

## AN $hp$ FINITE ELEMENT METHOD FOR A SINGULARLY PERTURBED REACTION-CONVECTION-DIFFUSION BOUNDARY VALUE PROBLEM WITH TWO SMALL PARAMETERS

IRENE SYKOPETRITOU AND CHRISTOS XENOPHONTOS

*This paper is dedicated to Manil Suri on the occasion of his 60<sup>th</sup> birthday*

**Abstract.** We consider a second order singularly perturbed boundary value problem, of reaction-convection-diffusion type with two small parameters, and the approximation of its solution by the  $hp$  version of the Finite Element Method on the so-called *Spectral Boundary Layer* mesh. We show that the method converges uniformly, with respect to both singular perturbation parameters, at an exponential rate when the error is measured in the energy norm. Numerical examples are also presented, which illustrate our theoretical findings as well as compare the proposed method with others found in the literature.

**Key words.** Singularly perturbed problem, reaction-convection-diffusion, boundary layers,  $hp$  finite element method, robust exponential convergence.

### 1. Introduction

The numerical solution of singularly perturbed problems has been studied extensively over the last few decades (see, e.g., the books [15], [16], [20] and the references therein). As is well known, a main difficulty in these problems is the presence of *boundary layers* in the solution, whose accurate approximation, independently of the singular perturbation parameter(s), is of great importance for the overall reliability of the approximate solution. In the context of the Finite Element Method (FEM), the robust approximation of boundary layers requires either the use of the  $h$  version on non-uniform, layer-adapted meshes (such as the Shishkin [24] or Bakhvalov [2] mesh), or the use of the high order  $p$  and  $hp$  versions on the so-called *Spectral Boundary Layer* mesh [11], [23].

Usually, problems of convection-diffusion or reaction-diffusion type are studied separately and several researchers have proposed and analyzed numerical schemes for the robust approximation of their solution (see, e.g., [20] and the references therein). When there are two singular perturbation parameters present in the differential equation, the problem becomes reaction-convection-diffusion and the relationship between the parameters determines the ‘regime’ we are in (see Table 1 ahead). In [6] this problem was addressed using the  $h$  version of the FEM as well as appropriate finite differences (see also [3], [5], [7], [17], [21], [28], [29]). In the present article we consider the  $hp$  version of the FEM on the *Spectral Boundary Layer* mesh (from [11]) and show that the method converges uniformly in the perturbation parameters at an exponential rate, when the error is measured in the energy norm.

The rest of the paper is organized as follows: in Section 2 we present the model problem and its regularity. Section 3 presents the discretization using the *Spectral Boundary Layer* mesh and contains our main result of uniform, exponential convergence. Finally, in Section 4 we show the results of numerical computations that illustrate and extend our theoretical findings.

With  $I \subset \mathbb{R}$  an interval with boundary  $\partial I$  and measure  $|I|$ , we will denote by  $C^k(I)$  the space of continuous functions on  $I$  with continuous derivatives up to order  $k$ . We will use the usual Sobolev spaces  $W^{k,m}(I)$  of functions on  $I$  with  $0, 1, 2, \dots, k$  generalized derivatives in  $L^m(I)$ , equipped with the norm and seminorm  $\|\cdot\|_{k,m,I}$  and  $|\cdot|_{k,m,I}$ , respectively. When  $m = 2$ , we will write  $H^k(I)$  instead of  $W^{k,2}(I)$ , and for the norm and seminorm, we will write  $\|\cdot\|_{k,I}$  and  $|\cdot|_{k,I}$ , respectively. The usual  $L^2(I)$  inner product will be denoted by  $\langle \cdot, \cdot \rangle_I$ , with the subscript omitted when there is no confusion. We will also use the space

$$H_0^1(I) = \{u \in H^1(I) : u|_{\partial I} = 0\}.$$

The norm of the space  $L^\infty(I)$  of essentially bounded functions is denoted by  $\|\cdot\|_{\infty,I}$ . Finally, the notation “ $a \lesssim b$ ” means “ $a \leq Cb$ ” with  $C$  being a generic positive constant, independent of any parameters (e.g. discretization, singular perturbation, etc.).

**2. The model problem and its regularity**

We consider the following model problem (cf. [14]): Find  $u$  such that

$$\begin{aligned} (1) \quad & -\varepsilon_1 u''(x) + \varepsilon_2 b(x)u'(x) + c(x)u(x) = f(x), \quad x \in I = (0, 1), \\ (2) \quad & u(0) = u(1) = 0, \end{aligned}$$

where  $0 < \varepsilon_1, \varepsilon_2 \leq 1$  are given parameters that can approach zero and the functions  $b, c, f$  are given and sufficiently smooth. In particular, we assume that they are analytic functions satisfying, for some positive constants  $\gamma_f, \gamma_c, \gamma_b$ , independent of  $\varepsilon_1, \varepsilon_2$ ,

$$(3) \quad \left\| f^{(n)} \right\|_{\infty,I} \lesssim n! \gamma_f^n, \quad \left\| c^{(n)} \right\|_{\infty,I} \lesssim n! \gamma_c^n, \quad \left\| b^{(n)} \right\|_{\infty,I} \lesssim n! \gamma_b^n \quad \forall n = 0, 1, 2, \dots$$

In addition, we assume that there exist positive constants  $\beta, \gamma, \rho$ , independent of  $\varepsilon_1, \varepsilon_2$ , such that  $\forall x \in \bar{I}$

$$(4) \quad b(x) \geq \beta > 0, \quad c(x) \geq \gamma > 0, \quad c(x) - \frac{\varepsilon_2}{2} b'(x) \geq \rho > 0.$$

The solution to (1), (2) satisfies (see, e.g., [6])

$$(5) \quad \|u\|_{\infty,I} \lesssim 1.$$

Moreover, the following result was shown in [27].

**Proposition 1.** *Let  $u$  be the solution of (1), (2). Then, there exists a positive constant  $K$ , independent of  $\varepsilon_1, \varepsilon_2$ , such that for  $n = 0, 1, 2, \dots$*

$$\left\| u^{(n)} \right\|_{\infty,I} \lesssim K^n \max \{n, \varepsilon_1^{-1}, \varepsilon_2^{-1}\}^n.$$

More details arise if one studies the structure of the solution to (1), which depends on the roots of the characteristic equation associated with the differential

operator. For this reason, we let  $\lambda_0(x), \lambda_1(x)$  be the solutions of the characteristic equation and set

$$(6) \quad \mu_0 = -\max_{x \in [0,1]} \lambda_0(x), \quad \mu_1 = \min_{x \in [0,1]} \lambda_1(x),$$

or equivalently,

$$\mu_{0,1} = \min_{x \in [0,1]} \frac{\mp \varepsilon_2 b(x) + \sqrt{\varepsilon_2^2 b^2(x) + 4\varepsilon_1 c(x)}}{2\varepsilon_1}.$$

The following hold true [21, 28]:

$$(7) \quad \left. \begin{aligned} 1 \ll \mu_0 \leq \mu_1, \quad \frac{\varepsilon_2}{\varepsilon_2 + \varepsilon_1^{1/2}} \lesssim \varepsilon_2 \mu_0 \lesssim 1, \quad \varepsilon_1^{1/2} \mu_0 \lesssim 1 \\ \max\{\mu_0^{-1}, \varepsilon_1 \mu_1\} \lesssim \varepsilon_1 + \varepsilon_2^{1/2}, \quad \varepsilon_2 \lesssim \varepsilon_1 \mu_1 \\ \text{for } \varepsilon_2^2 \geq \varepsilon_1 : \varepsilon_1^{-1/2} \lesssim \mu_1 \lesssim \varepsilon_1^{-1} \\ \text{for } \varepsilon_2^2 \leq \varepsilon_1 : \varepsilon_1^{-1/2} \lesssim \mu_1 \lesssim \varepsilon_1^{-1/2} \end{aligned} \right\}.$$

The values of  $\mu_0, \mu_1$  determine the strength of the boundary layers and since  $|\lambda_0(x)| < |\lambda_1(x)|$  the layer at  $x = 1$  is stronger than the layer at  $x = 0$ . Essentially, there are three regimes, as seen in Table 1 [6].

TABLE 1. Different regimes based on the relationship between  $\varepsilon_1$  and  $\varepsilon_2$ .

		$\mu_0$	$\mu_1$
convection-diffusion	$\varepsilon_1 \ll \varepsilon_2 = 1$	1	$\varepsilon_1^{-1}$
convection-reaction-diffusion	$\varepsilon_1 \ll \varepsilon_2^2 \ll 1$	$\varepsilon_2^{-1}$	$\varepsilon_2/\varepsilon_1$
reaction-diffusion	$1 \gg \varepsilon_1 \gg \varepsilon_2^2$	$\varepsilon_1^{-1/2}$	$\varepsilon_1^{-1/2}$

The above considerations suggest the following two cases:

- (1)  $\varepsilon_1$  is large compared to  $\varepsilon_2$ : this is similar to a ‘regular perturbation’ of reaction-diffusion type. If one considers the limiting case  $\varepsilon_2 = 0$ , then one sees that there are two boundary layers, one at each endpoint, of width  $O(\varepsilon_1^{1/2})$ . This situation has been studied in the literature (see, e.g., [8]) and will not be considered further in this article.
- (2)  $\varepsilon_1$  is small compared to  $\varepsilon_2$ : before discussing the different regimes, it is instructive to consider the limiting case  $\varepsilon_1 = 0$ . Then there is an exponential layer (of length scale  $O(\varepsilon_2)$ ) at the *left* endpoint. The homogeneous equation (with constant coefficients) suggests that the different regimes are  $\varepsilon_1 \ll \varepsilon_2^2, \varepsilon_1 \approx \varepsilon_2^2, \varepsilon_1 \gg \varepsilon_2^2$ .
  - (a) In the regime  $\varepsilon_1 \ll \varepsilon_2^2$ , we have  $\mu_0 = O(\varepsilon_2^{-1})$  and  $\mu_1 = O(\varepsilon_2 \varepsilon_1^{-1})$ . Hence  $\mu_1$  is much larger than  $\mu_0$  and the boundary layer in the vicinity of  $x = 1$  is stronger. Consequently, there is a layer of width  $O(\varepsilon_2)$  at the left endpoint (the one that arose from the analysis of the case  $\varepsilon_1 = 0$ ) and additionally there is another layer at the right endpoint, of width  $O(\varepsilon_1/\varepsilon_2)$ .
  - (b) In the regime  $\varepsilon_1 \approx \varepsilon_2^2$ , there are layers at both endpoints of width  $O(\varepsilon_2) = O(\varepsilon_1^{1/2})$ .

