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FAST GAUSS-RELATED QUADRATURE FOR HIGHLY OSCILLATORY INTEGRALS WITH LOGARITHM AND CAUCHY-LOGARITHMIC TYPE SINGULARITIES

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Abstract. This paper presents an efficient method for the computation of two highly oscillatory integrals having logarithmic and Cauchy-logarithmic singularities. This approach first requires the transformation of the original oscillatory integrals into a sum of line integrals with semi-infinite intervals. Afterwards, the coefficients of the three-term recurrence relation that satisfy the orthogonal polynomial are obtained by using the method based on moments, where classical Laguerre and Gautschi's logarithmic weight functions are employed. The algorithm reveals that with fixed n, the method is capable of achieving significant figures within a short time. Furthermore, the approach yields higher accuracy as the frequency increases. The results of numerical experiments are given to substantiate our theoretical analysis.

Key words. Highly oscillatory integrals, modified Chebyshev algorithm, steepest descent method, Cauchy principal value integrals, logarithmic weight function, algebraic and logarithm singular integrals.

1. Introduction

Logarithmic singular integrals have numerous applications in wave scattering, diffraction problems, aero and hydroacoustic problems, elasticity problems [1, 2, 3], etc. Boundary Element Methods, abbreviated as BEMs, are some of the popular methods for handling partial differential equations involving the singular boundary integral equations. Nevertheless, when the integral contains simultaneous integrands with singular factors and a very large frequency of oscillation, traditional numerical methods fail to achieve numerical accuracy for these types of integrals. Consequently, three kinds of singularities arise: (i) Weak singularity (ii) Strong singularity and (iii) Hyper singularity.

In this paper, we focus on the efficient computation of two highly oscillatory integrals having logarithmic and Cauchy-logarithmic singularities of the forms

$$Q\left[f\right]:=\int_{a}^{b}e^{ikx}\left(x-a\right)^{\alpha}\left(b-x\right)^{\beta}\left|x-c\right|^{\gamma}\ln\left(x-a\right)\ln\left(b-x\right)f\left(x\right)dx, \tag{1}$$

and

$$Q'\left[f\right]:=\int_{a}^{b}\frac{e^{ikx}\left(x-a\right)^{\alpha}\left(b-x\right)^{\beta}\ln\left(x-a\right)\ln\left(b-x\right)f\left(x\right)}{\left(x-\rho\right)}dx, \tag{2}$$

where $\alpha,\beta,\gamma>-1$, $-\infty<a<c<b<+\infty$, $|k|>>1$, $i^2=-1$, $a<\rho<b$ and $f\left(x\right)$ is a holomorphic function in an appropriate complex region consisting $[a,b]$. While observing the integral (1), one encounters many types of logarithmic oscillatory integrals such as

(1) $\int_a^b e^{ikx}\left(x-a\right)^{\alpha}\left(b-x\right)^{\beta}\ln\left(x-a\right)f\left(x\right)dx$,
(2) $\int_a^b e^{ikx}\left(x-a\right)^{\alpha}\left(b-x\right)^{\beta}\ln\left(x-a\right)\ln\left(b-x\right)f\left(x\right)dx$,

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which, previously, have been discussed extensively in [4] and [5]. The authors applied the truncated Chebyshev expansion for approximating the given smooth function, then computed the singular part using non-homogeneous recurrence relations of modified moments. However, their approach requires a lot of computation time and function evaluation when a very large value of frequency is applied, due to the error analysis on k (frequency) and stability of the recurrence relations. In the example section, we show the accuracy of the proposed method for efficient computation of the aforementioned types of integrals.

Normally, highly oscillatory integrands have the form of $H(x)\, e^{ikg(x)}$, where k is strictly large and well known as the wave number, or simply, as the frequency of oscillation, and where $H(x)$ and $g(x)$ are amplitude and phase functions, respectively. Moreover, $H(x)$ may contain singularities of weak and strong types whereas $g(x)$ has stationary points of a certain order. Generally, evaluation of highly oscillatory integrals is considered as a challenging task, in particular, the logarithmic types. Due to their wide range of applications in various branches of mathematics, applied computational sciences, and other areas of applied science and technology such as electromagnetic scattering, image processing, quantum mechanics, astronomy, seismology, many physical problems; phenomenal methods have been developed for solving oscillatory integrals. Much emphasis has been put in obtaining the best approximation of algebraic singular integrals. For instance, the integrals of types $\int_a^b e^{i\omega x} f(x)\, dx$ and $\int_a^b e^{i\omega x} (x-a)^{\alpha} (b-x)^{\beta} f(x)\, dx$, for more details, one can refer to [6, 7, 8, 9, 10, 11, 12, 13, 14, 30, 31], and the references therein. The integral type (2) exhibits severe shortcomings inasmuch as its integrands involve not only oscillatory but also weak and strong (Cauchy type) singularities. These integrals are also well known as Cauchy-type integrals and have recently become of great interest in the computational community. Commonly, the Cauchy principal value integral of type $\int_a^b f(x) / (x-\rho)\, dx$ with $-\infty \le a < b \le \infty$ such that $a < \rho < b$, is being recognized as Hilbert transformation. The sufficient condition for the existence of the Hilbert integral transform is that $f(x)$ satisfies Lipschitz and Hölder conditions. The integral (2) presents different types of Cauchy oscillatory integrals of which the following can be mentioned:

(1) $\int_a^b e^{ikx} \frac{f(x)}{x-\rho} dx$,
(2) $\int_a^b \frac{e^{ikx}(x-a)^{\alpha}(b-x)^{\beta}\ln(x-a) f(x)}{x-\rho} dx$.

When integrals of the above types arise, the classical Gauss rules cannot directly be applied, due to the fact that the integrands become unbounded at $x = \rho$. Special treatment is needed for their evaluation with a high order numerical accuracy. As a plausible solution, we propose a fast and accurate numerical method.

Herein, integrands are considered to be analytic in an appropriately large region containing the interval of integration. We then employ the Cauchy integral approach to transform integrals (1) and (2) into a sum of several lines integrals with the semi-infinite interval $[0, +\infty)$. Subsequently, we construct a Gaussian-type quadrature rule. While constructing this rule, we utilize a special type of weight function known as Gautchi-Logarithmic weight function [15]. Nonetheless, we use the method-based moments, the modified Chebyshev algorithm, and Jacobi matrix for the efficient computation of nodes and weights used for the construction of the related quadrature rule.

The rest of the paper is structured as follows: In the next section we evaluate the integral (1) and (2) using the Cauchy integral theorem approach. In Section 3, we construct the Gauss-type quadrature rule. Section 4 is dedicated to the numerical experiments in support of our theoretical analysis. Lastly, in Section 5, we give concluding remarks.

