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THE ARBITRARY LAGRANGIAN-EULERIAN FINITE ELEMENT METHOD FOR A TRANSIENT STOKES/PARABOLIC INTERFACE PROBLEM

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Abstract. In this paper, a type of nonconservative arbitrary Lagrangian-Eulerian (ALE) finite element method is developed and analyzed in the monolithic frame for a transient Stokes/parabolic moving interface problem with jump coefficients. The mixed and the standard finite element approximations are adopted for the transient Stokes equations and the parabolic equation on either side of the moving interface, respectively. The stability and optimal convergence properties of both semi- and full discretizations are analyzed in terms of the energy norm. The developed numerical method can be generally extended to the realistic fluid-structure interaction (FSI) problems in a time-dependent domain with a moving interface.

Key words. Arbitrary Lagrangian-Eulerian (ALE) method, mixed finite element method (FEM), fluid-structure interactions (FSI), Stokes/parabolic interface problem, stability, optimal convergence.

1. Introduction

In this paper, we study the following coupled system of partial differential equations (PDEs), which consists of the transient Stokes equations and a parabolic equation defined in respective time-dependent subdomains and separated by a moving interface:

$$
(1)\qquad \left\{\begin{array}{rcll}
\frac{\partial \boldsymbol{v}_1}{\partial t} - \nabla\cdot(\mu_1\nabla\boldsymbol{v}_1) + \nabla p_1 &=& \boldsymbol{f}_1, & \text{in } \Omega_t^1\times\mathcal{I}\\
\nabla\cdot\boldsymbol{v}_1 &=& 0, & \text{in } \Omega_t^1\times\mathcal{I}\\
\boldsymbol{v}_1 &=& 0, & \text{on } \partial\Omega_t^1\backslash\Gamma_t\times\mathcal{I}\\
\boldsymbol{v}_1(\boldsymbol{x},0) &=& \boldsymbol{v}_1^0, & \text{in } \hat{\Omega}^1=\Omega_0^1\\
\frac{\partial \boldsymbol{v}_2}{\partial t} - \nabla\cdot(\mu_2\nabla\boldsymbol{v}_2) &=& \boldsymbol{f}_2, & \text{in } \Omega_t^2\times\mathcal{I}\\
\boldsymbol{v}_2 &=& 0, & \text{on } \partial\Omega_t^2\backslash\Gamma_t\times\mathcal{I}\\
\boldsymbol{v}_2(\boldsymbol{x},0) &=& \boldsymbol{v}_2^0, & \text{in } \hat{\Omega}^2=\Omega_0^2\\
\boldsymbol{v}_1 &=& \boldsymbol{v}_2, & \text{on } \Gamma_t\times\mathcal{I}\\
(-p_1\boldsymbol{I}+\mu_1\nabla\boldsymbol{v}_1)\,\boldsymbol{n}_1 + \mu_2\nabla\boldsymbol{v}_2\boldsymbol{n}_2 &=& \boldsymbol{\tau}, & \text{on } \Gamma_t\times\mathcal{I}
\end{array}\right.
$$

where $\Omega\subset\mathbb{R}^d$ $(d=2,3)$, $\mathcal{I}=(0,T]$ $(T>0)$, and two subdomains, $\Omega_t^i := \Omega_i(t)\subset\Omega$ $(i=1,2)$ $(0\le t\le T)$, satisfy $\overline{\Omega_t^1}\cup\overline{\Omega_t^2}=\overline{\Omega}$, $\Omega_t^1\cap\Omega_t^2=\emptyset$ and are separated by a moving interface: $\Gamma_t := \Gamma(t) = \partial\Omega_t^1\cap\partial\Omega_t^2$. Γ_t may move/deform along with $t\in\mathcal{I}$, then may cause Ω_t^i $(i=1,2)$ to change with $t\in\mathcal{I}$ as well, which are thus termed as the current (Eulerian) domains with respect to $\boldsymbol{x}_i$ in contrast to their initial (reference/Lagrangian) domains, $\hat{\Omega}^i := \Omega_0^i$ with respect to $\hat{\boldsymbol{x}}_i$ $(i=1,2)$, where, a *flow map* is defined from $\hat{\Omega}^i$ to Ω_t^i, as: $\hat{\boldsymbol{x}}_i\mapsto\boldsymbol{x}_i(\hat{\boldsymbol{x}}_i,t)$ such that $\boldsymbol{x}_i(\hat{\boldsymbol{x}}_i,t)=\hat{\boldsymbol{x}}_i+\hat{\boldsymbol{u}}_i(\hat{\boldsymbol{x}}_i,t),\forall t\in\mathcal{I}$, where $\hat{\boldsymbol{u}}_i$ $(i=1,2)$ is the displacement field in the Lagrangian frame. In addition, μ_1 and μ_2 are jump constants. In what follows, we set $\hat{\psi}_i=\hat{\psi}_i(\hat{\boldsymbol{x}}_i,t)$ which equals $\psi_i(\boldsymbol{x}_i(\hat{\boldsymbol{x}}_i,t),t)$ $(i=1,2)$. Correspondingly,

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the deformation gradient tensor is defined as $\mathbf{F}^i := \nabla_{\hat{\boldsymbol{x}}_i} \boldsymbol{x}_i = \boldsymbol{I} + \nabla_{\hat{\boldsymbol{x}}_i} \hat{\boldsymbol{u}}_i$, and $\boldsymbol{J}^i = \det(\mathbf{F}^i)$ $(i = 1, 2)$.

The model problem (1) can be essentially considered as a linearized fluid-structure interaction (FSI) problem [5, 9, 16, 1, 2, 14, 17], where the transient Stokes equations describe the fluid motion in terms of the fluid velocity $\boldsymbol{v}_1$ and pressure p_1, and the parabolic equation just stands for a dynamic linear elasticity problem in terms of the structural velocity $\boldsymbol{v}_2$ [19]. In addition, μ_1 can represent the fluid viscosity, and μ_2 denotes the elastic parameter of the structure. Hence, (1) holds the essential characteristic of FSI problems at least partially, that is, two different types of time-dependent governing equations bearing with different primary unknowns and different compressibility/constitutive relations are defined on either side of the moving interface. FSI problems describe the coupled dynamics of fluid mechanics and structure mechanics through the moving interface. They are classical multiphysics problems and as such, have a diverse range of applications in engineering. A key factor in the simulation of such problems comes from the deformation of the domains due to the evolving fluid flow acting on the surface of structure and thus making the structure deformable. Specifically, we are looking at a two-way coupled system in which the fluid flow affects the structural deformation, at the same time, the motion of the structure impacts the fluid flow through their interfaces. The thing that every FSI problem has in common is that the subdomains in which the coupled system is defined will move with respect to time due to the interface motion, that is, the subdomains are no longer fixed. The movement of the domain/interface can be in the form of a rotation, translation and/or deformation.

In order to take the domain motion into consideration, the arbitrary Lagrangian-Eulerian (ALE) technique is always adopted to redescribe the moving boundary/interface problem, and then a conservative ALE-finite volume/element method [8, 13] is usually developed to discretize the corresponding moving boundary/interface problems in order to account for the preservation of geometric conservation law (GCL) [7]. In the case of finite element spatial discretizations, the relationships between GCL condition, stability and accuracy properties of the numerical scheme, have not been completely clarified yet [14]. Recently, GCL condition is proved to be neither necessary nor sufficient for the stability of ALE-finite element scheme [4]. Then, a nonconservative type of ALE-finite element discretization, which does not need to preserve the GCL condition, becomes promising due to its relatively simpler implementation and less storage since only one-level mesh is involved in the nonconservative ALE-finite element method [19], in contrast with the conservative ALE method in which two-level meshes must be employed.

Towards an effective and practical ALE-finite element approximation to a realistic also complicated FSI problem, in this paper we will start with a simplified FSI model – a transient Stokes/parabolic moving interface problem, develop its nonconservative ALE-finite element approximation in semi- and fully discrete schemes, and analyze their stability and optimal convergence properties. Afterwards, our method will be more likely extended to a realistic FSI problem that was first studied in [11, 12] where however a more complicated conservative scheme of ALE method is adopted, and, our simpler nonconservative ALE scheme will be still stable as well as possess an optimal error estimate for FSI problems, which will be studied in our next paper.

The structure of this paper is organized as follows: in Section 2 we introduce the ALE mapping as well as define the nonconservative weak form of the presented Stokes/parabolic interface problem. Then we define the semi-discrete ALE finite element approximation in the nonconservative form and analyze its stability and optimal convergence theorems in Section 3. The fully discrete nonconservative ALE-finite element scheme is defined in Section 4 and its stability and convergence properties are comprehensively analyzed as well. Numerical experiments are carried out in Section 5 to validate the theoretical results.

2. Model and weak form in ALE description

We first take some arbitrary invertible affine mapping from the initial (reference) domain to the current domain at any other time in the simulation. With this mapping we can define a domain velocity which allows the implementation of a mesh updating algorithm that follows the moving domain. With the model problem (1) in place, we now define the affine mapping that allows us to use the ALE description of the model problem. Assume there exists $\boldsymbol{X}_t^i \in H^1\left(0,T;W^{2,\infty}(\hat{\Omega}^i)^d\right)$ such that $\forall t \in \mathcal{I}$ and $i = 1,2$, the mapping:

$$\begin{aligned} \boldsymbol{X}_t^i \quad : \quad & \hat{\Omega}^i \to \Omega_t^i \\ & \hat{\boldsymbol{x}}_i \to \boldsymbol{x}_i(\hat{\boldsymbol{x}}_i, t) \end{aligned}$$

is invertible and $(\boldsymbol{X}_t^i)^{-1} \in W^{2,\infty}(\Omega_t^i)^d$, where $\hat{\boldsymbol{x}}_i \in \hat{\Omega}^i$ is known as the reference coordinate variable. The domain velocity is then defined as

$$\boldsymbol{\omega}_i : \Omega_t^i \times \mathcal{I} \to \mathbb{R}^d,\ \boldsymbol{\omega}_i(\boldsymbol{x}_i,t) = \hat{\boldsymbol{\omega}}_i \circ (\boldsymbol{X}_t^i)^{-1} = \frac{\partial \boldsymbol{X}_t^i}{\partial t} \circ (\boldsymbol{X}_t^i)^{-1},\ \text{for } i = 1,2.$$

We can now define a derivative which takes this domain velocity into account. It is known as the ALE time derivative and is defined as

$$\begin{aligned} \frac{d\boldsymbol{v}_i}{dt}\Big|_{\hat{\boldsymbol{x}}} : \Omega_t^i \times \mathcal{I} \quad &\to \quad \mathbb{R}^d \\ (\boldsymbol{x}_i, t) \quad &\to \quad \frac{d\boldsymbol{v}_i}{dt}\Big|_{\hat{\boldsymbol{x}}}(\boldsymbol{x}_i,t) = \frac{\partial \boldsymbol{v}_i}{\partial t}(\boldsymbol{x}_i,t) + (\boldsymbol{\omega}_i(\boldsymbol{x}_i,t)\cdot\nabla)\,\boldsymbol{v}_i(\boldsymbol{x}_i,t) \end{aligned} \tag{2}$$

Equipped with the domain velocity and ALE time derivative, we can proceed to rewrite our model problem (1) using the ALE description as follows.

$$\left\{ \begin{array}{rcll} \frac{d\boldsymbol{v}_1}{dt}\Big|_{\hat{\boldsymbol{x}}} - (\boldsymbol{\omega}_1\cdot\nabla)\,\boldsymbol{v}_1 - \nabla\cdot(\mu_1\nabla\boldsymbol{v}_1) + \nabla p_1 &=& \boldsymbol{f}_1, & \text{in } \Omega_t^1\times\mathcal{I} \\ \nabla\cdot\boldsymbol{v}_1 &=& 0, & \text{in } \Omega_t^1\times\mathcal{I} \\ \boldsymbol{v}_1 &=& 0, & \text{on } \partial\Omega_t^1\backslash\Gamma_t\times\mathcal{I} \\ \boldsymbol{v}_1(\boldsymbol{x},0) &=& \boldsymbol{v}_1^0, & \text{in } \hat{\Omega}^1 \\ \frac{d\boldsymbol{v}_2}{dt}\Big|_{\hat{\boldsymbol{x}}} - (\boldsymbol{\omega}_2\cdot\nabla)\,\boldsymbol{v}_2 - \nabla\cdot(\mu_2\nabla\boldsymbol{v}_2) &=& \boldsymbol{f}_2, & \text{in } \Omega_t^2\times\mathcal{I} \\ \boldsymbol{v}_2 &=& 0, & \text{on } \partial\Omega_t^2\backslash\Gamma_t\times\mathcal{I} \\ \boldsymbol{v}_2(\boldsymbol{x},0) &=& \boldsymbol{v}_2^0, & \text{in } \hat{\Omega}^2 \\ \boldsymbol{\omega}_1 &=& \boldsymbol{\omega}_2, & \text{on } \Gamma_t\times\mathcal{I} \\ \boldsymbol{v}_1 &=& \boldsymbol{v}_2, & \text{on } \Gamma_t\times\mathcal{I} \\ (-p_1 I + \mu_1\nabla\boldsymbol{v}_1)\,\boldsymbol{n}_1 + \mu_2\nabla\boldsymbol{v}_2\boldsymbol{n}_2 &=& \boldsymbol{\tau}, & \text{on } \Gamma_t\times\mathcal{I} \end{array} \right. \tag{3}$$

To define the weak form of (3), we need to introduce some Sobolev spaces.

$$\begin{array}{rcl} \boldsymbol{V} &:=& \{(\psi_1,\psi_2)\in H^1(\Omega_t^1)^d\times H^1(\Omega_t^2)^d \big| \psi_1 = \psi_2 \text{ on } \Gamma_t\} \\ \boldsymbol{V}_0 &:=& \{(\psi_1,\psi_2)\in \boldsymbol{V} \big| \psi_i = 0 \text{ on } \partial\Omega_t^i\backslash\Gamma_t, i = 1,2\} \\ Q^1 &:=& L^2\left(\Omega_t^1\right) \\ Q_0^1 &:=& \{q\in Q^1 \big| \int_{\Omega_t^1} q dx = 0\}. \end{array}$$

Then, the monolithic nonconservative ALE weak form of (3) can be defined as follows. Find $(\boldsymbol{v}_1, \boldsymbol{v}_2) \in (H^1 \cap L^\infty)(0, T; \boldsymbol{V}_0)$ and $p_1 \in L^2\left(0, T; Q_0^1\right)$ such that

$$\sum_{i=1}^{2}\left[\left(\left.\frac{d\boldsymbol{v}_i}{dt}\right|_{\hat{\boldsymbol{x}}}, \psi_i\right)_{\Omega_t^i} + (\mu_i \nabla \boldsymbol{v}_i, \nabla \psi_i)_{\Omega_t^i} - ((\boldsymbol{\omega}_i \cdot \nabla)\boldsymbol{v}_i, \psi_i)_{\Omega_t^i}\right] - (p_1, \nabla \cdot \psi_1)_{\Omega_t^1}$$

$$+ (\nabla \cdot \boldsymbol{v}_1, q_1)_{\Omega_t^1} = \sum_{i=1}^{2} (\boldsymbol{f}_i, \psi_i)_{\Omega_t^i} + \langle \boldsymbol{\tau}, \psi_1 \rangle_{\Gamma_t}, \forall (\psi_1, \psi_2) \in \boldsymbol{V}_0,\ q_1 \in Q^1, \tag{4}$$

which contains the incompressibility condition $\nabla \cdot \boldsymbol{v}_1 = 0$, i.e., $(\nabla \cdot \boldsymbol{v}_1, q_1)_{\Omega_t^1} = 0$ for any $q_1 \in Q^1$.