- (c) In the regime  $\varepsilon_2^2 \ll \varepsilon_1 \ll 1$ , there are layers at both endpoints of width  $O(\varepsilon_1^{1/2})$ .

**2.1. The asymptotic expansion.** We focus on Case 2 (a)–(c) above, i.e.  $\varepsilon_1 < \varepsilon_2$ , and describe an appropriate asymptotic expansion for  $u$ , in what follows. (The material also appears in [27].)

**2.1.1. The regime  $\varepsilon_1 \ll \varepsilon_2^2 \ll 1$ .** In this case we anticipate a layer of width  $O(\varepsilon_2)$  at the left endpoint and a layer of width  $O(\varepsilon_1/\varepsilon_2)$  at the right endpoint. To deal with this we define the *stretched variables*  $\tilde{x} = x/\varepsilon_2$  and  $\hat{x} = (1-x)\varepsilon_2/\varepsilon_1$  and make the formal ansatz

$$(8) \quad u \sim \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \varepsilon_1^i (\varepsilon_1/\varepsilon_2^2)^j (u_{i,j}(x) + \tilde{u}_{i,j}^{BL}(\tilde{x}) + \hat{u}_{i,j}^{BL}(\hat{x})),$$

with  $u_{i,j}, \tilde{u}_{i,j}^{BL}, \hat{u}_{i,j}^{BL}$  to be determined. Substituting (8) into (1), separating the slow (i.e.  $x$ ) and fast (i.e.  $\tilde{x}, \hat{x}$ ) variables, and equating like powers of  $\varepsilon_1$  and  $\varepsilon_2$ , we get<sup>1</sup>

$$(9) \quad \left. \begin{aligned} u_{0,0}(x) &= \frac{f(x)}{c(x)} \\ u_{i,0}(x) &= -\frac{b(x)}{c(x)} u'_{i-1,0}(x), \quad i \geq 1 \\ u_{0,j}(x) &= u_{1,j}(x) = 0, \quad j \geq 1 \\ u_{i,j}(x) &= \frac{1}{c(x)} (u''_{i-2,j-1}(x) - b(x)u'_{i-1,j}(x)), \quad i \geq 2, j \geq 1 \end{aligned} \right\},$$

$$(10) \quad \left. \begin{aligned} \tilde{b}_0 (\tilde{u}_{0,0}^{BL})' + \tilde{c}_0 \tilde{u}_{0,0}^{BL} &= 0 \\ \tilde{b}_0 (\tilde{u}_{i,0}^{BL})' + \tilde{c}_0 \tilde{u}_{i,0}^{BL} &= -\sum_{k=1}^i \left( \tilde{b}_k (\tilde{u}_{i-k,0}^{BL})' + \tilde{c}_k \tilde{u}_{i-k,0}^{BL} \right), \quad i \geq 1 \\ \tilde{b}_0 (\tilde{u}_{0,j}^{BL})' + \tilde{c}_0 \tilde{u}_{0,j}^{BL} &= (\tilde{u}_{0,j-1}^{BL})'', \quad j \geq 1 \\ \tilde{b}_0 (\tilde{u}_{i,j}^{BL})' + \tilde{c}_0 \tilde{u}_{i,j}^{BL} &= (\tilde{u}_{i,j-1}^{BL})'' - \sum_{k=1}^i \left( \tilde{b}_k (\tilde{u}_{i-k,j}^{BL})' + \tilde{c}_k \tilde{u}_{i-k,j}^{BL} \right), \quad i \geq 1, j \geq 1 \end{aligned} \right\},$$

$$(11) \quad \left. \begin{aligned} (\hat{u}_{i,0}^{BL})'' + \hat{b}_0 (\hat{u}_{i,0}^{BL})' &= 0, \quad i \geq 0 \\ (\hat{u}_{0,j}^{BL})'' + \hat{b}_0 (\hat{u}_{0,j}^{BL})' &= \hat{c}_0 \hat{u}_{0,j-1}^{BL}, \quad j \geq 1 \\ (\hat{u}_{i,1}^{BL})'' + \hat{b}_0 (\hat{u}_{i,1}^{BL})' &= \hat{c}_0 \hat{u}_{i,0}^{BL} - \hat{b}_1 (\hat{u}_{i-1,0}^{BL})', \quad i \geq 1 \\ (\hat{u}_{1,j}^{BL})'' + \hat{b}_0 (\hat{u}_{1,j}^{BL})' &= \hat{c}_0 \hat{u}_{1,j-1}^{BL} - \hat{b}_1 (\hat{u}_{0,j-1}^{BL})' + \hat{c}_1 \hat{u}_{0,j-2}^{BL}, \quad j \geq 2 \\ (\hat{u}_{i,j}^{BL})'' + \hat{b}_0 (\hat{u}_{i,j}^{BL})' &= \hat{c}_0 \hat{u}_{i,j-1}^{BL} - \hat{b}_j (\hat{u}_{i-j,0}^{BL})' + \\ \sum_{k=1}^{j-1} \left\{ -\hat{b}_k (\hat{u}_{i-k,j-k}^{BL})' + \hat{c}_k \hat{u}_{i-k,j-k-1}^{BL} \right\}, & \quad i \geq 2, j = 2, \dots, i \\ (\hat{u}_{i,j}^{BL})'' + \hat{b}_0 (\hat{u}_{i,j}^{BL})' &= \hat{c}_0 \hat{u}_{i,j-1}^{BL} + \\ \sum_{k=1}^i \left\{ -\hat{b}_k (\hat{u}_{i-k,j-k}^{BL})' + \hat{c}_k \hat{u}_{i-k,j-k-1}^{BL} \right\}, & \quad i \geq 2, j > i \end{aligned} \right\},$$

where the notation  $\tilde{b}_k(\tilde{x}) = \tilde{x}^k b^{(k)}(0)/k!$ ,  $\hat{b}_k(\hat{x}) = (-1)^k \hat{x}^k b^{(k)}(1)/k!$  is used, and analogously for the other terms. (We also adopt the convention that empty sums are 0.) The BVPs (10)–(11) are supplemented with the following boundary conditions (in order for (2) to be satisfied) for all  $i, j \geq 0$ :

$$(12) \quad \left. \begin{aligned} \tilde{u}_{i,j}^{BL}(0) &= -u_{i,j}(0), \quad \lim_{\tilde{x} \rightarrow \infty} \tilde{u}_{i,j}^{BL}(\tilde{x}) = 0 \\ \hat{u}_{i,j}^{BL}(0) &= -u_{i,j}(1), \quad \lim_{\hat{x} \rightarrow -\infty} \hat{u}_{i,j}^{BL}(\hat{x}) = 0 \end{aligned} \right\}.$$

<sup>1</sup>The constant coefficient case is considerably simpler – see [25].

Next, we define for some  $M \in \mathbb{N}$ ,

$$(13) \quad u_M(x) \quad : \quad = \sum_{i=0}^M \sum_{j=0}^M \varepsilon_2^i (\varepsilon_1/\varepsilon_2^2)^j u_{i,j}(x),$$

$$(14) \quad \tilde{u}_M^{BL}(\tilde{x}) \quad : \quad = \sum_{i=0}^M \sum_{j=0}^M \varepsilon_2^i (\varepsilon_1/\varepsilon_2^2)^j \tilde{u}_{i,j}^{BL}(\tilde{x}),$$

$$(15) \quad \hat{u}_M^{BL}(\hat{x}) \quad : \quad = \sum_{i=0}^M \sum_{j=0}^M \varepsilon_2^i (\varepsilon_1/\varepsilon_2^2)^j \hat{u}_{i,j}^{BL}(\hat{x}),$$

$$(16) \quad r_M^1 \quad : \quad = u - (u_M + \tilde{u}_M^{BL} + \hat{u}_M^{BL})$$

and we have the following decomposition

$$(17) \quad u = u_M + \tilde{u}_M^{BL} + \hat{u}_M^{BL} + r_M^1.$$

The following was shown in [27] (see also [26]) and it gives analytic regularity bounds on each term in the decomposition (17).

**Proposition 2.** *Assume (3), (4) hold. Then there exist positive constants  $K_1, K_2, \tilde{K}, \hat{K}, \tilde{\gamma}_1, \hat{\gamma}_1, \tilde{\gamma}_2, \hat{\gamma}_2, \delta$ , independent of  $\varepsilon_1, \varepsilon_2$ , such that the solution  $u$  of (1)–(2) can be decomposed as in (17), with*

$$(18) \quad \|u_M^{(n)}\|_{\infty, I} \lesssim n! K_1^n \quad \forall n \in \mathbb{N}_0,$$

$$(19) \quad \left| (\tilde{u}_M^{BL})^{(n)}(x) \right| \lesssim \tilde{K}^n \varepsilon_2^{-n} e^{-\text{dist}(x, \partial I)/\varepsilon_2} \quad \forall n \in \mathbb{N}_0,$$

$$(20) \quad \left| (\hat{u}_M^{BL})^{(n)}(x) \right| \lesssim \hat{K}^n \left( \frac{\varepsilon_1}{\varepsilon_2} \right)^{-n} e^{-\text{dist}(x, \partial I)\varepsilon_2/\varepsilon_1} \quad \forall n \in \mathbb{N}_0,$$

$$(21) \quad \|r_M^1\|_{E, I} \lesssim e^{-\delta/\varepsilon_2},$$

provided  $4\varepsilon_2 e^2 M \max\{1, K_2, \tilde{\gamma}_1, \tilde{\gamma}_2, \hat{\gamma}_1, \hat{\gamma}_2, \tilde{\gamma}_1^2\} < 1$  and  $\frac{\varepsilon_1}{\varepsilon_2} e^2 M \max\{1, K_2, \tilde{\gamma}_1, \tilde{\gamma}_2, \hat{\gamma}_1, \hat{\gamma}_2, \tilde{\gamma}_1^2\} < 1$ . The energy norm  $\|\cdot\|_{E, I}$  is defined by (29).

