INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 18, Number 3, Pages 311–323

A STABILIZER FREE WEAK GALERKIN FINITE ELEMENT METHOD FOR GENERAL SECOND-ORDER ELLIPTIC PROBLEM

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Abstract. This paper proposes a stabilizer free weak Galerkin (SFWG) finite element method for the convection-diffusion-reaction equation in the diffusion-dominated regime. The object of using the SFWG method is to obtain a simple formulation which makes the SFWG algorithm (9) more efficient and the numerical programming easier. The optimal rates of convergence of numerical errors of $\mathcal{O}(h^k)$ in H^1 and $\mathcal{O}(h^{k+1})$ in L^2 norms are achieved under conditions $\left(P_k(K), P_k(e), [P_j(K)]^2\right), j = k+1, k = 1, 2$ finite element spaces. Numerical experiments are reported to verify the accuracy and efficiency of the SFWG method.

Key words. Stabilizer free weak Galerkin methods, weak Galerkin finite element methods, weak gradient, error estimates.

1. Introduction

In this paper, we are concerned with the development of numerical methods for the following partial differential equation with boundary conditions using a stabilizer free weak Galerkin finite element method

$$-\nabla \cdot (\alpha \nabla u) + \boldsymbol{\beta} \cdot \nabla u + cu = f \quad \text{in } \Omega, \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where Ω is a polygonal or polyhedra domain in $\mathbb{R}^d (d = 2, 3)$, $\alpha = \alpha(x)$ is the diffusion coefficient matrix, $\boldsymbol{\beta} = \boldsymbol{\beta}(x)$ is the convection coefficient and $c = c(x)$ is the reaction coefficient in relevant applications. We suppose that $\alpha = (\alpha_{ij}(x))_{d\times d} \in [W^{1,\infty}(\Omega)]^{d\times d}, 0 \le c(x) \le M, \boldsymbol{\beta} \in [W^{1,\infty}(\Omega)]^d$ and $c - \frac{1}{2}\nabla \cdot \boldsymbol{\beta} > c_0 > 0$ for some constant c_0 and there exists positive constants $\alpha_m \le \alpha_M$ such that

$$\alpha_m \xi^T \xi \le \xi^T \alpha(x) \xi \le \alpha_M \xi^T \xi, \quad \forall \xi \in \mathbb{R}^d, x \in \Omega.$$

The convection-diffusion equation has numerous practical applications in many fields such as materials sciences, fluid flows, and image processing. There are several numerical methods in existing literature for solving the convection-diffusion equation.

The weak form of the problem (1)-(2) is to find $u \in H_0^1(\Omega)$ such that

$$(\alpha \nabla u, \nabla v) + (\boldsymbol{\beta} \cdot \nabla u, v) + (cu, v) = (f, v), \forall v \in H_0^1(\Omega). \tag{3}$$

The standard weak Galerkin method for the problem (1)-(2) seeks weak Galerkin finite element approximation $u_h = \{u_0, u_b\}$ satisfying

$$(\alpha \nabla_w u, \nabla_w v) + (\boldsymbol{\beta} \cdot \nabla_w u, v) + (cu, v) + s(u_h, v) = (f, v), \tag{4}$$

for all $v = \{v_0, v_b\}$ satisfying $v_b = 0$ on $\partial\Omega$, where ∇_w is the weak gradient operator and $s(u_h, v)$ in (4) is a stabilizer term that ensures a sufficient weak continuity for the numerical approximating. Recently, the weak Galerkin method has been developed to solve the elliptic equations [3, 6, 5], singularly perturbed reaction-diffusion

Received by the editors January 30, 2020 and, in revised form, November 16, 2020.
2000 *Mathematics Subject Classification.* Primary: 65N15, 65N30; Secondary: 35J50.

problems [1], the biharmonic problems [9], the Helmholtz equation [8], and the Maxwell equations [7]. More recently, Lin, et al. in [4], proposed a simple WG method for the convection-diffusion-reaction problem (1)-(2) with singular perturbation. One of the complexities of the WG methods and other discontinuous finite element methods is contained the stabilization terms. To reduce the programming complexity, the stabilizer free weak Galerkin finite element method, introduced by Ye and Zhang in [13], refers to the numerical techniques for solving Poisson equation on polytopal meshes in 2D or 3D, where there is a $j_0 > 0$ so that as long as the degree j of the weak gradient satisfies $j \geq j_0$, the new scheme will work and the optimal order of convergence can be achieved. In [2], Al-Taweel and Wang proved the optimal degree of weak gradient of the SFWG method to improve the efficiency of SFWG and to avoid the numerical difficulties associated with using high degree weak gradients. The benefits of using the SFWG method compared to the standard weak Galerkin method (4) are twofold: firstly, the SFWG method has a simple formulation which is closer to the weak form (3) and thus the implementation of the SFWG finite element method is easier than that of the standard weak Galerkin method; secondly and more importantly, it is more efficient than the standard WG method (4). The goal of this article is to study a stabilizer free weak Galerkin finite element method for solving convection-diffusion-reaction equations (1)-(2) on uniform triangular partitions and then establish the error analysis in the H^1 norm and L^2 norm.

This paper is organized as follows: In Section 2, we define weak gradient, weak divergence, and describe our SFWG finite element spaces and the SFWG scheme for the convection-diffusion-reaction equations (1)-(2). In Section 3, we will derive optimal order L^2 error estimates for the SFWG finite element method for solving the equations (1)-(2). Numerical experiment results are presented in Section 4 to validate the theoretical results. Finally, in Section 5, we present some concluding remarks.

2. Weak Galerkin Finite Element Schemes

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of convex polygons in 2D or polyhedra in 3D. Suppose that $\mathcal{T}_h$ is shape regular in the sense defined by (11)-(12). Let $\mathcal{E}_h$ be the set of all edges in $\mathcal{T}_h$, let let $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$ be the set of all interior edges. For each element $K \in \mathcal{T}_h$, denote by h_K the diameter of K, and $h = max_{K\in\mathcal{T}_h} h_K$ the mesh size of $\mathcal{T}_h$.

