}\left\|\partial_t^j\phi(s)\right\|_{2N+4-2j}ds\\
\lesssim&\sqrt{\mathscr{G}_{2N}(T_1)}+(T_2-T_1)(1+T_2)^{\frac{4N-8}{2}}\sup_{T_1\le r\le T_2}\sqrt{\mathscr{E}_{2N}(r)},
\end{aligned}$$

which yields

$$\begin{aligned}
&\sum_{0\le j\le N+2}\sup_{T_1\le t\le T_2}(1+t)^{4N-8}\left\|\partial_t^j\phi(t)\right\|_{2N+4-2j}^2\\
\lesssim&\mathscr{G}_{2N}(T_1)+(T_2-T_1)^2(1+T_2)^{4N-8}\sup_{T_1\le r\le T_2}\mathscr{E}_{2N}(r).
\end{aligned} \tag{128}$$

Similarly, we can get the estimate of u, η and p as follows

$$\begin{aligned}
&\sum_{0\le j\le N+2}\left[\sup_{T_1\le t\le T_2}(1+t)^{4N-8}\left\|\partial_t^j u(t)\right\|_{2N+4-2j}^2\right.\\
&\qquad\left.+\sup_{T_1\le t\le T_2}(1+t)^{4N-8}\left\|\partial_t^j\eta(t)\right\|_{2N+4-2j}^2\right]\\
&+\sum_{0\le j\le N+1}\sup_{T_1\le t\le T_2}(1+t)^{4N-8}\left\|\partial_t^j p(t)\right\|_{2N+3-2j}^2\\
\lesssim&\mathscr{G}_{2N}(T_1)+(T_2-T_1)^2(1+T_2)^{4N-8}\sup_{T_1\le r\le T_2}\mathscr{E}_{2N}(r).
\end{aligned} \tag{129}$$

Adding (128) and (129) together, we get

$$\sup_{T_1\le t\le T_2}(1+t)^{4N-8}\mathscr{E}_{N+2}(t)\lesssim\mathscr{G}_{2N}(T_1)+(T_2-T_1)^2(1+T_2)^{4N-8}\sup_{T_1\le r\le T_2}\mathscr{E}_{2N}(r),$$

and then combining (127) together, we immediately obtain the desired estimate (126). □

Then via Lemma 5.2, we immediately obtain

Corollary 5.2.1. *If we let $T_1 = 0$ and $T_2 = T$, in light of the definition of $\mathscr{G}_{2N}(T)$, we get the folllowing inequality*

$$\begin{aligned}
\mathscr{G}_{2N}(T)\le&\sup_{0\le r\le T}\mathscr{E}_{2N}(r)+\int_0^T\mathscr{D}_{2N}(r)dr+\sup_{0\le r\le T}\mathscr{F}_{2N}(r)\\
&+C_{10}(1+T)^{4N-6}\sup_{0\le r\le T}\mathscr{E}_{2N}(r).
\end{aligned} \tag{130}$$

With these estimates at hand, we give the proof of the main result Theorem 1.1 as follows.

Proof of Theorem 1.1. In view of (130), the solution, existing on the interval $[0,T]$ with $T<1$, obeys the estimates (124) and (125) in Proposition 5.1, so we get

$$\mathscr{G}_{2N}(T)\le C_9\kappa+\epsilon(2+C_{10}2^{4N-8}),$$

where the constant C_9 and C_{10} comes from (125) and (130) respectively. Choosing ϵ satisfying $\epsilon(2+C_{10}2^{4N-8})=\frac{\nu}{2}$ and κ small enough such that $C_9\kappa<\frac{\nu}{2}$ and

$\kappa < \delta_0(\epsilon)$ for $0 < \nu < 1$ and $\delta_0(\epsilon)$ coming from Proposition 5.1, we get that there exists a unique solution on the interval $[0, T]$, satisfying (124) and (125) which yields $\mathscr{G}_{2N}(T) < \nu$.

Because of the initial data satisfying the compatibility conditions and

$$\mathscr{E}_{2N}(0) + \mathscr{F}_{2N}(0) < \kappa,$$

we define that

$$T^*(\kappa) = \sup\{T > 0 | \text{ the solution of (17) exists on the interval } [0, T], \text{ satisfying } \mathscr{G}_{2N}(T) < \nu\}.$$

We know that $T^*(\kappa) > 0$ if κ is small enough, i.e. there exists a constant $\kappa_1 > 0$ such that $T^* : (0, \kappa_1] \to (0, \infty]$. It is easy to get that $T^*(\kappa)$ is non-increasing on $(0, \kappa_1]$. Denote

$$\kappa_0 = \min\{\kappa_1, \frac{\delta(\epsilon)}{C_0}, \frac{\nu}{3C_0(1 + C_9)}\}. \tag{131}$$

We claim that

$$T^*(\kappa_0) = \infty.$$

Once the claim is established, we complete the whole proof by choosing $\kappa \in (0, \kappa_0)$.

Now we prove the claim by the contradiction. If the claim doesn't hold, it gets that $T^*(\kappa_0) < \infty$ to the constant κ_0. The definition of $T^*(\kappa_0)$ gives that if the initial data satisfy the compatability conditions and the bound $\mathscr{E}_{2N}(0)+\mathscr{F}_{2N}(0) < \kappa_0$, the solution of (17) exists on $[0, T_1]$ for every $0 < T_1 < T^*(\kappa_0)$, satisfying $\mathscr{G}_{2N}(T_1) < \nu$. Then Proposition 4.9 implies

$$\mathscr{G}_{2N}(t) \le C_0(\mathscr{F}_{2N}(0) + \mathscr{E}_{2N}(0)) \le C_0\kappa_0 \quad \text{for} \quad 0 < t \le T_1. \tag{132}$$

Together with (131), it yields that

$$\mathscr{E}_{2N}(T_1) + \frac{\mathscr{F}_{2N}(T_1)}{1 + T_1} \le C_0\kappa_0 \le \delta(\epsilon) \quad \text{for all} \quad 0 < T_1 < T^*(\kappa_0). \tag{133}$$