**2.1.2. The regime  $\varepsilon_1 \approx \varepsilon_2^2$ .** Now there are layers at both endpoints of width  $O(\varepsilon_2)$ . So with  $\tilde{x} = x/\varepsilon_2, \bar{x} = (1-x)/\varepsilon_2$ , we make the formal ansatz

$$(22) \quad u \sim \sum_{i=0}^{\infty} \varepsilon_2^i (u_i(x) + \tilde{u}_i^{BL}(\tilde{x}) + \bar{u}_i^{BL}(\bar{x})),$$

with  $u_i, \tilde{u}_i^{BL}, \bar{u}_i^{BL}$  to be determined. Substituting (22) into (1), separating the slow (i.e.  $x$ ) and fast (i.e.  $\tilde{x}, \bar{x}$ ) variables, and equating like powers of  $\varepsilon_1 (= \varepsilon_2^2)$  and  $\varepsilon_2$  we get

$$\left. \begin{aligned} & \left. \begin{aligned} u_0(x) &= \frac{f(x)}{c(x)}, \quad u_1(x) = -\frac{b(x)}{c(x)} u_0'(x) \\ u_i(x) &= \frac{1}{c(x)} (u_{i-2}''(x) - b(x)u_{i-1}'(x)), \quad i \geq 2 \end{aligned} \right\}, \\ & \left. \begin{aligned} & -(\tilde{u}_0^{BL})'' + \tilde{b}_0 (\tilde{u}_0^{BL})' + \tilde{c}_0 \tilde{u}_0^{BL} = 0 \\ & -(\tilde{u}_i^{BL})'' + \tilde{b}_0 (\tilde{u}_i^{BL})' + \tilde{c}_0 \tilde{u}_i^{BL} = -\sum_{k=1}^i (\tilde{b}_k (\tilde{u}_{i-k}^{BL})' + \tilde{c}_k \tilde{u}_{i-k}^{BL}), \quad i \geq 1 \end{aligned} \right\}, \\ & \left. \begin{aligned} & -(\bar{u}_0^{BL})'' - \bar{b}_0 (\bar{u}_0^{BL})' + \bar{c}_0 \bar{u}_0^{BL} = 0 \\ & -(\bar{u}_i^{BL})'' - \bar{b}_0 (\bar{u}_i^{BL})' + \bar{c}_0 \bar{u}_i^{BL} = \sum_{k=1}^i (\bar{b}_k (\bar{u}_{i-k}^{BL})' - \bar{c}_k \bar{u}_{i-k}^{BL}), \quad i \geq 1 \end{aligned} \right\}, \end{aligned}$$

where the notation  $\tilde{b}_k(\tilde{x}) = \tilde{x}^k b^{(k)}(0)/k!$  etc., is used again. The above equations are supplemented with the following boundary conditions (in order to satisfy (2)):

$$\left. \begin{aligned} u_i(0) + \tilde{u}_i^{BL}(0) &= 0, \\ u_i(1) + \bar{u}_i^{BL}(0) &= 0, \\ \lim_{\tilde{x} \rightarrow \infty} \tilde{u}_i^{BL}(\tilde{x}) &= 0, \lim_{\bar{x} \rightarrow -\infty} \bar{u}_i^{BL}(\bar{x}) = 0. \end{aligned} \right\}$$

We then define for some  $M \in \mathbb{N}$ ,

$$u_M(x) = \sum_{i=0}^M \varepsilon_2^i u_i(x), \tilde{u}_M^{BL}(\tilde{x}) = \sum_{i=0}^M \varepsilon_2^i \tilde{u}_i^{BL}(\tilde{x}), \bar{u}_M^{BL}(\bar{x}) = \sum_{i=0}^M \varepsilon_2^i \bar{u}_i^{BL}(\bar{x}),$$

as well as

$$(23) \quad u = u_M + \tilde{u}_M^{BL} + \bar{u}_M^{BL} + r_M^2.$$

The following was proven in [9].

**Proposition 3.** *Assume (3), (4) hold. Then there exist positive constants  $K_1, K_2, \tilde{K}, \bar{K}, \delta$ , independent of  $\varepsilon_1, \varepsilon_2$ , such that the solution  $u$  of (1)–(2) can be decomposed as in (23), with*

$$\begin{aligned} \|u_M^{(n)}\|_{\infty, I} &\lesssim n! K_1^n \quad \forall n \in \mathbb{N}_0, \\ |(\tilde{u}_M^{BL})^{(n)}(x)| &\lesssim \tilde{K}^n \varepsilon_2^{-n} e^{-\text{dist}(x, \partial I)/\varepsilon_2} \quad \forall n \in \mathbb{N}_0, \\ |(\bar{u}_M^{BL})^{(n)}(x)| &\lesssim \bar{K}^n \varepsilon_2^{-n} e^{-\text{dist}(x, \partial I)/\varepsilon_2} \quad \forall n \in \mathbb{N}_0, \\ \|r_M^2\|_{E, I} &\lesssim e^{-\delta/\varepsilon_2}, \end{aligned}$$

provided  $\varepsilon_2 K_2 M < 1$ . The energy norm  $\|\cdot\|_{E, I}$  is defined by (29).

**2.1.3. The regime  $\varepsilon_2^2 \ll \varepsilon_1 \ll 1$ .** We anticipate layers at both endpoints of width  $O(\sqrt{\varepsilon_1})$ . So we define the stretched variables  $\tilde{x} = x/\sqrt{\varepsilon_1}$  and  $\check{x} = (1-x)/\sqrt{\varepsilon_1}$  and make the formal ansatz

$$(24) \quad u \sim \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \varepsilon_1^{i/2} (\varepsilon_2/\sqrt{\varepsilon_1})^j (u_{i,j}(x) + \check{u}_{i,j}^{BL}(\tilde{x}) + \check{u}_{i,j}^{BL}(\check{x})),$$

with  $u_{i,j}, \check{u}_{i,j}^{BL}, \check{u}_{i,j}^{BL}$  to be determined. Substituting (24) into (1), separating the slow (i.e.  $x$ ) and fast (i.e.  $\tilde{x}, \check{x}$ ) variables, and equating like powers of  $\varepsilon_1$  and  $\varepsilon_2$  we get

$$\left. \begin{aligned} u_{0,0}(x) &= \frac{f(x)}{c(x)}, u_{1,0}(x) = u_{0,j}(x) = 0, j \geq 1 \\ u_{i,0}(x) &= \frac{1}{c(x)} u''_{i-2,0}(x), i \geq 2 \\ u_{2i+1,0}(x) &= 0, i \geq 1 \\ u_{1,1}(x) &= -\frac{b(x)}{c(x)} u'_{0,0}(x), u_{1,j}(x) = 0, j \geq 2 \\ u_{i,j}(x) &= \frac{1}{c(x)} (u''_{i-2,j}(x) - b(x)u'_{i-1,j-1}(x)), i \geq 2, j \geq 1 \end{aligned} \right\},$$

$$\left. \begin{aligned} &-(\check{u}_{0,0}^{BL})'' + \check{c}_0 \check{u}_{0,0}^{BL} = 0 \\ &-(\check{u}_{i,0}^{BL})'' + \check{c}_0 \check{u}_{i,0}^{BL} = -\sum_{k=1}^i \check{c}_k \check{u}_{i-k,0}^{BL}, i \geq 1 \\ &-(\check{u}_{0,j}^{BL})'' + \check{c}_0 \check{u}_{0,j}^{BL} = -\check{b}_0 (\check{u}_{0,j-1}^{BL})', j \geq 1 \\ &-(\check{u}_{i,j}^{BL})'' + \check{c}_0 \check{u}_{i,j}^{BL} = -\check{b}_0 (\check{u}_{i,j-1}^{BL})' - \\ &\sum_{k=1}^i \check{b}_k (\check{u}_{i-k,j-1}^{BL})' + \check{c}_k \check{u}_{i-k,j}^{BL}, i \geq 1, j \geq 1 \end{aligned} \right\},$$

$$\left. \begin{aligned} & -(\check{u}_{0,0}^{BL})'' + \check{c}_0 \check{u}_{0,0}^{BL} = 0 \\ & -(\check{u}_{i,0}^{BL})'' + \check{c}_0 \check{u}_{i,0}^{BL} = -\sum_{k=1}^i \check{c}_k \check{u}_{i-k,0}^{BL}, i \geq 1 \\ & -(\check{u}_{0,j}^{BL})'' + \check{c}_0 \check{u}_{0,j}^{BL} = \check{b}_0 (\check{u}_{0,j-1}^{BL})', j \geq 1 \\ & -(\check{u}_{i,j}^{BL})'' + \check{c}_0 \check{u}_{i,j}^{BL} = \check{b}_0 (\check{u}_{i,j-1}^{BL})' + \\ & \sum_{k=1}^i \left\{ \check{b}_k (\check{u}_{i-k,j-1}^{BL})' - \check{c}_k \check{u}_{i-k,j}^{BL} \right\}, i \geq 1, j \geq 1 \end{aligned} \right\},$$

where the notation  $\check{b}_k(\check{x}) = \check{x}^k b^{(k)}(0)/k!$  etc., is used once more. The above equations are supplemented with the following boundary conditions (in order to satisfy (2)):

$$\left. \begin{aligned} & \check{u}_{i,j}^{BL}(0) = -u_{i,j}(0), \check{u}_{i,j}^{BL}(1) = -u_{i,j}(1), \\ & \lim_{\check{x} \rightarrow \infty} \check{u}_{i,j}^{BL}(\check{x}) = 0, \lim_{\check{x} \rightarrow -\infty} \check{u}_{i,j}^{BL}(\check{x}) = 0. \end{aligned} \right\}$$

We then define for some  $M \in \mathbb{N}$ ,

$$\begin{aligned} u_M(x) &= \sum_{i=0}^M \sum_{j=0}^M \varepsilon_1^{i/2} (\varepsilon_2/\sqrt{\varepsilon_1})^j u_{i,j}(x), \\ \check{u}_M^{BL}(\check{x}) &= \sum_{i=0}^M \sum_{j=0}^M \varepsilon_1^{i/2} (\varepsilon_2/\sqrt{\varepsilon_1})^j \check{u}_{i,j}^{BL}(\check{x}), \\ \check{u}_M^{BL}(\check{x}) &= \sum_{i=0}^M \sum_{j=0}^M \varepsilon_1^{i/2} (\varepsilon_2/\sqrt{\varepsilon_1})^j \check{u}_{i,j}^{BL}(\check{x}), \end{aligned}$$

and we have the following decomposition

$$(25) \quad u = u_M + \check{u}_M^{BL} + \check{u}_M^{BL} + r_M^3.$$

The proposition that follows is the analog of Proposition 2 (the proof is given in [30]).