On each K, let $P_k(K)$ be the space of all polynomials with degree k or less. Let V_h be the weak Galerkin finite element space associated with $K \in \mathcal{T}_h$ defined as follows:

$$V_h = \{v = \{v_0, v_b\} : v_0|_K \in P_k(K), v_b|_e \in P_k(e), K \in \mathcal{T}_h, e \in \partial K\}, \tag{5}$$

where $k \geq 1$ is a given integer. In this instance, the component v_0 symbolizes the interior value of v, and the component v_b symbolizes the edge value of v on each K and e, respectively. Let V_h^0 be the subspace of V_h defined as:

$$V_h^0 = \{v : v \in V_h, v_b = 0 \text{ on } \partial\Omega\}. \tag{6}$$

Definition 2.1. *(Weak Gradient) For any $v = \{v_0, v_b\}$, the weak gradient $\nabla_w v \in [P_j(K)]^d$, where $j > k$, is defined on K as the unique polynomial satisfying*

$$(\nabla_w v, \mathbf{q})_K = -(v_0, \nabla \cdot \mathbf{q})_K + \langle v_b, \mathbf{q} \cdot \mathbf{n}\rangle_{\partial K}, \quad \forall \mathbf{q} \in [P_j(K)]^d, \tag{7}$$

where $\mathbf{n}$ is the unit outward normal vector of ∂K.

Definition 2.2. *(Weak Divergence) For any $v = \{v_0, v_b\} \in V_h$, the weak divergence $\beta \cdot \nabla_w v \in P_k(K)$ is defined on K as the unique polynomial satisfying*

$$(\boldsymbol{\beta} \cdot \nabla_w v, w)_K \quad = \quad -(v_0, \nabla \cdot (\boldsymbol{\beta} w))_K + \langle v_b, \boldsymbol{\beta} \cdot \mathbf{n} w \rangle_{\partial K}, \quad \forall w \in P_k(K), \tag{8}$$

where $\mathbf{n}$ is the unit outward normal vector of ∂K.

Next, we define four global projections Q_0, Q_b, Q_h, and $\mathbb{Q}_h$ as follows.

Definition 2.3. *For each element $K \in \mathcal{T}_h$,*

$$Q_0 : L^2(K) \longrightarrow P_k(K),$$
$$Q_b : L^2(e) \longrightarrow P_k(e),$$
$$\mathbb{Q}_h : [L^2(K)]^d \longrightarrow [P_j(K)]^d,$$

are the L^2 projections onto the associated local polynomial spaces. Finally, we define a projection operator $Q_h v = \{Q_0 v, Q_b v\} \in V_h$ for $v \in H^1(\Omega)$.

For simplicity, we adopt the following notations,

$$(v, w)_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} (v, w)_K = \sum_{K \in \mathcal{T}_h} \int_K vw dx,$$
$$\langle v, w \rangle_{\partial \mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \langle v, w \rangle_{\partial K} = \sum_{K \in \mathcal{T}_h} \int_{\partial K} vw ds.$$

SFWG Method 1. *The SFWG scheme for (1)-(2) is to find $u_h = \{u_0, u_b\} \in V_h^0$, such that the following equation holds*

$$(\alpha \nabla_w u_h, \nabla_w v)_{\mathcal{T}_h} + (\boldsymbol{\beta} \cdot \nabla_w u_h, v_0)_{\mathcal{T}_h} + (c u_0, v_0)_{\mathcal{T}_h} = (f, v_0)_{\mathcal{T}_h}, \tag{9}$$

for all $v = \{v_0, v_b\} \in V_h^0$.

For any $v \in V_h$, we introduce an energy norm $|||\cdot|||$ as:

$$|||v|||^2 = (\alpha \nabla_w v, \nabla_w v)_{\mathcal{T}_h} + (c v_0, v_0)_{\mathcal{T}_h}. \tag{10}$$

An H^1 semi-norm is defined as follows:

$$\|v\|_{1,h} = \left(\sum_{K \in \mathcal{T}_h} \left(\|\nabla v_0\|_K^2 + h_K^{-1} \|v_0 - v_b\|_{\partial K}^2 \right) \right)^{\frac{1}{2}}.$$

It can be easily verified that $\|v\|_{1,h}$ is a norm in V_h^0.

The following lemma will be needed in the error estimation.

Lemma 1. *(see [2]) Suppose that $\forall K \in \mathcal{T}_h, K$ is convex with at most μ edges and satisfies the following regularity conditions: for all edges e_t and e_s of K*

$$|e_s| < \alpha_0 |e_t|; \tag{11}$$

for any two adjacent edges e_t and e_s the angle θ between them satisfies

$$\theta_0 < \theta < \pi - \theta_0, \tag{12}$$

where $1 \leq \alpha_0$ and $\theta_0 > 0$ are independent of K and h. Let $j_0 = k + \mu - 2$ or $j_0 = k + \mu - 3$ when each edge of K is parallel to another edge of K. Denote $deg \nabla_w v$ be the degree of weak gradient when $deg \nabla_w v = j \geq j_0$, then there exist two constants $C_1, C_2 > 0$, such that for each $v = \{v_0, v_b\} \in V_h$, the following hold true

$$C_1 \|v\|_{1,h} \leq (\nabla_w v, \nabla_w v)_{\mathcal{T}_h} \leq C_2 \|v\|_{1,h},$$

where C_1 and C_2 depend only on α_0 and θ_0.

Remark 1. *When* $\mu = 3$ *then* $j_0 = k+1$. *If all* K*'s are parallelograms, then* $\mu = 4$, *and* $j_0 = k+1$.

Next, we list important inequalities which will be needed in error estimates.

Lemma 2. *(Trace inequality, see* [11]*) On each element* $K \in \mathcal{T}_h$*, the following trace inequality holds true:*

$$\|\varphi\|_e^2 \leq C\left(h_K^{-1}\|\varphi\|_K^2 + h_K\|\nabla\varphi\|_{1,K}^2\right), \qquad \varphi \in H^1(K), \tag{13}$$

for some constant C.