Since $\mathscr{E}_{2N}(T_1) \le \delta(\epsilon)$, by Proposition 5.1, we can view $u(T_1)$, $\phi(T_1)$, $p(T_1)$ and $\eta(T_1)$ as the initial data which satisfy the compatability. Proposition 5.1 gives that the solution can be extended to $[0, T_2]$ for some positive constant $T_2 > T_1$ satisfying

$$0 < T_2 - T_1 \le T_0 := C(\epsilon) \min\left\{1, \frac{1}{\|\eta(T_1)\|^2_{4N+\frac{1}{2}}}\right\}.$$

Let

$$\epsilon = \frac{\nu}{3} \min\left\{\frac{1}{2}, \frac{1}{C_{10}}\right\} \tag{134}$$

and denote

$$\hat{T} = C(\epsilon) \min\left\{1, \frac{1}{\delta(\epsilon)(1 + T^*(\kappa_0))}\right\}. \tag{135}$$

According to (133), it is easy to get $\hat{T} \le T_0$. Denote

$$\gamma = \min\left\{\hat{T}, T^*(\kappa_0), \frac{1}{(1 + 2T^*(\kappa_0))^{\frac{4N-8}{2}}}\right\}.$$

Choosing $T_1 = T^*(\kappa_0) - \frac{\gamma}{2}$, by the above analysis, we can extend the solution to $[0, T_2]$ for $T_2 = T^*(\kappa_0) + \frac{\gamma}{2}$ and Proposition 5.1 gives the estimate

$$\sup_{T_1 \le r \le T_2} \mathscr{E}_{2N}(r) + \int_{T_1}^{T_2} \mathscr{D}_{2N}(r)dr \le \epsilon \quad \text{and} \quad \sup_{T_1 \le r \le T_2} \mathscr{F}_{2N}(r) \le C_9\mathscr{F}_{2N}(T_1) + \epsilon,$$

which combines (132), (133) and(126) together, yields

$$\begin{aligned}\mathscr{G}_{2N}(T_2) \leq & \mathscr{G}_{2N}(T_1) + \sup_{T_1\leq r\leq T_2} \mathscr{E}_{2N}(r) + \int_{T_1}^{T_2} \mathscr{D}_{2N}(r)dr + \frac{1}{1+T_1} \sup_{T_1\leq r\leq T_2} \mathscr{F}_{2N}(r) \\ & + C_{10}(T_2-T_1)^2(1+T_2)^{4N-8} \sup_{T_1\leq r\leq T_2} \mathscr{E}_{2N}(r) \\ \leq & C_0\kappa_0 + \epsilon + \frac{C_9C_0\kappa_0(1+T_1)+\epsilon}{1+T_1} + \epsilon C_{10}\gamma^2(1+2T^*(\kappa_0))^{4N-8} \\ \leq & C_0\kappa_0(1+C_9) + 2\epsilon + \epsilon C_{10}\gamma^2(1+2T^*(\kappa_0))^{4N-8} \\ \leq & \frac{\nu}{3} + \frac{\nu}{3} + \frac{\nu}{3} = \nu,\end{aligned}$$

which contradicts with the definition of $T^*(\kappa_0)$. Then we obtain the claim and complete the proof. □

6. Numerical Simulations.

In this section, an efficient explicit discrete scheme is established based on finite-volume method for the free interface system in two dimension. Our numerical simulations demonstrate similar conclusions with the main results of Theorem 1.1.

6.1. Discrete schemes. To begin with, we give here our notations for the discretization. To this end, we discretize the spatial domain by placing a grid over the domain $\Omega(t)$ with the uniform small grid size $\Delta x = \Delta y = \frac{1}{N}$ (N is a positive integer). The mesh is centered at $x_i = i\Delta x$, $y_j = j\Delta y$ with endpoints $x_{i+\frac{1}{2}} = \left(i+\frac{1}{2}\right)\Delta x$ and $y_{j+\frac{1}{2}} = \left(j+\frac{1}{2}\right)\Delta y$ for $i,j = 1,\cdots,N$, which is finally divided into the cells

$$C_{i,j} := \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right] \times \left[y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}\right], \qquad i,j = 1,\cdots,N.$$

For the time discretization, we set $t^n = n\Delta t$ for $n \in \mathbb{N}$, where we assume that Δt is a small temporal scale. The discrete approximation of the velocity $\widetilde{\mathbf{u}} = (u,v)$ in the finite volume sense is denoted by

$$\begin{aligned}\bar{u}_{i,j}^n &\approx \frac{1}{\Delta x\Delta y}\iint_{C_{i,j}} u(x,y,t^n)dxdy, \qquad i,j=1,\cdots,N, \quad n\in\mathbb{N},\\ \bar{v}_{i,j}^n &\approx \frac{1}{\Delta x\Delta y}\iint_{C_{i,j}} v(x,y,t^n)dxdy, \qquad i,j=1,\cdots,N, \quad n\in\mathbb{N},\\ \bar{\eta}_i^n &\approx \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \eta(x,t^n)dx, \qquad i=1,\cdots,N, \quad n\in\mathbb{N}.\end{aligned}$$

As for the discretization of spacial domain, Ω^n consist of two moving regions Ω_-^n and Ω_+^n defined as

$$\Omega_-^n = \left\{\left(x_i, y_j, t^n\right)|0\leq x_i\leq L, -b\leq y_j\leq \eta_i^n\right\}$$
$$\Omega_+^n = \left\{\left(x_i, y_j, t^n\right)|0\leq x_i\leq L, \eta_i^n\leq y_j\leq 1\right\}$$

respectively. Ω_-^n and Ω_+^n is separated by the free interface Σ_-^n given by

$$\Sigma_-^n = \left\{(x_i, y_j, t^n)|y_j = \eta_i^n\right\}$$

where L, b are positive constant and the discretized interface at each discrete time is presented as $\eta_i^n = \eta(x_i, t^n)$, $i = 1,\cdots,N, n\in\mathbb{N}$. We consider the left side point values at the interface. Let

$$\Sigma_+ = \left\{\left(x_i, y_{N+\frac{1}{2}}\right)|y_{N+\frac{1}{2}} = 1\right\} \quad and \quad \Sigma_{-b} = \left\{\left(x_i, y_{\frac{1}{2}}\right)|y_{\frac{1}{2}} = -b\right\}$$

denote the given upper and lower boundary of Ω^n. Therefore, an explicit discrete scheme for the incompressible Navier-Stokes equations (2) and the porous medium model (3) with $\gamma = 1$ can be written into the following form as