**Proposition 4.** *Assume (3), (4) hold. Then there exist positive constants  $K_1, K_2, \check{K}, \check{K}$  and  $\delta$ , independent of  $\varepsilon_1, \varepsilon_2$ , such that the solution  $u$  of (1)–(2) can be decomposed as in (25), with*

$$\begin{aligned} & \|u_M^{(n)}\|_{\infty, I} \lesssim n! K_1^n \quad \forall n \in \mathbb{N}_0, \\ & \left| (\check{u}_M^{BL})^{(n)}(x) \right| \lesssim \check{K}^n \varepsilon_1^{-n/2} e^{-\text{dist}(x, \partial I)/\sqrt{\varepsilon_1}} \quad \forall n \in \mathbb{N}_0, \\ & \left| (\check{u}_M^{BL})^{(n)}(x) \right| \lesssim \check{K}^n \varepsilon_1^{-n/2} e^{-\text{dist}(x, \partial I)/\sqrt{\varepsilon_1}} \quad \forall n \in \mathbb{N}_0, \\ & \|r_M^3\|_{E, I} \lesssim e^{-\delta/\sqrt{\varepsilon_1}}, \end{aligned}$$

provided  $\sqrt{\varepsilon_1} K_2 M < 1$ . The energy norm  $\|\cdot\|_{E, I}$  is defined by (29).

### 3. Discretization by an hp-FEM

**3.1. Discrete formulation and definition of the mesh.** The variational formulation of (1)–(2) reads: Find  $u \in H_0^1(I)$  such that

$$(26) \quad \mathcal{B}(u, v) = \mathcal{F}(v) \quad \forall v \in H_0^1(I),$$

where

$$(27) \quad \mathcal{B}(u, v) = \varepsilon_1 \langle u', v' \rangle_I + \varepsilon_2 \langle bu', v \rangle_I + \langle cu, v \rangle_I,$$

$$(28) \quad \mathcal{F}(v) = \langle f, v \rangle_I.$$

The bilinear form  $\mathcal{B}(\cdot, \cdot)$  given by (27) is coercive (due to (4)) with respect to the energy norm

$$(29) \quad \|v\|_{E,I}^2 := \varepsilon_1 |v|_{1,I}^2 + \|v\|_{0,I}^2,$$

i.e.,

$$(30) \quad \mathcal{B}(v, v) \geq \|v\|_{E,I}^2 \quad \forall v \in H_0^1(I).$$

With  $S \subset H_0^1(I)$  a finite dimensional subspace that will be defined shortly, the discrete version of (26) reads: find  $u_N \in S$  such that

$$(31) \quad \mathcal{B}(u_N, v) = \mathcal{F}(v) \quad \forall v \in S.$$

In order to define the subspace  $S$ , let  $\hat{I} = [-1, 1]$  be the reference element and denote by  $\mathcal{P}_p(\hat{I})$  the space of polynomials on  $\hat{I}$ , of degree  $\leq p$ . Then, with  $\Delta = \{x_j\}_{j=0}^N$  an arbitrary subdivision of  $I$ , we define

$$(32) \quad S \equiv S^p(\Delta) = \{u \in H_0^1(I) : u(Q_j(\xi)) \in \mathcal{P}_p(\hat{I}), j = 1, \dots, N\},$$

where the linear element mapping is given by  $Q_j(\xi) = (2\xi - x_{j-1} - x_j)/(x_j - x_{j-1})$ .

We next give the definition of the *Spectral Boundary Layer Mesh* we will use (cf. [11]):

**Definition 1** (Spectral Boundary Layer mesh). *Let  $\mu_0, \mu_1$  be given by (6). For  $\kappa > 0$ ,  $p \in \mathbb{N}$  and  $0 < \varepsilon_1, \varepsilon_2 \leq 1$ , define the Spectral Boundary Layer mesh  $\Delta_{BL}(\kappa, p)$  as*

$$\Delta_{BL}(\kappa, p) := \begin{cases} \Delta = \{0, 1\} & \text{if } \kappa p \varepsilon_1 \geq 1/2 \\ \Delta = \{0, \kappa p \mu_0^{-1}, 1 - \kappa p \mu_1^{-1}, 1\} & \text{if } \kappa p \varepsilon_2 < 1/2 \end{cases}.$$

The spaces  $S(\kappa, p)$  and  $S_0(\kappa, p)$  of piecewise polynomials of degree  $p$  are given by

$$S(\kappa, p) := S^p(\Delta_{BL}(\kappa, p)),$$

$$S_0(\kappa, p) := S_0^p(\Delta_{BL}(\kappa, p)) = S(\kappa, p) \cap H_0^1(I).$$

The following tool from [22] will be used in the next subsection for the construction of the approximation.

**Proposition 5.** *Let  $I = (a, b)$ . Then for any  $u \in C^\infty(I)$  there exists  $\mathcal{I}_p u \in \mathcal{P}_p(I)$  such that*

$$(33) \quad u(a) = \mathcal{I}_p u(a) \quad , \quad u(b) = \mathcal{I}_p u(b) \quad ,$$

$$(34) \quad \|u - \mathcal{I}_p u\|_{0,I}^2 \leq (b-a)^{2s} \frac{1}{p^2} \frac{(p-s)!}{(p+s)!} \left\| u^{(s+1)} \right\|_{0,I}^2 \quad , \quad 0 \leq s \leq p,$$

$$(35) \quad \|(u - \mathcal{I}_p u)'\|_{0,I}^2 \leq (b-a)^{2s} \frac{(p-s)!}{(p+s)!} \left\| u^{(s+1)} \right\|_{0,I}^2 \quad , \quad 0 \leq s \leq p.$$

The following auxiliary result will be used repeatedly in the proofs that follow.

**Lemma 1.** *For every  $t \in (0, 1]$ , there exists a constant  $C$  (depending on  $t \in (0, 1]$ ) such that for every  $q \in \mathbb{N}$ , there holds*

$$\frac{(q-tq)!}{(q+ tq)!} \leq C \left[ \frac{(1-t)^{(1-t)}}{(1+t)^{(1+t)}} \right]^q q^{-2tq} e^{2tq}.$$



*Proof.* We have  $(q \pm tq)! = \Gamma(q \pm tq + 1)$ , and as  $q \rightarrow \infty$  [1],

$$\begin{aligned} \frac{(q - tq)!}{(q + tq)!} &= \frac{\Gamma(q - tq + 1)}{\Gamma(q + tq + 1)} \leq C \frac{(q(1 - t) + 1)^{q - tq + 1/2} e^{-(q - tq + 1)}}{(q(1 + t) + 1)^{q + tq + 1/2} e^{-(q + tq + 1)}} \\ &\leq C \left[ \frac{(1 - t)^{(1 - t)}}{(1 + t)^{(1 + t)}} \right]^q q^{-2tq} e^{2tq}. \end{aligned}$$

□

**Remark 1.** *In the proofs that follow, we will be using derivatives and norms of fractional order, as well as non-integer factorials. The corresponding error estimates may be obtained by classical interpolation arguments.*

**3.2. Error estimates.** We begin with the following lemma, which provides an estimate for the interpolation error.

**Lemma 2.** *Let  $u$  be the solution of (1), (2) and let  $\mathcal{I}_p$  be the approximation operator of Proposition 5. Then there exists a constant  $\sigma > 0$ , independent of  $\varepsilon_1, \varepsilon_2$ , such that*

$$\|u - \mathcal{I}_p u\|_{E,I} \lesssim e^{-\sigma p}.$$

*Proof.* The proof is separated into two cases:

*Case 1:*  $\kappa p \varepsilon_1 \geq 1/2$  (asymptotic case)

In this case the mesh consists of only one element and by Proposition 1, there holds

$$\|u^{(n)}\|_{0,I} \lesssim K^n \max\{n, \varepsilon_1^{-1}, \varepsilon_2^{-1}\}^n = K^n \max\{n, \varepsilon_1^{-1}\}^n,$$

since we assumed  $\varepsilon_1 < \varepsilon_2$ . By Proposition 5, there exists  $\mathcal{I}_p u \in \mathcal{P}_p(I)$  such that

$$\|u - \mathcal{I}_p u\|_{0,I}^2 \lesssim \frac{1}{p^2} \frac{(p - s)!}{(p + s)!} \|u^{(s+1)}\|_{0,I}^2, \quad 0 \leq s \leq p.$$

Choose  $s = \lambda p$ , with  $\lambda \in (0, 1)$  to be chosen shortly. Then

$$\|u - \mathcal{I}_p u\|_{0,I}^2 \lesssim \frac{1}{p^2} \frac{(p - s)!}{(p + s)!} K^{2(\lambda p + 1)} \max\{\lambda p + 1, \varepsilon_1^{-1}\}^{2(\lambda p + 1)}$$

and since  $\kappa p \varepsilon_1 \geq 1/2$ ,

$$\max\{\lambda p + 1, \varepsilon_1^{-1}\}^{2(\lambda p + 1)} = (\lambda p + 1)^{2(\lambda p + 1)},$$

provided the constant  $\kappa$  satisfies  $\kappa \leq \lambda/2$ . Lemma 1 gives

$$\begin{aligned} \|u - \mathcal{I}_p u\|_{0,I}^2 &\lesssim \frac{1}{p^2} \frac{(p - \lambda p)!}{(p + \lambda p)!} K^{2(\lambda p + 1)} (\lambda p + 1)^{2(\lambda p + 1)} \\ &\lesssim \frac{1}{p^2} \left[ \frac{(1 - \lambda)^{(1 - \lambda)}}{(1 + \lambda)^{(1 + \lambda)}} \right]^p e^{2\lambda p + 1} K^{2(\lambda p + 1)} (\lambda p + 1)^2 \left( \frac{\lambda p + 1}{p} \right)^{2\lambda p} \\ &\lesssim e K^2 \left[ \frac{(1 - \lambda)^{(1 - \lambda)}}{(1 + \lambda)^{(1 + \lambda)}} (eK)^{2\lambda} \right]^p \left( \frac{1}{p} + \lambda \right)^{2\lambda p}. \end{aligned}$$

Since  $\left(\frac{1}{p} + \lambda\right)^{2\lambda p} = \lambda^{2\lambda p} \left[ \left(1 + \frac{1}{\lambda p}\right)^{\lambda p} \right]^2 \leq e^2 \lambda^{2\lambda p}$ , we further have

$$\|u - \mathcal{I}_p u\|_{0,I}^2 \lesssim \left[ \frac{(1 - \lambda)^{(1 - \lambda)}}{(1 + \lambda)^{(1 + \lambda)}} (eK\lambda)^{2\lambda} \right]^p$$

If we choose  $\lambda = (eK)^{-1} \in (0, 1)$  then we obtain

$$\|u - \mathcal{I}_p u\|_{0,I}^2 \lesssim \left[ \frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}} \right]^p \lesssim e^{-\beta_1 p},$$

where

$$\beta_1 = |\ln q_1|, \quad q_1 = \frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}} < 1.$$

We note that the choice of  $\lambda$  implies that the constant  $\kappa$  in the definition of the mesh, satisfies  $\kappa < \frac{1}{2eK}$ .