2. Evaluation of (1) and (2) by Steepest descent method

The focal point here is to choose a suitable contour; the steepest descent path, in the complex plane so that the original integrals with an oscillatory kernel function can be transformed into a sum of non-oscillatory integrals with an exponentially decaying weight function on $[0, +\infty)$. Afterwards, each of these integrals is computed efficiently by an appropriate Gauss integration rule. Based on Cauchys theorem [28], the value of a contour integral of a holomorphic function $f(z)$ from a fixed point z_1 to another fixed point z_2, does not always depend on the path that is taken. For instance, consider the Fourier oscillatory integral $\int_{-1}^{1} f(x)\, e^{ikh(x)} dx$, where f and h are non-oscillatory, nonsingular smooth functions, and $k >> 1$ is an oscillatory parameter, respectively. In order to acquire the steepest descent method, we follow the path where h has a constant real part and growing imaginary part, which makes the Fourier integral non-oscillatory and exponentially decaying. This method is well known to have the highest asymptotic order. However, before it is utilized f has to be sufficiently holomorphic in the large enough region of the complex plane accommodating the interval of integration, then we can put Cauchy's theorem into application. For more about steepest descent method, refer to [27, 6, 26].

Theorem 1 **([[29], p. 115]).** Assume that $f(z)$ is sufficiently analytic in the region $\Re$ and on its boundary C. Then

$$\oint_C f(z)\, dz = 0.$$

This theorem is known as the Cauchy-Goursat theorem and is employed to prove Theorem 2 and 3.

Herein, we assume that $y = \ln z$ and can be defined as $y = \ln z = \ln|z| + i(\theta + 2k\pi)$, $k = 0, \pm 1, ...$ where $z = |z|\, e^{i(\theta+2k\pi)}$. Since $\ln(z)$ is a multi-valued function, we define our principal branch as $\ln|z| + i\theta$, where $0 \le \theta < 2\pi$, in order to get a single valued function.

2.1. Evaluation of logarithmic highly oscillatory integral (1).

Theorem 2. Assume that f is analytic inside and on a simple closed path in the z plane with the property that $Re(z) \in [a, c] \bigcup [c, b]$, $Im(z) \in [0, \infty)$ and suppose that with R sufficiently large enough, there exist two non-negative constants M and k_0 such that for all $x \in [a, b]$

$$|f(x + iR)| \le M e^{k_0 R}.$$

Then $Q[f]$ can be expressed as

$$Q[f] = P_{1\gamma} Q_1[f] + P_{1\alpha} Q_2[f] + P_{2\beta} Q_3[f] + P_{2\gamma} Q_4[f], \tag{3}$$

where

$$\begin{aligned} Q_1[f] = \int_0^\infty & \left(c - a + \tfrac{i}{k}t\right)^\alpha \left(b - c - \tfrac{i}{k}t\right)^\beta \ln\left(c - a + \tfrac{i}{k}t\right) \\ & \times \ln\left(b - c - \tfrac{i}{k}t\right) f\left(c + \tfrac{i}{k}t\right) t^\gamma e^{-t} dt, \\ Q_2[f] = \int_0^\infty & \left(b - a - \tfrac{i}{k}t\right)^\beta \left(c - a - \tfrac{i}{k}t\right)^\gamma \ln\left(\tfrac{i}{k}t\right) \\ & \times \ln\left(b - a - \tfrac{i}{k}t\right) f\left(a + \tfrac{i}{k}t\right) t^\alpha e^{-t} dt, \end{aligned} \tag{4}$$

$$P_{1\gamma} = \frac{(-i)^{\gamma+1} e^{ikc}}{k^{\gamma+1}}, P_{1\alpha} = \left(\frac{i}{k}\right)^{\alpha+1} e^{ika}, \tag{5}$$

and

$$\begin{aligned} Q_3[f] = \int_0^\infty & \left(b-a+\tfrac{i}{k}t\right)^\alpha \left(b-c+\tfrac{i}{k}t\right)^\gamma \ln\left(b-a+\tfrac{i}{k}t\right) \\ & \times \ln\left(-\tfrac{i}{k}t\right) f\left(b+\tfrac{i}{k}t\right) t^\beta e^{-t} dt, \\ Q_4[f] = \int_0^\infty & \left(c-a+\tfrac{i}{k}t\right)^\alpha \left(b-c-\tfrac{i}{k}t\right)^\beta \ln\left(c-a+\tfrac{i}{k}t\right) \\ & \times \ln\left(b-c-\tfrac{i}{k}t\right) f\left(c+\tfrac{i}{k}t\right) t^\gamma e^{-t} dt, \end{aligned} \tag{6}$$

$$P_{2\beta} = \frac{(-i)^{\beta+1} e^{ikb}}{k^{\beta+1}}, P_{2\gamma} = \left(\frac{i}{k}\right)^{\gamma+1} e^{ikc}. \tag{7}$$

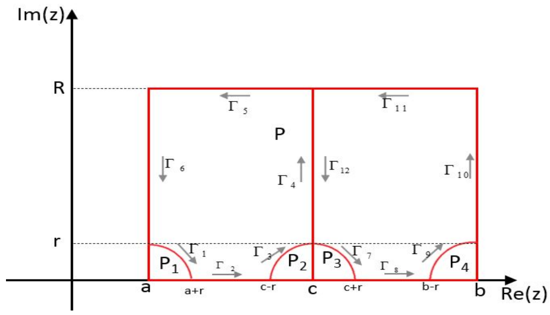

FIGURE 1. Integration path for the integral (1).

Proof: Since the integral (1) contains an absolute-valued integrand function, it can be split into a sum of two integrals of the forms

$$Q[f] = \int_a^c \varphi_1(x)\,dx + \int_c^b \varphi_2(x)\,dx, \tag{8}$$

where

$$\varphi_1(x) = (x-a)^\alpha (b-x)^\beta (c-x)^\gamma \ln(x-a) \ln(b-x) f(x) e^{ikx}, \tag{9}$$

and

$$\varphi_2(x) = (x-a)^\alpha (b-x)^\beta (x-c)^\gamma \ln(x-a) \ln(b-x) f(x) e^{ikx}. \tag{10}$$

Then we define the regions $P = \{z \in \mathbb{C} : a \le Re(z) \le b, 0 \le Im(z) \le R\}$, $P_1 = \{z \in \mathbb{C} : |z-a| \le r,\ 0 \le \theta \le \frac{\pi}{2}\}$, $P_2 = \left\{z \in \mathbb{C} : |z-c| \le r, \frac{\pi}{2} \le \theta \le \pi\right\}$, $P_3 = \left\{z \in \mathbb{C} : |z-c| \le r, 0 \le \theta \le \frac{\pi}{2}\right\}$ and $P_4 = \{z \in \mathbb{C} : |z-b| \le r, \frac{\pi}{2} \le \theta \le \pi\}$, where P contains P_1, P_2, P_3 and P_4 for sufficiently small r. Inasmuch as the integrands are considered analytic in an appropriately large region containing P_1, P_2, P_3 and P_4, by applying the Cauchy-Goursat theorem, we have

$$\begin{aligned} & \int_{a+r}^{c-r} \varphi_1(x)\,dx + \int_{c+r}^{b-r} \varphi_2(x)\,dx \\ & = -\left(\oint_{\Gamma_1} \varphi_1(z)\,dz + \sum_{i=3}^{6} \oint_{\Gamma_i} \varphi_1(z)\,dz + \oint_{\Gamma_7} \varphi_2(z)\,dz + \sum_{j=9}^{12} \oint_{\Gamma_j} \varphi_2(z)\,dz\right). \end{aligned} \tag{11}$$