Remark 2.1. *In this paper we adopt the terminology of "nonconservative" from the reference* [7, Section 2.2]*, where two weak forms within the ALE frame are defined: one is called the nonconservative formulation and the other one belongs to the conservative type. The so-called "conservative" formulation is to consider the weak form involves* $\frac{d}{dt}(\boldsymbol{v}, \psi)_{V(t)}$*, i.e., the time differentiation acts on the entire inner product over a moving domain* $V(t)$*. As addressed in* [11, Equation (2.26)]*, the "conservative" formulation actually states that "in absence of source terms, the variation of the primary unknown,* $\boldsymbol{v}$*, over a control volume* V *is due only to contribution coming from the boundary of* V*. It can be noted that also the contribution of the ALE term to the conservation reduces to a boundary term, which is indeed related to the additional "flux" of* $\boldsymbol{v}$ *through the boundary as a consequence of its movement". Essentially, the Reynold's Transport Theorem (RTT) shown below plays a key role for the concept of "conservation" over a moving domain.*
Reynold's Transport Theorem (RTT) [15, 10]*:*

$$\frac{d}{dt}\int_{V(t)} \boldsymbol{\alpha}(\boldsymbol{x}, t) d\boldsymbol{x} = \int_{V(t)} \frac{\partial \boldsymbol{\alpha}}{\partial t}(\boldsymbol{x}, t) d\boldsymbol{x} + \int_{\partial V(t)} \boldsymbol{\alpha}(\boldsymbol{x}, t)\boldsymbol{\omega}(\boldsymbol{x}, t) \cdot \boldsymbol{n} ds$$

$$= \int_{V(t)} \left[\frac{\partial \boldsymbol{\alpha}}{\partial t}(\boldsymbol{x}, t) + \nabla \cdot (\boldsymbol{\alpha}(\boldsymbol{x}, t)\boldsymbol{\omega}^T(\boldsymbol{x}, t))\right] d\boldsymbol{x} \tag{5}$$

$$= \int_{V(t)} \left[\left.\frac{\partial \boldsymbol{\alpha}}{\partial t}\right|_{\hat{\boldsymbol{x}}} + \nabla \cdot \boldsymbol{\omega}(\boldsymbol{x}, t)\boldsymbol{\alpha}(\boldsymbol{x}, t)\right] d\boldsymbol{x}, \tag{6}$$

where $\boldsymbol{\omega}$ *is the domain (grid) velocity of the control volume* $V(t)$*. Taking* $\boldsymbol{\alpha} = \boldsymbol{v}\psi$ *in (5) with the test function* ψ *that is associated with the grid point only, RTT demonstrates the fact that in the weak form of momentum equation, the summation of the physical time derivative term of* $\boldsymbol{v}$ *and the convection term of* $\boldsymbol{v}$ *carried over by the domain (grid) velocity* $\boldsymbol{\omega}$ *can be represented as just the time differentiation on the inner product of* $\boldsymbol{v}$ *and* ψ *over the moving domain* $V(t)$*, only. It explains that the time derivative term* $\frac{d}{dt}(\boldsymbol{v}, \psi)_{V(t)}$*, together with the divergence of the physical flux term,* $-(\nabla \cdot \boldsymbol{F}, \psi)_{V(t)}$*, fully contributes to the momentum conservation in a moving domain* $V(t)$*. In contrast, instead of* $\frac{d}{dt}(\boldsymbol{v}, \psi)_{V(t)}$*, if only the term* $\left(\left.\frac{\partial \boldsymbol{v}}{\partial t}\right|_{\hat{\boldsymbol{x}}}, \psi\right)_{V(t)}$ *remains in the weak form, where* $\left.\frac{\partial \boldsymbol{v}}{\partial t}\right|_{\hat{\boldsymbol{x}}} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{v}$ *is the ALE time derivative of* $\boldsymbol{v}$ *associated with the domain (grid) velocity* $\boldsymbol{\omega}$*, then according to (6), an extra term* $(\nabla \cdot \boldsymbol{\omega}\boldsymbol{v}, \psi)_{V(t)}$ *must be counted in the weak form in order to conserve the momentum together with the divergence of the physical flux term. However, this extra term can be removed from the weak form, which then induces the so-called "nonconservative" formulation defined in a moving domain* $V(t)$ *that is associated with the domain (grid) velocity* $\boldsymbol{\omega}$ *.*

In view of the weak form (4) in which the ALE time derivative $\frac{d\boldsymbol{v}_i}{dt}\big|_{\hat{\boldsymbol{x}}}$ *remains inside the inner product and no extra term* $(\nabla\cdot\boldsymbol{\omega}_i\boldsymbol{v}_i,\psi_i)_{\Omega_t^i}$ *is added to the scheme, we know such weak form does not follow the meaning of "conservation" explained above, it is thus called the "nonconservative" formulation. This demonstration can be similarly applied to the following semi- and fully discrete ALE-finite element discretizations shown in (9) and (26), respectively, to account for the terminology of "nonconservative" therein.*

3. Semi-discrete ALE–finite element discretization

First, we construct a quasi-uniform and interface-fitted triangulation $\mathcal{T}^i_{h,0}$ in the initial domain $\hat{\Omega}^i$ $(i=1,2)$, where no triangle of $\mathcal{T}^i_{h,0}$ has two edges on $\partial\Omega^i_0$ and that no triangle crosses the interface Γ_t.

3.1. Discretized ALE mapping and the semi-discrete ALE scheme. For any $t\in\mathcal{I}$ consider the following discretization of ALE mapping $\boldsymbol{X}^i_t$, $\boldsymbol{X}^i_{h,t}:\hat{\Omega}^i\to\Omega^i_t$, by means of piecewise linear Lagrangian finite elements, where $\boldsymbol{X}^i_{h,t}$ $(i=1,2)$ is smooth and invertible. Likewise, the discrete ALE velocity is defined as follows:

$$\boldsymbol{\omega}_{i,h}:\Omega^i_t\times\mathcal{I}\to\mathbb{R}^d,\quad \boldsymbol{\omega}_{i,h}(\boldsymbol{x}_i,t)=\frac{\partial\boldsymbol{X}^i_{h,t}}{\partial t}\circ(\boldsymbol{X}^i_{h,t})^{-1},\ i=1,2,$$

which leads to the discrete ALE time derivative:

$$\begin{aligned}\frac{d\boldsymbol{v}_i}{dt}\Big|^h_{\hat{\boldsymbol{x}}}:\Omega^i_t\times\mathcal{I}\quad&\to\quad\mathbb{R}^d\\ (\boldsymbol{x}_i,t)\quad&\to\quad\frac{d\boldsymbol{v}_i}{dt}\Big|^h_{\hat{\boldsymbol{x}}}(\boldsymbol{x}_i,t)=\frac{\partial\boldsymbol{v}_i}{\partial t}(\boldsymbol{x}_i,t)+(\boldsymbol{\omega}_{i,h}(\boldsymbol{x}_i,t)\cdot\nabla)\,\boldsymbol{v}_i(\boldsymbol{x}_i,t).\end{aligned}$$

In practice, such discrete ALE mapping $\boldsymbol{X}^i_{h,t}$ $(i=1,2)$ can map $\mathcal{T}^i_{h,0}$ to $\mathcal{T}^i_{h,t}$ $(i=1,2)$ for $t\in\mathcal{I}$ that is non-degenerate with time. Then, $\boldsymbol{X}^i_{h,t}$ $(i=1,2)$ represents a moving mesh that adapts to the moving interface/boundary. $\boldsymbol{X}^i_{h,t}$ $(i=1,2)$ can be arbitrarily defined, for instance, by the following harmonic mapping:

$$(7)\qquad\left\{\begin{array}{rcll}-\Delta\boldsymbol{X}^i_{h,t}&=&0,&\text{in }\hat{\Omega}^i,\\ \boldsymbol{X}^i_{h,t}&=&0,&\text{on }\partial\hat{\Omega}^i\backslash\hat{\Gamma},\\ \boldsymbol{X}^i_{h,t}&=&\boldsymbol{x}_\Gamma(\boldsymbol{x}(\hat{\boldsymbol{x}},t),t),&\text{on }\hat{\Gamma},\end{array}\right.$$

where $\boldsymbol{x}_\Gamma$ denotes a prescribed interface motion.

We now proceed to the definition of our finite element spaces using the classical P^2 elements to approximate $\boldsymbol{v}_i$ $(i=1,2)$ and P^1 element for Q^1. Then the discrete ALE FEM spaces are defined as follows:

$$(8)\qquad\begin{array}{rcl}\boldsymbol{V}_h&=&\{(\psi_{1,h},\psi_{2,h})\in\boldsymbol{V}_0\big|\psi_{i,h}|_K\in P^2(K),\forall K\in\mathcal{T}^i_{h,t}\ (i=1,2)\},\\ Q_h&=&\{q_h\in Q^1\big|q_h|_K\in P^1(K),\forall K\in\mathcal{T}^1_{h,t}\},\\ Q^0_h&=&\{q_h\in Q^1_0\big|q_h|_K\in P^1(K),\forall K\in\mathcal{T}^1_{h,t}\},\end{array}$$

where $P^n(K)$ is the set of piecewise polynomials of degree n on the element K.

Hence, the nonconservative semi-discrete ALE finite element discretization corresponding to the weak form (4) can be defined as: find $(\boldsymbol{v}_{1,h},\boldsymbol{v}_{2,h})\in\boldsymbol{V}_h$, $p_{1,h}\in Q^0_h$

such that

$$\sum_{i=1}^{2}\left[\left(\frac{d\boldsymbol{v}_{i,h}}{dt}\Big|_{\hat{\boldsymbol{x}}}^{h},\psi_{i,h}\right)_{\Omega_t^i}+(\mu_i\nabla\boldsymbol{v}_{i,h},\nabla\psi_{i,h})_{\Omega_t^i}-((\boldsymbol{\omega}_{i,h}\cdot\nabla)\,\boldsymbol{v}_{i,h},\psi_{i,h})\right]$$

$$-(p_{1,h},\nabla\cdot\psi_{1,h})_{\Omega_t^1}+(\nabla\cdot\boldsymbol{v}_{1,h},q_{1,h})_{\Omega_t^1}=\sum_{i=1}^{2}(\boldsymbol{f}_i,\psi_{i,h})_{\Omega_t^i}+\langle\boldsymbol{\tau},\psi_{1,h}\rangle_{\Gamma_t},$$

$$\forall\,(\psi_{1,h},\psi_{2,h})\in\boldsymbol{V}_h,\ q_{1,h}\in Q_h, \tag{9}$$

which implies that $(\nabla\cdot\boldsymbol{v}_{1,h},q_{1,h})_{\Omega_t^1}=0$ for any $q_{1,h}\in Q_h$.

We assume that the following error estimates hold for approximations to ALE mapping and ALE velocity [8]:

$$\begin{aligned}\|\boldsymbol{X}_t^i-\boldsymbol{X}_{h,t}^i\|_{0,\infty,\Omega_t^i}+h\|\boldsymbol{X}_t^i-\boldsymbol{X}_{h,t}^i\|_{1,\infty,\Omega_t^i}&\le Ch^2|\ln h|\|\boldsymbol{X}_t^i\|_{2,\infty,\Omega_t^i},\\ \|\boldsymbol{\omega}_i-\boldsymbol{\omega}_{i,h}\|_{0,\infty,\Omega_t^i}+h\|\boldsymbol{\omega}_i-\boldsymbol{\omega}_{i,h}\|_{1,\infty,\Omega_t^i}&\le Ch^2|\ln h|\|\boldsymbol{\omega}_i\|_{2,\infty,\Omega_t^i},\end{aligned} \tag{10}$$

where we assume $\boldsymbol{\omega}_i\in W^{2,\infty,\Omega_t^i}(\Omega_t^i)^d$ $(i=1,2)$. Thus, we have the boundedness for $\boldsymbol{\omega}_{i,h}$ as follows

$$\|\boldsymbol{\omega}_{i,h}\|_{0,\infty,\Omega_t^i}\le C,\quad \|\boldsymbol{\omega}_{i,h}\|_{1,\infty,\Omega_t^i}\le C,\quad i=1,2. \tag{11}$$

3.2. Stability analysis.

Theorem 3.1. *Assume the conditions for the semi-discrete ALE finite element scheme (9) hold. Then the following stability result holds for any $t\in\mathcal{I}$:*

$$\sum_{i=1}^{2}\left(\|\boldsymbol{v}_{i,h}\|_{L^\infty\left(0,t;L^2(\Omega_t^i)^d\right)}+\|\boldsymbol{v}_{i,h}\|_{L^2(0,t;H^1(\Omega_t^i)^d)}\right)$$

$$\le C\left(\sum_{i=1}^{2}\left(\|\boldsymbol{f}_i\|_{L^2\left(0,t;L^2(\Omega_t^i)^d\right)}+\|\boldsymbol{v}_i(0)\|_{L^2(\hat{\Omega}^i)}\right)+\|\boldsymbol{\tau}\|_{L^2(0,t:L^2(\Gamma_t)^d)}\right). \tag{12}$$

Proof. Let $\psi_{i,h}=\boldsymbol{v}_{i,h},\ q_{1,h}=p_{1,h}$ in (9). Then

$$\sum_{i=1}^{2}\left[\left(\frac{d\boldsymbol{v}_{i,h}}{dt}\Big|_{\hat{\boldsymbol{x}}}^{h},\boldsymbol{v}_{i,h}\right)_{\Omega_t^i}+(\mu_i\nabla\boldsymbol{v}_{i,h},\nabla\boldsymbol{v}_{i,h})_{\Omega_t^i}-((\boldsymbol{\omega}_{i,h}\cdot\nabla)\,\boldsymbol{v}_{i,h},\boldsymbol{v}_{i,h})\right]$$

$$=\sum_{i=1}^{2}(\boldsymbol{f}_i,\boldsymbol{v}_{i,h})_{\Omega_t^i}+\langle\boldsymbol{\tau},\boldsymbol{v}_{1,h}\rangle_{\Gamma_t}. \tag{13}$$