Lemma 3. *(Inverse Inequality, see* [11]*) There exists a constants* C *such that for any piecewise polynomial* $\varphi \in P_k(K)$.

$$\|\nabla\varphi\|_K \leq C h_K^{-1}\|\varphi\|_K, \qquad \forall K \in \mathcal{T}_h. \tag{14}$$

The following lemma presents estimates for the projection operator Q_0 and $\mathbb{Q}_h$.

Lemma 4. [10] *Let* $\mathcal{T}_h$ *be a finite element partition of* Ω *satisfying the shape regularity conditions (11)-(12), and* $u \in H^{k+1}(\Omega)$*. Then, the* L^2 *projections* Q_0 *and* $\mathbb{Q}_h$ *satisfy*

$$\sum_{K\in\mathcal{T}_h}\left(\|\varphi - Q_0\varphi\|_K^2 + h_K^2\|\nabla(\varphi - Q_0\varphi)\|_K^2\right) \leq Ch^{2(s+1)}\|\varphi\|_{s+1}^2, \quad 0 \leq s \leq k,$$

$$\sum_{K\in\mathcal{T}_h}\left(\|\nabla\varphi - \mathbb{Q}_h\nabla\varphi\|_K^2 + h_K^2|\nabla\varphi - \mathbb{Q}_h\nabla\varphi)|_{1,K}^2\right) \leq Ch^{2s}\|\varphi\|_{s+1}^2, \quad 0 \leq s \leq k.$$

The following lemma will also be needed in error estimates.

Lemma 5. *Let* Q_h *and* $\mathbb{Q}_h$ *be the projection operators defined in definition (2.3) and* $\phi \in H^{k+1}(\Omega)$*. Then for each element* $K \in \mathcal{T}_h$*, we have*

$$\|\mathbb{Q}_h(\nabla\phi) - \nabla_w Q_h\phi\|_K \leq Ch_K^k|\phi|_{k+1,K}. \tag{15}$$

Proof. By definition (2.1) and integration by parts, we have

$$(\mathbb{Q}_h(\nabla\phi), \mathbf{q})_K = -(\phi, \nabla\cdot\mathbf{q})_K + \langle\phi, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial K}, \tag{16}$$

$$(\nabla_w Q_h\phi, \mathbf{q})_K = -(Q_0\phi, \nabla\cdot\mathbf{q})_K + \langle Q_b\phi, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial K}, \tag{17}$$

for any $\mathbf{q} \in [P_j(K)]^d$. Subtracting (17) from (16), using integration by parts, trace inequality (13) and inverse inequality (14), we get

$$\begin{aligned}(\mathbb{Q}_h(\nabla\phi) - \nabla_w Q_h\phi, \mathbf{q})_K &= -(\phi - Q_0\phi, \nabla\cdot\mathbf{q})_K + \langle\phi - Q_b\phi, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial K}\\ &= (\nabla(\phi - Q_0\phi), \mathbf{q})_K + \langle Q_0\phi - Q_b\phi, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial K}\\ &\leq \|\nabla(\phi - Q_0\phi)\|_K\|\mathbf{q}\|_K + Ch_K^{-\frac{1}{2}}\|\phi - Q_0\phi\|_{\partial K}\|\mathbf{q}\|_K\\ &\leq Ch_K^k|\phi|_{k+1,K}\|\mathbf{q}\|_K.\end{aligned}$$

Letting $\mathbf{q} = \mathbb{Q}_h(\nabla\phi) - \nabla_w Q_h\phi$ in the above equation yields

$$\|\mathbb{Q}_h(\nabla\phi) - \nabla_w Q_h\phi\|_K \leq Ch_K^k|\phi|_{k+1,K},$$

which completes the proof. □

Lemma 6. *Let* $\phi \in H^1(\Omega)$*. Then for all* $v \in V_h$*, we have*

$$\begin{aligned}(\alpha\nabla\phi, \nabla v_0)_K = (\alpha\nabla_w(Q_h\phi), \nabla_w v)_K &+ \langle(\mathbb{Q}_h(\alpha\nabla\phi)\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial K}\\ &+ (\mathbb{Q}_h(\alpha\nabla\phi) - \alpha\nabla_w Q_h\phi, \nabla_w v)_K.\end{aligned} \tag{18}$$

Proof. Let $\mathbb{Q}_h(\alpha\nabla\phi) = \mathbf{q}$ and $Q_h\phi = P$. Then

$$(\mathbf{q}, \nabla v_0)_K = -(\nabla\cdot\mathbf{q}, v_0)_K + \langle \mathbf{q}\cdot\mathbf{n}, v_0\rangle_{\partial K}, \tag{19}$$

$$(\mathbf{q}, \nabla_w v)_K = -(\nabla\cdot\mathbf{q}, v_0)_K + \langle \mathbf{q}\cdot\mathbf{n}, v_b\rangle_{\partial K}. \tag{20}$$

Subtracting (20) from (19) yields

$$\begin{aligned}(\mathbf{q}, \nabla v_0)_K &= (\mathbf{q}, \nabla_w v)_K + \langle \mathbf{q}\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial K}\\ &= (\alpha\nabla_w P, \nabla_w v)_K + (\mathbf{q} - \alpha\nabla_w P, \nabla v)_K + \langle \mathbf{q}\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial K},\end{aligned}$$

which completes the proof. □

Lemma 7. *For all $v \in V_h$, we have*

$$(\boldsymbol{\beta}\cdot\nabla u, v_0)_{\mathcal{T}_h} = (\boldsymbol{\beta}\cdot\nabla_w Q_h u, v_0)_{\mathcal{T}_h} - \ell_{\boldsymbol{\beta}}(u, v), \tag{21}$$

where

$$\ell_{\boldsymbol{\beta}}(u, v) = (u - Q_0 u, \nabla\cdot(\boldsymbol{\beta} v_0))_{\mathcal{T}_h} - \langle u - Q_b u, \boldsymbol{\beta}\cdot\mathbf{n}(v_0 - v_b)\rangle_{\partial\mathcal{T}_h}$$