$$\begin{cases} \rho_\omega \bar{u}_{i,j}^{n+1} = \rho_\omega \bar{u}_{i,j}^n - \frac{\rho_\omega \Delta t}{\Delta x}\left(\bar{u}_{i,j}^n\left(\bar{u}_{i+\frac{1}{2},j}^n - \bar{u}_{i-\frac{1}{2},j}^n\right)\right) & \\ \quad -\frac{\rho_\omega \Delta t}{\Delta y}\left(\bar{v}_{i,j}^n\left(\bar{u}_{i,j+\frac{1}{2}}^n - \bar{u}_{i,j-\frac{1}{2}}^n\right)\right) - \frac{\Delta t}{\Delta x}\left(p_{i+1,j}^n - p_{i,j}^n\right) & \\ \quad +\frac{\mu \Delta t}{(\Delta x)^2}\left(\bar{u}_{i+1,j}^n - 2\bar{u}_{i,j}^n + \bar{u}_{i-1,j}^n\right) + \frac{\mu \Delta t}{(\Delta y)^2}\left(\bar{u}_{i,j+1}^n - 2\bar{u}_{i,j}^n + \bar{u}_{i,j-1}^n\right) & \text{in } \Omega_+^n, \\ \rho_\omega \bar{v}_{i,j}^{n+1} = \rho_\omega \bar{v}_{i,j}^n - \frac{\rho_\omega \Delta t}{\Delta x}\left(\bar{u}_{i,j}^n\left(\bar{v}_{i+\frac{1}{2},j}^n - \bar{v}_{i-\frac{1}{2},j}^n\right)\right) & \\ \quad -\frac{\rho_\omega \Delta t}{\Delta y}\left(\bar{v}_{i,j}^n\left(\bar{v}_{i,j+\frac{1}{2}}^n - \bar{v}_{i,j-\frac{1}{2}}^n\right)\right) - \frac{\Delta t}{\Delta y}\left(p_{i,j+1}^n - p_{i,j}^n\right) & \\ \quad +\frac{\mu \Delta t}{(\Delta x)^2}\left(\bar{v}_{i+1,j}^n - 2\bar{v}_{i,j}^n + \bar{v}_{i-1,j}^n\right) + \frac{\mu \Delta t}{(\Delta y)^2}\left(\bar{v}_{i,j+1}^n - 2\bar{v}_{i,j}^n + \bar{v}_{i,j-1}^n\right) - \rho_\omega g & \text{in } \Omega_+^n, \\ \frac{1}{\Delta x}\left(\bar{u}_{i+\frac{1}{2},j}^n - \bar{u}_{i-\frac{1}{2},j}^n\right) + \frac{1}{\Delta y}\left(\bar{v}_{i,j+\frac{1}{2}}^n - \bar{v}_{i,j-\frac{1}{2}}^n\right) = 0 & \text{in } \Omega_+^n, \\ (\phi_{i,j}^{n+1})^\gamma = (\phi_{i,j}^n)^\gamma + \frac{\Delta t}{(\Delta x)^2}\left((\phi_{i+1,j}^n)^\gamma - 2(\phi_{i,j}^n)^\gamma + (\phi_{i-1,j}^n)^\gamma\right) & \\ \quad +\frac{\Delta t}{(\Delta y)^2}\left((\phi_{i,j+i}^n)^\gamma - 2(\phi_{i,j}^n)^\gamma + (\phi_{i,j-1}^n)^\gamma\right) & \text{in } \Omega_-^n. \end{cases}$$

where $\phi_{i,j}^n = \tilde{\phi}(x_i, y_j, t^n)$ is the discrete approximation of $\widetilde{\phi}$ and $p_{i,j}^n = \tilde{p}(x_i, y_j, t^n)$ is the discrete approximation of $\widetilde{p}$. The values $\bar{u}_{i+\frac{1}{2},j}^n - \bar{u}_{i-\frac{1}{2},j}^n$, $\bar{u}_{i,j+\frac{1}{2}}^n - \bar{u}_{i,j-\frac{1}{2}}^n$, $\bar{v}_{i+\frac{1}{2},j}^n - \bar{v}_{i-\frac{1}{2},j}^n$ and $\bar{v}_{i,j+\frac{1}{2}}^n - \bar{v}_{i,j-\frac{1}{2}}^n$ are computed in an up-wind manner:

$$\bar{u}_{i+\frac{1}{2},j}^n - \bar{u}_{i-\frac{1}{2},j}^n = \begin{cases} \bar{u}_{i,j}^n - \bar{u}_{i-1,j}^n, & if \quad \bar{u}_{i,j}^n > 0, \\ \bar{u}_{i+1,j}^n - \bar{u}_{i,j}^n, & otherwise. \end{cases};$$

$$\bar{u}_{i,j+\frac{1}{2}}^n - \bar{u}_{i,j-\frac{1}{2}}^n = \begin{cases} \bar{u}_{i,j}^n - \bar{u}_{i,j-1}^n, & if \quad \bar{v}_{i,j}^n > 0, \\ \bar{u}_{i,j+1}^n - \bar{u}_{i,j}^n, & otherwise. \end{cases};$$

and

$$\bar{v}_{i+\frac{1}{2},j}^n - \bar{v}_{i-\frac{1}{2},j}^n = \begin{cases} \bar{v}_{i,j}^n - \bar{v}_{i-1,j}^n, & if \quad \bar{u}_{i,j}^n > 0, \\ \bar{v}_{i+1,j}^n - \bar{v}_{i,j}^n, & otherwise. \end{cases};$$

$$\bar{v}_{i,j+\frac{1}{2}}^n - \bar{v}_{i,j-\frac{1}{2}}^n = \begin{cases} \bar{v}_{i,j}^n - \bar{v}_{i,j-1}^n, & if \quad \bar{v}_{i,j}^n > 0, \\ \bar{v}_{i,j+1}^n - \bar{v}_{i,j}^n, & otherwise. \end{cases}.$$

Here we use the standard five-points stencil to obtain a second-order approximate Laplace operator

$$\Delta_{i,j} c = \frac{c_{i+1,j} - 2c_{i,j} + c_{i-1,j}}{(\Delta x)^2} + \frac{c_{i,j+1} - 2c_{i,j} + c_{i,j-1}}{(\Delta y)^2}$$