The direction of the integrals in (11) is counterclockwise as depicted in the Figure 1. It is rather simple to demonstrate that the integrals over the quarter circles $\Gamma_1, \Gamma_3, \Gamma_7$ and Γ_9 tend to zero as $r \to 0$ for both integrands $\varphi_1(z)$ and $\varphi_2(z)$. In fact, for example, by considering the integral on the quarter circle $\Gamma_1 : z = a + re^{i\theta}$, for $\theta \in \left[0, \frac{\pi}{2}\right]$, we have

$$\begin{aligned} \left|\oint_{\Gamma_1} \varphi_1(z)\,dz\right| &= \left|-\int_0^{\frac{\pi}{2}} \varphi_1\left(a + re^{i\theta}\right) d\theta\right| \\ &\leq r^{1+\alpha}\left(|\ln r| \int_0^{\frac{\pi}{2}} |F(r,\theta)|\,d\theta + \int_0^{\frac{\pi}{2}} |\theta||F(r,\theta)|\,d\theta\right), \end{aligned} \tag{12}$$

where

$$F(r,\theta) = \left(b - a - re^{i\theta}\right)^{\beta}\left(c - a - re^{i\theta}\right)^{\gamma} \ln\left(b - a - re^{i\theta}\right) f\left(a + re^{i\theta}\right) e^{ik\left(a + re^{i\theta}\right)}.$$

It is easy to see that the function $F(t,\theta)$ is continuous for all $t \in [0, r]$ and $\theta \in \left[0, \frac{\pi}{2}\right]$. Taking the limit in the both sides as $r \to 0$, this results $\oint_{\Gamma_1} \varphi_1(z)\,dz \to 0$. A similar technique can be implemented to obtain $\oint_{\Gamma_3} \varphi_1(z)\,dz \to 0$, $\oint_{\Gamma_7} \varphi_2(z)\,dz \to 0$ and $\oint_{\Gamma_9} \varphi_2(z)\,dz \to 0$. We proceed by considering $\Gamma_5 : z = x + iR$, for $x \in [a, c]$, we find

$$\begin{aligned} \left|\oint_{\Gamma_5} \varphi_1(z)\,dz\right| &= \left|-\int_a^c \varphi_1(x + iR)\,dx\right| \\ &\leq e^{-kR}\int_a^c |f(x + iR)|\,|\phi(x + iR)|\,dx \\ &\leq MM_1 e^{R(k_0 + k_1 - k)}(c - a), \end{aligned}$$

where $\phi(x + iR)$ can be defined as

$$\begin{aligned} \phi(x + iR) = &(x - a + iR)^{\alpha}(b - iR - x)^{\beta}(c - iR - x)^{\gamma} \\ &\cdot \ln(x - a + iR)\ln(b - x - iR). \end{aligned}$$

In the above expressions we can choose $M_1 > 0$, $k_1 > 0$, such that $|\phi(x + iR)| \leq M_1 e^{k_1 R}$, for sufficiently large R. Then it follows that if $k > k_0 + k_1$, $\oint_{\Gamma_5} \varphi_1(z)\,dz \to 0$ as $R \to \infty$. We can apply the same procedure to conclude that as $R \to \infty$, $\oint_{\Gamma_{11}} \varphi_1(z)\,dz \to 0$, when $\Gamma_{11} : z = x + iR$, for $x \in [c, b]$. Moreover, we can also observe that by considering $\Gamma_4 : z = c + ix$, for $x \in [r, R]$, as $r \to 0$, $R \to \infty$, and letting $x = t/k$, respectively. The integral over Γ_4 can be written as

$$\begin{aligned} \oint_{\Gamma_4} \varphi_1(z)\,dz = &P_{1\gamma}\int_0^{\infty} \left(c - a + \tfrac{i}{k}t\right)^{\alpha}\left(b - c - \tfrac{i}{k}t\right)^{\beta} t^{\gamma} \ln\left(c - a + \tfrac{i}{k}t\right) \\ &\times \ln\left(b - c - \tfrac{i}{k}t\right) f\left(c + \tfrac{i}{k}t\right) e^{-t}\,dt, \end{aligned} \tag{13}$$

same as

$$\begin{aligned} \oint_{\Gamma_6} \varphi_1(z)\,dz = &P_{1\alpha}\int_0^{\infty} t^{\alpha}\left(b - a - \tfrac{i}{k}t\right)^{\beta}\left(c - a - \tfrac{i}{k}t\right)^{\gamma} \ln\left(\tfrac{i}{k}t\right) \\ &\times \ln\left(b - a - \tfrac{i}{k}t\right) f\left(a + \tfrac{i}{k}t\right) e^{-t}\,dt. \end{aligned} \tag{14}$$

For the second integrand $\varphi_2(z)$, same procedure yields

$$\begin{aligned} \oint_{\Gamma_{10}} \varphi_2(z)\,dz = &P_{2\beta}\int_0^{\infty} t^{\beta}\left(b - a + \tfrac{i}{k}t\right)^{\alpha}\left(b - c + \tfrac{i}{k}t\right)^{\gamma} \ln\left(-\tfrac{i}{k}t\right) \\ &\times \ln\left(b - a + \tfrac{i}{k}t\right) f\left(b + \tfrac{i}{k}t\right) e^{-t}\,dt, \end{aligned} \tag{15}$$

and

$$\oint_{\Gamma_{12}} \varphi_2(z)\,dz = P_{2\gamma}\int_0^\infty \left(c-a+\tfrac{i}{k}t\right)^\alpha \left(b-c-\tfrac{i}{k}t\right)^\beta t^\gamma \ln\left(c-a+\tfrac{i}{k}t\right) \times \ln\left(b-c-\tfrac{i}{k}t\right) f\left(c+\tfrac{i}{k}t\right) e^{-t}dt. \tag{16}$$

Taking the sum of the above evaluated integrals and substituting them into (11) complete the proof.

2.2. Evaluation of logarithmic-Cauchy integral (2). The results of evaluating the integral (2) are summarized in the following theorem:

Theorem 3. Let f be analytic inside and on a simple closed path in the z plane with the property that $C = \{z \in \mathbb{C} : a \le Re(z) \le b, Im(z) \in [0,\infty)\}$. If there exist two non-negative constants M and λ_0 such that $|f(z)| \le Me^{\lambda_0|z|}$, we have

$$Q'[f] = Q_1'[f] + Q_2'[f] + i\pi e^{ik\rho}G(\rho), \tag{17}$$

where

$$Q_1'[f] = \left(\frac{i}{k}\right)^{1+\alpha} e^{ika}\int_0^\infty \frac{\left(b-a-\frac{i}{k}t\right)^\beta \ln\left(\frac{i}{k}t\right)\ln\left(b-a-\frac{i}{k}t\right) f\left(a+\frac{i}{k}t\right)t^\alpha e^{-t}}{\left(a-\rho+\frac{i}{k}t\right)}dt, \tag{18}$$

$$Q_2'[f] = \frac{(-i)^{1+\beta}}{k^{1+\beta}} e^{ikb}\int_0^\infty \frac{\left(b-a+\frac{i}{k}t\right)^\alpha \ln\left(-\frac{i}{k}t\right)\ln\left(b-a+\frac{i}{k}t\right) f\left(b+\frac{i}{k}t\right)t^\beta e^{-t}}{\left(b-\rho+\frac{i}{k}t\right)}dt, \tag{19}$$

and

$$G(\rho) = (\rho-a)^\alpha (b-\rho)^\beta \ln(\rho-a)\ln(b-\rho) f(\rho). \tag{20}$$