Proof. From integration by parts and definition 2.2 we get

$$\begin{aligned}(\boldsymbol{\beta}\cdot\nabla u, v_0)_{\mathcal{T}_h} &= -(u, \nabla\cdot(\boldsymbol{\beta} v_0))_{\mathcal{T}_h} + \langle u, \boldsymbol{\beta}\cdot\mathbf{n} v_0\rangle_{\partial\mathcal{T}_h}\\ &= -(Q_0 u, \nabla\cdot(\boldsymbol{\beta} v_0))_{\mathcal{T}_h} + (Q_0 u - u, \nabla\cdot(\boldsymbol{\beta} v_0))_{\mathcal{T}_h}\\ &+ \langle Q_b u, \boldsymbol{\beta}\cdot\mathbf{n} v_0\rangle_{\partial\mathcal{T}_h} + \langle u - Q_b u, \boldsymbol{\beta}\cdot\mathbf{n} v_0\rangle_{\partial\mathcal{T}_h}\\ &= (\boldsymbol{\beta}\cdot\nabla_w(Q_h u), v_0)_{\mathcal{T}_h} - \ell_{\boldsymbol{\beta}}(u, v)\end{aligned}$$

where in the last equality we have used the fact that $\langle u - Q_b u.\boldsymbol{\beta}\cdot\mathbf{n} v_b\rangle_{\partial\mathcal{T}_h}$. This concludes the proof □

Lemma 8. *Let $u \in H^{k+1}(\Omega)$. Then for any $v \in V_h^0$, we have*

$$(-\nabla\cdot(\alpha\nabla u), v_0) = (\alpha\nabla_w Q_h u, \nabla_w v)_{\mathcal{T}_h} - \ell_\alpha(u, v), \tag{22}$$

where

$$\ell_\alpha(u, v) = \langle(\alpha\nabla u - \mathbb{Q}_h(\alpha\nabla u)\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial\mathcal{T}_h} + (\mathbb{Q}_h(\alpha\nabla u) - \alpha\nabla_w Q_h u, \nabla_w v)_{\mathcal{T}_h}.$$

Proof. Using integration by parts and the fact that $\sum_{K\in\mathcal{T}_h}\langle\nabla u\cdot\mathbf{n}, v_b\rangle_{\partial K} = 0$, we obtain

$$\begin{aligned}(-\nabla\cdot(\alpha\nabla u), v_0) &= \sum_{K\in\mathcal{T}_h}(\alpha\nabla u, \nabla v_0)_K - \sum_{K\in\mathcal{T}_h}\langle\alpha\nabla u\cdot\mathbf{n}, v_0\rangle_{\partial K}\\ &= \sum_{K\in\mathcal{T}_h}(\alpha\nabla u, \nabla v_0)_K - \sum_{K\in\mathcal{T}_h}\langle\alpha\nabla u\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial K}, \qquad (23)\end{aligned}$$

By letting $\phi = u$ in (18) and substituting it into (23), we get

$$\begin{aligned}(-\nabla\cdot(\alpha\nabla u), v_0) &= \sum_{K\in\mathcal{T}_h}(\alpha\nabla_w(Q_h u), \nabla_w v)_K\\ &- \sum_{K\in\mathcal{T}_h}\langle(\alpha\nabla u - \mathbb{Q}_h(\alpha\nabla u))\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial K}\\ &+ \sum_{K\in\mathcal{T}_h}(\mathbb{Q}_h(\alpha\nabla u) - \alpha\nabla_w Q_h u, \nabla_w v)_K.\end{aligned}$$

This concludes the proof. □

Assume that $\ell_c(u, v) = -(cu - cQ_0 u, v_0)$. Then $e_h = Q_h u - u_h \in V_h^0$ satisfies the following error equation.

Lemma 9. *For all $v \in V_h^0$, we have*

$$(\alpha \nabla_w e_h, \nabla_w v)_{\mathcal{T}_h} + (\boldsymbol{\beta} \cdot \nabla_w e_h, v_0) + (ce_0, v_0) = \ell_\alpha(u,v) + \ell_{\boldsymbol{\beta}}(u,v) + \ell_c(u,v). \quad (24)$$

Proof. Testing (1) against v_0 we find that

$$(-\nabla \cdot (\alpha \nabla u), v_0) + (\boldsymbol{\beta} \cdot \nabla u, v_0) + (cu, v_0) = (f, v_0).$$

Using Lemmas 7, 8 and properties of the projections, we obtain

$$\alpha(\nabla_w Q_h u, \nabla_w v)_{\mathcal{T}_h} + (\boldsymbol{\beta} \cdot \nabla_w Q_h u, v_0)_{\mathcal{T}_h} + (cQ_0 u, v_0) = (f, v_0) + \ell_\alpha(u,v) + \ell_{\boldsymbol{\beta}}(u,v) + \ell_c(u,v). \quad (25)$$

Subtracting equation (9) from (25) generates (24), which completes the proof. □

Lemma 10. *Let $u \in H^{k+1}(\Omega)$. If α is a piecewise constant matrix, then, for any $v \in V_h^0$, the following estimates hold*

$$|\ell_\alpha(u,v)| \le Ch^k |u|_{k+1} |||v|||, \quad (26)$$

$$|\ell_{\boldsymbol{\beta}}(u,v)| \le Ch^k |u|_{k+1} |||v|||, \quad (27)$$

$$|\ell_c(u,v)| \le Ch^k |u|_{k+1} |||v|||. \quad (28)$$

Proof. For the first estimate (26), by applying Cauchy-Schwarz inequality and Lemma 5 , we obtain

$$\begin{aligned} |\ell_\alpha(u,v)| &\le \sum_{K \in \mathcal{T}_h} |\langle(\alpha \nabla u - \mathbb{Q}_h(\alpha \nabla u) \cdot \mathbf{n}, v_0 - v_b\rangle_{\partial K}| \\ &+ \sum_{K \in \mathcal{T}_h} |(\mathbb{Q}_h(\alpha \nabla u) - \alpha \nabla_w Q_h u, \nabla_w v)_K| \\ &\le C \sum_{K \in \mathcal{T}_h} \|\nabla u - \mathbb{Q}_h \nabla u)\|_{\partial K} \|v_0 - v_b\|_{\partial K} + Ch^k |u|_{k+1} |||v||| \\ &\le C \left(\sum_{K \in \mathcal{T}_h} h_K \|\nabla u - \mathbb{Q}_h \nabla u\|_{\partial K}^2 \right)^{1/2} \left(\sum_{K \in \mathcal{T}_h} h_K^{-1} \|v_0 - v_b\|_{\partial K}^2 \right)^{1/2} \\ &+ Ch^k |u|_{k+1} |||v|||. \end{aligned}$$