The kinematic boundary condition (7) and the Beavers-Joesph-Saffman's interface condition (6) with $\gamma = 1$, which is vital in our schemes to assign value to $\widetilde{u}$ and $\widetilde{\phi}$

at each time step size on $\Sigma_-(t)$, take the following form

$$\begin{cases} \bar{\eta}_i^{n+1} = \bar{\eta}_i^n + \Delta t \bar{v}_{i,j}^n - \frac{\Delta t}{\Delta x}\bar{u}_{i,j}^n \left(\bar{\eta}_{i+\frac{1}{2}}^n - \bar{\eta}_{i-\frac{1}{2}}^n\right) & \text{on } \Sigma_-^n, \\ \frac{1}{\Delta x}\left(\bar{\eta}_{i+\frac{1}{2}}^n - \bar{\eta}_{i-\frac{1}{2}}^n\right)\bar{u}_{i,j}^n - \bar{v}_{i,j}^n & \\ \quad = \frac{\gamma}{\gamma-1}\{-\frac{1}{(\Delta x)^2}\left((\phi_{i+1,j}^n)^{\gamma-1} - (\phi_{i,j}^n)^{\gamma-1}\right)\left(\bar{\eta}_{i+\frac{1}{2}}^n - \bar{\eta}_{i-\frac{1}{2}}^n\right) & \\ \qquad + \frac{1}{\Delta y}\left((\phi_{i,j}^n)^{\gamma-1} - (\phi_{i,j-1}^n)^{\gamma-1}\right)\} & \text{on } \Sigma_-^n, \\ \frac{1}{\Delta x}\left(\bar{\eta}_{i+\frac{1}{2}}^n - \bar{\eta}_{i-\frac{1}{2}}^n\right)\left(p_{i,j}^n - \frac{2}{\Delta x}\left(\bar{u}_{i+\frac{1}{2},j}^n - \bar{u}_{i-\frac{1}{2},j}^n\right)\right) & \\ \qquad + \frac{1}{\Delta y}\left(\bar{u}_{i,j+\frac{1}{2}}^n - \bar{u}_{i,j-\frac{1}{2}}^n\right) + \frac{1}{\Delta x}\left(\bar{v}_{i+\frac{1}{2},j}^n - \bar{v}_{i-\frac{1}{2},j}^n\right) & \\ \quad = \frac{\gamma}{\gamma-1}\frac{(\phi_{i,j}^n)^{\gamma-1}}{\Delta x}\left(\bar{\eta}_{i+\frac{1}{2}}^n - \bar{\eta}_{i-\frac{1}{2}}^n\right) - \frac{\rho_s g \eta_i^n}{\Delta x}\left(\bar{\eta}_{i+\frac{1}{2}}^n - \bar{\eta}_{i-\frac{1}{2}}^n\right) & \text{on } \Sigma_-^n, \\ \frac{1}{\Delta x}\left(\bar{\eta}_{i+\frac{1}{2}}^n - \bar{\eta}_{i-\frac{1}{2}}^n\right)\left(\frac{1}{\Delta y}\left(\bar{u}_{i,j+\frac{1}{2}}^n - \bar{u}_{i,j-\frac{1}{2}}^n\right) + \frac{1}{\Delta x}\left(\bar{v}_{i+\frac{1}{2},j}^n - \bar{v}_{i-\frac{1}{2},j}^n\right)\right) & \\ \qquad + p_{i,j}^n - \frac{2}{\Delta y}\left(\bar{v}_{i,j+\frac{1}{2}}^n - \bar{v}_{i,j-\frac{1}{2}}^n\right) = \frac{\gamma}{\gamma-1}(\phi_{i,j}^n)^{\gamma-1} - \rho_s g \eta_i^n & \text{on } \Sigma_-^n. \end{cases}$$

where the value $\bar{\eta}_{i+\frac{1}{2}}^n - \bar{\eta}_{i-\frac{1}{2}}^n$ is given similarly as above in an up-wind manner. The boundary condition on Σ_+ and Σ_{-b} satisfies

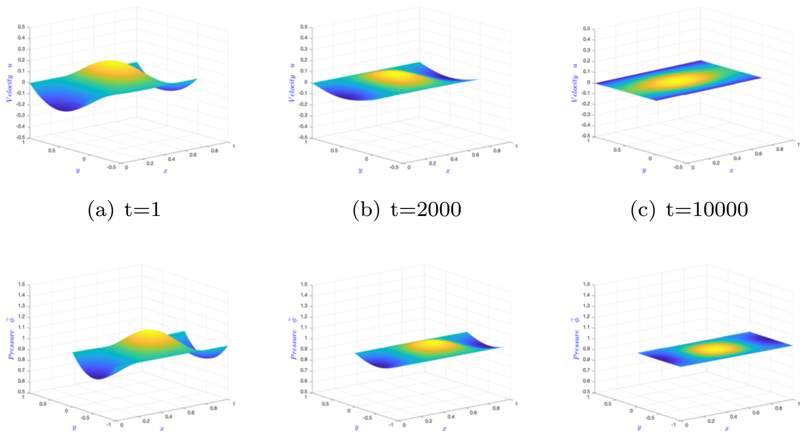
$$\begin{cases} \bar{u}_{i,N+\frac{1}{2}}^n = 0 & \text{on } \Sigma_+^n, \\ \bar{v}_{i,N+\frac{1}{2}}^n = 0 & \text{on } \Sigma_+^n, \\ \phi_{i,\frac{1}{2}}^n = \widetilde{\phi}_b & \text{on } \Sigma_{-b}^n, \end{cases}$$

where $i = 1, \cdots, N, n \in \mathbb{N}$. Finally, we define the initial data $\left(\bar{u}_{i,j}^0, \bar{v}_{i,j}^0, \widetilde{\phi}_{i,j}^0, \bar{\eta}_i^0\right)$ as a small perturbation of the constant steady state $\left(\bar{u}, \bar{v}, \bar{\phi}, \tilde{\eta}\right)$ in (11).

6.2. Numerical results. We have obtained the global well-posedness of the solution perturbed around the constant steady state, that is, if the initial data of our model is a small perturbation of the constant steady state, the solution to the free interface problem exists globally in time and converges to the constant steady state at the almost exponential time rate as shown in Theorem 1.1 in Section 1. In this section, we perform several numerical experiments to illustrate the theoretical analysis of the long time behavior of the Navier-Stokes-Darcy equations. The numerical experiments agree with the theoretical results demonstrated by the following figures.