By using the following equality and estimate:

$$\begin{aligned}\left(\frac{d\boldsymbol{v}_{i,h}}{dt}\Big|_{\hat{\boldsymbol{x}}}^{h},\boldsymbol{v}_{i,h}\right)_{\Omega_t^i}&=\frac{1}{2}\left(\frac{d}{dt}\|\boldsymbol{v}_{i,h}\|_{0,\Omega_t^i}^2-(\nabla\cdot\boldsymbol{\omega}_h\boldsymbol{v}_{i,h},\boldsymbol{v}_{i,h})_{\Omega_t^i}\right),\\ (\mu_i\nabla\boldsymbol{v}_{i,h},\nabla\boldsymbol{v}_{i,h})_{\Omega_t^i}&=\mu_i\|\nabla\boldsymbol{v}_{i,h}\|_{0,\Omega_t^i}^2\ge C\|\boldsymbol{v}_{i,h}\|_{1,\Omega_t^i}^2,\end{aligned}$$

we then have,

$$\sum_{i=1}^{2}\left[\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}_{i,h}\|_{0,\Omega_t^i}^2+C\|\boldsymbol{v}_{i,h}\|_{1,\Omega_t^i}^2\right]\le\sum_{i=1}^{2}\left[(\boldsymbol{f}_i,\boldsymbol{v}_{i,h})_{\Omega_t^i}+\frac{1}{2}(\nabla\cdot\boldsymbol{\omega}_{i,h}\boldsymbol{v}_{i,h},\boldsymbol{v}_{i,h})_{\Omega_t^i}\right.$$

$$\left.+((\boldsymbol{\omega}_{i.h}\cdot\nabla)\,\boldsymbol{v}_{i,h},\boldsymbol{v}_{i,h})_{\Omega_t^i}\right]+\langle\boldsymbol{\tau},\boldsymbol{v}_{1,h}\rangle_{\Gamma_t}.$$

Applying Cauchy-Schwarz inequality, Young's inequality with ϵ, the boundedness in (11) and the trace theorem, we have

$$\begin{aligned}
\left(\left(\boldsymbol{\omega}_{i,h}\cdot\nabla\right)\boldsymbol{v}_{i,h},\boldsymbol{v}_{i,h}\right)_{\Omega_t^i} &\leq \|\boldsymbol{\omega}_{i,h}\|_{0,\infty,\Omega_t^i}\|\nabla\boldsymbol{v}_{i,h}\|_{0,\Omega_t^i}\|\boldsymbol{v}_{i,h}\|_{0,\Omega_t^i}\\
&\leq \epsilon\|\boldsymbol{v}_{i,h}\|_{1,\Omega_t^i}^2 + C\|\boldsymbol{v}_{i,h}\|_{0,\Omega_t^i}^2,\\
\left(\nabla\cdot\boldsymbol{\omega}_{i,h}\boldsymbol{v}_{i,h},\boldsymbol{v}_{i,h}\right)_{\Omega_t^i} &\leq C\|\boldsymbol{v}_{i,h}\|_{0,\Omega_t^i}^2,\\
\left(\boldsymbol{f}_i,\boldsymbol{v}_{i,h}\right)_{\Omega_t^i} &\leq C\left(\|\boldsymbol{f}_i\|_{0,\Omega_t^i}^2 + \|\boldsymbol{v}_{i,h}\|_{0,\Omega_t^i}^2\right),\\
\langle\boldsymbol{\tau},\boldsymbol{v}_{1,h}\rangle_{\Gamma_t} &\leq \|\boldsymbol{\tau}\|_{L^2(\Gamma_t)}\|\boldsymbol{v}_{1,h}\|_{L^2(\Gamma_t)} \leq C\|\boldsymbol{\tau}\|_{L^2(\Gamma_t)}\|\boldsymbol{v}_{1,h}\|_{1,,\Omega_t^1}\\
&\leq C\|\boldsymbol{\tau}\|_{L^2(\Gamma_t)}^2 + \epsilon\|\boldsymbol{v}_{1,h}\|_{1,\Omega_t^1}^2.
\end{aligned}$$

Choosing a sufficiently small ϵ, we attain

$$\begin{aligned}
&\sum_{i=1}^{2}\left[\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}_{i,h}\|_{0,\Omega_t^i}^2 + C\|\boldsymbol{v}_{i,h}\|_{1,\Omega_t^i}^2\right]\\
&\leq C\left(\sum_{i=1}^{2}\left(\|\boldsymbol{f}_i\|_{0,\Omega_t^i}^2 + \|\boldsymbol{v}_{i,h}\|_{0,\Omega_t^i}^2\right) + \|\boldsymbol{\tau}\|_{L^2(\Gamma_t)}^2\right).
\end{aligned}$$

Integrating over time from 0 to t, then

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{2}\left(\|\boldsymbol{v}_{i,h}(t)\|_{0,\Omega_t^i}^2 - \|\boldsymbol{v}_{i,h}(0)\|_{0,\Omega_t^i}^2\right) + \sum_{i=1}^{2}\int_0^t\|\boldsymbol{v}_{i,h}\|_{1,\Omega_t^i}^2dt\\
&\leq C\left(\sum_{i=1}^{2}\int_0^t\left(\|\boldsymbol{f}_i\|_{0,\Omega_t^i}^2 + \|\boldsymbol{v}_{i,h}\|_{0,\Omega_t^i}^2\right)dt + \int_0^t\|\boldsymbol{\tau}\|_{L^2(\Gamma_t)}^2dt\right).
\end{aligned}\tag{14}$$

We choose $\boldsymbol{v}_{i,h}(0) = \Pi_h\boldsymbol{v}_i^0 \in \boldsymbol{V}_h$ where $\Pi_h : \boldsymbol{V}_0 \to \boldsymbol{V}_h$ is the interpolation operator, and apply Grönwall's inequality to (14), then the desired stability result (12) is attained. □

3.3. Semi-discrete error analysis. We begin by introducing the following lemmas which will help us through the error analysis.

Lemma 3.2. [13] *Assume $\alpha,\beta,\gamma : \Omega_t \to \mathbb{R}$ are smooth functions. Then we have*

$$\begin{aligned}
\frac{d}{dt}\left(\alpha\nabla\beta,\nabla\gamma\right)_{\Omega_t} =& \left(\left.\frac{d\alpha}{dt}\right|_{\hat{\boldsymbol{x}}}^h\nabla\beta,\nabla\gamma\right)_{\Omega_t} + \left(\alpha\nabla\left.\frac{d\beta}{dt}\right|_{\hat{\boldsymbol{x}}}^h,\nabla\gamma\right)_{\Omega_t} + \left(\alpha\nabla\beta,\nabla\left.\frac{d\gamma}{dt}\right|_{\hat{\boldsymbol{x}}}^h\right)_{\Omega_t}\\
&- \left(\left(\nabla\boldsymbol{\omega}_h + \nabla\boldsymbol{\omega}_h^T\right)\alpha\nabla\beta,\nabla\gamma\right)_{\Omega_t} + \left(\left(\nabla\cdot\boldsymbol{\omega}_h\right)\alpha\nabla\beta,\nabla\gamma\right)_{\Omega_t},\\
\frac{d}{dt}\left(\alpha,\nabla\cdot\beta\right)_{\Omega_t} =& \left(\left.\frac{d\alpha}{dt}\right|_{\hat{\boldsymbol{x}}}^h,\nabla\cdot\beta\right)_{\Omega_t} + \left(\alpha,\nabla\cdot\left.\frac{d\beta}{dt}\right|_{\hat{\boldsymbol{x}}}^h\right)_{\Omega_t} + \left(\left(\nabla\cdot\boldsymbol{\omega}_h\right)\alpha,\nabla\cdot\beta\right)_{\Omega_t}\\
&- \left(\alpha\nabla\boldsymbol{\omega}_h,\nabla\beta^T\right)_{\Omega_t}.
\end{aligned}$$

Lemma 3.3. [18] *Assume $v \in \boldsymbol{V}_h$ and $q \in Q_h$, then the following inf-sup condition holds*

$$\inf_{q\in Q_h}\sup_{(\boldsymbol{v}_1,\boldsymbol{v}_2)\in\boldsymbol{V}_h}\frac{(\nabla\cdot\boldsymbol{v},q)}{\|(\boldsymbol{v}_1,\boldsymbol{v}_2)\|_1\|q\|_0} \geq C > 0.$$

Lemma 3.4. [8] *For all $t \in \mathcal{I}$, let $\Pi_h u \in \boldsymbol{V}_h$ be the interpolation of $u(t) \in H^2(\Omega_t)$, then there exists a constant C, independent of h, such that*

(15)
$$\left\| \frac{\partial (u - \Pi_h u)}{\partial t}\bigg|_{\hat{\boldsymbol{x}}}^h \right\|_{L^2(\Omega_t)} \leq C h^r \left\| \frac{\partial u}{\partial t}\bigg|_{\hat{\boldsymbol{x}}}^h \right\|_{H^r(\Omega_t)}, \quad \text{for } r \geq 1,$$

(16)
$$\left\| \frac{\partial (u - \Pi_h u)}{\partial t}\bigg|_{\hat{\boldsymbol{x}}}^h \right\|_{L^2(\Omega_t)} \leq C h \left(\left\| \frac{\partial u}{\partial t}\bigg|_{\hat{\boldsymbol{x}}} \right\|_{H^1(\Omega_t)} + h |\ln h| \|\boldsymbol{\omega}\|_{W^{2,\infty}(\Omega_t)} \|u\|_{H^2(\Omega_t)} \right).$$

We can now proceed to the main theorem of the section as follows.

Theorem 3.5. *Suppose $(\boldsymbol{v}_1, p_1, \boldsymbol{v}_2)$ is the solution to (4) satisfying the following regularity properties* [13]*:*

$$\begin{aligned} &\boldsymbol{v}_i \in L^\infty\left(0, T; H^r(\Omega_t^i)^d\right), \ \tfrac{d\boldsymbol{v}_i}{dt}\big|_{\hat{\boldsymbol{x}}}^h \in L^2(0; T; H^r(\Omega_t^i)^d) \quad (i = 1, 2), \\ &p_1 \in L^\infty\left(0, T; H^{r-1}(\Omega_t^1)\right), \ \tfrac{dp_1}{dt}\big|_{\hat{\boldsymbol{x}}}^h \in L^2(0; t; H^{r-1}(\Omega_t^1)), \ \textit{for } r \geq 3, \end{aligned} \tag{17}$$

and $(\boldsymbol{v}_{1,h}, p_{1,h}, \boldsymbol{v}_{2,h})$ is the solution to (9). Then we have the following error estimate for any $t \in \mathcal{I}$:

(18)
$$\begin{aligned} &\sum_{i=1}^{2} \left[\left\| \frac{d\boldsymbol{v}_i}{dt}\bigg|_{\hat{\boldsymbol{x}}} - \frac{d\boldsymbol{v}_{i,h}}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h \right\|_{L^2\left(0,t;L^2(\Omega_t^i)^d\right)} + \|\boldsymbol{v}_i - \boldsymbol{v}_{i,h}\|_{L^\infty\left(0,t;H^1(\Omega_t^i)^d\right)} \right. \\ &\left. + \|p_1 - p_{1,h}\|_{L^2(0;t;L^2(\Omega_t^1))} \right] \leq C h^{r-1} \left(\sum_{i=1}^{2} \left[\|\boldsymbol{v}_i\|_{L^\infty\left(0,t;H^r(\Omega_t^i)^d\right)} \right. \right. \\ &\left. \left. + \left\| \frac{d\boldsymbol{v}_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h \right\|_{L^2(0;t;H^r(\Omega_t^i)^d)} \right] + \|p_1\|_{L^\infty(0,t;H^{r-1}(\Omega_t^1))} + \left\| \frac{dp_1}{dt}\bigg|_{\hat{x}}^h \right\|_{L^2(0;t;H^{r-1}(\Omega_t^1))} \right). \end{aligned}$$

Proof. Subtracting (9) from (4) and using the identity $\frac{d\boldsymbol{v}_i}{dt}\big|_{\hat{\boldsymbol{x}}} = \frac{d\boldsymbol{v}_i}{dt}\big|_{\hat{\boldsymbol{x}}}^h + (\boldsymbol{\omega}_i - \boldsymbol{\omega}_{i,h}) \cdot \nabla \boldsymbol{v}_i$, we get the error equation:

$$\begin{aligned} &\sum_{i=1}^{2} \left[\left(\frac{d\boldsymbol{v}_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h - \frac{d\boldsymbol{v}_{i,h}}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h, \psi_{i,h} \right)_{\Omega_t^i} + \left(\mu_i \nabla \left(\boldsymbol{v}_i - \boldsymbol{v}_{i,h}\right), \nabla \psi_{i,h}\right)_{\Omega_t^i} - \right. \\ &\left. \left(\left(\boldsymbol{\omega}_{i,h} \cdot \nabla\right) \left(\boldsymbol{v}_i - \boldsymbol{v}_{i,h}\right), \psi_{i,h}\right)_{\Omega_t^i} \right] - \left(p_1 - p_{1,h}, \nabla \cdot \psi_{1,h}\right)_{\Omega_t^1} + \\ &\left(\nabla \cdot \left(\boldsymbol{v}_1 - \boldsymbol{v}_{i,h}\right), q_{1,h}\right)_{\Omega_t^1} = 0, \end{aligned} \tag{19}$$

which implies that $\left(\nabla \cdot \left(\boldsymbol{v}_1 - \boldsymbol{v}_{1,h}\right), q_{1,h}\right)_{\Omega_t^1} = 0$ for any $q_{1,h} \in Q_h$. To proceed, we need to introduce the discrete kernel space $\boldsymbol{K}_h$ as

$$\boldsymbol{K}_h := \{(\psi_{1,h}, \psi_{2,h}) \in \boldsymbol{V}_h \big| \left(\nabla \cdot \psi_{1,h}, q_{1,h}\right)_{\Omega_t^1} = 0, \forall q_{1,h} \in Q_h^0\},$$

thus $\boldsymbol{v}_{1,h} \in \boldsymbol{K}_h$. Pick arbitrary discrete functions $\tilde{\boldsymbol{v}} = (\tilde{\boldsymbol{v}}_1, \tilde{\boldsymbol{v}}_2) \in \boldsymbol{K}_h$ and $\tilde{p}_1 \in Q_h^0$. Let $\boldsymbol{v}_i - \boldsymbol{v}_{i,h} = \boldsymbol{v}_i - \tilde{\boldsymbol{v}}_i + \tilde{\boldsymbol{v}}_i - \boldsymbol{v}_{i,h} = \eta_i + \xi_i$, and $p_1 - p_{1,h} = p_1 - \tilde{p}_1 + \tilde{p}_1 - p_{1,h} = \alpha + \beta$, and choose $\psi_{i,h} = \frac{d\xi_i}{dt}\big|_{\hat{\boldsymbol{x}}}^h$, $q_{1,h} = \beta$ in (19), yield

$$\sum_{i=1}^{2}\left[\left(\frac{d\xi_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h, \frac{d\xi_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h\right)_{\Omega_t^i} + \left(\mu_i \nabla\xi_i, \nabla\frac{d\xi_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h\right)_{\Omega_t^i}\right] = \sum_{i=1}^{2}\left[-\left(\frac{d\eta_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}, \frac{d\xi_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h\right)_{\Omega_t^i}\right.$$
$$\left. - \left(\mu_i\nabla\eta_i, \nabla\frac{d\xi_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h\right)_{\Omega_t^i} + \left((\boldsymbol{\omega}_i\cdot\nabla)(\eta_i+\xi_i), \frac{d\xi_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h\right)\right] + \left(\alpha+\beta, \nabla\cdot\frac{d\xi_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h\right)_{\Omega_t^1}$$
$$= \sum_{j=1}^{4} G_j. \tag{20}$$