From the trace inequality (13) and Lemma 4, we have

$$\begin{aligned} |\ell_\alpha(u,v)| &\le Ch^k |u|_{k+1} \left[\left(\sum_{K \in \mathcal{T}_h} h_K^{-1} \|v_0 - v_b\|_{\partial K}^2 \right)^{1/2} + |||v||| \right] \\ &\le Ch^k |u|_{k+1} |||v|||. \end{aligned}$$

The estimate for $\ell_{\boldsymbol{\beta}}(u,v)$ can be obtained from Lemma 2.5 in [12]. The last estimate (28) is resulting from the Cauchy-Schwarz inequality, and Lemma 4

$$\begin{aligned} |\ell_c(u,v)| &= |(cu - cQ_0 u, v_0)| \\ &\le c_M \, |(Q_0 u - u, v_0)| \\ &\le Ch^k |u|_{k+1} |||v|||, \end{aligned}$$

which completes the proof. □

Lemma 11. *Let $e_h = \{e_0, e_b\} = \{Q_0 u - u_0, Q_b u - u_b\}$. Then there exists a constant C such that*

$$|||e_h|||^2 \le (\alpha \nabla_w e_h, \nabla_w e_h)_{\mathcal{T}_h} + (\boldsymbol{\beta} \cdot \nabla_w e_h, e_h) + (ce_h, e_h). \quad (29)$$

Proof. By using $|\!|\!| \cdot |\!|\!|$ defined in (10), we obtain

$$\begin{aligned}(\alpha\nabla_w e_h, \nabla_w e_h)_{\mathcal{T}_h} + (\boldsymbol{\beta}\cdot\nabla_w e_h, e_h) + (ce_h, e_h) &= |\!|\!|e_h|\!|\!|^2 + (\beta\cdot\nabla_w e_h, e_h)\\ &\geq |\!|\!|e_h|\!|\!|^2 - C\|e_h\|\|\nabla_w e_h\|\\ &\geq (C_1 - C_2 h)|\!|\!|e_h|\!|\!|^2\\ &\geq C|\!|\!|e_h|\!|\!|^2,\end{aligned}$$

when h is sufficiently small. □

Lemma 12. *The weak Galerkin scheme (9) has one and only one solution.*

Proof. It suffices to verify the uniqueness for the homogeneous equation. Assume that $u_h^{(1)}$ and $u_h^{(2)}$ are two solutions of (9). Then $e_h = u_h^{(1)} - u_h^{(2)}$ would satisfy the forthcoming equation

$$(\alpha\nabla_w e_h, v)_{\mathcal{T}_h} + (\boldsymbol{\beta}\cdot\nabla_w e_h, v) + (ce_h, v) = 0, \qquad \forall v \in V_h^0. \tag{30}$$

Note that $e_h \in V_h^0$. Suppose that $v = e_h$, in the equation (30) we obtain

$$(\alpha\nabla_w e_h, e_h)_{\mathcal{T}_h} + (\boldsymbol{\beta}\cdot\nabla_w e_h, e_h) + (ce_h, e_h) = 0.$$

From Lemma 11, we have

$$|\!|\!|e_h|\!|\!| \leq (\alpha\nabla_w e_h, e_h)_{\mathcal{T}_h} + (\boldsymbol{\beta}\cdot\nabla_w e_h, e_h) + (ce_h, e_h) = 0.$$

Which implies $e_h \equiv 0$. Consequently, $u_h^{(1)} \equiv u_h^{(2)}$. □

Theorem 1. *Let $u_h = \{u_0, u_b\}$ be the solution to the formulation of (9). Assume the exact solution $u \in H^{k+1}(\Omega)$. If α is a piecewise constant matrix, then there exists a constant C independent of h, k such that*

$$|\!|\!|Q_h u - u_h|\!|\!| \leq Ch^k|u|_{k+1}. \tag{31}$$

Proof. It follows from (29) that

$$|\!|\!|e_h|\!|\!|^2 \leq (\alpha\nabla_w e_h, \nabla_w e_h)_{\mathcal{T}_h} + (\boldsymbol{\beta}\cdot\nabla_w e_h, e_h) + (ce_h, e_h). \tag{32}$$

Letting $v = e_h$ in (24) gives

$$(\alpha\nabla_w e_h, \nabla_w e_h)_{\mathcal{T}_h} + (\boldsymbol{\beta}\cdot\nabla_w e_h, e_0) + (ce_0, e_0) = \ell_\alpha(u, e_h) + \ell_{\boldsymbol{\beta}}(u, e_h) + \ell_c(u, e_h).$$

Then (31) follows from Lemma 10. □

3. Error Estimates in L^2 norm

The duality argument is utilized to get L^2 error estimate. Let $e_h = \{e_0, e_b\} = Q_h u - u_h$. The considered dual problem seek $\Phi \in H_0^1(\Omega)$ satisfying

$$\begin{aligned}-\nabla\cdot(\alpha\nabla\Phi) + \boldsymbol{\beta}\cdot\nabla\Phi + c\Phi &= e_0, \quad \text{in } \Omega\\ \Phi &= 0, \quad \text{on } \partial\Omega.\end{aligned} \tag{33}$$

suppose that the following H^2-regularity holds true

$$\|\Phi\|_2 \leq C\|e_0\|. \tag{34}$$

Theorem 2. *Let $u_h = \{u_0, u_b\}$ be the SFWG finite element solution of (9). Assume that the exact solution $u \in H^{k+1}(\Omega)$ and (34) holds true. If α is a piecewise constant matrix, then, there exists a constant C such that*