Proof: Let us define the regions $C = \{z \in \mathbb{C} : a \le Re(z) \le b, 0 \le Im(z) \le R\}$, $C_1 = \{z \in \mathbb{C} : |z-a| \le r,\ 0 \le \theta \le \frac{\pi}{2}\}$, $C_2 = \{z \in \mathbb{C} : |z-\rho| \le r, 0 \le \theta \le \pi\}$, and $C_3 = \left\{z \in \mathbb{C} : |z-b| \le r, \frac{\pi}{2} \le \theta \le \pi\right\}$, where the length R is large enough and r is a small positive quantity such that C contains C_1, C_2 and C_3. Since the integrand $\psi(x) = (x-a)^\alpha (b-x)^\beta \ln(x-a)\ln(b-x) f(x) e^{ikx}/(x-\rho)$ is considered analytic in an appropriately large region containing C then by Cauchy-Goursat theorem, we have

$$\int_{a+r}^{\rho-r}\psi(x)\,dx + \int_{\rho+r}^{b-r}\psi(x)\,dx = -\left(\oint_{\Gamma_1}\psi(z)\,dz + \oint_{\Gamma_3}\psi(z)\,dz + \sum_{m=5}^{8}\oint_{\Gamma_m}\psi(z)\,dz\right), \tag{21}$$

where the orientation is in the counterclockwise direction as depicted in the Figure 2. We can easily demonstrate that the integrals over quarter circles Γ_1 and Γ_5 result in zero as $r \to 0$, using the same techniques applied in the previous sub-section. On the quarter circle $\Gamma_1 : z = a + re^{i\theta}$, for $\theta \in \left[0, \frac{\pi}{2}\right]$, we have

$$\begin{aligned}\left|\oint_{\Gamma_1}\psi(z)\,dz\right| &= \left|-\int_0^{\frac{\pi}{2}}\psi\left(a+re^{i\theta}\right)d\theta\right| \\ &\le r^{1+\alpha}\left(|\ln(r)|\int_0^{\frac{\pi}{2}}|H(r,\theta)|\,d\theta + \int_0^{\frac{\pi}{2}}|\theta|\,|H(r,\theta)|\,d\theta\right),\end{aligned}$$

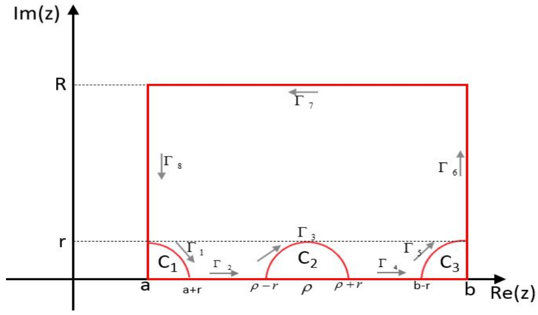

FIGURE 2. Integration path for the integral (2).

where

$$|H(r,\theta)| = \frac{\left|b-a-re^{i\theta}\right|^{\beta}\left|\ln\left(b-a-re^{i\theta}\right)\right|\left|f\left(a+re^{i\theta}\right)\right|}{\left|a-\rho+re^{i\theta}\right|}.$$

It can be shown that the function $H(t,\theta)$ is continuous on $t\in[0,r]$ and $\theta\in[0,\frac{\pi}{2}]$. Thus taking the limit in both sides, we have $\oint_{\Gamma_1}\psi(z)\,dz\to 0$, as $r\to 0$. Similarly, $\oint_{\Gamma_5}\psi(z)\,dz\to 0$, as $r\to 0$. On Γ_7 since $z=x+iR$, for $x\in[a,b]$, we get

$$\begin{aligned}\left|\oint_{\Gamma_7}\psi(z)\,dz\right| &= \left|-\int_a^b\psi(x+iR)\,dx\right|\\ &\le e^{-kR}\int_a^b|f(x+iR)|\,|\psi_1(x+iR)|dx\\ &\le MM_2e^{-R(k-\Lambda)}(b-a),\end{aligned}$$

where $\Lambda=\lambda_0+\lambda$, and $\psi_1(x+iR)$ can be defined as

$$\psi_1(x+iR) = \frac{(x-a+iR)^{\alpha}(b-iR-x)^{\beta}\ln(x-a+iR)\ln(b-iR-x)}{(x-\rho+iR)}.$$

Note that $M_2>0$, $\lambda>0$, since $|\psi_1(x+iR)|\le M_2e^{\lambda R}$ for sufficiently large R. Thus it follows that the integral $\oint_{\Gamma_7}\psi(z)\,dz\to 0$, as $R\to\infty$. We observe that by considering Γ_8 since $z=a+ix$, for $x\in[r,R]$, as $r\to 0, R\to\infty$, and assuming $x=t/k$. The integral over Γ_8 yields

$$\oint_{\Gamma_8}\psi(z)\,dz = \left(\frac{i}{k}\right)^{1+\alpha}e^{ika}$$

$$\times\int_0^{\infty}\frac{\left(b-a-\frac{i}{k}t\right)^{\beta}t^{\alpha}\ln\left(\frac{i}{k}t\right)\ln\left(b-a-\frac{i}{k}t\right)f\left(a+\frac{i}{k}t\right)e^{-t}}{\left(a-\rho+\frac{i}{k}t\right)}idt. \tag{22}$$

Same as integral over Γ_6 yields

$$\oint_{\Gamma_6}\psi(z)\,dz = \frac{(-i)}{k^{1+\beta}}^{1+\beta}e^{ikb}$$

$$\times\int_0^{\infty}\frac{\left(b-a+\frac{i}{k}t\right)^{\alpha}t^{\beta}\ln\left(-\frac{i}{k}t\right)\ln\left(b-a+\frac{i}{k}t\right)f\left(b+\frac{i}{k}t\right)e^{-t}}{\left(b-\rho+\frac{i}{k}t\right)}dt. \tag{23}$$

The calculation of integral over Γ_3 results in

$$\oint_{\Gamma_3}\psi(z)\,dz$$

$$= \int_0^{\pi}\frac{t^{\alpha}\ln\left(\rho-a+re^{i\theta}\right)\ln\left(b-\rho-re^{i\theta}\right)f\left(\rho+re^{i\theta}\right)e^{ik\rho}e^{ikre^{i\theta}}}{\left(\rho-a+re^{i\theta}\right)^{-\alpha}\left(b-\rho-re^{i\theta}\right)^{-\beta}\left(re^{i\theta}\right)}ire^{i\theta}d\theta, \tag{24}$$

as $r \to 0$, the integral is reduced to the form

$$\oint_{\Gamma_3} \psi(z)\, dz = i\pi e^{ik\rho} (\rho - a)^{\alpha} (b-\rho)^{\beta} \ln(\rho - a) \ln(b-\rho) f(\rho). \tag{25}$$

Substituting the above-evaluated integrals into (21) produces a result that proves Theorem 3.