Indeed, (a)-(c) of Figure 1 demonstrate time evolution to the velocity of the upper fluid in domain $\Omega_+(t)$ and (d)-(f) of Figure 1 show time evolution to the pressure of lower fluid in domain $\Omega_-(t)$. It is clear that the velocity with a small initial data will decay to zero as time goes to infinite. Meanwhile, the initial data of the lower fluid as a small perturbation of $\tilde{\phi}_b$ will converge to $\tilde{\phi}_b$ when time goes to infinite. These numerical simulation results are consistent with what we have shown in Theorem 1.1. Especially, for the free interface $\Sigma_-(t)$, Figure 2 shows the evolution trend of the interface when time changes. In fact, the time-dependent $\eta(t)$ starts with a curve and ends with a line, that is, the free interface $\Sigma_-(t)$ with the nonzero initial data will converge to zero as $t \to \infty$.

In Figure 3, We perform numerical experiments to test the different decay rate for varying exponent of the porous medium equation provided that others are chosen to be the same. We observe that the decay rate of the hydraulic head $\tilde{\phi}$ towards the equilibrium $\tilde{\phi}_b$ is faster when the exponent for the porous medium equation is bigger. But this is not proved in the theoretical analysis.

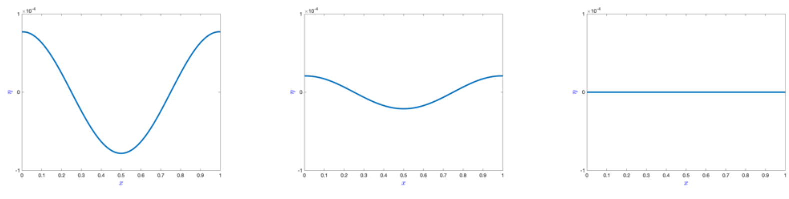


(a) t=1 (b) t=2000 (c) t=10000

(d) t=1 (e) t=2000 (f) t=10000

FIGURE 1. Time evolution of velocity u (up pictures) and pressure $\widetilde{\phi}$ (low pictures) on a uniform mesh with $\Delta x = \Delta y = \frac{1}{100}$ and $\Delta t = 1 \times 10^{-5}$. Here we assume the initial data $u^0 = v^0 = \frac{7}{10} y(y-1)\cos 2\pi x$ and $\widetilde{\phi}^0 = \widetilde{\phi}_b + \frac{7}{10} y(y-1)\cos 2\pi x$ with $\widetilde{\phi}_b = 1$ as a small perturbation of steady state . The positive constant L is fixed as $L = 1$. The density of two fluids are given as $\rho_\omega = 1$ and $\rho_s = 2$.

Moreover, a numerical decay rate with $\rho_\omega = 1$ is exhibited in Figure 4. We compare the decay rate of fluid with different density $\rho_\omega = 0.5$ and $\rho_\omega = 1.5$. In these three cases, the fluid with density $\rho_\omega = 0.5$ has the faster decay rate than other two cases, while the fluid with density $\rho_\omega = 1.5$ takes more time to converge to the steady-state solution.



(a) t=1 (b) t=2000 (c) t=10000

FIGURE 2. Time evolution of the free interface $\Sigma_-(t)$.

7. Appendix.

In this section, for the completeness of this paper, we will give the local existence result and some analytic tools which are used in this paper.

7.1. Local existence. For the completion of this paper, we record the results of the local existence, which has been obtained in our forthcoming paper. Firstly, we

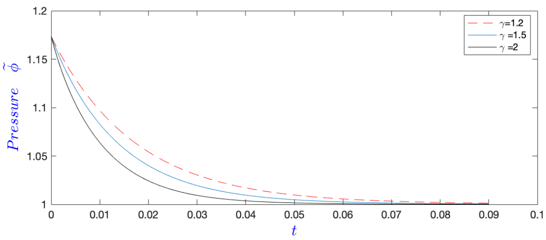


FIGURE 3. Decay rate varying with the different power of diffusion for the porous medium.

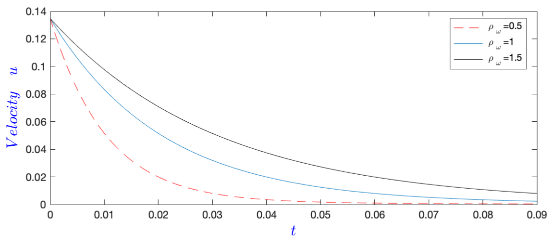


FIGURE 4. Decay rate of velocity with different density ρ_w.

introduce some related notations. Set

$$
\begin{aligned}
{}^0H^1(\Omega_+) &:= \{u \in H^1(\Omega+) \mid u\,|_{\Sigma_+} = 0\},\\
{}^0H^1_\sigma(\Omega_+) &:= \{u \in H^1(\Omega+) \mid u\,|_{\Sigma_+} = 0 \text{ and } \operatorname{div} u = 0\},\\
{}_0H^1(\Omega_-) &:= \{\phi \in H^1(\Omega-) \mid \phi\,|_{\Sigma_b} = 0\}.
\end{aligned}
$$

Then the inner-product on ${}^0H^1(\Omega_+)$ is defined by

$$(u, v)_{\mathscr{H}^1_{(1)}} := \int (\mathcal{D}_{\mathscr{A}} u : \mathcal{D}_{\mathscr{A}} v) J(t),$$

where u and v denote the vector fields. Similarly, the inner-product on ${}_0H^1(\Omega_-)$ for scalar value is also defined by

$$(\phi, \psi)_{\mathscr{H}^1_{(2)}} := \int (\nabla_{\mathscr{A}} \phi \cdot \nabla_{\mathscr{A}} \psi) J.$$

Next we can define the space$\mathscr{H}^1_{(1)} := \{u \in {}^0H^1(\Omega_+) \mid \|u\|_{\mathscr{H}^1_{(1)}} < \infty, \operatorname{div}_{\mathscr{A}} u = 0\}$ and $\mathscr{H}^1_{(2)} := \{\phi \in {}_0H^1(\Omega_-) \mid \|\phi\|_{\mathscr{H}^1_{(2)}} < \infty\}$, also we use $(\mathscr{H}^1_{(1)})^*$ and $(\mathscr{H}^1_{(2)})^*$ to represent the dual space of $\mathscr{H}^1_{(1)}$ and $\mathscr{H}^1_{(2)}$.