$$\|Q_0 u - u_0\| \leq Ch^{k+1}|u|_{k+1}. \tag{35}$$

Proof. Testing (33) by e_0 and using the fact that $\sum_{K\in\mathcal{T}_h} \langle \nabla\Phi\cdot\mathbf{n}, e_b\rangle_{\partial\mathcal{T}_h} = 0$ obtain

$$\|e_0\|^2 = (-\nabla\cdot(\alpha\nabla\Phi), e_0) + (\boldsymbol{\beta}\cdot\nabla\Phi, e_0) + (c\Phi, e_0)$$
$$= (\alpha\nabla\Phi, \nabla e_0)_{\mathcal{T}_h} - \langle \alpha\nabla\Phi\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial\mathcal{T}_h} + (\boldsymbol{\beta}\cdot\nabla\Phi, e_0)_{\mathcal{T}_h} + (c\Phi, e_0)_{\mathcal{T}_h}. \tag{36}$$

Setting $\phi = \Phi$ and $v = e_h$ in (18) yields

$$(\alpha\nabla\Phi, \nabla\epsilon_0)_{\mathcal{T}_h} = (\alpha\nabla_w(Q_h\Phi), \nabla_w\epsilon_h)_{\mathcal{T}_h} + \langle \alpha\mathbb{Q}_h(\nabla\Phi)\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial\mathcal{T}_h}$$
$$+ (\mathbb{Q}_h(\nabla\Phi) - \nabla_w Q_h\phi, \nabla_w e_h)_{\mathcal{T}_h}. \tag{37}$$

Substituting (37) into (36) and using Lemma 7 gives

$$\|e_0\|^2 = (\alpha\nabla_w(Q_h\Phi), \nabla_w e_h)_{\mathcal{T}_h} + \ell_\alpha(\Phi, e_h) + (\boldsymbol{\beta}\cdot\nabla_w Q_h\Phi, e_0)_{\mathcal{T}_h}$$
$$- \ell_{\boldsymbol{\beta}}(\Phi, v) + (c\Phi, e_0)_{\mathcal{T}_h}. \tag{38}$$

Using equation 9 and the error equation (24), we have

$$(\alpha\nabla_w(Q_h\Phi), \nabla_w e_h)_{\mathcal{T}_h} + (\boldsymbol{\beta}\cdot\nabla_w Q_h\Phi, e_0)_{\mathcal{T}_h}$$
$$= \ell_\alpha(u, Q_h\Phi) + \ell_{\boldsymbol{\beta}}(u, Q_h\Phi) + \ell_c(u, Q_h\Phi) - (ce_0, Q_0\Phi). \tag{39}$$

By combining (38) with (39), we obtain

$$\|e_0\|^2 = (c(u - Q_0u), Q_0\Phi) + \ell_\alpha(u, Q_h\Phi) + \ell_{\boldsymbol{\beta}}(u, Q_h\Phi)$$
$$+ \ell_\alpha(\Phi, e_h) - \ell_{\boldsymbol{\beta}}(\Phi, e_h) + (ce_0, \Phi - Q_0\Phi). \tag{40}$$

To bound the terms on the right-hand side of equation (40). We use the Cauchy-Schwarz inequality, the trace inequality (13) and the definition of Q_h and $\mathbb{Q}_h$ to get

$$\begin{aligned}
|\ell_a(u, Q_h\Phi)| &= \Bigg| \sum_{K\in\mathcal{T}_h} \langle \alpha\nabla u - \mathbb{Q}_h(\alpha\nabla u)\cdot\mathbf{n}, Q_0\Phi - Q_b\Phi\rangle_{\partial K} \\
&+ (\mathbb{Q}_h(\alpha\nabla u) - \alpha\nabla_w Q_h u, \nabla_w Q_h\Phi)_K \Bigg| \\
&\le \left(\sum_{K\in\mathcal{T}_h} \|\nabla u - \mathbb{Q}_h\nabla u\|_{\partial K}^2\right)^{\frac{1}{2}} \left(\sum_{K\in\mathcal{T}_h} \|Q_0\Phi - Q_b\Phi\|_{\partial K}^2\right)^{\frac{1}{2}} \\
&+ Ch^{k+1}|u|_{k+1}|\phi|_2 \\
&\le C\left(\sum_{K\in\mathcal{T}_h} h\|\nabla u - \mathbb{Q}_h\nabla u\|_{\partial K}^2\right)^{\frac{1}{2}} \left(\sum_{K\in\mathcal{T}_h} h^2\|Q_0\Phi - Q_b\Phi\|_{\partial K}^2\right)^{\frac{1}{2}} \\
&+ Ch^{k+1}|u|_{k+1}|\phi|_2 \\
&\le Ch^{k+1}|u|_{k+1}|\Phi|_2,
\end{aligned}$$

which implies

$$|\ell_a(u, Q_h\Phi)| \le Ch^{k+1}|u|_{k+1}|\Phi|_2. \tag{41}$$

The estimate (27), and Lemma 4 give

$$|\ell_{\boldsymbol{\beta}}(u, Q_h\Phi)| = \big|(u - Q_0u, \nabla\cdot(\boldsymbol{\beta}Q_0\Phi))_{\mathcal{T}_h} - \langle u - Q_bu, \boldsymbol{\beta}\cdot\mathbf{n}(Q_0\Phi - Q_b\Phi)\rangle_{\partial\mathcal{T}_h}\big|$$
$$\le Ch^{k+1}|u|_{k+1}|\Phi|_2. \tag{42}$$

The estimates (26), (31), and Lemma 4 give

$$\begin{aligned} & |\ell_a(\Phi, e_h)| \\ = & \left|\langle \alpha\nabla\Phi - \mathbb{Q}_h(\alpha\nabla\Phi)\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial\mathcal{T}_h} + (\mathbb{Q}_h(\alpha\nabla\Phi) - \alpha\nabla_w Q_h\Phi, \nabla_w e_h)_{\mathcal{T}_h}\right| \\ \le & Ch^{k+1}|u|_{k+1}|\Phi|_2. \end{aligned} \tag{43}$$