Following the same steps as above and using Propositions 1 and 5, we may show

$$\|(u - \mathcal{I}_p u)'\|_{0,I}^2 \lesssim p^2 e^{-\beta_1 p},$$

so that combining the two, gives the desired result (note that the  $p^2$  term above may be absorbed into the exponential by adjusting the constants).

*Case 2:  $\kappa p \varepsilon_2 < 1/2$  (pre-asymptotic case)*

In this case the mesh is given by

$$\Delta_{BL}(\kappa, p) := \{0, \kappa p \mu_0^{-1}, 1 - \kappa p \mu_1^{-1}, 1\},$$

and the solution is decomposed based on the relationship between  $\varepsilon_1$  and  $\varepsilon_2$ . We will consider the first regime (see Section 2.1.1) and note that the approximation for the other two regimes (see Sections 2.1.2 and 2.1.3) is analogous (see also [8]).

So we assume  $\varepsilon_1 \ll \varepsilon_2^2 \ll 1$  and we have the decomposition (17):

$$u = u_M + \tilde{u}_M^{BL} + \hat{u}_M^{BL} + r_M^1,$$

with each term satisfying the bounds presented in Proposition 2. We will construct a different approximation for each part, using Proposition 5.

For the smooth part  $u_M$ , we have that there exists  $\mathcal{I}_p u_M \in \mathcal{P}_p(I)$  such that

$$\|u_M - \mathcal{I}_p u_M\|_{0,I}^2 + \|(u_M - \mathcal{I}_p u_M)'\|_{0,I}^2 \lesssim \frac{(p-s)!}{(p+s)!} \|u_M^{(s+1)}\|_{0,I}^2, \quad 0 \leq s \leq p.$$

Choose  $s = \bar{\lambda}p$ , with  $\bar{\lambda} \in (0, 1)$  to be chosen shortly. Then, utilizing the estimate (18) and Lemma 1, we arrive at

$$\begin{aligned} \|u_M - \mathcal{I}_p u_M\|_{0,I}^2 + \|(u_M - \mathcal{I}_p u_M)'\|_{0,I}^2 &\lesssim \frac{(p-s)!}{(p+s)!} K_1^{2(\bar{\lambda}p+1)} (\bar{\lambda}p+1)^{2(\bar{\lambda}p+1)} \\ &\lesssim \left[ \frac{(1-\bar{\lambda})^{(1-\bar{\lambda})}}{(1+\bar{\lambda})^{(1+\bar{\lambda})}} \right]^p e^{2\bar{\lambda}p+1} K_1^{2(\bar{\lambda}p+1)} (\bar{\lambda}p+1)^2 \left( \frac{\bar{\lambda}p+1}{p} \right)^{2\bar{\lambda}p} \\ &\lesssim p^2 e K_1^2 \left[ \frac{(1-\bar{\lambda})^{(1-\bar{\lambda})}}{(1+\bar{\lambda})^{(1+\bar{\lambda})}} (eK_1)^{2\bar{\lambda}} \right]^p \left( \frac{1}{p} + \bar{\lambda} \right)^{2\bar{\lambda}p} \\ &\lesssim p^2 \left[ \frac{(1-\bar{\lambda})^{(1-\bar{\lambda})}}{(1+\bar{\lambda})^{(1+\bar{\lambda})}} (eK_1 \bar{\lambda})^{2\bar{\lambda}} \right]^p. \end{aligned}$$

Following the same reasoning as in *Case 1* above, i.e. choosing  $\bar{\lambda} = (eK_1)^{-1}$  etc., we obtain

$$\|u_M - \mathcal{I}_p u_M\|_{E,I} \lesssim p e^{-\sigma p}.$$

For the left boundary layer  $\tilde{u}_M^{BL}$ , we will construct different approximations on the intervals

$$\tilde{I}_1 = [0, \kappa p \mu_0^{-1}], \quad \tilde{I}_2 = [\kappa p \mu_0^{-1}, 1].$$

On  $\tilde{I}_1$ , Proposition 5 gives the existence of  $\mathcal{I}_p \tilde{u}_M^{BL} \in \mathcal{P}_p(\tilde{I}_1)$  such that

$$\left\| (\tilde{u}_M^{BL} - \mathcal{I}_p \tilde{u}_M^{BL})' \right\|_{0, \tilde{I}_1}^2 \lesssim (\kappa p \mu_0^{-1})^{2s} \frac{(p-s)!}{(p+s)!} \left\| (\tilde{u}_M^{BL})^{(s+1)} \right\|_{0, \tilde{I}_1}^2, \quad 0 \leq s \leq p.$$

Choose  $s = \tilde{\lambda}p$ , with  $\tilde{\lambda} \in (0, 1)$  arbitrary. Then, with the aid of Lemma 1, we have

$$\begin{aligned} & \left\| (\tilde{u}_M^{BL} - \mathcal{I}_p \tilde{u}_M^{BL})' \right\|_{0, \tilde{I}_1}^2 \\ & \lesssim (\kappa p \mu_0^{-1})^{2\tilde{\lambda}p} \frac{(p - \tilde{\lambda}p)!}{(p + \tilde{\lambda}p)!} \left\| (\tilde{u}_M^{BL})^{(\tilde{\lambda}p+1)} \right\|_{0, \tilde{I}_1}^2 \\ & \lesssim (\kappa p \mu_0^{-1})^{2\tilde{\lambda}p} \left[ \frac{(1 - \tilde{\lambda})^{(1-\tilde{\lambda})}}{(1 + \tilde{\lambda})^{(1+\tilde{\lambda})}} \right]^p p^{-2\tilde{\lambda}p} e^{2\tilde{\lambda}p+1} \left\| (\tilde{u}_M^{BL})^{(\tilde{\lambda}p+1)} \right\|_{0, \tilde{I}_1}^2. \end{aligned}$$

By (19),

$$\begin{aligned} \left\| (\tilde{u}_M^{BL})^{(\tilde{\lambda}p+1)} \right\|_{0, \tilde{I}_1}^2 &= \int_0^{\kappa p \mu_0^{-1}} \left[ (\tilde{u}_M^{BL})^{(\tilde{\lambda}p+1)}(x) \right]^2 dx \\ &\lesssim \int_0^{\kappa p \mu_0^{-1}} \tilde{K}^{-2(\tilde{\lambda}p+1)} \varepsilon_2^{-2(\tilde{\lambda}p+1)} e^{-2\text{dist}(x, \partial I)/\varepsilon_2} dx \\ &\lesssim \kappa p \mu_0^{-1} \tilde{K}^{-2(\tilde{\lambda}p+1)} \varepsilon_2^{-2(\tilde{\lambda}p+1)}, \end{aligned}$$

so that

$$\begin{aligned} & \left\| (\tilde{u}_M^{BL} - \mathcal{I}_p \tilde{u}_M^{BL})' \right\|_{0, \tilde{I}_1}^2 \\ & \lesssim (\kappa p \mu_0^{-1})^{2\tilde{\lambda}p} \left[ \frac{(1 - \tilde{\lambda})^{(1-\tilde{\lambda})}}{(1 + \tilde{\lambda})^{(1+\tilde{\lambda})}} \right]^p p^{-2\tilde{\lambda}p} e^{2\tilde{\lambda}p+1} \kappa p \mu_0^{-1} \tilde{K}^{-2(\tilde{\lambda}p+1)} \varepsilon_2^{-2(\tilde{\lambda}p+1)} \\ & \lesssim p (\mu_0^{-1})^{2\tilde{\lambda}p+1} \varepsilon_2^{-2(\tilde{\lambda}p+1)} \left[ \frac{(1 - \tilde{\lambda})^{(1-\tilde{\lambda})}}{(1 + \tilde{\lambda})^{(1+\tilde{\lambda})}} \right]^p (\kappa e \tilde{K})^{2\tilde{\lambda}p} \\ & \lesssim p \varepsilon_2^{-1} (\mu_0 \varepsilon_2)^{-(2\tilde{\lambda}p+1)} \left[ \frac{(1 - \tilde{\lambda})^{(1-\tilde{\lambda})}}{(1 + \tilde{\lambda})^{(1+\tilde{\lambda})}} \right]^p, \end{aligned}$$

by the choice of  $\kappa < 1/(e\tilde{K})$ . Since in this regime there holds  $\mu_0 \varepsilon_2 = O(1)$ , we get

$$\left\| (\tilde{u}_M^{BL} - \mathcal{I}_p \tilde{u}_M^{BL})' \right\|_{0, \tilde{I}_1}^2 \lesssim p \varepsilon_2^{-1} e^{-\beta_2 p},$$

with

$$\beta_2 = |\ln q_2|, \quad q_2 = \frac{(1 - \tilde{\lambda})^{(1-\tilde{\lambda})}}{(1 + \tilde{\lambda})^{(1+\tilde{\lambda})}} < 1.$$

On the interval  $\tilde{I}_2 = [\kappa p \mu_0^{-1}, 1]$ , we approximate  $\tilde{u}_M^{BL}$  by its linear interpolant  $\mathcal{I}_1 \tilde{u}_M^{BL}$  and we have

$$\begin{aligned} \left\| (\tilde{u}_M^{BL} - \mathcal{I}_1 \tilde{u}_M^{BL})' \right\|_{0, \tilde{I}_2}^2 &\lesssim \left\| (\tilde{u}_M^{BL})' \right\|_{0, \tilde{I}_2}^2 + \left\| (\mathcal{I}_1 \tilde{u}_M^{BL})' \right\|_{0, \tilde{I}_2}^2 \\ &\lesssim \int_{\kappa p \mu_0^{-1}}^1 \left[ (\tilde{u}_M^{BL})'(x) \right]^2 dx \\ &\lesssim \int_{\kappa p \mu_0^{-1}}^1 \tilde{K}^2 \varepsilon_2^{-2} e^{-2 \operatorname{dist}(x, \partial I) / \varepsilon_2} dx \lesssim \varepsilon_2^{-1} e^{-2 \kappa p \mu_0^{-1} / \varepsilon_2} \\ &\lesssim \varepsilon_2^{-1} e^{-2 \kappa p}, \end{aligned}$$

by (7). Therefore,

$$\left\| (\tilde{u}_M^{BL} - \mathcal{I}_p \tilde{u}_M^{BL})' \right\|_{0, I} \lesssim \varepsilon_2^{-1/2} e^{-\sigma p},$$

for some  $\sigma > 0$ , independent of  $\varepsilon_1, \varepsilon_2$ . Repeating the argument for the  $L^2$  norm of the error and using the definition of the energy norm, we get

$$\begin{aligned} \left\| (\tilde{u}_M^{BL} - \mathcal{I}_p \tilde{u}_M^{BL}) \right\|_{E, I} &\lesssim \varepsilon_1^{1/2} \left\| (\tilde{u}_M^{BL} - \mathcal{I}_p \tilde{u}_M^{BL})' \right\|_{0, I} + \left\| (\tilde{u}_M^{BL} - \mathcal{I}_p \tilde{u}_M^{BL}) \right\|_{0, I} \\ &\lesssim \varepsilon_1^{1/2} \varepsilon_2^{-1/2} e^{-\sigma p} + e^{-\sigma p} \\ &\lesssim e^{-\sigma p}, \end{aligned}$$

since  $\varepsilon_1^{1/2} \varepsilon_2^{-1/2} = O(1)$  due to  $\varepsilon_1 < \varepsilon_2$ .