Theorem 7.1. *Assume $N \geq 3$ be an interger,$\gamma = 1$ or $\gamma > 1$ and the initial data (u_0, ϕ_0, η_0) satisfy $\|u_0\|^2_{4N} + \|\phi_0\|^2_{4N} + \|\eta_0\|^2_{4N+\frac{1}{2}} < \infty$ and the compatibility conditions* (18). *Then there exists $0 < \delta_0$, $T_0 < 1$, such that if $0 < T \leq T_0 min\{1, \frac{1}{\|\eta_0\|^2_{4N+\frac{1}{2}}}\}$ and $\|u_0\|^2_{4N} + \|\phi_0\|^2_{4N} + \|\eta_0\|^2_{4N} \leq \delta_0$, there exists a unique* *solution (u,ϕ,p,η) on interval $[0,T]$ and the solution obeys the estimates*

$$\begin{aligned}
&\sum_{0\le m\le 2N}\big(\|\partial_t^m u\|^2_{L^\infty_T H^{4N-2j}}+\|\partial_t^m \phi\|^2_{L^\infty_T H^{4N-2j}}+\|\partial_t^m \eta\|^2_{L^\infty_T H^{4N-2j}}\big)\\
&+\sum_{0\le m\le 2N-1}\|\partial_t^m p\|^2_{L^\infty_T H^{4N-2j-1}}\\
&+\sum_{0\le m\le 2N}\big(\|\partial_t^m u\|^2_{L^2_T H^{4N-2j+1}}+\|\partial_t^m \phi\|^2_{L^2_T H^{4N-2j+1}}\big)\\
&+\sum_{0\le m\le 2N-1}\|\partial_t^m p\|^2_{L^2_T H^{4N-2j}}+\|\eta\|^2_{L^2_T H^{4N+\frac12}}+\|\partial_t\eta\|^2_{L^2_T H^{4N-\frac12}}\\
&+\sum_{2\le m\le 2N+1}\|\partial_t^m \eta\|^2_{L^2_T H^{4N-2j+\frac52}}+\big\|\partial_t^{2N+1}u\big\|^2_{L^2_T(\mathscr{H}^1_{(1)})^*}+\big\|\partial_t^{2N+1}\phi\big\|^2_{L^2_T(\mathscr{H}^1_{(2)})^*}\\
\le& C\big(\|u_0\|^2_{4N}+\|\phi_0\|^2_{4N}+\|\eta_0\|^2_{4N}+T\,\|\eta_0\|^2_{4N+\frac12}\big)
\end{aligned}$$

and

$$\|\eta\|^2_{L^\infty\left(0,T;H^{4N+\frac12}(\Sigma_-)\right)}\le C\left(\|u_0\|^2_{4N}+\|\phi_0\|^2_{4N}+(1+T)\,\|\eta_0\|^2_{4N+\frac12}\right),$$

where $C>0$ is a universal constant depending on N and δ_0, but not depending on time.

7.2. Poisson extension. Firstly, Poission extension is listed here, which is used to extend the free boundary to the interal domain and keep the regularity of the boundary matching with one in the internal domain. The related reference can be referred [43].

Denote that $\Sigma_-=\mathbb{T}^2\times\{0\}$, where $\mathbb{T}^2:=(2\pi L_1\mathbb{T})\times(2\pi L_2\mathbb{T})$ and define the Poisson integral in $\mathbb{T}^2\times(-\infty,0)$ by

$$\mathcal{P}_-\eta(x)=\sum_{n\in(L_1^{-1}\mathbb{Z})\times(L_2^{-1}\mathbb{Z})}\frac{e^{in\cdot x'}}{2\pi\sqrt{L_1L_2}}e^{|n|x_3}\hat\eta(n).$$

Here $\hat\eta(n)=\int_{\Sigma_-}\eta(x')\frac{e^{-in\cdot x'}}{2\pi\sqrt{L_1L_2}}dx'$ for $n\in(L_1^{-1}\mathbb{Z})\times(L_2^{-1}\mathbb{Z})$; Via the Fourier analysis, we have

Lemma 7.2. *Let $\mathcal{P}_-\eta$ be a Poisson extension of a function η that is either in $\dot H^q(\Sigma_-)$ or in $\dot H^{q-\frac12}(\Sigma_-)$ for $q\in\mathbb{N}$, where $\dot H^q(\Sigma_-)$ and $\dot H^{q-\frac12}(\Sigma_-)$ denote the homogeneous Sobolev space. Then we have*

$$\|\nabla^q\mathcal{P}_-\eta\|_0^2\lesssim\|\eta\|^2_{\dot H^q(\mathbb{T}^2)}\quad\text{and}\quad\|\nabla^q\mathcal{P}_-\eta\|_0^2\lesssim\|\eta\|^2_{\dot H^{q-\frac12}(\mathbb{T}^2)}$$

Proof. We omit the proof here, which can be seen in the appendix of [17] □

Next, we extend the free interface η to Ω_+ and Ω_-. Let $0<\lambda_0<\lambda_1<\cdots<\lambda_m<\infty$ for $m\in\mathbb{N}$ and define the $(m+1)\times(m+1)$ Vandermonde matrix $V(\lambda_0,\lambda_1,\ldots,\lambda_m)$ by $(V(\lambda_0,\lambda_1,\ldots,\lambda_m))_{ij}=(-\lambda_j)^i$ for $i,j=0,\ldots,m$. Note that the Vandermonde matrix is invertible, so define $\alpha=(\alpha_0,\alpha_1,\ldots,\alpha_m)^T$ is the solution to

$$V(\lambda_0,\lambda_1,\ldots,\lambda_m)\,\alpha=q_m \tag{136}$$

for $q_{m+1}=(1,1,\ldots,1)^T$. Then define the specialized Poisson integral in $\mathbb{T}^2\times(0,\infty)$ by

$$\mathcal{P}_+\eta(x)=\sum_{n\in(L_1^{-1}\mathbb{Z})\times(L_2^{-1}\mathbb{Z})}\frac{e^{in\cdot x'}}{2\pi\sqrt{L_1L_2}}\sum_{j=0}^m\alpha_je^{-|n|\lambda_jx_3}\hat\eta(n).$$

Via the equation (136), it is easy to check that

$$\partial^\alpha \mathcal{P}_+\eta(x',0) = \partial^\alpha \mathcal{P}_-\eta(x',0) \qquad \forall\, \alpha \in \mathbb{N}^3 \text{ with } 0 \le |\alpha| \le m.$$

These facts allow us to extend η to the whole domain Ω by

$$\bar\eta(x',x_3) = \mathcal{P}\eta(x',x_3) := \begin{cases} \mathcal{P}_+\eta(x',x_3) & x_3 > 0, \\ \mathcal{P}_-\eta(x',x_3) & x_3 \le 0. \end{cases} \tag{137}$$

It is easy to know that if $\eta \in H^{s-1/2}(\Sigma_-)$ for $0 \le s \le m$, then $\bar\eta \in H^s(\Omega)$.