The estimates (27), (31), and Lemma 4 give

$$\begin{aligned} & |\ell_{\boldsymbol{\beta}}(\Phi, e_h)| \\ = & \left|(\Phi - Q_0\Phi, \nabla\cdot(\boldsymbol{\beta} e_0))_{\mathcal{T}_h} - \langle \Phi - Q_b\Phi, \boldsymbol{\beta}\cdot\mathbf{n}(e_0 - e_b)\rangle_{\partial\mathcal{T}_h}\right| \\ \le & Ch^{k+1}|u|_{k+1}|\Phi|_2. \end{aligned} \tag{44}$$

It follows from the Cauchy-Schwarz inequality and Lemma 4 that

$$|c(e_0, \Phi - Q_0\Phi)| \le Ch^{k+1}|u|_{k+1}\|e_0\| \le Ch^{k+1}|u|_{k+1}|\Phi|_2. \tag{45}$$

Similarly, it follows from the Cauchy-Schwarz inequality and lemma 4 that

$$|(c(u - Q_0u), Q_0\Phi)| \le Ch^{k+1}|u|_{k+1}|\Phi|_2. \tag{46}$$

Now combining (40) with the estimates (41)-(46), we obtain

$$\|e_0\|^2 \le Ch^{k+1}|u|_{k+1}|\Phi|_2, \tag{47}$$

which combined with (34) and the triangle inequality, provides the required error estimate (35). □

4. Numerical Experiments

The goal of this section is to present four of the numerical examples to verify the theoretical results derived in previous sections. The stabilizer free weak Galerkin finite element scheme (9) has been applied for the polynomial degrees $k = 1, 2$ on a square domain and an L-shaped domain with uniform triangular partitions.

For polynomial degree $k = 1$, the numerical solution $u_h = \{u_0, u_b\}$ is obtained from setting $j = 2$ in the weak gradient (7) and the following finite element space:

$$V_h = \{v = \{v_0, v_b\} : v_0|_K \in P_1(K), v_b|_e \in P_1(e), K \in \mathcal{T}_h, e \in \partial K\}.$$

For polynomial degree $k = 2$, the finite element space is given as follows:

$$V_h = \{v = \{v_0, v_b\} : v_0|_K \in P_2(K), v_b|_e \in P_2(e), K \in \mathcal{T}_h, e \in \partial K\},$$

and setting $j = 3$ in the weak gradient (7) to find the SFWG solution $u_h = \{u_0, u_b\}$. All numerical experiments are carried out on a Laptop computer with 12.0 GB memory and Intel(R) Core (TM) i7-8550U CPU @ 1.80 GHz.

Example 4.1. *In this example, we consider the problem (1)-(2) posed on the domain* $\Omega = (0, 1)^2$ *with the following data: the diffusion coefficient matrix* $\alpha = I_2$, *the convection coefficient* $\boldsymbol{\beta} = (1, 2)^T$, *the reaction coefficient* $c = \sin(2xy)$, *and the* $f(x, y)$ *term is given such that the exact solution is* $u(x, y) = \sin(2\pi x)\sin(2\pi y)$. *Table 1 lists errors and convergence rates in* $|\!|\!|\cdot|\!|\!|$*-norm and* L^2*-norm. Numerical results show that the SFWG method with* P_k *elements has convergence rate of* $\mathcal{O}(h^k)$ *in* H^1 *norm and* $\mathcal{O}(h^{k+1})$ *in* L^2*-norm. Although the numerical rates of convergence of the standard weak Galerkin scheme (4), in* H^1 *and* L^2 *norms, are the same as those of the SFWG method, the SFWG method is more efficient and easier to implement for the convection-diffusion equation (1)-(2).*

Table 2 shows the computational time (in seconds) comparison between SFWG finite element scheme (9) and weak Galerkin finite element scheme (4). As we can see in Table 2 that the SFWG method is running faster than the standard weak

TABLE 1. Error profiles and convergence rates for $\left(P_k(K), P_k(e), [P_{k+1}(K)]^2\right), k = 1, 2$ finite element spaces.

		SFWG elements				*WG elements*			
k	h	$\lVert\lvert Q_h u - u_h\rvert\rVert$	**Rate**	$\lVert Q_0 u - u_0\rVert$	**Rate**	$\lVert\lvert Q_h u - u_h\rvert\rVert$	**Rate**	$\lVert Q_0 u - u_0\rVert$	**Rate**
	1/2	3.5541E-00	-	2.4091E-01	-	3.9679E-00	-	2.5892E-01	-
	1/4	1.7763E-00	1.00	8.4718E-02	1.51	1.9790E-00	1.00	9.5420E-02	1.44
1	1/8	8.7169E-01	1.03	2.6767E-02	1.66	9.7728E-01	1.02	2.9762E-02	1.68
	1/16	4.3122E-01	1.02	7.1492E-03	1.90	4.8343E-01	1.02	7.8973E-03	1.91
	1/32	2.1492E-01	1.00	1.8184E-03	1.98	2.4059E-01	1.00	2.0032E-03	1.98
	1/64	1.0737E-01	1.00	4.5661E-04	1.99	1.2006E-01	1.00	5.0242E-04	2.00
	1/2	1.5302E-00	-	8.3164E-03	-	1.5369E-00	-	2.0213E-02	-
	1/4	5.2384E-01	1.55	1.0947E-02	-0.39	5.7207E-01	1.42	1.1749E-02	0.78
2	1/8	1.3121E-01	2.00	1.2747E-03	3.10	1.4378E-01	1.99	1.3705E-03	3.01
	1/16	3.2542E-02	2.01	1.5431E-04	3.05	3.5654E-02	2.01	1.6613E-04	3.04
	1/32	8.1111E-03	2.00	1.9095E-05	3.02	8.8770E-03	2.00	2.0542E-05	3.02
	1/64	2.0262E-03	2.00	2.3779E-06	3.00	2.2155E-03	2.00	2.5572E-06	3.00