For the right boundary layer  $\hat{u}_M^{BL}$ , we will construct different approximations on the intervals

$$\hat{I}_1 = [0, 1 - \kappa p \mu_1^{-1}], \quad \hat{I}_2 = [1 - \kappa p \mu_1^{-1}, 1].$$

The steps are the same as for the left boundary layer. On  $\hat{I}_1$  we use the linear interpolant  $\mathcal{I}_1 \hat{u}_M^{BL}$  for the approximation, getting with the help of (20),

$$\begin{aligned} \left\| (\hat{u}_M^{BL} - \mathcal{I}_1 \hat{u}_M^{BL})' \right\|_{0, \hat{I}_1}^2 &\lesssim \left\| (\hat{u}_M^{BL})' \right\|_{0, \hat{I}_1}^2 + \left\| (\mathcal{I}_1 \hat{u}_M^{BL})' \right\|_{0, \hat{I}_1}^2 \\ &\lesssim \int_0^{1 - \kappa p \mu_1^{-1}} \left[ (\hat{u}_M^{BL})'(x) \right]^2 dx \\ &\lesssim \int_0^{1 - \kappa p \mu_1^{-1}} \hat{K}^2 \left( \frac{\varepsilon_1}{\varepsilon_2} \right)^{-2} e^{-2 \operatorname{dist}(x, \partial I) \varepsilon_2 / \varepsilon_1} dx \\ &\lesssim \left( \frac{\varepsilon_1}{\varepsilon_2} \right)^{-1} e^{-2 \kappa p}. \end{aligned}$$

On  $\hat{I}_2$ , we have by Proposition 5 that there exists  $\mathcal{I}_p \hat{u}_M^{BL} \in \mathcal{P}_p(\hat{I}_2)$  such that

$$\left\| (\hat{u}_M^{BL} - \mathcal{I}_p \hat{u}_M^{BL})' \right\|_{0, \hat{I}_2}^2 \lesssim (\kappa p \mu_1^{-1})^{2s} \frac{(p-s)!}{(p+s)!} \left\| (\hat{u}_M^{BL})^{(s+1)} \right\|_{0, \hat{I}_2}^2, \quad 0 \leq s \leq p.$$

Choose  $s = \hat{\lambda}p$ , with  $\hat{\lambda} \in (0, 1)$  arbitrary. Then, with the aid of Lemma 1, we have

$$\begin{aligned} & \left\| (\hat{u}_M^{BL} - \mathcal{I}_p \hat{u}_M^{BL})' \right\|_{0, \hat{I}_2}^2 \\ & \lesssim (\kappa p \mu_1^{-1})^{2\hat{\lambda}p} \frac{(p - \hat{\lambda}p)!}{(p + \hat{\lambda}p)!} \left\| (\hat{u}_M^{BL})^{(\hat{\lambda}p+1)} \right\|_{0, \hat{I}_2}^2 \\ & \lesssim (\kappa p \mu_1^{-1})^{2\hat{\lambda}p} \left[ \frac{(1 - \hat{\lambda})^{(1-\hat{\lambda})}}{(1 + \hat{\lambda})^{(1+\hat{\lambda})}} \right]^p p^{-2\hat{\lambda}p} e^{2\hat{\lambda}p+1} \left\| (\hat{u}_M^{BL})^{(\hat{\lambda}p+1)} \right\|_{0, \hat{I}_2}^2. \end{aligned}$$

By (20),

$$\begin{aligned} \left\| (\hat{u}_M^{BL})^{(\hat{\lambda}p+1)} \right\|_{0, \hat{I}_2}^2 &= \int_{1-\kappa p \mu_1^{-1}}^1 \left[ (\hat{u}_M^{BL})^{(\hat{\lambda}p+1)}(x) \right]^2 dx \\ &\lesssim \int_{1-\kappa p \mu_1^{-1}}^1 \hat{K}^{2(\hat{\lambda}p+1)} \left( \frac{\varepsilon_1}{\varepsilon_2} \right)^{-2(\hat{\lambda}p+1)} e^{-2\text{dist}(x, \partial I) \varepsilon_2 / \varepsilon_1} dx \\ &\lesssim \kappa p \mu_1^{-1} \hat{K}^{2(\hat{\lambda}p+1)} \left( \frac{\varepsilon_1}{\varepsilon_2} \right)^{-2\hat{\lambda}p-2}, \end{aligned}$$

so that

$$\begin{aligned} & \left\| (\hat{u}_M^{BL} - \mathcal{I}_p \hat{u}_M^{BL})' \right\|_{0, \hat{I}_2}^2 \\ & \lesssim (\kappa p \mu_1^{-1})^{2\hat{\lambda}p+1} \left[ \frac{(1 - \hat{\lambda})^{(1-\hat{\lambda})}}{(1 + \hat{\lambda})^{(1+\hat{\lambda})}} \right]^p p^{-2\hat{\lambda}p} e^{2\hat{\lambda}p+1} \hat{K}^{2(\hat{\lambda}p+1)} \left( \frac{\varepsilon_1}{\varepsilon_2} \right)^{-2\hat{\lambda}p-2} \\ & \lesssim p (\mu_1^{-1})^{2\hat{\lambda}p+1} \left( \frac{\varepsilon_1}{\varepsilon_2} \right)^{-2\hat{\lambda}p-2} \left[ \frac{(1 - \hat{\lambda})^{(1-\hat{\lambda})}}{(1 + \hat{\lambda})^{(1+\hat{\lambda})}} \right]^p (\kappa e \hat{K})^{2\hat{\lambda}p} \\ & \lesssim p \left( \frac{\varepsilon_1}{\varepsilon_2} \right)^{-1} \left( \mu_1^{-1} \frac{\varepsilon_2}{\varepsilon_1} \right)^{2\hat{\lambda}p+1} \left[ \frac{(1 - \hat{\lambda})^{(1-\hat{\lambda})}}{(1 + \hat{\lambda})^{(1+\hat{\lambda})}} \right]^p, \end{aligned}$$

by the choice of  $\kappa < 1/(e\hat{K})$ . Since in this regime there holds  $\mu_1^{-1} \frac{\varepsilon_2}{\varepsilon_1} = O(1)$ , we get

$$(36) \quad \left\| (\hat{u}_M^{BL} - \mathcal{I}_p \hat{u}_M^{BL})' \right\|_{0, \hat{I}_2}^2 \lesssim p \left( \frac{\varepsilon_1}{\varepsilon_2} \right)^{-1} e^{-\beta_3 p},$$

with

$$\beta_3 = |\ln q_3|, \quad q_3 = \frac{(1 - \hat{\lambda})^{(1-\hat{\lambda})}}{(1 + \hat{\lambda})^{(1+\hat{\lambda})}} < 1.$$

For the  $L^2$  error, we have in an analogous fashion

$$\left\| \hat{u}_M^{BL} - \mathcal{I}_p \hat{u}_M^{BL} \right\|_{0, \hat{I}_2} \lesssim e^{-\sigma p},$$

so that the above considerations yield

$$\left\| \hat{u}_M^{BL} - \mathcal{I}_p \hat{u}_M^{BL} \right\|_{E, I} \lesssim \left( \varepsilon_1^{1/2} (\varepsilon_2^{1/2} / \varepsilon_1^{1/2}) + 1 \right) e^{-\sigma p} \lesssim e^{-\sigma p},$$

with  $\sigma > 0$  a constant independent of  $\varepsilon_1, \varepsilon_2$ .

We finally consider the remainder,  $r_M^1$  which satisfies (21):

$$\left\| r_M^1 \right\|_{E, I} \lesssim e^{-\delta / \varepsilon_2}.$$

Since the remainder is already exponentially small, it will not be approximated. Due to  $\kappa p \varepsilon_2 < 1/2$ , we have

$$\|r_M^1\|_{E,I} \lesssim e^{-\sigma \kappa p},$$

with  $\sigma > 0$  a constant independent of  $\varepsilon_1, \varepsilon_2$ . Combining all the above we obtain the desired result.  $\square$

We next estimate the error between the finite element solution  $u_{FEM}$  and the interpolant  $\mathcal{I}_p u$ .

**Lemma 3.** *Let  $u$  be the solution of (1)–(2),  $u_{FEM} \in S_0(\kappa, p)$  be its approximation based on the Spectral Boundary Layer Mesh, and let  $\mathcal{I}_p$  be the approximation operator of Proposition 5. Then there exists a constant  $\sigma > 0$ , independent of  $\varepsilon_1, \varepsilon_2$ , such that*

$$\|\mathcal{I}_p u - u_{FEM}\|_{E,I} \lesssim e^{-\sigma p}.$$

*Proof.* By coercivity of the bilinear form  $\mathcal{B}_\varepsilon$  (eq. (30)), there holds with  $\xi := \mathcal{I}_p u - u_{FEM}$ ,

$$\|\xi\|_{E,I}^2 \leq \mathcal{B}_\varepsilon(\xi, \xi) = -\mathcal{B}_\varepsilon(u - \mathcal{I}_p u, \xi),$$

where we also used Galerkin orthogonality. Hence

$$\|\xi\|_{E,I}^2 \leq -\varepsilon_1 \langle (u - \mathcal{I}_p u)', \xi' \rangle_I - \varepsilon_2 \langle b(u - \mathcal{I}_p u)', \xi \rangle_I - \langle c(u - \mathcal{I}_p u), \xi \rangle_I.$$

Using integration by parts for the second term above, we obtain

$$\|\xi\|_{E,I}^2 \leq -\varepsilon_1 \langle (u - \mathcal{I}_p u)', \xi' \rangle_I + \varepsilon_2 \langle b(u - \mathcal{I}_p u), \xi' \rangle_I + \langle \tilde{c}(u - \mathcal{I}_p u), \xi \rangle_I,$$

where  $\tilde{c} = \varepsilon_2 b' - c$ .