Next we also need some estimates of the Poission extension $\mathcal{P}\eta$.

Lemma 7.3. *Let $\mathcal{P}\eta$ be the Poisson integral of the interface function η that is either in $\dot H^q(\Sigma_-)$ or $\dot H^{q-1/2}(\Sigma_-)$ for $q \in \mathbb{N}$. Then*

$$\|\nabla^q \mathcal{P}\eta\|_0^2 \lesssim \|\eta\|^2_{\dot H^q(\Sigma_-)} \ \ \textit{or}\ \ \|\nabla^q \mathcal{P}\eta\|_0^2 \lesssim \|\eta\|^2_{\dot H^{q-1/2}(\Sigma_-)}\,. \tag{138}$$

Proof. Let $\Omega := \Omega_+ \cup \Omega_-$, where $\Omega_+ := \mathbb{T}^2 \times (0,1)$ and $\Omega_- := \mathbb{T}^2 \times (-b, 0)$. Denote $\bar b = \max_{\mathbb{T}^2} b$ and $\tilde\Omega_- := \mathbb{T} \times (-\bar b, 0)$. Since $\mathcal{P}\eta$ is defined on $\mathbb{T}^2 \times (-\infty,\infty)$, it suffices to prove the estimates on $\tilde\Omega := \Omega_+ \cup \tilde\Omega_-$ for $\Omega \subset \tilde\Omega$. By Fubini and Parseval Theorem, it yields

$$\begin{aligned}
\|\nabla^q \mathcal{P}\eta\|_{H^0(\tilde\Omega)} &\lesssim \sum_{n \in (L_1^{-1}\mathbb{Z})\times(L_2^{-1}\mathbb{Z})} \int_{-\bar b}^0 |n|^{2q} \left|\hat\eta(n)\right|^2 e^{2|n|x_3} dx_3 \\
&\quad + \sum_{n \in (L_1^{-1}\mathbb{Z})\times(L_2^{-1}\mathbb{Z})} \sum_{0\le j\le m} \int_0^1 |n|^{2q} \left|\hat\eta(n)\right|^2 \alpha_j^2 e^{-2|n|\lambda_j x_3} dx_3 \\
&\lesssim \sum_{n \in (L_1^{-1}\mathbb{Z})\times(L_2^{-1}\mathbb{Z})} |n|^{2q} \left|\hat\eta(n)\right|^2 \left(\frac{1-e^{-2\bar b|n|}}{|n|}\right) \\
&\quad + \sum_{n \in (L_1^{-1}\mathbb{Z})\times(L_2^{-1}\mathbb{Z})} \sum_{0\le j\le m} |n|^{2q} \left|\hat\eta(n)\right|^2 \alpha_j^2 \left(\frac{1-e^{-2|n|\lambda_j}}{|n|\,\lambda_j}\right).
\end{aligned} \tag{139}$$

However,

$$\frac{1-e^{-4\pi\bar b|n|}}{|n|} \le \min\left\{2\bar b, \frac{1}{|n|}\right\} \quad \text{and} \quad \frac{1-e^{-2|n|\lambda_j}}{|n|\,\lambda_j} \le \min\left\{2, \frac{1}{|n|\,\lambda_j}\right\},$$

which means that the right hand side of (139) is bounded by either $\|\eta\|^2_{\dot H^{q-1/2}(\Sigma)}$ or $\|\eta\|^2_{\dot H^q(\Sigma)}$. □

We will also need the L^∞ estimate.

Lemma 7.4. *Let $\mathcal{P}\eta$ be the Poisson integral of the interface function η that is in $\dot H^{q+s}(\Sigma_-)$ for $q \ge 1$ an integer and $s > 1$. Then*

$$\|\nabla^q \mathcal{P}\eta\|^2_{L^\infty} \lesssim \|\eta\|^2_{\dot H^{q+s}}\,.$$

The same estimate holds for $q = 0$ if η satisfies $\int_{\mathbb{T}^2} \eta = 0$.

Proof. Via the definition of $\mathcal{P}\eta$, we have

$$\begin{aligned}
&\|\nabla^q \mathcal{P}\eta\|_{L^\infty(\Omega)} \\
&\lesssim \sum_{n\in(L_1^{-1}\mathbb{Z})\times(L_2^{-1}\mathbb{Z})} |\hat{\eta}(n)|\,|n|^q + \sum_{n\in(L_1^{-1}\mathbb{Z})\times(L_2^{-1}\mathbb{Z})} \sum_{0\le j\le m} \bar{\lambda}^q\,|n|^q\,|\alpha_j|\,|\hat{\eta}(n)| \\
&\lesssim \sum_{n\in(L_1^{-1}\mathbb{Z})\times(L_2^{-1}\mathbb{Z})} |\hat{\eta}(n)|\,|n|^q \\
&\lesssim \|\eta\|_{\dot{H}^{q+s}} \left(\sum_{n\in(L_1^{-1}\mathbb{Z})\times(L_2^{-1}\mathbb{Z})\setminus\{0\}} |n|^{-2s} \right)^{1/2} \lesssim \|\eta\|_{\dot{H}^{q+s}}\,.
\end{aligned}$$

Here $\bar{\lambda} = \max\{1, \lambda_1, \lambda_2, \lambda_3, ..., \lambda_m\}$. The same estimate works with $q = 0$ if $\hat{\eta}(0) = 0$. □

7.3. Properties of $\mathscr{A}$. The following lemma gives some properties of matrix $\mathscr{A}$ which will be used through the whole of this paper.