In the following, we analyze each term in (20) by using Lemma 3.2, Cauchy-Schwarz inequality, Young's inequality with ϵ and (11).

$$\begin{aligned}
\left(\frac{d\xi_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h, \frac{d\xi_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h\right)_{\Omega_t^i} =& \left\|\frac{d\xi_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h\right\|_{0,\Omega_t^i}^2,\\
\left(\mu_i\nabla\xi_i, \nabla\frac{d\xi_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h\right)_{\Omega_t^i} =& \frac{1}{2}\bigg(\frac{d}{dt}(\mu_i\nabla\xi_i,\nabla\xi_i)_{\Omega_t^i} + \left(\mu_i\nabla\xi_i\left(\nabla\boldsymbol{\omega}_{i,h}+\nabla\boldsymbol{\omega}_{i,h}^T\right),\nabla\xi_i\right)_{\Omega_t^i}\\
&-\left((\nabla\cdot\boldsymbol{\omega}_{i,h})\,\mu_i\nabla\xi_i,\nabla\xi_i\right)_{\Omega_t^i}\bigg)\\
=&\frac{\mu_i}{2}\frac{d}{dt}\|\nabla\xi_i\|_{0,\Omega_t^i}^2 + H_1 + H_2,\\
|H_1|+|H_2| \le& C\|\nabla\xi_i\|_{0,\Omega_t^i}^2,\\
G_1 \le& C\left\|\frac{d\eta_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h\right\|_{0,\Omega_t^i}^2 + \epsilon\left\|\frac{d\xi_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h\right\|_{0,\Omega_t^i}^2,\\
G_2 =& -\frac{d}{dt}(\mu_i\nabla\eta_i,\nabla\xi_i)_{\Omega_t^i} + \left(\mu_i\nabla\frac{d\eta_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h, \nabla\xi_i\right)_{\Omega_t^i}\\
&-\left(\mu_i\nabla\eta_i\left(\nabla\boldsymbol{\omega}_{i,h}+\nabla\boldsymbol{\omega}_{i,h}^T\right),\nabla\xi_i\right)_{\Omega_t^i} + \left((\nabla\cdot\boldsymbol{\omega}_{i,h})\,\mu_i\nabla\eta_i,\nabla\xi_i\right)_{\Omega_t^i},\\
\le& -\frac{d}{dt}(\mu_i\nabla\eta_i,\nabla\xi_i)_{\Omega_t^i} + C\bigg(\left\|\frac{d\eta_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h\right\|_{1,\Omega_t^i}^2 + \|\eta_i\|_{1,\Omega_t^i}^2 + \|\nabla\xi_i\|_{0,\Omega_t^i}^2\bigg)\\
G_3 \le& C\left(\|\eta_i\|_{1,\Omega_t^i}^2 + \|\nabla\xi_i\|_{0,\Omega_t^i}^2\right) + \epsilon\|\frac{d\xi_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h\|_{0,\Omega_t^i}^2\\
G_4 =& \frac{d}{dt}(\alpha+\beta,\nabla\cdot\xi_1)_{\Omega_t^1} - \left(\frac{d(\alpha+\beta)}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h, \nabla\cdot\xi_1\right)_{\Omega_t^1} -\\
&\left((\nabla\cdot\boldsymbol{\omega}_{1,h})(\alpha+\beta),\nabla\cdot\xi_1\right)_{\Omega_t^1} + \left((\alpha+\beta)\nabla\boldsymbol{\omega}_{1,h},\nabla\xi_1^T\right)_{\Omega_t^1}\\
\le& \frac{d}{dt}(\alpha,\nabla\cdot\xi_1)_{\Omega_t^1} + C\bigg(\left\|\frac{d\alpha}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h\right\|_{0,\Omega_t^1}^2 + \|\nabla\xi_1\|_{0,\Omega_t^1}^2 + \|\alpha\|_{0,\Omega_t^1}^2\bigg)\\
&+\epsilon_1\|\beta\|_{0,\Omega_t^1}^2,
\end{aligned}$$

where we apply the facts, $(\beta,\nabla\cdot\xi_1) = \left(\frac{d\beta}{dt}\big|_{\hat{\boldsymbol{x}}}^h, \nabla\cdot\xi_1\right) = 0$, to G_4 due to the definition of $\boldsymbol{K}_h$. Choose a sufficiently small ϵ in above error estimates, resulting in

$$\sum_{i=1}^{2}\left[\left\|\frac{d\xi_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^{h}\right\|_{0,\Omega_t^i}^2+\frac{\mu_i}{2}\frac{d}{dt}\|\nabla\xi_i\|_{0,\Omega_t^i}^2\right]\leq-\frac{d}{dt}\left(\mu_i\nabla\eta_i,\nabla\xi_i\right)_{\Omega_t^i}+\frac{d}{dt}\left(\alpha,\nabla\cdot\xi_1\right)_{\Omega_t^1}$$
$$+C\Bigg(\sum_{i=1}^{2}\left[\|\nabla\xi_i\|_{0,\Omega_t^i}^2+\|\eta_i\|_{1,\Omega_t^i}^2+\left\|\frac{d\eta_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^{h}\right\|_{1,\Omega_t^i}^2\right]+\left\|\frac{d\alpha}{dt}\bigg|_{\hat{\boldsymbol{x}}}^{h}\right\|_{0,\Omega_t^1}^2+\|\alpha\|_{0,\Omega_t^1}^2\Bigg)$$
$$+\epsilon_1\|\beta\|_{0,\Omega_t^1}^2.$$

Integrating in time from 0 to t, applying Grönwall's inequality and Poincaré inequality for $\|\nabla\xi_i\|_{0,\Omega_t^i}^2$, and Young's inequality to have $\epsilon\|\nabla\xi_i\|_{0,\Omega_t^i}^2$ with a sufficiently small ϵ, we obtain

$$\sum_{i=1}^{2}\left[\left\|\frac{d\xi_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^{h}\right\|_{L^2\left(0,t;L^2(\Omega_t^i)^d\right)}^2+\|\xi_i\|_{L^\infty\left(0,t;H^1(\Omega_t^i)^d\right)}^2\right]$$
$$\leq C\Bigg(\sum_{i=1}^{2}\left[\|\frac{d\eta_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^{h}\|_{L^2\left(0,t;H^1(\Omega_t^i)^d\right)}^2+\|\eta_i\|_{L^\infty\left(0,t;H^1(\Omega_t^i)^d\right)}^2+\|\xi_i(0)\|_{H^1(\hat{\Omega}^i)^d}^2\right]\tag{21}$$
$$+\|\alpha\|_{L^\infty(0,t;L^2(\Omega_t^1))}^2+\|\frac{d\alpha}{dt}\bigg|_{\hat{\boldsymbol{x}}}^{h}\|_{L^2(0,t;L^2(\Omega_t^1))}^2\Bigg)+\epsilon_1\|\beta\|_{L^2(0,t;L^2(\Omega_t^1))}^2.$$

For the error estimate on pressure term, $\|\beta\|_{L^2(0,t;L^2(\Omega_t^1))}^2$, we utilize the discrete inf-sup condition as shown in Lemma 3.3, then

$$C\|\beta\|_{0,\Omega_t^1}^2\leq\sup_{(\boldsymbol{\psi}_{1,h},\boldsymbol{\psi}_{2,h})\in\boldsymbol{V}_h}\frac{(\nabla\cdot\boldsymbol{\psi}_{1,h},\beta)_{\Omega_t^1}}{\|(\boldsymbol{\psi}_{1,h},\boldsymbol{\psi}_{2,h})\|_1}$$
$$=\sup_{(\boldsymbol{\psi}_{1,h},\boldsymbol{\psi}_{2,h})\in\boldsymbol{V}_h}\frac{(\nabla\cdot\boldsymbol{\psi}_{1,h},\alpha+\beta)_{\Omega_t^1}-(\nabla\cdot\boldsymbol{\psi}_{1,h},\alpha)_{\Omega_t^1}}{\|(\boldsymbol{\psi}_{1,h},\boldsymbol{\psi}_{2,h})\|_1}$$
$$\leq\sup_{(\boldsymbol{\psi}_{1,h},\boldsymbol{\psi}_{2,h})\in\boldsymbol{V}_h}\frac{\sum_{i=1}^{2}\left[\left(\frac{d(\boldsymbol{v}_i-\boldsymbol{v}_{i,h})}{dt}\bigg|_{\hat{\boldsymbol{x}}}^{h},\boldsymbol{\psi}_{i,h}\right)_{\Omega_t^i}+\left(\mu_i\nabla\left(\boldsymbol{v}_i-\boldsymbol{v}_{i,h}\right),\nabla\boldsymbol{\psi}_{i,h}\right)_{\Omega_t^i}\right]}{\|(\boldsymbol{\psi}_{1,h},\boldsymbol{\psi}_{2,h})\|_1}$$
$$+\sup_{(\boldsymbol{\psi}_{1,h},\boldsymbol{\psi}_{2,h})\in\boldsymbol{V}_h}\frac{-\sum_{i=1}^{2}\left(\left(\boldsymbol{\omega}_{i,h}\cdot\nabla\right)\left(\boldsymbol{v}_i-\boldsymbol{v}_{i,h}\right),\boldsymbol{\psi}_{i,h}\right)_{\Omega_t^i}-(\nabla\cdot\boldsymbol{\psi}_{1,h},\alpha)_{\Omega_t^1}}{\|(\boldsymbol{\psi}_{1,h},\boldsymbol{\psi}_{2,h})\|_1}$$
$$\leq C\sum_{i=1}^{2}\left[\left\|\frac{d\eta_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^{h}\right\|_{0,\Omega_t^i}^2+\left\|\frac{d\xi_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^{h}\right\|_{0,\Omega_t^i}^2+\|\eta_i\|_{1,\Omega_t^i}^2+\|\xi_i\|_{1,\Omega_t^i}^2\right]+\|\alpha\|_{0,\Omega_t^1}^2,\tag{22}$$

where we use (19). Integrating (22) in time from 0 to t, we have

$$\|\beta\|_{L^2(0,t;L^2(\Omega_t^1))}^2\leq C\Big(\sum_{i=1}^{2}\left[\|\frac{d\eta_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^{h}\|_{L^2\left(0,t;L^2(\Omega_t^i)^d\right)}^2+\|\eta_i\|_{L^2\left(0,t;H^1(\Omega_t^i)^d\right)}^2+\right.$$
$$\left.\|\frac{d\xi_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^{h}\|_{L^2\left(0,t;L^2(\Omega_t^i)^d\right)}^2+\|\xi_i\|_{L^2\left(0,t;H^1(\Omega_t^i)^d\right)}^2\right]+\|\alpha\|_{L^2(0,t;L^2(\Omega_t^1))}^2\Big).\tag{23}$$

Substituting (23) for the last term on the right hand side of (21), taking an sufficiently small ϵ_1, and considering that $\tilde{\boldsymbol{v}}_i$ $(i=1,2)$ and $\tilde{p}_1$ are arbitrary discrete functions in $\boldsymbol{K}_h$ and Q_h^0, respectively, we obtain

$$\sum_{i=1}^{2}\left[\left\|\frac{d\xi_i}{dt}\Big|_{\hat{\boldsymbol{x}}}^{h}\right\|^2_{L^2\left(0,t;L^2(\Omega_t^i)^d\right)}+\|\xi_i\|^2_{L^\infty\left(0,t;H^1(\Omega_t^i)^d\right)}\right]$$
$$\leq C\inf_{\substack{\tilde{\boldsymbol{v}}\in\boldsymbol{K}_h\setminus\{0\},\\ \tilde{p}\in Q_h^0\setminus\{0\}}}\left(\sum_{i=1}^{2}\left[\|\frac{d\eta_i}{dt}\Big|_{\hat{\boldsymbol{x}}}^{h}\|^2_{L^2\left(0,t;H^1(\Omega_t^i)^d\right)}+\|\eta_i\|^2_{L^\infty\left(0,t;H^1(\Omega_t^i)^d\right)}+\|\xi_i(0)\|^2_{H^1(\hat{\Omega}^i)^d}\right]\right.$$
$$\left.+\|\alpha\|^2_{L^\infty(0,t;L^2(\Omega_t^1))}+\|\frac{d\alpha}{dt}\Big|_{\hat{\boldsymbol{x}}}^{h}\|^2_{L^2(0,t;L^2(\Omega_t^1))}\right).$$

Applying the classic Brezzi theory [3, 6], then we can take the infimum over the entire finite element space $\boldsymbol{V}_h$ instead of the kernel space $\boldsymbol{K}_h$, only. And, we choose $\boldsymbol{v}_{i,h}(0)=\tilde{\boldsymbol{v}}_i(0)\in\boldsymbol{V}_h$, then $\xi_i(0)=0$ $(i=1,2)$, which gives the following:

$$\sum_{i=1}^{2}\left[\left\|\frac{d\xi_i}{dt}\Big|_{\hat{\boldsymbol{x}}}^{h}\right\|^2_{L^2\left(0,t;L^2(\Omega_t^i)^d\right)}+\|\xi_i\|^2_{L^\infty\left(0,t;H^1(\Omega_t^i)^d\right)}\right]$$
$$\leq C\inf_{\substack{\tilde{\boldsymbol{v}}\in\boldsymbol{V}_h\setminus\{0\},\\ \tilde{p}\in Q_h^0\setminus\{0\}}}\left(\sum_{i=1}^{2}\left[\left\|\frac{d\eta_i}{dt}\Big|_{\hat{\boldsymbol{x}}}^{h}\right\|^2_{L^2\left(0,t;H^1(\Omega_t^i)^d\right)}+\|\eta_i\|^2_{L^\infty\left(0,t;H^1(\Omega_t^i)^d\right)}\right]\right.\tag{24}$$
$$\left.+\|\alpha\|^2_{L^\infty(0,t;L^2(\Omega_t^1))}+\left\|\frac{d\alpha}{dt}\Big|_{\hat{\boldsymbol{x}}}^{h}\right\|^2_{L^2(0,t;L^2(\Omega_t^1))}\right).$$