The first and last term may be estimated using Cauchy Schwarz:

$$\begin{aligned} |-\varepsilon_1 \langle (u - \mathcal{I}_p u)', \xi' \rangle_I| + |\langle \tilde{c}(u - \mathcal{I}_p u), \xi \rangle_I| &\lesssim \varepsilon_1 \|(u - \mathcal{I}_p u)'\|_{0,I} \|\xi'\|_{0,I} + \\ &+ \|\tilde{c}\|_{\infty,I} \|u - \mathcal{I}_p u\|_{0,I} \|\xi\|_{0,I} \lesssim \max\{1, \|\tilde{c}\|_{\infty,I}\} \|u - \mathcal{I}_p u\|_{E,I} \|\xi\|_{E,I}. \end{aligned}$$

For the second term, we will consider the two ranges of  $p$  separately: in the asymptotic range of  $p$ , i.e.  $\kappa p \varepsilon_1 \geq 1/2$ , we have

$$\begin{aligned} |\varepsilon_2 \langle b(u - \mathcal{I}_p u), \xi' \rangle_I| &\lesssim \varepsilon_2 \|b\|_{\infty,I} \|u - \mathcal{I}_p u\|_{0,I} \|\xi'\|_{0,I} \\ &\lesssim \varepsilon_2 \varepsilon_1^{-1/2} \|u - \mathcal{I}_p u\|_{E,I} \|\xi\|_{E,I} \\ &\lesssim \varepsilon_1^{1/2} \kappa p \|u - \mathcal{I}_p u\|_{E,I} \|\xi\|_{E,I} \\ &\lesssim e^{-\sigma p} \|\xi\|_{E,I}. \end{aligned}$$

In the pre-asymptotic range of  $p$ , i.e.  $\kappa p \varepsilon_2 < 1/2$ , we consider the three intervals of the Spectral Boundary Layer mesh

$$[0, \kappa p \mu_0^{-1}] \cup [\kappa p \mu_0^{-1}, 1 - \kappa p \mu_1^{-1}] \cup [1 - \kappa p \mu_1^{-1}, 1].$$

On the first subinterval we have

$$\begin{aligned} \left| \varepsilon_2 \langle b(u - \mathcal{I}_p u), \xi' \rangle_{[0, \kappa p \mu_0^{-1}]} \right| &\lesssim \varepsilon_2 \|b\|_{\infty, [0, \kappa p \mu_0^{-1}]} \left| \langle u - \mathcal{I}_p u, \xi' \rangle_{[0, \kappa p \mu_0^{-1}]} \right| \\ &\lesssim \varepsilon_2 \|u - \mathcal{I}_p u\|_{0, [0, \kappa p \mu_0^{-1}]} \|\xi'\|_{0, [0, \kappa p \mu_0^{-1}]} \\ &\lesssim \frac{\varepsilon_2}{\kappa p \mu_0^{-1}} \|u - \mathcal{I}_p u\|_{0, [0, \kappa p \mu_0^{-1}]} \|\xi\|_{0, [0, \kappa p \mu_0^{-1}]} \\ &\lesssim \frac{\varepsilon_2 \mu_0}{\kappa p} \|u - \mathcal{I}_p u\|_{0, [0, \kappa p \mu_0^{-1}]} \|\xi\|_{0, [0, \kappa p \mu_0^{-1}]}, \end{aligned}$$

where we used an inverse inequality (see, e.g. [22, Thm. 3.91]). Thus, (7) and Lemma 2 give

$$\left| \varepsilon_2 \langle b(u - \mathcal{I}_p u), \xi' \rangle_{[0, \kappa p \mu_0^{-1}]} \right| \lesssim e^{-\beta p} \|\xi\|_{E,I}.$$

Similarly, on the second subinterval we have

$$\begin{aligned} & \left| \varepsilon_2 \langle b(u - \mathcal{I}_p u), \xi' \rangle_{[\kappa p \mu_0^{-1}, 1 - \kappa p \mu_1^{-1}]} \right| \\ & \lesssim \varepsilon_2 \|b\|_{\infty, [\kappa p \mu_0^{-1}, 1 - \kappa p \mu_1^{-1}]} \left| \langle u - \mathcal{I}_p u, \xi' \rangle_{[\kappa p \mu_0^{-1}, 1 - \kappa p \mu_1^{-1}]} \right| \\ & \lesssim \varepsilon_2 \|u - \mathcal{I}_p u\|_{0, [\kappa p \mu_0^{-1}, 1 - \kappa p \mu_1^{-1}]} \|\xi\|_{0, [\kappa p \mu_0^{-1}, 1 - \kappa p \mu_1^{-1}]} \\ & \lesssim e^{-\beta p} \|\xi\|_{E,I}. \end{aligned}$$

Finally, on the third subinterval we have

$$\begin{aligned} \left| \varepsilon_2 \langle b(u - \mathcal{I}_p u), \xi' \rangle_{[1 - \kappa p \mu_1^{-1}, 1]} \right| & \lesssim \varepsilon_2 \|b\|_{\infty, [1 - \kappa p \mu_1^{-1}, 1]} \left| \langle u - \mathcal{I}_p u, \xi' \rangle_{[1 - \kappa p \mu_1^{-1}, 1]} \right| \\ & \lesssim \frac{\varepsilon_2}{\varepsilon_1^{1/2}} \|u - \mathcal{I}_p u\|_{0, [1 - \kappa p \mu_1^{-1}, 1]} \|\xi\|_{E,I} \\ & \lesssim \frac{\varepsilon_2}{\varepsilon_1^{1/2}} \frac{\varepsilon_1^{1/2}}{\varepsilon_2^{1/2}} e^{-\beta p} \|\xi\|_{E,I} \\ & \lesssim e^{-\beta p} \|\xi\|_{E,I}, \end{aligned}$$

where Poincaré’s inequality was used. Therefore,

$$\left| \varepsilon_2 \langle b(u - \mathcal{I}_p u), \xi' \rangle_I \right| \lesssim e^{-\beta p} \|\xi\|_{E,I}$$

and

$$\|\xi\|_{E,I}^2 \lesssim e^{-\beta p} \|\xi\|_{E,I}$$

which completes the proof. □

We conclude with the main result of the article.

**Theorem 1.** *Let  $u$  be the solution of (1)–(2) and let  $u_{FEM} \in S_0(\kappa, p)$  be its approximation based on the Spectral Boundary Layer Mesh. Then there exists a constant  $\sigma > 0$ , independent of  $\varepsilon_1, \varepsilon_2$ , such that*

$$\|u - u_{FEM}\|_{E,I} \lesssim e^{-\sigma p}.$$

*Proof.* We begin with the triangle inequality:

$$\|u - u_{FEM}\|_{E,I} \leq \|u - \mathcal{I}_p u\|_{E,I} + \|\mathcal{I}_p u - u_{FEM}\|_{E,I},$$

where  $\mathcal{I}_p$  is the approximation operator of Proposition 5. The first term is handled by Lemma 2 and the second by Lemma 3. □

#### 4. Numerical results

In this section we present the results of numerical computations in order to illustrate the theory, for two examples, using the values

$$(37) \quad \varepsilon_1 = 10^{-9}, \varepsilon_2 = 10^{-4}; \quad \varepsilon_1 = 10^{-10}, \varepsilon_2 = 10^{-5}; \quad \varepsilon_1 = 10^{-12}, \varepsilon_2 = 10^{-12},$$

(hence we cover all three regimes).

We also consider a third example, in which a comparison is made between the proposed method and others found in the literature. This is meant to show the advantages of the present approach.

**Example 1:** We consider (1), (2) with  $b(x) = c(x) = f(x) = 1$ . An exact solution is available, hence our results are reliable. We take  $\kappa = 1$  in the definition of the mesh and we use polynomials of degree  $p = 1, \dots, 11$  for the approximation. Figure 1 shows the percentage relative error measured in the energy norm, versus the number of degrees of freedom  $DOF = 3p - 1$ , in a semi-log scale. The fact that we see straight lines indicates the exponential convergence of the method, while the robustness is visible since the method does not deteriorate as the singular perturbation parameters tend to 0.

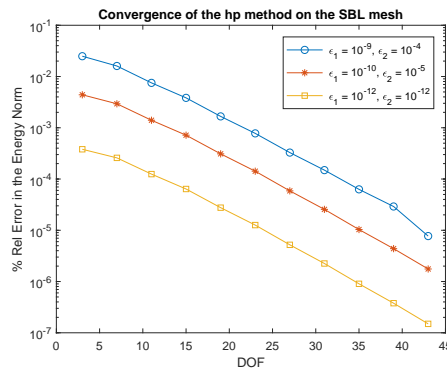


FIGURE 1. Energy norm convergence for Example 1.

In order to get a ‘clearer’ picture of the performance of the method, we show in Figures 2–4 the convergence in each regime separately. In regime 1 ( $\varepsilon_1 \ll \varepsilon_2^2$ ), we see from Figure 2 that the method converges exponentially (we get straight lines) and independently of  $\varepsilon_1, \varepsilon_2$  (the lines coincide). In regime 2 ( $\varepsilon_1 \approx \varepsilon_2^2$ ), however, the lines do not coincide, even though we have exponential convergence. This is due to the fact that the energy norm is not *balanced* for reaction-diffusion problems (see, e.g. [18], [19]) and this manifests itself as the method performing better as  $\varepsilon_1, \varepsilon_2 \rightarrow 0$ . The same is true in regime 3 ( $\varepsilon_1 \gg \varepsilon_2^2$ ), as seen in Figure 4, since in this regime we again have a reaction-diffusion problem.

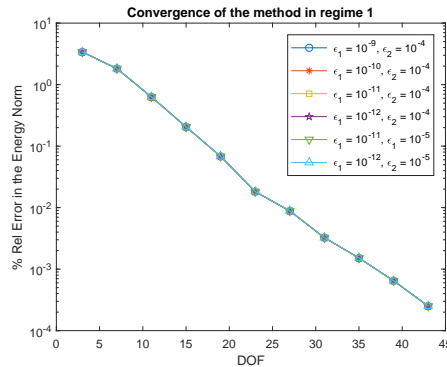


FIGURE 2. Energy norm convergence for Example 1, when  $\varepsilon_1 \ll \varepsilon_2^2$ .