3. Computation of (3) and (17) by Gauss-type quadrature rule

Before we start the construction of the Gaussian-related quadrature rule, let us introduce the following notations for $Q[f]$;

$$\begin{aligned} F_{1\gamma}(t) =& F_{2\gamma}(t) \\ &= \left(c - a + \tfrac{i}{k}t\right)^{\alpha} \left(b - c - \tfrac{i}{k}t\right)^{\beta} \ln\left(c - a + \tfrac{i}{k}t\right) \\ &\quad \cdot \ln\left(b - c - \tfrac{i}{k}t\right) f\left(c + \tfrac{i}{k}t\right), \\ F_{1\alpha}(t) &= \left(b - a - \tfrac{i}{k}t\right)^{\beta} \left(c - a - \tfrac{i}{k}t\right)^{\gamma} \ln\left(b - a - \tfrac{i}{k}t\right) f\left(a + \tfrac{i}{k}t\right), \\ F_{2\beta}(t) &= \left(b - a + \tfrac{i}{k}t\right)^{\alpha} \left(b - c + \tfrac{i}{k}t\right)^{\gamma} \ln\left(b - a + \tfrac{i}{k}t\right) f\left(b + \tfrac{i}{k}t\right). \end{aligned} \tag{26}$$

Similarly, for $Q'[f]$, let

$$P'_{1\alpha} = \left(\frac{i}{k}\right)^{1+\alpha} e^{ika}, P'_{2\beta} = \frac{(-i)}{k^{1+\beta}}^{1+\beta} e^{ikb}, \tag{27}$$

$$F'_{1\alpha}(t) = \frac{\left(b - a - \frac{i}{k}t\right)^{\beta} \ln\left(b - a - \frac{i}{k}t\right) f\left(a + \frac{i}{k}t\right)}{\left(a - \rho + \frac{i}{k}t\right)} \tag{28}$$

and

$$F'_{2\beta}(t) = \frac{\left(b - a + \frac{i}{k}t\right)^{\alpha} \ln\left(b - a + \frac{i}{k}t\right) f\left(b + \frac{i}{k}t\right)}{\left(b - \rho + \frac{i}{k}t\right)}. \tag{29}$$

Then $Q[f]$ and $Q'[f]$ can be written in the simple forms of

$$\begin{aligned} Q[f] = (P_{1\gamma} + P_{2\gamma}) \int_0^{\infty} F_{1\gamma}(t)\, t^{\gamma} e^{-t} dt + P_{1\alpha} \int_0^{\infty} \ln\left(\tfrac{i}{k}t\right) F_{1\alpha}(t)\, t^{\alpha} e^{-t} dt \\ + P_{2\beta} \int_0^{\infty} \ln\left(-\tfrac{i}{k}t\right) F_{2\beta}(t)\, t^{\beta} e^{-t} dt, \end{aligned} \tag{30}$$

and

$$Q'[f] = P'_{1\alpha} \int_0^{\infty} \ln\left(\tfrac{i}{k}t\right) F'_{1\alpha}(t)\, t^{\alpha} e^{-t} dt + P'_{2\beta} \int_0^{\infty} \ln\left(-\tfrac{i}{k}t\right) F'_{2\beta}(t)\, t^{\beta} e^{-t} dt. \tag{31}$$

In the equations (30) and (31) one encounters logarithmic singular factors $\ln\left(\frac{i}{k}t\right)$ and $\ln\left(-\frac{i}{k}t\right)$ at $t = 0$, which cannot be computed directly. Indeed, some modifications are required. Using the same idea of Gautschi [15], we can expand those factors into the forms below

$$\begin{aligned} \ln\left(\tfrac{i}{k}t\right) &= \left(\ln\left(\tfrac{i}{k}\right) - 1\right) + t - (t - 1 - \ln t), \\ \ln\left(-\tfrac{i}{k}t\right) &= \left(\ln\left(-\tfrac{i}{k}\right) - 1\right) + t - (t - 1 - \ln t). \end{aligned} \tag{32}$$

Here $\ln i = \frac{i\pi}{2}$, and $\ln(-i) = -\frac{i\pi}{2}$, since our principal branch was defined as $\ln|z| + i\theta$, for $\theta \in [0, 2\pi)$. Moreover, by substituting the expression (32) into (30) and (31) we observe that several Generalized Gauss-Laguerre and Logarithmic Gautschi's weight functions emerge over the semi-infinite interval $[0, \infty)$. These weight functions are presented in Table 1 below.

Weight functions for $Q[f]$	Weight functions for $Q'[f]$
$\omega^{(\gamma)}(t) = t^{\gamma}e^{-t}$ $\omega^{(\alpha)}(t) = t^{\alpha}e^{-t}, \omega^{(\alpha+1)}(t) = t^{\alpha+1}e^{-t}$ $\omega^{(\alpha,L)}(t) = t^{\alpha}(t-1-\ln t)e^{-t}$ $\omega^{(\beta)}(t) = t^{\beta}e^{-t}, \omega^{(\beta+1)}(t) = t^{\beta+1}e^{-t}$ $\omega^{(\beta,L)}(t) = t^{\beta}(t-1-\ln t)e^{-t}$	$\omega'^{(\alpha)}(t) = t^{\alpha}e^{-t}, \omega'^{(\alpha+1)}(t) = t^{\alpha+1}e^{-t}$ $\omega'^{(\beta)}(t) = t^{\beta}e^{-t}, \omega'^{(\beta+1)}(t) = t^{\beta+1}e^{-t}$ $\omega'^{(\alpha,L)}(t) = t^{\alpha}(t-1-\ln t)e^{-t}$ $\omega'^{(\beta,L)}(t) = t^{\beta}(t-1-\ln t)e^{-t}$.

Table 1. Generalized Gauss-Laguerre and Logarithmic Gauss-Laguerre weights functions.

In Table 1, $\omega^{(y)}(t;y) = t^{y}e^{-t}, (y = \{\alpha,\beta,\gamma\} > -1)$ represents ordinal Gauss-Laguerre weight functions over $[0,\infty)$, whereas $\omega^{(y,L)}(t;y) = t^{y}e^{-t}(t-1-\ln t), (y = \{\alpha,\beta\} > -1)$ represents Logarithmic Gauss-Laguerre weight functions over $[0,\infty)$.