Applying (24) to (23), choosing corresponding interpolation functions of the solution as our arbitrary discrete functions $((\tilde{\boldsymbol{v}}_1,\tilde{\boldsymbol{v}}_2),\tilde{p}_1)\in\boldsymbol{V}_h\times Q_h^0$, and employing standard a priori interpolation error estimates for η_i, α and their ALE time derivatives [8], we attain

$$\sum_{i=1}^{2}\left[\left\|\frac{d\xi_i}{dt}\Big|_{\hat{\boldsymbol{x}}}^{h}\right\|_{L^2\left(0,t;L^2(\Omega_t^i)^d\right)}+\|\xi_i\|_{L^\infty\left(0,t;H^1(\Omega_t^i)^d\right)}+\|\beta\|_{L^2(0;t;L^2(\Omega_t^1))}\right]$$
$$\leq Ch^{r-1}\left(\sum_{i=1}^{2}\left[\left\|\frac{d\boldsymbol{v}_i}{dt}\Big|_{\hat{\boldsymbol{x}}}^{h}\right\|_{L^2\left(0,t;H^r(\Omega_t^i)^d\right)}+\|\boldsymbol{v}_i\|_{L^\infty\left(0,t;H^r(\Omega_t^i)^d\right)}\right]\right.\tag{25}$$
$$\left.+\|p_1\|_{L^\infty(0,t;H^{r-1}(\Omega_t^1))}+\left\|\frac{dp_1}{dt}\Big|_{\hat{\boldsymbol{x}}}^{h}\right\|_{L^2(0,t;H^{r-1}(\Omega_t^1))}\right)$$

Then, adding the a priori interpolation error estimates of $\left\|\frac{d\eta_i}{dt}\Big|_{\hat{\boldsymbol{x}}}^{h}\right\|_{L^2(\Omega_t^i)^d}$, $\|\eta_i\|_{H^1(\Omega_t^i)^d}$ and $\|\alpha\|_{L^2(\Omega_t^1)}$ back in and applying the triangular inequality, we have the result (18). □

4. Fully discrete ALE–finite element discretization

With the semi-discrete scheme taken care of, we can now move on to the fully-discrete scheme. Let $\Delta t>0$ be the time step and $t_n=n\Delta t$ for $n=0,...,N$, where $N=\frac{T}{\Delta t}$, and denote $\phi^n=\phi(t_n)$ for any function $\phi(t)$. We introduce the following notation to account for the backward Euler scheme that is used to discretize the time derivative:

$$\partial_t\boldsymbol{v}_{i,h}^{n+1}=\frac{\boldsymbol{v}_{i,h}^{n+1}-\boldsymbol{v}_{i,h}^{n}\circ\boldsymbol{X}_{n+1,n}^{i}}{\Delta t},$$

where $\boldsymbol{X}_{n+1,n}^{i}=\boldsymbol{X}_{h,n}^{i}\circ(\boldsymbol{X}_{h,n+1}^{i})^{-1}$ for $i=1,2$.

The nonconservative fully discrete scheme of (4) can now be defined as: find $(\boldsymbol{v}_{1,h}^{n+1}, \boldsymbol{v}_{2,h}^{n+1}) \in \boldsymbol{V}_h^{n+1}$, $p_{1,h}^{n+1} \in Q_h^{0,n+1}$ such that

(26)
$$\sum_{i=1}^{2}\left[\left(\partial_t \boldsymbol{v}_{i,h}^{n+1}, \psi_{i,h}\right)_{\Omega_{n+1}^i} + \left(\mu_i \nabla \boldsymbol{v}_{i,h}^{n+1}, \nabla\psi_{i,h}\right)_{\Omega_{n+1}^i} - \left((\boldsymbol{\omega}_{i,h}^{n+1}\cdot\nabla)\boldsymbol{v}_{i,h}^{n+1}, \psi_{i,h}\right)_{\Omega_{n+1}^i}\right]$$
$$- \left(p_{1,h}^{n+1}, \nabla\cdot\psi_{1,h}\right)_{\Omega_{n+1}^1} + \left(\nabla\cdot\boldsymbol{v}_{1,h}^{n+1}, q_{1,h}\right)_{\Omega_{n+1}^1} = \sum_{i=1}^{2}\left(\boldsymbol{f}_i^{n+1}, \psi_{i,h}\right)_{\Omega_{n+1}^i}$$
$$+ \langle \boldsymbol{\tau}, \psi_{1,h}\rangle_{\Gamma_{n+1}}, \qquad \forall(\psi_{1,h}, \psi_{2,h}) \in \boldsymbol{V}_h^{n+1},\ q_{1,h} \in Q_h^{n+1}.$$

To perform the required error analysis for (26), we need to introduce some lemmas as follows.

Lemma 4.1. *Let $\phi_{i,h}^{n+1} \in \boldsymbol{V}_h^{n+1}$ be a discrete function defined in Ω_{n+1}^i $(i=1,2)$. Then*
$$\|\phi_{i,h}^{n+1}\circ \boldsymbol{X}_{n,n+1}^i\|_{0,\Omega_n^i}^2 = \|\phi_{i,h}^{n+1}\|_{0,\Omega_{n+1}^i}^2 - \int_{t_n}^{t_{n+1}}\left(\int_{\Omega_t^i} |\phi_{i,h}^{n+1}\circ \boldsymbol{X}_{t,n+1}^i|^2 \nabla\cdot\boldsymbol{\omega}_{i,h} d\boldsymbol{x}\right)dt.$$

Proof.
$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t^i} |\phi_{i,h}^{n+1}\circ \boldsymbol{X}_{t,n+1}^i|^2 d\boldsymbol{x} &= \frac{d}{dt}\int_{\hat\Omega^i} |\hat\phi_{i,h}^{n+1}|^2 \boldsymbol{J}_t^i d\hat{\boldsymbol{x}} = \int_{\hat\Omega^i} |\hat\phi_{i,h}^{n+1}|^2 \frac{d\boldsymbol{J}_t^i}{dt} d\hat{\boldsymbol{x}} \\ &= \int_{\Omega_t} |\phi^{n+1}\circ \boldsymbol{X}_{t,n+1}|^2 \nabla\cdot\boldsymbol{\omega}_{i,h} dx,\end{aligned} \tag{27}$$
where we employ the identity $\frac{d\boldsymbol{J}_t^i}{dt} = \boldsymbol{J}_t^i \nabla\cdot\boldsymbol{\omega}_{i,h}$ [8]. Integrating (27) in time from t_n to t_{n+1}, we get

(28)
$$\int_{t_n}^{t_{n+1}}\int_{\Omega_t^i} |\phi_i^{n+1}\circ \boldsymbol{X}_{t,n+1}^i|^2 \nabla\cdot\boldsymbol{\omega}_{i,h} d\boldsymbol{x}dt = \int_{t_n}^{t_{n+1}} \frac{d}{dt}\int_{\Omega_t^i} |\phi_i^{n+1}\circ \boldsymbol{X}_{t,n+1}^i|^2 d\boldsymbol{x}dt$$
$$= \int_{\Omega_{n+1}^i} |\phi_i^{n+1}|^2 d\boldsymbol{x} - \int_{\Omega_n^i} |\phi_i^{n+1}\circ \boldsymbol{X}_{n,n+1}^i|^2 d\boldsymbol{x},$$
where the rearrangement gives the result. □

The following lemma considers the classical Taylor expansion technique in the context of the ALE description.

Lemma 4.2. *For any $\boldsymbol{v}_i \in H^2(0,T;H^1(\Omega_t^i))$ where Ω_t^i is mapped from $\hat\Omega^i$ by the discrete ALE mapping $\boldsymbol{X}_{h,t}^i$, we have*

(29)
$$\begin{aligned}&\partial_t \boldsymbol{v}_i^{n+1} = \tfrac{\boldsymbol{v}_i(\boldsymbol{x}^{n+1}, t_{n+1}) - \boldsymbol{v}_i(\boldsymbol{x}^n, t_n)}{\Delta t} \\ &= \left(\tfrac{d\boldsymbol{v}_i}{dt}\Big|_{\hat{\boldsymbol{x}}}^h\right)^{n+1} - \tfrac{\Delta t}{2}\left[\left(\tfrac{d^2\boldsymbol{v}_i}{dt^2}\Big|_{\hat{\boldsymbol{x}}}^h\right)^{n+1} - \boldsymbol{\omega}_{i,h}^{n+1}(\nabla\boldsymbol{\omega}_{i,h})^{n+1}(\nabla\boldsymbol{v}_i)^{n+1}\right] + O((\Delta t)^2),\end{aligned}$$

where, $\boldsymbol{\omega}_{i,h} = \frac{\partial \boldsymbol{x}}{\partial t}$ denotes the ALE moving mesh velocity.

Proof. Expanding $\boldsymbol{v}_i(\boldsymbol{x}^n, t_n)$ at $\boldsymbol{x}^{n+1}$, we get
$$\begin{aligned}\boldsymbol{v}_i(\boldsymbol{x}^n, t_n) &= \boldsymbol{v}_i(\boldsymbol{x}^{n+1}, t_n) - \Delta\boldsymbol{x}\left(\tfrac{\partial \boldsymbol{v}_i}{\partial \boldsymbol{x}}\right)(\boldsymbol{x}^{n+1}, t_n) \\ &\quad + \tfrac{(\Delta\boldsymbol{x})^2}{2}\left(\tfrac{\partial^2\boldsymbol{v}_i}{\partial\boldsymbol{x}^2}\right)(\boldsymbol{x}^{n+1}, t_n) + O((\Delta\boldsymbol{x})^2),\end{aligned} \tag{30}$$

where $\Delta \boldsymbol{x} = \boldsymbol{x}^{n+1} - \boldsymbol{x}^n = \boldsymbol{x}(\hat{\boldsymbol{x}}, t_{n+1}) - \boldsymbol{x}(\hat{\boldsymbol{x}}, t_n)$. If expanding $\boldsymbol{x}(\hat{\boldsymbol{x}}, t_n)$ at t_{n+1}, then

$$\boldsymbol{x}(\hat{\boldsymbol{x}}, t_n) = \boldsymbol{x}(\hat{\boldsymbol{x}}, t_{n+1}) - \Delta t \left(\frac{\partial \boldsymbol{x}}{\partial t}\right)^{n+1} + \frac{(\Delta t)^2}{2}\left(\frac{\partial^2 \boldsymbol{x}}{\partial t^2}\right)^{n+1} + O((\Delta t)^3),$$

thus,

$$\Delta \boldsymbol{x} = \Delta t \left(\frac{\partial \boldsymbol{x}}{\partial t}\right)^{n+1} - \frac{(\Delta t)^2}{2}\left(\frac{\partial^2 \boldsymbol{x}}{\partial t^2}\right)^{n+1} + O((\Delta t)^3). \tag{31}$$

Further by Taylor expansion, we have

$$\begin{aligned}
\left(\frac{\partial \boldsymbol{v}_i}{\partial \boldsymbol{x}}\right)(\boldsymbol{x}^{n+1}, t_n) &= \left(\frac{\partial \boldsymbol{v}_i}{\partial \boldsymbol{x}}\right)(\boldsymbol{x}^{n+1}, t_{n+1}) \\
&\quad - \Delta t \left(\frac{\partial^2 \boldsymbol{v}_i}{\partial t \partial \boldsymbol{x}}\right)(\boldsymbol{x}^{n+1}, t_{n+1}) + O((\Delta t)^2), \\
\left(\frac{\partial^2 \boldsymbol{v}_i}{\partial \boldsymbol{x}^2}\right)(\boldsymbol{x}^{n+1}, t_n) &= \left(\frac{\partial^2 \boldsymbol{v}_i}{\partial \boldsymbol{x}^2}\right)(\boldsymbol{x}^{n+1}, t_{n+1}) \\
&\quad - \Delta t \left(\frac{\partial^3 \boldsymbol{v}_i}{\partial t \partial \boldsymbol{x}^2}\right)(\boldsymbol{x}^{n+1}, t_{n+1}) + O((\Delta t)^2).
\end{aligned} \tag{32}$$

Then, we can rewrite (30) as

$$\begin{aligned}
\boldsymbol{v}_i(\boldsymbol{x}^n, t_n) &= \boldsymbol{v}_i(\boldsymbol{x}^{n+1}, t_n) - \Delta \boldsymbol{x}\left(\frac{\partial \boldsymbol{v}_i}{\partial \boldsymbol{x}}\right)(\boldsymbol{x}^{n+1}, t_{n+1}) + \Delta \boldsymbol{x} \Delta t \left(\frac{\partial^2 \boldsymbol{v}_i}{\partial t \partial \boldsymbol{x}}\right)(\boldsymbol{x}^{n+1}, t_{n+1}) \\
&\quad + \frac{(\Delta \boldsymbol{x})^2}{2}\left(\frac{\partial^2 \boldsymbol{v}_i}{\partial \boldsymbol{x}^2}\right)(\boldsymbol{x}^{n+1}, t_{n+1}) + O(\Delta \boldsymbol{x}(\Delta t)^2) + O((\Delta \boldsymbol{x})^2 \Delta t).
\end{aligned} \tag{33}$$

Since

$$\frac{\boldsymbol{v}_i(\boldsymbol{x}^{n+1}, t_{n+1}) - \boldsymbol{v}_i(\boldsymbol{x}^n, t_n)}{\Delta t} = \frac{\boldsymbol{v}_i(\boldsymbol{x}^{n+1}, t_{n+1}) - \boldsymbol{v}_i(\boldsymbol{x}^{n+1}, t_n)}{\Delta t} + \frac{\boldsymbol{v}_i(\boldsymbol{x}^{n+1}, t_n) - \boldsymbol{v}_i(\boldsymbol{x}^n, t_n)}{\Delta t}, \tag{34}$$

which, when expanded, gives

$$\frac{\boldsymbol{v}_i(\boldsymbol{x}^{n+1}, t_{n+1}) - \boldsymbol{v}_i(\boldsymbol{x}^{n+1}, t_n)}{\Delta t} = \left(\frac{\partial \boldsymbol{v}_i}{\partial t}\right)^{n+1} - \frac{\Delta t}{2}\left(\frac{\partial^2 \boldsymbol{v}_i}{\partial t^2}\right)^{n+1} + O((\Delta t)^2), \tag{35}$$

and due to (33), we have

$$\begin{aligned}
\frac{\boldsymbol{v}_i(\boldsymbol{x}^{n+1}, t_n) - \boldsymbol{v}_i(\boldsymbol{x}^n, t_n)}{\Delta t} &= \frac{\Delta \boldsymbol{x}}{\Delta t}\left(\frac{\partial \boldsymbol{v}_i}{\partial \boldsymbol{x}}\right)^{n+1} - \frac{(\Delta \boldsymbol{x})^2}{2\Delta t}\left(\frac{\partial^2 \boldsymbol{v}_i}{\partial \boldsymbol{x}^2}\right)^{n+1} - \Delta \boldsymbol{x}\left(\frac{\partial^2 \boldsymbol{v}_i}{\partial t \partial \boldsymbol{x}}\right)^{n+1} \\
&\quad + O(\Delta \boldsymbol{x} \Delta t) + O((\Delta \boldsymbol{x})^2).
\end{aligned} \tag{36}$$