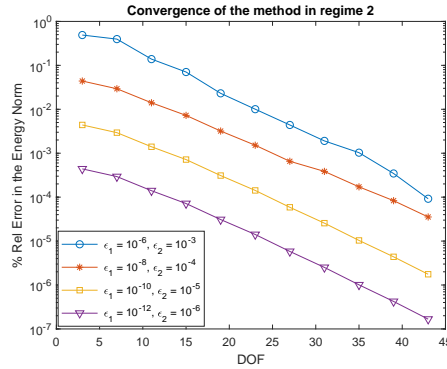


FIGURE 3. Energy norm convergence for Example 1, when  $\epsilon_1 \approx \epsilon_2^2$ .

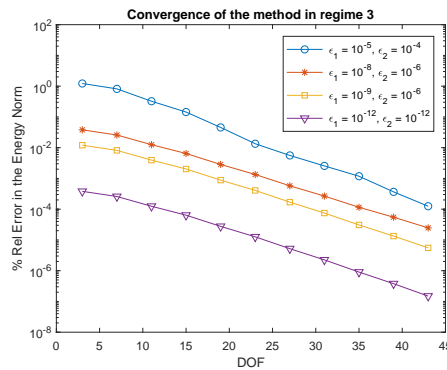


FIGURE 4. Energy norm convergence for Example 1, when  $\epsilon_1 \gg \epsilon_2^2$ .

**Example 2:** We now consider (1), (2) with  $b(x) = e^x, c(x) = x, f(x) = 1$ . An exact solution is not available, so we use a reference solution obtained with twice as many DOF. In Figure 5 we show the convergence of the method for the values of  $\epsilon_1, \epsilon_2$  given by (37).

Once again we observe robust exponential convergence as  $DOF$  is increased.

**Example 3:** We consider a final example, namely (1), (2) with  $b(x) = c(x) = 1, f(x) = \cos(\pi x)$ . This example is taken from [3] with a known exact solution, and our goal is to compare the following methods:

- The  $hp$ -FEM on the SBL mesh (proposed method),  $p = 1, \dots, 4$
- The  $p$ -FEM on a single element,  $p = 1, \dots, 7$ , i.e. we use polynomials of degree  $p = 1, \dots, 7$  defined on  $[0, 1]$  to approximate the solution
- The  $h$ -FEM on a Shishkin mesh with  $p = 1, 2, 3$
- The  $h$ -FEM on an exponential graded (eXp) mesh from [4], with  $p = 1, 2, 3$
- The  $hp$ -FEM on the eXp mesh,  $p = 1, \dots, 8$ , i.e.  $p$  is increased linearly and  $h$  is decreased (in an exponential fashion, see [4])

We use the values  $\epsilon_1 = 10^{-6}, \epsilon_2 = 10^{-2}$  (i.e., we are in the first regime) noting that different choices of these parameters gave qualitatively the same results. Figure 6 shows the convergence of each method using a log-log scale. As expected, the

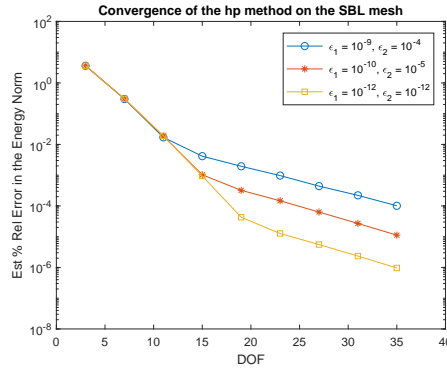


FIGURE 5. Energy norm convergence for Example 2.

$p$ -FEM on a single element does not perform well (for this reasonable range of  $p$ ), while the Shishkin and exponential mesh  $h$ -FEMs yield almost optimal and optimal, respectively, algebraic rates of convergence which are independent of  $\varepsilon_1, \varepsilon_2$ . The proposed method and the  $hp$ -FEM on the eXp mesh, are the only ones converging at an exponential rate. However, the proposed  $hp$ -FEM on the SBLM outperforms all others, in the sense that it produces a small error at a lower number of degrees of freedom.

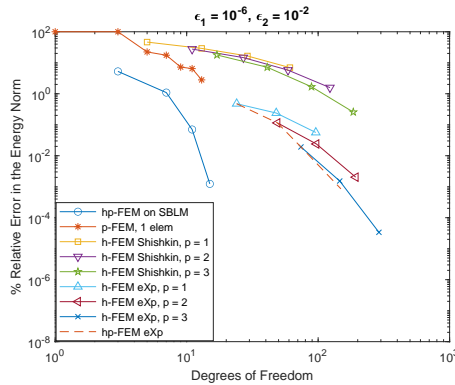


FIGURE 6. Energy norm convergence for Example 3.

**References**

- [1] G. E. Andrews, R. Askey and R. Roy, Special functions, Cambridge University Press, Cambridge (1999).
- [2] N. S. Bakhvalov, Towards optimization of methods for solving boundary value problems in the presence of boundary layers (in Russian), Zh. Vychisl. Mat. Mat. Fiz., 9 (1969) 841–859.
- [3] M. Brdar and H. Zarin, A singularly perturbed problem with two parameters on a Bakhvalov-type mesh, J. Comp. Appl. Math., 292 (2016) 307–319.
- [4] P. Constantinou and C. Xenophontos, Finite element analysis of an exponentially graded mesh for singularly perturbed problems, Comp. Meth. Appl. Math., 150 (2015) 135–143.
- [5] J. L. Garcia, E. O’Riordan and M. L. Pickett, A parameter robust high order numerical method for a singularly perturbed two parameter problem, Appl. Num. Math., 56 (2006) 962–980.

- [6] T. Linß, Layer-adapted meshes for reaction-convection-diffusion problems, Lecture Notes in Mathematics 1985, Springer-Verlag, 2010.
- [7] T. Linß and H. G. Roos, Analysis of a finite difference scheme for a singularly perturbed problem with two small parameters, *Comput. Meth. Appl. Math.*, 289 (2004) 355–366.
- [8] J. M. Melenk, *hp Finite Element Methods for Singular Perturbations*, Lecture Notes in Mathematics 1796, Springer-Verlag, 2002.
- [9] J. M. Melenk, On the robust exponential convergence of hp finite element methods for problems with boundary layers, *IMA J. Numer. Anal.*, 17 (1997) 577–601.
- [10] J. M. Melenk and C. Schwab, An hp Finite Element Method for convection-diffusion problems in one-dimension, *IMA J. Num. Anal.*, 19 (1999) 425–453.
- [11] J. M. Melenk, C. Xenophontos and L. Oberbroeckling, Robust exponential convergence of hp-FEM for singularly perturbed systems of reaction-diffusion equations with multiple scales, *IMA J. Num. Anal.*, 33 (2013) 609–628.
- [12] J. M. Melenk, C. Xenophontos and L. Oberbroeckling, Analytic regularity for singularly perturbed systems of reaction-diffusion equations with multiple scales, *Adv. Comp. Math.*, 39 (2013) 367–394.
- [13] J. M. Melenk, C. Xenophontos and L. Oberbroeckling, Analytic regularity for singularly perturbed systems of reaction-diffusion equations with multiple scales: proofs, arXiv:1108.2002v2 (2012).
- [14] R. E. O’Malley, *Singular Perturbation Methods for Ordinary Differential Equations*, Springer-Verlag (1991).
- [15] J. J. H. Miller, E. O’Riordan and G. I. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems*, World Scientific, 1996.
- [16] K. W. Morton, *Numerical Solution of Convection-Diffusion Problems*, Volume 12 of Applied Mathematics and Mathematical Computation, Chapman & Hall, 1996.
- [17] E. O’Riordan, M. L. Pickett and G. I. Shishkin, Singularly perturbed problems modelling reaction-convection-diffusion processes, *Comput. Meth. Appl. Math.*, 3 (2003) 424–442.
- [18] H. G. Roos and S. Franz, Error estimation in a balanced norm for a convection-diffusion problems with two different boundary layers, *Calcolo*, 51 (2014) 423–440.
- [19] H. G. Roos and M. Schopf, Convergence and stability in balanced norms of finite element methods on Shishkin meshes for reaction-diffusion problems, *ZAMM*, 95 (2015) 551–565.
- [20] H.-G. Roos, M. Stynes, and L. Tobiska. *Robust numerical methods for singularly perturbed differential equations*, volume 24 of Springer Series in Computational Mathematics. Springer-Verlag, Berlin, second edition, 2008. Convection-diffusion-reaction and flow problems.
- [21] H.-G. Roos and Z. Uzelac, The SDFEM for a Convection-Diffusion Problem with Two Small Parameters, *Comput. Methods Appl. Math.*, 3 (2003) 443–458.
- [22] C. Schwab, *p - and hp -Finite Element Methods*, Oxford Science Publications, 1998.
- [23] C. Schwab and M. Suri, The p and hp versions of the finite element method for problems with boundary layers, *Math. Comp.*, 65 (1996) 1403–1429.
- [24] G. I. Shishkin, Grid approximation of singularly perturbed boundary value problems with a regular boundary layer, *Sov. J. Numer. Anal. Math. Model.* 4 (1989) 397–417.
- [25] I. Sykopetritou, An hp finite element method for a second order singularly perturbed boundary value problem with two small parameters, M.Sc. Thesis, Department of Mathematics & Statistics, University of Cyprus (2018).
- [26] I. Sykopetritou, The hp finite element method for singularly perturbed boundary value problem with two small parameters, Ph.D. Dissertation, Department of Mathematics & Statistics, University of Cyprus, in preparation (2021).
- [27] I. Sykopetritou and C. Xenophontos, Analytic regularity for a singularly perturbed reaction-convection-diffusion boundary value problem with two small parameters, <http://arxiv.org/abs/1901.09397>.
- [28] Lj. Teofanov and H.-G. Roos, An elliptic singularly perturbed problem with two parameters I: Solution decomposition, *J. Comput. Appl. Math.*, 206 (2007) 1802–1097.
- [29] Lj. Teofanov and H.-G. Roos, An elliptic singularly perturbed problem with two parameters II: Robust finite element solution, *J. Comput. Appl. Math.*, 212 (2008) 374–389.
- [30] C. Xenophontos and I. Sykopetritou, Isogeometric analysis for singularly perturbed problems: error estimates, *ETNA*, 52 (2020) 1–25.

*E-mail:* sykopetritou.eirini@ucy.ac.cy and xenophontos@ucy.ac.cy

Department of Mathematics and Statistics, University of Cyprus, Nicosia, 1678 CYPRUS