To deal with (30) and (31), let $\left\{t_j^{(\alpha)},\omega_j^{(\alpha)}\right\}_{j=1}^{n}$, $\left\{t_j^{(\alpha+1)},\omega_j^{(\alpha+1)}\right\}_{j=1}^{n}$, $\left\{t_j^{(\beta)},\omega_j^{(\beta)}\right\}_{j=1}^{n}$, $\left\{t_j^{(\beta+1)},\omega_j^{(\beta+1)}\right\}_{j=1}^{n}$, $\left\{t_j^{(\alpha,L)},\omega_j^{(\alpha,L)}\right\}_{j=1}^{n}$, $\left\{t_j^{(\beta,L)},\omega_j^{(\beta,L)}\right\}_{j=1}^{n}$, $\left\{t_j^{(\gamma)},\omega_j^{(\gamma)}\right\}_{j=1}^{n}$, $\left\{t_j^{'(\alpha)},\omega_j^{'(\alpha)}\right\}_{j=1}^{n}$, $\left\{t_j^{'(\alpha+1)},\omega_j^{'(\alpha+1)}\right\}_{j=1}^{n}$, $\left\{t_j^{'(\beta)},\omega_j^{'(\beta)}\right\}_{j=1}^{n}$, $\left\{t_j^{'(\beta+1)},\omega_j^{'(\beta+1)}\right\}_{j=1}^{n}$, $\left\{t_j^{'(\alpha,L)},\omega_j^{'(\alpha,L)}\right\}_{j=1}^{n}$, $\left\{t_j^{'(\beta,L)},\omega_j^{'(\beta,L)}\right\}_{j=1}^{n}$ be the nodes and weights of the n-point Generalized Gauss-Laguerre [24] and Gauss-Gautschi-Laguerre quadrature rules for $Q[f]$ and $Q'[f]$ that are associated with the weight functions mentioned in Table 1. Then the related quadrature rule can be given by

$$
\begin{aligned}
Q_n(f) = {} & (P_{1\gamma}+P_{2\gamma})\sum_{j=1}^{n}\omega_j^{(\gamma)}F_{1\gamma}\left(t_j^{(\gamma)}\right) + P_{1\alpha}\left(\ln\left(\tfrac{i}{k}\right)-1\right)\sum_{j=1}^{n}\omega_j^{(\alpha)}F_{1\alpha}\left(t_j^{(\alpha)}\right) \\
& + P_{1\alpha}\sum_{j=1}^{n}\omega_j^{(\alpha+1)}F_{1\alpha}\left(t_j^{(\alpha+1)}\right) - P_{1\alpha}\sum_{j=1}^{n}\omega_j^{(\alpha,L)}F_{1\alpha}\left(t_j^{(\alpha,L)}\right) \\
& + P_{2\beta}\left(\ln\left(-\tfrac{i}{k}\right)-1\right)\sum_{j=1}^{n}\omega_j^{(\beta)}F_{2\beta}\left(t_j^{(\beta)}\right) + P_{2\beta}\sum_{j=1}^{n}\omega_j^{(\beta+1)}F_{2\beta}\left(t_j^{(\beta+1)}\right) \\
& - P_{2\beta}\sum_{j=1}^{n}\omega_j^{(\beta,L)}F_{2\beta}\left(t_j^{(\beta,L)}\right) + R_n(f),
\end{aligned}
\tag{33}
$$

and

$$
\begin{aligned}
Q_n^{'}(f) = {} & P_{1\alpha}^{'}\left(\ln\left(\tfrac{i}{k}\right)-1\right)\sum_{j=1}^{n}\omega_j^{'(\alpha)}F_{1\alpha}^{'}\left(t_j^{'(\alpha)}\right) + P_{1\alpha}^{'}\sum_{j=1}^{n}\omega_j^{'(\alpha+1)}F_{1\alpha}^{'}\left(t_j^{'(\alpha+1)}\right) \\
& - P_{1\alpha}^{'}\sum_{j=1}^{n}\omega_j^{'(\alpha,L)}F_{1\alpha}^{'}\left(t_j^{'(\alpha,L)}\right) + P_{2\beta}^{'}\left(\ln\left(-\tfrac{i}{k}\right)-1\right)\sum_{j=1}^{n}\omega_j^{'(\beta)}F_{2\beta}^{'}\left(t_j^{'(\beta)}\right) \\
& + P_{2\beta}^{'}\sum_{j=1}^{n}\omega_j^{'(\beta+1)}F_{2\beta}^{'}\left(t_j^{'(\beta+1)}\right) - P_{2\beta}^{'}\sum_{j=1}^{n}\omega_j^{'(\beta,L)}F_{2\beta}^{'}\left(t_j^{'(\beta,L)}\right) + R_n^{'}(f).
\end{aligned}
\tag{34}
$$

j	α_j^L	β_j^L
0	0.32822320338703947023971687791117	1.33367222431043314496521628609 04
1	4.40438612155818996017443174768 79	0.89056615210958550059998886037 33
2	6.33694099857093585615934184225 96	4.19348096644097562993821438482 46
3	7.91470004566468868074971040596 92	10.7287859007371181292238422714 70
4	9.48153016379823408676121229618 04	20.3585488569654405218482399591 06

Table 2. Coefficients $\left\{\alpha_j^L, \beta_j^L\right\}_{j=0}^{4}$ corresponding to $t^{\alpha}\left(t-1-\ln t\right)e^{-t}$, for $\alpha = -\frac{1}{3}$.

Here $t_j^{(s)}, t_j^{\prime(s)}, \omega_j^{(s)}$ and $\omega_j^{\prime(s)}$ for $s \in \{\alpha, \alpha+1, (\alpha, L), \beta, \beta+1, (\beta, L)\}$ are the nodes and weight coefficients for the rules, whereas $R_n\left(f\right)$ and $R_n^{\prime}\left(f\right)$ are the remainder terms. Computation of nodes and weight require $O\left(n\right)$ operations.

A recursive routine is required to automatically synthesize the coefficients of the three-term recurrence relation that satisfy the orthogonal polynomial. As soon as the coefficients are known, the generation of nodes and weights needed to compute the Gauss related quadrature rule becomes a straightforward matter. The method used in this paper is based on moment information. It is well known that the first n recursion coefficients α_j and β_j for $j = 0, 1, ..., 2n-1$ can be calculated using the first $2n$ moments $\mu_j = \int_{\mathrm{R}} x^j \omega\left(x\right) dx,\ j = 0, 1, ..., 2n-1$, when α_j's and β_j's are being expressed in terms of Hankel determinants. However, the problem is that as n becomes large the formulas become increasingly sensitive to small errors unless we use a sufficiently high precision arithmetic. One can employ the so called modified moments $\mu_j^{\prime} = \int_{\mathrm{R}} L_j\left(x\right)\omega\left(x\right)dx$, where L_j is an appropriate orthogonal polynomial and $\omega\left(x\right)$ is its corresponding weight function. Moreover, the system $\{L_j\}$ are assumed to be monic polynomials that satisfy a three term recurrence relation that is for $a_j \in \mathbb{R}$, $b_j \in [0, \infty)$, $L_{j+1}\left(x\right) = \left(x - a_j\right)L_j\left(x\right) - b_j L_{j-1}\left(x\right),\ j = 0, 1, 2, ...,$ $L_{-1}\left(x\right) = 0$, and $L_0\left(x\right) = 1$. Thanks to the modified Chebyshev algorithm; which takes $2n$ modified moments and the $2n-1$ coefficients $\{a_j, b_j\}_{j=0}^{2n-2}$ to generate the coefficients α_j and β_j for which we strongly desired; where $j = 0, ..., n-1$. For more about these check ([15, 16, 17, 18, 19] and the references therein). For example, using our code written in Mathematica we can generate the coefficients according to n. Generated coefficients for $n = 5$ are tabulated in Table 2.