Further, (31) yields

$$\frac{\Delta \boldsymbol{x}}{\Delta t} = \left(\frac{\partial \boldsymbol{x}}{\partial t}\right)^{n+1} - \frac{\Delta t}{2}\left(\frac{\partial^2 \boldsymbol{x}}{\partial t^2}\right)^{n+1} + O((\Delta t)^2),$$

then we have

$$
\begin{aligned}
&\frac{\boldsymbol{v}_i(\boldsymbol{x}^{n+1},t_{n+1})-\boldsymbol{v}_i(\boldsymbol{x}^n,t_n)}{\Delta t} = \left(\frac{\partial \boldsymbol{v}_i}{\partial t}\right)^{n+1} - \frac{\Delta t}{2}\left(\frac{\partial^2 \boldsymbol{v}_i}{\partial t^2}\right)^{n+1} + \left(\frac{\partial \boldsymbol{x}}{\partial t}\right)^{n+1}\left(\frac{\partial \boldsymbol{v}_i}{\partial \boldsymbol{x}}\right)^{n+1} \\
&- \frac{\Delta t}{2}\left(\frac{\partial^2 \boldsymbol{x}}{\partial t^2}\right)^{n+1}\left(\frac{\partial \boldsymbol{v}_i}{\partial \boldsymbol{x}}\right)^{n+1} - \frac{\Delta t}{2}\left[\left(\frac{\partial \boldsymbol{x}}{\partial t}\right)^{n+1}\right]^2\left(\frac{\partial^2 \boldsymbol{v}_i}{\partial \boldsymbol{x}^2}\right)^{n+1} \\
&- \Delta t\left(\frac{\partial \boldsymbol{x}}{\partial t}\right)^{n+1}\left(\frac{\partial^2 \boldsymbol{v}_i}{\partial t\partial \boldsymbol{x}}\right)^{n+1} + O((\Delta t)^2) \\
&= \left[\left(\frac{\partial \boldsymbol{v}_i}{\partial t}\right)^{n+1} + \left(\frac{\partial \boldsymbol{x}}{\partial t}\right)^{n+1}\left(\frac{\partial \boldsymbol{v}_i}{\partial \boldsymbol{x}}\right)^{n+1}\right] \\
&- \frac{\Delta t}{2}\left[\left(\frac{\partial^2 \boldsymbol{v}_i}{\partial t^2}\right)^{n+1} + \left(\frac{\partial^2 \boldsymbol{x}}{\partial t^2}\right)^{n+1}\left(\frac{\partial \boldsymbol{v}_i}{\partial \boldsymbol{x}}\right)^{n+1}\right. \\
&\left. + 2\left(\frac{\partial \boldsymbol{x}}{\partial t}\right)^{n+1}\left(\frac{\partial^2 \boldsymbol{v}_i}{\partial t\partial \boldsymbol{x}}\right)^{n+1} + \left(\left(\frac{\partial \boldsymbol{x}}{\partial t}\right)^{n+1}\right)^2\left(\frac{\partial^2 \boldsymbol{v}_i}{\partial \boldsymbol{x}^2}\right)^{n+1}\right] + O((\Delta t)^2) \\
&= \left(\left.\frac{d\boldsymbol{v}_i}{dt}\right|_{\hat{\boldsymbol{x}}}^h\right)^{n+1} - \frac{\Delta t}{2}\left[\left(\left.\frac{d^2\boldsymbol{v}_i}{dt^2}\right|_{\hat{\boldsymbol{x}}}^h\right)^{n+1} - \left(\frac{\partial \boldsymbol{x}}{\partial t}\right)^{n+1}\left(\frac{\partial^2 \boldsymbol{x}}{\partial \boldsymbol{x}\partial t}\right)^{n+1}\left(\frac{\partial \boldsymbol{v}_i}{\partial x}\right)^{n+1}\right] \\
&+ O((\Delta t)^2) \\
&= \left(\left.\frac{d\boldsymbol{v}_i}{dt}\right|_{\hat{\boldsymbol{x}}}^h\right)^{n+1} - \frac{\Delta t}{2}\left[\left(\left.\frac{d^2\boldsymbol{v}_i}{dt^2}\right|_{\hat{\boldsymbol{x}}}^h\right)^{n+1} - \boldsymbol{\omega}_{i,h}^{n+1}\left(\nabla\boldsymbol{\omega}_{i,h}\right)^{n+1}\left(\nabla\boldsymbol{v}_i\right)^{n+1}\right] + O((\Delta t)^2).
\end{aligned}
\tag{37}
$$

Then the result is finally proved. □

Lemma 4.3. [13] *There exists C_1 and C_2 depending on the discrete ALE mapping $\boldsymbol{X}_{h,t}^i$ $(i=1,2)$ such that*

$$
\|\boldsymbol{J}_t^i\|_{L^\infty(\hat{\Omega}^i)} \le C_1, \quad \|(\boldsymbol{J}_t^i)^{-1}\|_{L^\infty(\Omega_t^i)} \le C_2, \quad \forall t\in[0,T], \tag{38}
$$

where, $\boldsymbol{J}_t^i = det\left(\mathbf{F}_t^i \circ \left(\boldsymbol{X}_{h,t}^i\right)^{-1}\right)$, $(\boldsymbol{J}_t^i)^{-1} = det\left((\mathbf{F}_t^i)^{-1}\circ \boldsymbol{X}_{h,t}^i\right)$. *And,*

$$
\|\boldsymbol{J}_t^i - \boldsymbol{J}_n^i\|_{L^\infty(\hat{\Omega}^i)} \le C\Delta t, \quad \forall t\in[t_n,t_{n+1}]. \tag{39}
$$

We can now proceed to the following main theorem of the section.

Theorem 4.4. *Suppose $(\boldsymbol{v}_1, p_1, \boldsymbol{v}_2)$ is the solution to (4) satisfying the following regularity properties*

(40)

$$
\begin{aligned}
&\boldsymbol{v}_i \in L^\infty\left(0,T;H^r(\Omega_t^i)^d\right), \left.\tfrac{d\boldsymbol{v}_i}{dt}\right|_{\hat{\boldsymbol{x}}}^h \in L^\infty(0,T;H^{r-1}(\Omega_t^i)^d), \\
&\left.\tfrac{d^2\boldsymbol{v}_i}{dt^2}\right|_{\hat{\boldsymbol{x}}}^h \in L^\infty(0,T;L^2(\Omega_t^i)^d), p_1 \in L^\infty\left(0,T;H^{r-1}(\Omega_t^1)\right), \ \textit{for } r\ge 3,\ i=1,2,
\end{aligned}
$$

and $\left(\boldsymbol{v}_{1,h}^{n+1}, p_{1,h}^{n+1}, \boldsymbol{v}_{2,h}^{n+1}\right)$ *is the solution to (26), then we have the following error estimate:*

$$\begin{aligned}
&\sum_{i=1}^{2}\left[\|\boldsymbol{v}_i^N-\boldsymbol{v}_{i,h}^N\|_{L^2(\Omega_N^i)^d}+\Delta t\sum_{j=1}^{N}\|\boldsymbol{v}_i^j-\boldsymbol{v}_{i,h}^j\|_{H^1(\Omega_j^i)^d}\right]\\
&\le C(h^{r-1}+\Delta t)\Bigg(\sum_{i=1}^{2}\Bigg[\|\boldsymbol{v}_i\|_{L^\infty(0,T;H^r(\Omega_t^i)^d)}+\left\|\frac{d\boldsymbol{v}_i}{dt}\Big|_{\hat{\boldsymbol{x}}}^{h}\right\|_{L^\infty(0,T;H^{r-1}(\Omega_t^i)^d)}\\
&+\left\|\frac{d^2\boldsymbol{v}_i}{dt^2}\Big|_{\hat{\boldsymbol{x}}}^{h}\right\|_{L^\infty(0,T;L^2(\Omega_t^i)^d)}\Bigg]+\|p_1\|_{L^\infty(0,T;H^{r-1}(\Omega_t^1))}\Bigg).
\end{aligned}\tag{41}$$

Proof. Subtracting (26) from (4) at t_{n+1}, we have

$$\begin{aligned}
&\sum_{i=1}^{2}\Bigg[\left(\left(\frac{d\boldsymbol{v}_i}{dt}\Big|_{\hat{\boldsymbol{x}}}^{h}\right)^{n+1}-\partial_t\boldsymbol{v}_i^{n+1},\psi_{i,h}\right)_{\Omega_{n+1}^i}+\left(\partial_t\boldsymbol{v}_i^{n+1}-\partial_t\boldsymbol{v}_{i,h}^{n+1},\psi_{i,h}\right)_{\Omega_{n+1}^i}\\
&+\left(\mu_i\nabla\left(\boldsymbol{v}_i^{n+1}-\boldsymbol{v}_{i,h}^{n+1}\right),\nabla\psi_{i,h}\right)_{\Omega_{n+1}^i}-\left(\boldsymbol{\omega}_{i,h}^{n+1}\cdot\nabla\left(\boldsymbol{v}_i^{n+1}-\boldsymbol{v}_{i,h}^{n+1}\right),\psi_{i,h}\right)_{\Omega_{n+1}^i}\Bigg]\\
&-\left(p_1^{n+1}-p_{1,h}^{n+1},\nabla\cdot\psi_{1,h}\right)_{\Omega_{n+1}^1}+\left(\nabla\cdot\left(\boldsymbol{v}_1^{n+1}-\boldsymbol{v}_{1,h}^{n+1}\right),q_{1,h}\right)_{\Omega_{n+1}^1}=0.
\end{aligned}\tag{42}$$

Pick arbitrary discrete functions $\tilde{\boldsymbol{v}}=(\tilde{\boldsymbol{v}}_1,\tilde{\boldsymbol{v}}_2)\in\boldsymbol{K}_h$ and $\tilde{p}\in Q_h^0$, and adopt the same notations ξ,η,α and β which are defined in Section 3.3. Let $\psi_{i,h}=\xi_i^{n+1}\in\boldsymbol{K}_h,\ q_{1,h}=\beta^{n+1}$ in (42), leading to

$$\begin{aligned}
&\sum_{i=1}^{2}\Bigg[\left(\left(\frac{d\boldsymbol{v}_i}{dt}\Big|_{\hat{\boldsymbol{x}}}^{h}\right)^{n+1}-\partial_t\boldsymbol{v}_i^{n+1},\xi_i^{n+1}\right)_{\Omega_{n+1}^i}+\left(\partial_t\xi_i^{n+1},\xi_i^{n+1}\right)_{\Omega_{n+1}^i}+\\
&\left(\mu_i\nabla\xi_i^{n+1},\nabla\xi_i^{n+1}\right)_{\Omega_{n+1}^i}\Bigg]-\left(\beta^{n+1},\nabla\cdot\xi_1^{n+1}\right)_{\Omega_{n+1}^1}+\left(\nabla\cdot\xi_1^{n+1},\beta^{n+1}\right)_{\Omega_{n+1}^1}\\
&=\sum_{i=1}^{2}\Bigg[-\left(\partial_t\eta_i^{n+1},\xi_i^{n+1}\right)_{\Omega_{n+1}^i}-\left(\mu_i\nabla\eta_i^{n+1},\nabla\xi_i^{n+1}\right)_{\Omega_{n+1}^i}+\\
&\left(\left(\boldsymbol{\omega}_{i,h}^{n+1}\cdot\nabla\right)\left(\eta_i^{n+1}+\xi_i^{n+1}\right),\xi_i^{n+1}\right)_{\Omega_{n+1}^i}\Bigg]+\left(\alpha^{n+1},\nabla\cdot\xi_1^{n+1}\right)_{\Omega_{n+1}^1}\\
&-\left(\nabla\cdot\eta_i^{n+1},\beta^{n+1}\right)_{\Omega_{n+1}^1}.
\end{aligned}\tag{43}$$

First, we notice that $-\left(\beta^{n+1},\nabla\cdot\xi_1^{n+1}\right)_{\Omega_{n+1}^1}+\left(\nabla\cdot\xi_1^{n+1},\beta^{n+1}\right)_{\Omega_{n+1}^1}=0$, and

$$\left(\nabla\cdot\eta_1^{n+1},\beta^{n+1}\right)_{\Omega_{n+1}^1}=0,$$

which is due to the incompressibility condition $\nabla\cdot\boldsymbol{v}_1=0$ and $\tilde{\boldsymbol{v}}_1\in\boldsymbol{K}_h$.

Using Cauchy-Schwarz inequality, Poincaré inequality and Young's inequality with ϵ, we have error estimates for the following terms in (43):

$$\begin{aligned}
&\left(\mu_i \nabla \xi_i^{n+1}, \nabla \xi_i^{n+1}\right)_{\Omega^i_{n+1}} \geq C\|\xi_i^{n+1}\|^2_{1,\Omega^i_{n+1}},\\
&\left(\mu_i \nabla \eta_i^{n+1}, \nabla \xi_i^{n+1}\right)_{\Omega^i_{n+1}} \leq C\|\eta_i^{n+1}\|^2_{1,\Omega^i_{n+1}} + \epsilon\|\xi_i^{n+1}\|^2_{1,\Omega^i_{n+1}},\\
&\left(\left(\boldsymbol{\omega}^{n+1}_{i,h}\cdot\nabla\right)\left(\eta_i^{n+1}+\xi_i^{n+1}\right), \xi_i^{n+1}\right)_{\Omega^i_{n+1}} \leq C\left(\|\eta_i^{n+1}\|^2_{1,\Omega^i_{n+1}} + \|\xi_i^{n+1}\|^2_{0,\Omega^i_{n+1}}\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \epsilon\|\xi_i^{n+1}\|^2_{1,\Omega^i_{n+1}},\\
&\left(\alpha^{n+1}, \nabla\cdot\xi_1^{n+1}\right)_{\Omega^1_{n+1}} \leq C\|\alpha^{n+1}\|^2_{0,\Omega^1_{n+1}} + \epsilon\|\xi_i^{n+1}\|^2_{1,\Omega^1_{n+1}}.
\end{aligned}$$

The term $\left(\partial_t \xi_i^{n+1}, \xi_i^{n+1}\right)$ in (43) is handled in the following way.