Lemma 7.5. *Let $\mathscr{A}$ be the matrix defined by* (15)*. Then we have*
(1) $\partial_j(J\mathscr{A}_i j) = 0$ for $i = 1, 2, 3$.
(2) $J\mathscr{A}\vec{e}_3 = \vec{N}$ on Σ_-, while $J\mathscr{A}\vec{e}_3 = \vec{e}_3$ on Σ_b.

Proof. The first item can be verified by the direct calculation.
For the second item, we compute to get $J\mathscr{A}\vec{e}_3 = -A\vec{e}_1 - B\vec{e}_2 + \vec{e}_3 = -\partial_1\bar{\eta}\vec{e}_1 - \partial_2\bar{\eta}\vec{e}_2 + \vec{e}_3 = -\partial_1\eta\vec{e}_1 - \partial_2\eta\vec{e}_2 + \vec{e}_3 = \vec{N}$ and then these two equalities hold since $\theta(0) = 1$ and $\theta(-b) = 0$. □

7.4. Products in Sobolev spaces. In this subsection, we will list some inequalities which are used to estimate the nonlinear terms. The proofs can be found in [4] [17] and the referrence therein.

Lemma 7.6. *The following inequalities hold for sufficiently smooth subsets of $\mathbb{R}^n$.*

(1) Let $0 \le r \le s_1 \le s_2$ be such that $s_1 > n/2$. Let $f \in H^{s_1}$, $g \in H^{s_2}$. Then $fg \in H^r$ and

$$\|fg\|_{H^r} \lesssim \|f\|_{H^{s_1}}\,\|g\|_{H^{s_2}}\,. \tag{140}$$

(2) Let $0 \le r \le s_1 \le s_2$ be such that $s_2 > r + n/2$. Let $f \in H^{s_1}$, $g \in H^{s_2}$. Then $fg \in H^r$ and

$$\|fg\|_{H^r} \lesssim \|f\|_{H^{s_1}}\,\|g\|_{H^{s_2}}\,. \tag{141}$$

(3) Let $0 \le r \le s_1 \le s_2$ be such that $s_2 > r+n/2$. Let $f \in H^{-r}(\Sigma)$, $g \in H^{s_2}(\Sigma)$. Then $fg \in H^{-s_1}(\Sigma)$ and

$$\|fg\|_{-s_1} \lesssim \|f\|_{-r}\,\|g\|_{s_2}\,. \tag{142}$$

We will also need the following lemma.

Lemma 7.7. *Suppose that $f \in C^1(\Sigma)$ and $g \in H^{1/2}(\Sigma)$. Then*

$$\|fg\|_{1/2} \lesssim \|f\|_{C^1}\,\|g\|_{1/2}\,.$$

In the following lemma, some special Poincaré-type inequalities are also used in this paper. Denote U by a periodic domain of the form $\Omega_\pm$ and Σ_u and Σ_l by the flat upper boundary and the lower boundary which may not be flat, respectively.

Lemma 7.8. *The following hold.*

(1) $\|f\|^2_{L^2(U)} \lesssim \|f\|^2_{L^2(\Sigma_u)} + \|\partial_3 f\|^2_{L^2(U)}$ *for all* $f \in H^1(U)$.
(2) $\|f\|_{L^2(\Sigma_u)} \lesssim \|\partial_3 f\|_{L^2(U)}$ *for* $f \in H^1(U)$ *and* $f = 0$ *on* Σ_l.
(3) $\|f\|_0 \lesssim \|f\|_1 \lesssim \|\nabla f\|_{L^2(U)}$ *for all* $f \in H^1(U)$ *and* $f = 0$ *on* Σ_l.

Next Korn's inequality is also used to the relation between $\mathbb{D}u$ and ∇u.

Lemma 7.9. *It holds that* $\|u\|_1 \lesssim \|\mathbb{D}u\|_0$, *for all* $u \in {}^0H^1(U)$.

7.5. Stokes estimates. Let U be the horizontally periodic slab and Γ_+ and Γ_- be the smooth boundaries which may not be flat. We consider the Stokes problem with the following boundary.

$$\begin{cases} -\mu\Delta u + \nabla p = f^1 & \text{in} \quad U, \\ \operatorname{div} u = f^2 & \text{in} \quad U, \\ u = 0 & \text{on} \quad \Gamma_+, \\ (pI - \mu\mathbb{D}u)\nu = f^3 & \text{on} \quad \Gamma_-. \end{cases}$$

Lemma 7.10. *Assume that* $s \geq 2$, $f^1 \in H^{s-2}(U)$, $f^2 \in H^{s-1}(U)$ *and* $f^3 \in H^{s-\frac{3}{2}}(\Gamma_-)$. *Then the above Stokes problem has an unique solutions* (u, p), *satisfying the following inequality*

$$\|u\|_s + \|p\|_{s-1} \lesssim \left\|f^1\right\|_{H^s(U)} + \left\|f^2\right\|_{H^{s-1}(U)} + \left\|f^3\right\|_{H^{s-\frac{3}{2}}(\Gamma_-)}.$$

Proof. Seeing [4]. □

Next, we present the Stokes problem with Dirichlet boundary condition on both Γ_+ and Γ_-.

$$\begin{cases} -\mu\Delta u + \nabla p = h^1 & \text{in} \quad U, \\ \operatorname{div} u = h^2 & \text{in} \quad U, \\ u = h^3 & \text{on} \quad \Gamma_+, \\ u = h^4 & \text{on} \quad \Gamma_-. \end{cases}$$

Lemma 7.11. *Assume that* $s \geq 2$, $h^1 \in H^{s-2}(U)$, $h^2 \in H^{s-1}(U)$, $h^3 \in H^{s-\frac{3}{2}}(\Gamma_-)$ *and* $h^4 \in H^{s-\frac{3}{2}}(\Gamma_-)$, *satisfying*

$$\int_u h^2 = \int_{\Gamma_+} h^3 \cdot \nu + \int_{\Gamma_-} h^4 \cdot \nu.$$

Then there exists an unique solution (u, p), *satisfying*

$$\|u\|_s + \|\nabla p\|_{s-2} \lesssim \left\|h^1\right\|_{H^s(U)} + \left\|h^2\right\|_{H^{s-1}(U)} + \left\|h^3\right\|_{H^{s-\frac{3}{2}}(\Gamma_+)} + \left\|h^4\right\|_{H^{s-\frac{3}{2}}(\Gamma_-)}.$$

Proof. Seeing [21], [41] and [43]. □

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