The nodes t_j, $j = 1, ..., n$ are the eigenvalues of the known (Jacobi matrix) symmetric tridiagonal matrix J_n whose diagonal elements are $\{\alpha_j\}_{j=0}^{n-1}$ and sub-diagonal elements are $\left\{\sqrt{\beta_j}\right\}_{j=1}^{n-1}$ [20]. The QR method proposed by Francis in [21, 22] is the best method available to calculate eigenvalues of that tridiagonal system. The method converges very rapidly for symmetric matrices. Moreover, the QR method is about ten times as fast as the widely used Jacobi method when eigenvalues are desired and four times as fast if only eigenvectors are desired. Consequently, in order to determine the weights $\{\omega_j\}_{j=1}^{n}$, one has to compute the eigenvectors corresponding to the eigenvalues t_j. Computation of eigenvalues and eigenvectors of the Jacobi matrix J_n of order n requires only $O\left(n^2\right)$ operations. As demonstrated by Golup and Welsch in [23], only the first component of the orthonormal eigenvector needs to be computed in order to obtain the weights of the quadrature rule. Therefore, the weights are given by $\omega_j = \beta_0\left(v_{j,1}\right)^2,\ j = 1, ..., n$ where $v_{j,1}$ is the normalized eigenvector corresponding to eigenvalue t_j. By definition, β_0 is calculated by taking $\beta_0 = \mu_0 = \int_a^b \omega(x)dx$. For example from the weight function

j	$t_j^{(\alpha,L)}$	$\omega_j^{(\alpha,L)}$
1	13.957575204455362126945003075789	0.0000277567751626219469638065128
2	8.1674247997771366523812331801089	0.0032492052331344293859393891194
3	4.4282634869888655590793168724134	0.0429201348512020415744680538662
4	1.8475250172211450267292602511115	0.0769017081450287624161493640741
5	0.0649920245365786889495997905852	1.2105734193059052896416956725177

Table 3. Nodes and weights $\left\{t_j^{(\alpha,L)}, \omega_j^{(\alpha,L)}\right\}_{j=1}^{5}$ corresponding to $t^{\alpha}\,(t-1-\ln t)\,e^{-t}$, for $\alpha = -\frac{1}{3}$.

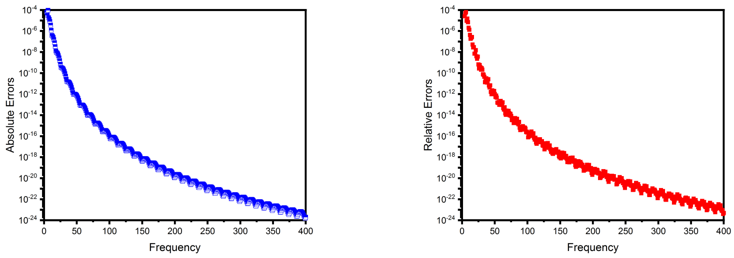

Figure 3. Convergence rate for the integral (35).

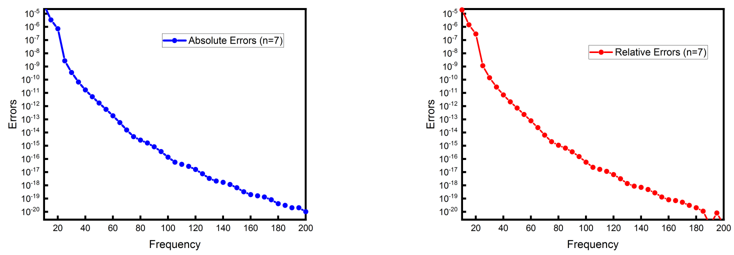

Figure 4. Convergence rate for the Cauchy integral (41).

$\omega^{(\alpha,L)}\,(t) = t^{\alpha}e^{-t}(t-1-\ln t)$, the modified moments can be calculated as

$$\mu_j^{'} = \int_0^{\infty} \omega^{(\alpha,L)}\,(t)\,L_j^{(\alpha)}(t)dt = \begin{cases} \Gamma(\alpha+1)[\alpha-\psi(\alpha+1)], & \text{if } j=0 \\ \alpha\Gamma(\alpha+1), & \text{if } j=1 \\ (-1)^j(j-1)!\Gamma(\alpha+1), & \text{if } j\geq 2, \end{cases}$$

where $\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt$ is the Gamma function and $\psi(z)$ is the logarithmic derivative of the Gamma function known as the Digamma function [25]. Therefore, the coefficient β_0 can be obtained by $\beta_0 = \alpha\Gamma(\alpha+1) - \Gamma(\alpha+1)\psi(\alpha+1)$. In Table 3, there are the weights and nodes computed using the modified moments calculation.

4. Numerical examples

Herein, selected concrete numerical experiments are exhibited for the purpose of demonstrating the effectiveness of the proposed algorithm. The assumed-to-be-exact values are obtained using Mathematica 9.0 with 30 digits precision. The

k	$E_n(Q)$	Computation time
10^2	$2.88\text{x}10^{-16}$	0.125
10^3	$0.11\text{x}10^{-28}$	0.093
10^4	$0.01\text{x}10^{-28}$	0.093
10^5	$0.01\text{x}10^{-29}$	0.093
10^6	$0.03\text{x}10^{-29}$	0.093

TABLE 4. Absolute errors and executed time in seconds for the integral (35) with $n = 6$.

experiments are conducted on a personal computer running the operating system Windows 8, 4GB RAM and 2.4 GHz CPU clock speed. In some examples, the computation time in seconds is also given to support our analysis.
Example 1. We compute the integral

$$Q[f] = \int_{-1}^{1} \frac{e^{ikx}(x+1)^{-\frac{1}{2}}(1-x)^{-\frac{1}{3}}\left|x-\frac{1}{2}\right|^{-\frac{1}{3}}\ln(x+1)\ln(1-x)}{\left(1+e^{-\left(x+\frac{7}{2}\right)}\right)}dx, \tag{35}$$

by the proposed method with n fixed. In addition, Table 4 exhibits the results (errors and computation time) of the above integral with the frequency $k = 10^l, l = 2, 3, ..., 6$. The exact value was computed using Mathematica's NIntegrate command with a working precision equal to 30 to efficiently compute these integrals. Taking the magnitude difference between the exact value and the approximation one, our algorithm was able to achieve $|errors| \leq 10^{-16}$ on our PC. Moreover, Table 4 as well as Figure 3 exhibit that the proposed algorithm implemented on the integral of the form (1) is capable of achieving significant decimal digits within zero seconds as the frequency increases while n remains fixed.

Example 2. We investigate the computation of the integral

$$Q'[f] = \int_{-1}^{1} \frac{e^{ikx}(x+1)^{-\frac{1}{2}}(1-x)^{-\frac{1}{3}}\ln(x+1)\ln(1-x)}{\left(1+e^{-\left(x+\frac{7}{2}\right)}\right)(x-0.68)}dx, \tag{36}$$

by the proposed algorithm with different values of k.

The related quadrature rule for efficiently approximating integral (36) is given in (34). Table 5 shows the obtained absolute errors and computation time in seconds for the above Cauchy Principal Value integral (CPV) with $n = 7$, $\rho = 0.68$ fixed and $k = 10^2, 10^3, 10^4, 10^5, 10^6$, respectively. The approximated value was computed using our program written in Mathematica with a 30-digits precision. Furthermore, it can be easily seen that the algorithm proposed is more accurate for moderate as well as very high-frequency values for computation of CPV integrals with highly oscillatory kernels.
Example 3. Computation of the integral

$$\begin{aligned} Q[f] = \int_{1}^{10} & \frac{e^{ikx}(x-1)^{-\frac{1}{2}}(10-x)^{-\frac{1}{4}}|x-5|^{-\frac{2}{5}}}{1+x} \\ & \times \ln(x-1)\ln(10-x)\,100\ln(x+1)\,dx, \end{aligned} \tag{37}$$

is considered by the proposed method with n fixed.