$$\begin{aligned}
&\left(\frac{\xi_i^{n+1}-\xi_i^n\circ \boldsymbol{X}^i_{n+1,n}}{\Delta t}, \xi_i^{n+1}\right)_{\Omega^i_{n+1}} = \frac{1}{\Delta t}\left[\left(\xi_i^{n+1},\xi_i^{n+1}\right)_{\Omega^i_{n+1}} - \left(\xi_i^n\circ \boldsymbol{X}^i_{n+1,n}, \xi_i^{n+1}\right)_{\Omega^i_{n+1}}\right]\\
&= \frac{1}{\Delta t}\left[\left(\xi_i^{n+1},\xi_i^{n+1}\right)_{\Omega^i_{n+1}} - \left(\xi_i^n, \xi_i^{n+1}\frac{\boldsymbol{J}^i_{n+1}}{\boldsymbol{J}^i_n}\right)_{\Omega^i_n}\right]\\
&= \frac{1}{\Delta t}\Bigg[\|\xi_i^{n+1}\|^2_{0,\Omega^i_{n+1}} - \left(\xi_i^n, \xi_i^{n+1}\circ \boldsymbol{X}^i_{n,n+1}\right)_{\Omega^i_n}\\
&\quad - \left(\xi_i^n, \xi_i^{n+1}\circ \boldsymbol{X}^i_{n,n+1}\left(\frac{\boldsymbol{J}^i_{n+1}-\boldsymbol{J}^i_n}{\boldsymbol{J}^i_n}\right)\right)_{\Omega^i_n}\Bigg],
\end{aligned}$$

where, applying Lemma 4.1 and (38) to the second term, we have

$$\begin{aligned}
&\left(\xi_i^n, \xi_i^{n+1}\circ \boldsymbol{X}^i_{n,n+1}\right)_{\Omega^i_n} \leq \frac{1}{2}\|\xi_i^n\|^2_{0,\Omega^i_n} + \frac{1}{2}\|\xi_i^{n+1}\circ \boldsymbol{X}^i_{n,n+1}\|^2_{0,\Omega^i_n}\\
&\leq \frac{1}{2}\|\xi_i^n\|^2_{0,\Omega^i_n} + \frac{1}{2}\|\xi_i^{n+1}\|^2_{0,\Omega^i_{n+1}} - \frac{1}{2}\int_{t_n}^{t_{n+1}}\left(\int_{\Omega^i_t}|\xi_i^{n+1}\circ \boldsymbol{X}^i_{t,n+1}|^2\nabla\cdot\boldsymbol{\omega}_{i,h}d\boldsymbol{x}\right)dt\\
&\leq \frac{1}{2}\|\xi_i^n\|^2_{0,\Omega^i_n} + \frac{1}{2}\|\xi_i^{n+1}\|^2_{0,\Omega^i_{n+1}}\\
&+ \frac{1}{2}\sup_{t\in[t_n,t_{n+1}]}\|\nabla\cdot\boldsymbol{\omega}_{i,h}\|_{\infty,\Omega^i_t}\int_{t_n}^{t_{n+1}}\left(\int_{\Omega^i_{n+1}}|\xi_i^{n+1}\circ \boldsymbol{X}^i_{t,n+1}|^2\frac{\boldsymbol{J}^i_t}{\boldsymbol{J}^i_{n+1}}d\boldsymbol{x}\right)dt\\
&\leq \frac{1}{2}\|\xi_i^n\|^2_{0,\Omega^i_n} + \frac{1}{2}\|\xi_i^{n+1}\|^2_{0,\Omega^i_{n+1}} + C\Delta t\|\xi_i^{n+1}\|^2_{0,\Omega^i_{n+1}}.
\end{aligned}$$

Following similarly, and applying (38) and (39), we also have:

$$\begin{aligned}
&\left(\xi_i^n, \xi_i^{n+1}\circ \boldsymbol{X}^i_{n,n+1}\left(\frac{\boldsymbol{J}^i_{n+1}-\boldsymbol{J}^i_n}{\boldsymbol{J}^i_n}\right)\right)_{\Omega^i_n}\\
&\leq C\Delta t\left(\frac{1}{2}\|\xi_i^n\|^2_{0,\Omega^i_n} + \frac{1}{2}\|\xi_i^{n+1}\circ \boldsymbol{X}^i_{n,n+1}\|^2_{0,\Omega^i_n}\right)\\
&\leq C\Delta t\left(\|\xi_i^n\|^2_{0,\Omega^i_n} + \|\xi_i^{n+1}\|^2_{0,\Omega^i_{n+1}} + \Delta t\|\xi_i^{n+1}\|^2_{0,\Omega^i_{n+1}}\right).
\end{aligned} \tag{44}$$

Thus,

$$\left(\partial_t \xi_i^{n+1}, \xi_i^{n+1}\right) \geq \frac{1}{2\Delta t}\left(\|\xi_i^{n+1}\|^2_{0,\Omega^i_{n+1}} - \|\xi_i^n\|^2_{0,\Omega^i_n}\right) - C\left(\|\xi_i^n\|^2_{0,\Omega^i_n} + \|\xi_i^{n+1}\|^2_{0,\Omega^i_{n+1}}\right).$$

The remaining terms will be handled as follows using Lemma 4.2.

$$\begin{aligned}
&\left(\left(\frac{d\boldsymbol{v}_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h\right)^{n+1} - \partial_t \boldsymbol{v}_i^{n+1}, \xi_i^{n+1}\right)_{\Omega_{n+1}^i} \\
&= \left(\frac{\Delta t}{2}\left(\left(\frac{d^2\boldsymbol{v}}{dt^2}\bigg|_{\hat{\boldsymbol{x}}}^h\right)^{n+1} - \boldsymbol{\omega}_{i,h}^{n+1}(\nabla\boldsymbol{\omega}_{i,h})^{n+1}\nabla\boldsymbol{v}_i^{n+1}\right), \xi_i^{n+1}\right)_{\Omega_{n+1}^i} \\
&\le C(\Delta t)^2\left(\left\|\left(\frac{d^2\boldsymbol{v}_i}{dt^2}\bigg|_{\hat{\boldsymbol{x}}}^h\right)^{n+1}\right\|_{0,\Omega_{n+1}^i}^2 + \|\boldsymbol{v}_i^{n+1}\|_{1,\Omega_{n+1}^i}^2\right) + \frac{1}{2}\|\xi_i^{n+1}\|_{0,\Omega_{n+1}^i}^2,
\end{aligned}$$

and

$$\begin{aligned}
&\left(\partial_t \eta_i^{n+1}, \xi_i^{n+1}\right)_{\Omega_{n+1}^i} \\
&= \left(\left(\frac{d\eta_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h\right)^{n+1} - \frac{\Delta t}{2}\left(\left(\frac{d^2\eta_i}{dt^2}\bigg|_{\hat{\boldsymbol{x}}}^h\right)^{n+1} - \boldsymbol{\omega}_{i,h}^{n+1}(\nabla\boldsymbol{\omega}_{i,h})^{n+1}\nabla\eta_i^{n+1}\right), \xi_i^{n+1}\right)_{\Omega_{n+1}^i} \\
&\le C\Bigg[\left\|\left(\frac{d\eta_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h\right)^{n+1}\right\|_{0,\Omega_{n+1}^i}^2 + (\Delta t)^2\left(\left\|\left(\frac{d^2\eta_i}{dt^2}\bigg|_{\hat{\boldsymbol{x}}}^h\right)^{n+1}\right\|_{0,\Omega_{n+1}^i}^2 + \|\eta_i^{n+1}\|_{1,\Omega_{n+1}^i}^2\right)\Bigg] \\
&+ \frac{1}{2}\|\xi_i^{n+1}\|_{0,\Omega_{n+1}^i}^2.
\end{aligned}$$

Combining error estimates of all terms at above, and choosing ϵ sufficiently small, we have

$$\begin{aligned}
&\sum_{i=1}^{2}\left(\frac{1}{2\Delta t}\left(\|\xi_i^{n+1}\|_{0,\Omega_{n+1}^i}^2 - \|\xi_i^{n}\|_{0,\Omega_{n}^i}^2\right) + \|\xi_i^{n+1}\|_{1,\Omega_{n+1}^i}^2\right) \\
&\le C\Bigg[\sum_{i=1}^{2}\left(\left\|\left(\frac{d\eta_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h\right)^{n+1}\right\|_{0,\Omega_{n+1}^i}^2 + \|\eta_i^{n+1}\|_{1,\Omega_{n+1}^i}^2 + \|\xi_i^{n+1}\|_{0,\Omega_{n+1}^i}^2 + \|\xi_i^{n}\|_{0,\Omega_{n}^i}^2\right) \\
&+ \|\alpha^{n+1}\|_{0,\Omega_{n+1}^1}^2 + (\Delta t)^2\sum_{i=1}^{2}\left(\left\|\left(\frac{d^2\boldsymbol{v}_i}{dt^2}\bigg|_{\hat{\boldsymbol{x}}}^h\right)^{n+1}\right\|_{0,\Omega_{n+1}^i}^2 + \|\boldsymbol{v}_i^{n+1}\|_{1,\Omega_{n+1}^i}^2\right)\Bigg].
\end{aligned}$$

To achieve the global error estimate we sum over n from 0 to $N-1$ on both sides of the above inequality, then apply the discrete Grönwall's inequality, and choose $\boldsymbol{v}_{i,h}(0) = \tilde{\boldsymbol{v}}_i(0) \in \boldsymbol{V}_h$ to have $\xi_i(0) = 0$ $(i = 1, 2)$, leading to

(45)

$$\begin{aligned}
&\sum_{i=1}^{2}\left(\|\xi_i^N\|_{0,\Omega_N^i}^2 + \Delta t\sum_{j=1}^{N}\|\xi_i^j\|_{1,\Omega_j^i}^2\right) \\
&\le C\Bigg[\inf_{\substack{\tilde{\boldsymbol{v}}\in\boldsymbol{K}_h\setminus\{0\},\\ \tilde{p}\in Q_h^0\setminus\{0\}}}\left(\sum_{i=1}^{2}\left(\Delta t\sum_{j=1}^{N}\left\|\left(\frac{d\eta_i}{dt}\bigg|_{\hat{\boldsymbol{x}}}^h\right)^{j}\right\|_{0,\Omega_j^i}^2 + \Delta t\sum_{j=1}^{N}\|\eta_i^j\|_{1,\Omega_j^i}^2\right)\right. \\
&\left. + \Delta t\sum_{j=1}^{N}\|\alpha^j\|_{0,\Omega_j^1}^2\right) + (\Delta t)^2\sum_{i=1}^{2}\left(\left\|\left(\frac{d^2\boldsymbol{v}_i}{dt^2}\bigg|_{\hat{\boldsymbol{x}}}^h\right)^{n+1}\right\|_{0,\Omega_{n+1}^i}^2 + \|\boldsymbol{v}_i^{n+1}\|_{1,\Omega_{n+1}^i}^2\right)\Bigg].
\end{aligned}$$

By the classic Brezzi theory [3, 6], we extend the infimum over the entire finite element spaces, resulting in

(46)

$$\sum_{i=1}^{2}\left(\|\xi_i^N\|_{0,\Omega_N^i}^2+\Delta t\sum_{j=1}^{N}\|\xi_i^j\|_{1,\Omega_j^i}^2\right)$$
$$\leq C\Bigg[\inf_{\substack{\tilde{\boldsymbol{v}}\in\boldsymbol{V}_h\backslash\{0\},\\ \tilde{p}\in Q_h^0\backslash\{0\}}}\left(\sum_{i=1}^{2}\left(\Delta t\sum_{j=1}^{N}\left\|\left(\frac{d\eta_i}{dt}\Big|_{\hat{\boldsymbol{x}}}^h\right)^j\right\|_{0,\Omega_j^i}^2+\Delta t\sum_{j=1}^{N}\|\eta_i^j\|_{1,\Omega_j^i}^2\right)\right.$$
$$\left.+\Delta t\sum_{j=1}^{N}\|\alpha^j\|_{0,\Omega_j^1}^2\right)+(\Delta t)^2\sum_{i=1}^{2}\left(\left\|\left(\frac{d^2\boldsymbol{v}_i}{dt^2}\Big|_{\hat{\boldsymbol{x}}}^h\right)^{n+1}\right\|_{0,\Omega_{n+1}^i}^2+\|\boldsymbol{v}_i^{n+1}\|_{1,\Omega_{n+1}^i}^2\right)\Bigg].$$

Choosing corresponding interpolation functions of the solution as our arbitrary discrete functions $((\tilde{\boldsymbol{v}}_1,\tilde{\boldsymbol{v}}_2),\tilde{p}_1)\in\boldsymbol{V}_h\times Q_h^0$, and employing standard a priori interpolation error estimates [8] for η_i, $\frac{d\eta_i}{dt}\big|_{\hat{\boldsymbol{x}}}$ and α, we obtain

(47)

$$\sum_{i=1}^{2}\left(\|\xi_i^N\|_{0,\Omega_N^i}^2+\Delta t\sum_{j=1}^{N}\|\xi_i^j\|_{1,\Omega_j^i}^2\right)$$
$$\leq C(h^{2(r-1)}+(\Delta t)^2)\Bigg[\Delta t\sum_{j=0}^{N}\left(\sum_{i=1}^{2}\left(\|\boldsymbol{v}_i^j\|_{H^r(\Omega_j^i)^d}^2+\left\|\left(\frac{d\boldsymbol{v}_i}{dt}\Big|_{\hat{\boldsymbol{x}}}^h\right)^j\right\|_{H^{r-1}(\Omega_j^i)^d}^2\right)\right.$$
$$\left.+\|p_1^j\|_{H^{r-1}(\Omega_j^1)}^2\right)+\sum_{i=1}^{2}\left(\left\|\left(\frac{d^2\boldsymbol{v}_i}{dt^2}\Big|_{\hat{\boldsymbol{x}}}^h\right)^j\right\|_{L^2(\Omega_j^i)^d}^2+\|\boldsymbol{v}_i^j\|_{H^1(\Omega_j^i)^d}^2\right)\Bigg]$$
$$\leq C(h^{2(r-1)}+(\Delta t)^2)\Bigg(\sum_{i=1}^{2}\Bigg[\|\boldsymbol{v}_i\|_{L^\infty(0,T;H^r(\Omega_t^i)^d)}^2+\left\|\frac{d\boldsymbol{v}_i}{dt}\Big|_{\hat{\boldsymbol{x}}}^h\right\|_{L^\infty(0,T;H^{r-1}(\Omega_t^i)^d)}^2$$
$$+\left\|\frac{d^2\boldsymbol{v}_i}{dt^2}\Big|_{\hat{\boldsymbol{x}}}^h\right\|_{L^\infty(0,T;L^2(\Omega_t^i)^d)}^2\Bigg]+\|p_1\|_{L^\infty(0,T;H^{r-1}(\Omega_t^1))}^2\Bigg)$$

Adding the a priori error estimates of η_i in L^2- and H^1 norm back in and using the triangular inequality, we have our result (41). □

5. Numerical Experiments

5.1. The case of a globally smooth real solution. We first consider the case of a smooth real solution. By appropriately choosing the functions $\boldsymbol{f}_1,\boldsymbol{f}_2,\boldsymbol{\tau},\boldsymbol{v}_1^0,\boldsymbol{v}_2^0$, we can let the following smooth functions

$$\begin{cases}\boldsymbol{v} &= \begin{pmatrix}\sin(2\pi y)(\cos(2\pi x)-1)\sin(t)\\ \sin(2\pi x)(-\cos(2\pi y)+1)\sin(t)\end{pmatrix},\\ p &= -2\pi\sin(2\pi x)\sin(2\pi y)\sin(t),\end{cases}$$

be the real solution to (1) in two dimension. Clearly, $\boldsymbol{v}\in\left(H^3(\Omega_t^1\cup\Omega_t^2)^2\right)\cap H^2(\Omega_t)^2$, $p\in H^2(\Omega_t^1)$. In their definitions, $\boldsymbol{x}=(x,y)^T\in\bar{\Omega}=[-1,1]\times[-1,1]$ which immerses the initial subdomains $\hat{\Omega}^2=(-0.25,0.25)\times(-0.25,0.25)$, $\hat{\Omega}^1=\Omega\backslash\bar{\hat{\Omega}}^2$, $T=1$. Then the interface $\Gamma_t=\partial\Omega_t^2$. In addition, we prescribe the interface motion as follows in terms of its position function $\boldsymbol{x}_\Gamma$:

$$\boldsymbol{x}_\Gamma = \left(1 + \frac{t}{10}\right)\boldsymbol{x}_2,\ \forall \boldsymbol{x}_2 \in \Gamma_t = \partial\Omega_t^2,\ \forall t \in [0,1], \tag{48}$$

which induces two slowly deformed subdomains Ω_t^1 and Ω_t^2. According to the prescribed $\boldsymbol{x}_\Gamma$, we solve the discrete ALE mapping $\boldsymbol{X}_{h,t}^i$ on $\hat{\Omega}^i$ for the moving meshes $\mathcal{T}_{h,t}^i$ $(i = 1, 2)$ and $t \in [0,1]$. Fig. 1 shows the initial and terminal domains and their meshes.