Absolute errors and computation time achieved are tabulated in Table 6. Moreover, computing the same integral by using our algorithm with WorkingPrecision− >200, PrecisionGoal− >100, MaxRecursion− >100 and

k	$E_n(Q')$	Computation time
10^2	$1.54\text{x}10^{-15}$	0.078
10^3	$0.12\text{x}10^{-18}$	0.078
10^4	$0.01\text{x}10^{-19}$	0.078
10^5	$4.08\text{x}10^{-20}$	0.078
10^6	$5.58\text{x}10^{-18}$	0.078

Table 5. Errors and computation time in seconds for the integral (36) with $n = 7$ fixed.

k	$E_n(Q)$	Computation time
10^2	$3.22\text{x}10^{-15}$	0.046
10^3	$1.52\text{x}10^{-23}$	0.017
10^4	$0.01\text{x}10^{-27}$	0.283
10^5	$0.01\text{x}10^{-27}$	0.140
10^6	$0.01\text{x}10^{-28}$	0.156

Table 6. Absolute errors and executed time in seconds for the integral (37) with $n = 4$.

k	$E_n(Q')$	Computation time
10^2	$1.61\text{x}10^{-10}$	0.062
10^3	$5.02\text{x}10^{-17}$	0.046
10^4	$1.43\text{x}10^{-18}$	0.213
10^5	$1.46\text{x}10^{-19}$	0.171
10^6	$1.80\text{x}10^{-20}$	0.203

Table 7. Absolute errors and executed time in seconds for the integral (38) with $n = 5$.

applying the Timing function in Mathematica version 9.0 to evaluate the integral in Example 3, with frequency $k = 10^4$ and nodes $n = 20$, our algorithm achieved 121 exact decimal digits in 1.98475 seconds. In contrast, the same integral computed by employing Mathematica's NIntegrate command will produce results in 470.695 seconds. This proves the robustness of the presented algorithm for the computation of logarithmic highly oscillatory Fourier type integrals.

Example 4. Computation of the integral

$$Q'[f] = \int_{-1}^{1} \frac{e^{ikx}(x+1)^{-\frac{1}{4}}(1-x)^{-\frac{1}{5}}\ln(x+1)\ln(1-x)\tan x}{x-0.8}dx, \tag{38}$$

is considered by the proposed method with n fixed. Absolute errors and computation time in seconds are shown in Table 7. Furthermore, while calculating the errors we required the exact value of the integral of type (2) which was obtained by employing Mathematica's NIntegrate strategy with the Method$-$ >PrincipalValue because the other methods may fail since the integral of type (2) considered in Examples 2, 4 and 7 is unbounded at a point in the interval. Note that all singularities need to be specified in the interval of integration's section or utilize the Exclusions option.

k	$E_n(Q)$	Computation time
10^2	$4.01\text{x}10^{-15}$	0.031
10^3	$5.92\text{x}10^{-22}$	0.109
10^4	$9.65\text{x}10^{-29}$	0.125
10^5	$1.98\text{x}10^{-29}$	0.109
10^6	$2.92\text{x}10^{-30}$	0.109

TABLE 8. Errors and computation time in seconds for the integral (39) with $n = 3$ fixed.

k	$E_n(Q)$	Computation time
10^2	$3.43\text{x}10^{-13}$	0.078
10^3	$3.38\text{x}10^{-21}$	0.078
10^4	$9.35\text{x}10^{-26}$	0.062
10^5	$7.18\text{x}10^{-26}$	0.062
10^6	$2.56\text{x}10^{-26}$	0.125

TABLE 9. Errors and computation time in seconds for the integral (40) with $n = 4$ fixed.

Example 5. Computation of the integral type (1) where $\gamma = 0$

$$Q[f] = \int_{-1}^{1/2} e^{ikx} (1+x)^{\frac{99}{100}} \left(\tfrac{1}{2} - x\right)^{\frac{1}{300}} \ln(1+x) \ln\left(\tfrac{1}{2} - x\right) \cos x dx, \tag{39}$$

by the proposed algorithm with different values of k and n fixed. Errors and computation time are shown in Table 8.

From these results it is clear that the algorithm presented in this paper is efficient and fast especially for high-frequency values. In order to avoid the computation time errors, we computed each frequency independently by employing ClearSystemCache[] command or Quit Kernel then Local option.

Example 6. Computation of the integral

$$Q[f] = \int_0^1 \frac{e^{ikx} x^{-0.8} (1-x)^{-0.4} \left|x - \frac{1}{3}\right|^{-0.6} \ln x \ln(1-x)}{\left(5 + 100(x+1)^2\right)} dx, \tag{40}$$

by the proposed algorithm with different values of k and n fixed.

Computation of integrals type (1) considered in the examples 1,3,5 and 6 first requires the split of integrals (1) into the sum of two different integrals inasmuch as the integral contains absolute value integrand kernel function. Secondly, the related quadrature rule is given in (33). After successfully calculating the nodes and weights for the rule (33), it was an easy matter to efficiently compute integrals of type (1). Absolute Errors and computation time are shown in Table 9. Results in Table 9 also show that the convergence of the proposed algorithm is fast even when small nodes $n = 4$ are considered. We computed the integral in Example 6 with precision=100, nodes $n = 12$, and $k = 1000$, and obtained an approximation of 47 digits' precision.

	$E_n(Q')$		
$k\backslash N$	2	3	4
10^2	9.2x10^{-7}	2.5x10^{-9}	1.3x10^{-11}
10^3	8.0x10^{-11}	3.9x10^{-14}	4.0x10^{-14}
10^4	7.0x10^{-15}	2.0x10^{-21}	0.1x10^{-22}
10^5	5.6x10^{-19}	0.1x10^{-22}	0.2x10^{-22}

Table 10. Absolute Errors for the integral (41) with different value of n.

Example 7. Computation of the integral

$$Q'[f] = \int_0^1 \frac{e^{ikx} x^{-0.19} (1-x)^{-0.81} \ln x \ln(1-x)}{(1+x^2)(x-0.5)} dx, \tag{41}$$

by the proposed algorithm with different values of k and n. Absolute Errors and different values of n are displayed in Table 10. Furthermore, by examining the results in Table 10, it is easy to see that as n increases, the convergence also improves. We can drawn the same conclusion also in Figure 4 for low frequencies.

5. Conclusions

In this paper, we have proposed a method for the efficient computation of two highly oscillatory integrals, having logarithmic and Cauchy-logarithmic singularities. For both integrals, the presented method exhibited an astonishing characteristic that can easily achieve higher precision approximation for n fixed as the frequency increases. Moreover, the method was shown to be accurate and efficient for moderate and also for very large frequencies. The examples and tables presented in Section 4 provide the substantiation of performance for the proposed methods.

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