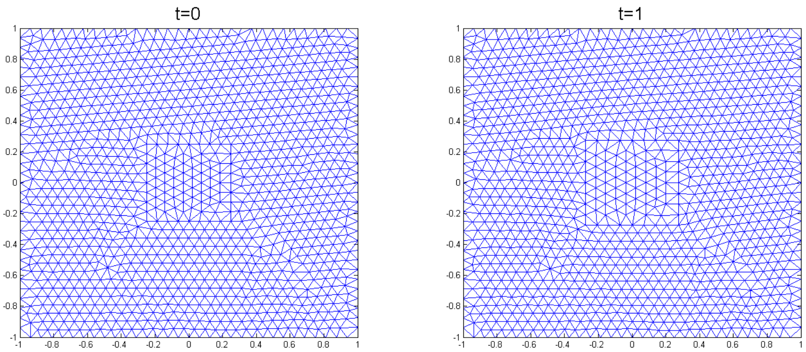

FIGURE 1. A growing square is immersed in the domain at different time and their meshes with $h = \frac{1}{16}$.

We utilize the fully discrete ALE finite element approximation (26) within the finite element spaces (8), i.e., P^2P^1 element, to solve the Stokes/parabolic interface problem (1) defined as above for $(\boldsymbol{v}_1, p_1, \boldsymbol{v}_2)$ with a grid doubling. In order to possibly observe the optimal convergence rate of velocity errors in L^2 norm, $\|\boldsymbol{v} - \boldsymbol{v}_h\|_{L^2(\Omega_T^1 \cup \Omega_T^2)^d}$ that may be $O(h^3 + \Delta t)$, which is however not included in Theorem 4.4, we choose the time step size Δt that is proportional to h^3. The convergence performances are illustrated in Table 1, where, we denote $\|\boldsymbol{v} - \boldsymbol{v}_h\|_{H^1(\Omega_T^1 \cup \Omega_T^2)^d}$ by $e_{\boldsymbol{v},1}$, $\|\boldsymbol{v} - \boldsymbol{v}_h\|_{L^2(\Omega_T^1 \cup \Omega_T^2)^d}$ by $e_{\boldsymbol{v},0}$, and $\|p - p_h\|_{L^2(\Omega_T^1)}$ by $e_{p,0}$, the convergence "*rate*" is calculated by $\log_2\left(\frac{e_{2h}}{e_h}\right)$. Fig. 2 illustrates the convergence history for each error via a log-log plot. We can see from Table 1 and Fig. 2 that the convergence rates

TABLE 1. Convergence performance of the smooth real solution case.

h	$e_{\boldsymbol{v},1}$	rate	$e_{\boldsymbol{v},0}$	rate	$e_{p,0}$	rate
$\frac{1}{4}$	.10561E+01		.68015E-01		.19884E+01	
$\frac{1}{8}$	.14801E+00	2.83	.41524E-02	4.03	.45018E+00	2.14
$\frac{1}{16}$	.22697E-01	2.71	.31383E-03	3.73	.10089E+00	2.16
$\frac{1}{32}$	.46903E-02	2.27	.33040E-04	3.25	.24995E-01	2.01

of both the velocity in H^1-norm and the pressure in L^2-norm are of the second order. Additionally, errors of velocity in L^2-norm even has the third convergence order, which means, all convergence rates are optimal regarding the adopted P^2P^1 finite element and relatively high local regularity properties of the chosen smooth real solutions that shall be no lower than $(H^3)^d$ for the velocity in $\Omega_t^1 \cup \Omega_t^2$ and H^2

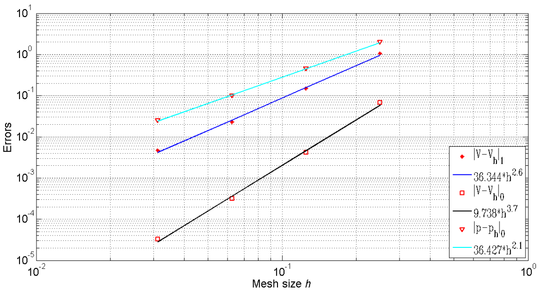

FIGURE 2. Convergence history of the smooth real solution case.

for the pressure in Ω_t^1. Thus, numerical results validate Theorem 4.4 for $r = 3$, and more beyond, we see that the convergence rate of velocity in L^2-norm is one order higher than its convergence rate in H^1-norm, which is not included in Theorem 4.4, may be considered as a superconvergence phenomenon for a globally smooth real solution case. It will be illustrated in the next example that such superconvergence on the convergence of velocity in L^2-norm is deteriorated for a globally non-smooth real solution case.

5.2. The case of a globally non-smooth real solution. Now we consider a more general numerical example for the Stokes/parabolic interface problem with a globally low regularity for the velocity $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2)^T \in H^2(\Omega_t^1 \cup \Omega_t^2)^2 \cap H^1(\Omega)^2$ by appropriately choosing the following real solution functions:

$$\begin{cases} \boldsymbol{v} &= \begin{pmatrix} (y - 0.3 - \boldsymbol{w}t)\left((x - 0.3 - \boldsymbol{w}t)^2 + (y - 0.3 - \boldsymbol{w}t)^2 - 0.01\right) t/\beta \\ -(x - 0.3 - \boldsymbol{w}t)\left((x - 0.3 - \boldsymbol{w}t)^2 + (y - 0.3 - \boldsymbol{w}t)^2 - 0.01\right) t/\beta \end{pmatrix}, \\ p &= 0.1\left(x^3 - y^3\right)\left((-\boldsymbol{w}t)^2 + (y - \boldsymbol{w}t)^2 - 0.01\right) t. \end{cases}$$

where, $\beta = \beta_i(\boldsymbol{x})$, $\forall \boldsymbol{x} \in \Omega_t^i$ $(i = 1, 2)$, are chosen as a piecewise constant, $\boldsymbol{x} = (x, y)^T \in \bar{\Omega} = [0, 1] \times [0, 1]$ that immerses $\hat{\Omega}^2 = \{(x, y)\,|(x-0.3)^2 + (y-0.3)^2 \leq 0.01\}$ and $\hat{\Omega}^1 = \Omega \backslash \bar{\hat{\Omega}}^2$. The interface $\Gamma_t = \partial\Omega_t^2$ satisfies the equation of a circle:

$$(x - 0.3 - \boldsymbol{w}t)^2 + (y - 0.3 - \boldsymbol{w}t)^2 = 0.01, \quad \forall t \in [0, T],$$

where $\boldsymbol{w}$ is a prescribed moving velocity of Γ_t, The interface motion, $\boldsymbol{x}_\Gamma$, is thus defined as $\boldsymbol{x}_\Gamma = \boldsymbol{w}t + \boldsymbol{x}_2$, $\forall \boldsymbol{x}_2 \in \Gamma_t$, $\forall t \in [0, T]$. By defining the real solution $\boldsymbol{v}$ and the interface Γ_t this way, we know $\nabla \boldsymbol{v} \in L^2(\Omega)^4$, only, leading to $\boldsymbol{v} \in H^1(\Omega)^2$.

According to the prescribed interface motion, $\boldsymbol{x}_\Gamma$, we solve the discrete ALE mapping $\boldsymbol{X}_{h,t}^i$ on $\hat{\Omega}^i$ for the moving meshes $\mathcal{T}_{h,t}^i$, $i = 1, 2$. The initial and terminal domains and their triangulations are shown in Fig. 3, respectively.

In the following numerical experiments, we pick $\beta_1 = 10$, $\beta_2 = 1$, $\boldsymbol{w} = 0.1$, $T = 1$ and still choose Δt is proportional to h^3 along with a grid doubling to include velocity errors in L^2 norm in our convergence test. By carrying out the same numerical approach as we do for Case 1, we obtain the following numerical results as shown in Table 2 and Fig. 4, illustrating that the convergence rates of the velocity in H^1- and L^2-norm, the pressure in L^2-norm are all of the second order, but the convergence rate of the velocity in L^2 norm is decreased from previous third-order in Section 5.1 down to the second order around. Thus Theorem 4.4 is fully validated for the case of a globally low regularity of the solution.

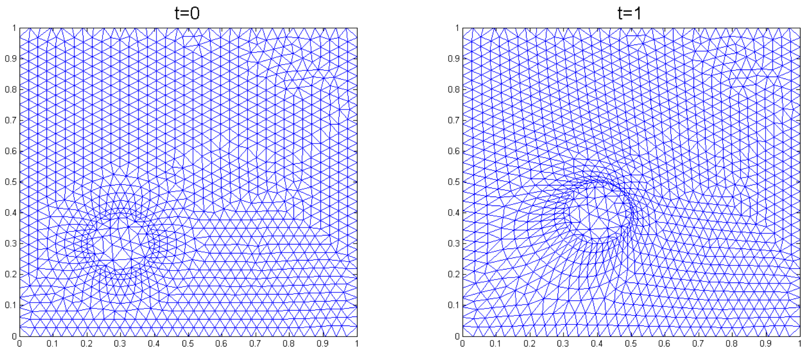

FIGURE 3. A shifting circle is immersed in the domain at different time and their meshes with $h = \frac{1}{32}$.

TABLE 2. Convergence performance of the non-smooth real solution case.

h	$e_{\boldsymbol{v},1}$	rate	$e_{\boldsymbol{v},0}$	rate	$e_{p,0}$	rate
$\frac{1}{8}$	2.82590E-04		1.30050E-05		8.46200E-04	
$\frac{1}{16}$	6.09940E-05	2.21	1.73470E-06	2.91	1.94010E-04	2.12
$\frac{1}{32}$	2.31040E-05	1.40	3.24730E-07	2.42	3.75110E-05	2.37
$\frac{1}{64}$	4.78590E-06	2.27	6.63540E-08	2.29	8.11640E-06	2.21

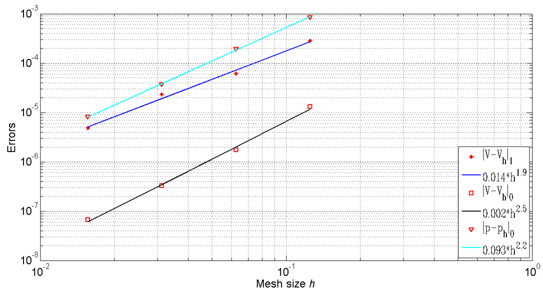

FIGURE 4. Convergence history of the non-smooth real solution case.

5.3. The case of less regularity with a larger jump ratio. In this case, we choose the real solution as:

$$\begin{cases} \boldsymbol{v} &= \begin{pmatrix} (y-\boldsymbol{w}t)\sin\left((x-\boldsymbol{w}t)^2+(y-\boldsymbol{w}t)^2-0.0625\right)\sin(t)/\beta \\ -(x-\boldsymbol{w}t)\sin\left((x-\boldsymbol{w}t)^2+(y-\boldsymbol{w}t)^2-0.0625\right)\sin(t)/\beta \end{pmatrix}, \\ p &= (\pi\cos(2\pi\,(x-\boldsymbol{w}t))\cos(2\pi\,(y-\boldsymbol{w}t))+0.080716)\,t. \end{cases}$$

where, $\beta = \beta_i(\boldsymbol{x})$, $\forall \boldsymbol{x} \in \Omega_t^i$ $(i = 1, 2)$ are chosen as a piecewise constant, $\boldsymbol{x} = (x, y)^T \in \bar{\Omega} = [-1, 1] \times [-1, 1]$ that immerses $\hat{\Omega}^2 = \{(x, y)\,|x^2 + y^2 \leq 0.0625\}$ and $\hat{\Omega}^1 = \Omega\backslash\bar{\hat{\Omega}}^2$. In addition, we prescribe the interface motion, $\boldsymbol{x}_\Gamma$, as follows

$$\boldsymbol{x}_\Gamma = \boldsymbol{w}t + \boldsymbol{x}_2,\ \forall \boldsymbol{x}_2 \in \Gamma_t = \partial\Omega_t^2,\ \forall t \in [0, 1]. \tag{49}$$

So, it is still the case of a shifting circle immersed in a square domain but initially being centered at the barycenter of the square this time.

We take a larger jump ratio, $\beta_1 = 1, \beta_2 = 1000$, for this case with other parameters set up as the same ones in Section 5.2. Numerical results are illustrated in Table 3 and Fig. 5, where we can see that the second-order convergence rate is obtained again for both the velocity in H^1- and L^2-norm, and the pressure in L^2-norm, on the average. And, the convergence rate of the velocity in L^2 norm is kept deteriorating down to about the second order. Theorem 4.4 is thus validated one more time for the case of globally low solution regularity with a larger jump ratio.

TABLE 3. Convergence performance of the less regularity case: $\beta_1 = 1, \beta_2 = 1000$.

h	$e_{\boldsymbol{v},1}$	rate	$e_{\boldsymbol{v},0}$	rate	$e_{p,0}$	rate
$\frac{1}{4}$	0.038158806		0.001895841		0.757454821	
$\frac{1}{8}$	0.00872165	2.13	0.000279706	2.76	0.21057848	1.85
$\frac{1}{16}$	0.002256687	1.95	6.63E-05	2.08	0.055024071	1.94
$\frac{1}{32}$	0.000652103	1.79	1.77E-05	1.91	0.013839607	1.99

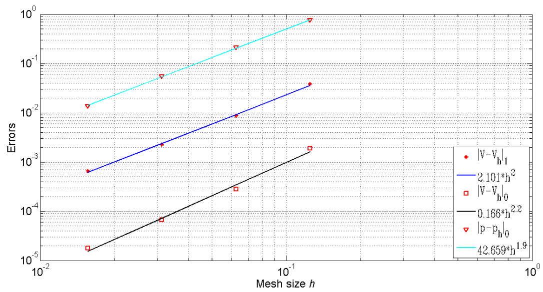

FIGURE 5. Convergence history of the less regularity case: $\beta_1 = 1, \beta_2 = 1000$.

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