TABLE 2. Comparison of computation time (in seconds) of the SFWG and the WG methods on the uniform triangular grid with a different number of the elements.

k	h	*No. of elements*	*SFWG method*	*WG method*
	1/2	8	0.3281	0.4219
	1/4	32	0.0469	0.0469
1	1/8	128	0.2344	0.2344
	1/16	512	1.3281	1.3750
	1/32	2048	4.5938	5.2813
	1/64	8192	29.5625	30.5156

Galerkin method (4). It can be observed in Table 2 that the computation time with 2048 *elements by using the SFWG is* 29.5625*, which is less than* 30.5156*, needed by using the standard weak Galerkin method. Therefore, when a large number of elements are used the computation time becomes a significant factor. The SFWG method is more efficient in accuracy and computation time.*

Example 4.2. Interior layer-continuous boundary condition. *This example is adopted from* [12]. *Let the problem (1)-(2) be posed on square domain* $\Omega = (0,1) \times (0,1)$ *with the following data:* $\boldsymbol{\beta} = (1, 0)$, *and* $c = 1$ *and the exact solution is given by*

$$u(x, y) = 0.5x(1-x)y(1-y)\left(1 - \tanh\frac{\eta - x}{\gamma}\right), \tag{48}$$

where the parameters η *and* γ *control the location and thickness of the interior layer. Figures 1 shows the numerical solution, the exact solution and the error of the equation (48) on the uniform triangular meshes. Table 3 shows that the SFWG scheme (9) with* $P_k(k), k = 1, 2$ *elements has convergence rate of* $\mathcal{O}(h^{k+1})$ *in* $\|Q_0 u - u_0\|$ *and* $\mathcal{O}(h^k)$ *in* $\lVert\lvert Q_h u - u_h\rvert\rVert$ *for the convection-diffusion-reaction problem with the diffusion coefficient matrix* $\alpha = 0.1I_2$ *and* $\alpha = 0.01I_2, \eta = 0.5,$ *and* $\gamma = 0.05$. *We can capture the interior layer accurately.*

Example 4.3. L-shaped domain. *In this example, we solve the problem (1)-(2) on a L-shaped domain* $\Omega = [-1, 1]^2 \setminus (0, 1) \times (-1, 0)$ *partitioned into triangles with the following data:* $\alpha = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \boldsymbol{\beta} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}, c = 2xy,$ *and the* $f(x, y)$ *term is given such that the exact solution is* $u(x, y) = x(1-x)y(1-y)$. *The results reported in Table 4 shows the errors and the numerical convergence rates in the* L^2 *norm and*

TABLE 3. Error profiles and convergence rates for $\left(P_k(K), P_k(e), [P_{k+1}(K)]^2\right), k = 1, 2$ finite element spaces.

		When $\alpha = 0.1I_2$				When $\alpha = 0.0I_2$			
k	h	$\|\|\|Q_h u - u_h\|\|\|$	*Rate*	$\|\|Q_0 u - u_0\|\|$	*Rate*	$\|\|\|Q_h u - u_h\|\|\|$	*Rate*	$\|\|Q_0 u - u_0\|\|$	*Rate*
1	1/2	2.2770E-02	-	3.7286E-03	-	9.1971E-03	-	3.6434E-03	-
	1/4	1.6433E-02	0.47	1.6279E-03	1.11	5.2288E-03	0.81	9.3178E-04	1.97
	1/8	1.5076E-02	0.12	9.5359E-04	0.77	4.2468E-03	0.30	4.6206E-04	1.01
	1/16	7.8536E-03	0.94	2.5914E-04	1.88	2.3020E-03	0.88	1.5014E-04	1.62
	1/32	3.3953E-03	1.21	5.7185E-05	2.18	1.0588E-03	1.12	3.0678E-05	2.29
	1/64	1.6808E-03	1.01	1.4411E-05	1.98	5.3007E-04	1.00	7.0847E-06	2.11
	1/128	8.3836E-04	1.00	3.6166E-06	2.00	2.6495E-04	1.00	1.7257E-06	2.04
2	1/2	1.0284E-02	-	1.8411E-03	-	6.9487E-03	-	2.8192E-03	-
	1/4	9.0976E-03	0.18	6.1503E-04	1.58	2.7526E-03	1.34	5.6747E-04	2.31
	1/8	3.9788E-03	1.19	1.2294E-04	2.32	1.1827E-03	1.22	1.0512E-04	2.43
	1/16	7.4666E-04	2.41	8.4507E-06	3.86	2.3511E-04	2.83	7.8587E-06	3.74
	1/32	3.3477E-04	1.16	2.5129E-06	1.75	1.0489E-04	1.16	2.3771E-06	1.73
	1/64	8.3991E-05	1.99	3.2724E-07	2.94	2.6461E-05	1.99	3.2317E-07	2.88
	1/128	2.0896E-05	2.00	4.1188E-08	2.99	6.6013E-06	2.00	4.1069E-08	2.98

TABLE 4. Error analysis and convergence rates for $P_k(K), (k = 1, 2)$ elements with $[P_{k+1}(K)]^2$ weak gradient ($j = k+1$ in (7)) on L-shape domain.

k	h	$\|\|\|Q_h u - u_h\|\|\|$	*Rate*	$\|\|Q_0 u - u_0\|\|$	*Rate*
1	1/2	1.0407E-00	-	5.2124E-02	-
	1/4	5.2665E-01	0.98	1.3218E-02	1.98
	1/8	2.6407E-01	1.00	3.3182E-03	1.99
	1/16	1.3212E-01	1.00	8.3043E-04	2.00
	1/32	6.6073E-02	1.00	2.0766E-04	2.00
	1/64	3.3038E-02	1.00	5.1911E-05	2.00
2	1/2	1.1351E-01	-	3.1497E-03	-
	1/4	2.8516E-02	1.99	3.7752E-04	3.06
	1/8	7.1405E-03	1.00	4.6264E-05	3.03
	1/16	1.7864E-03	1.00	5.7278E-06	3.01
	1/32	4.4675E-04	1.00	7.1258E-07	3.00
	1/64	1.1171E-04	1.00	8.8864E-08	3.00

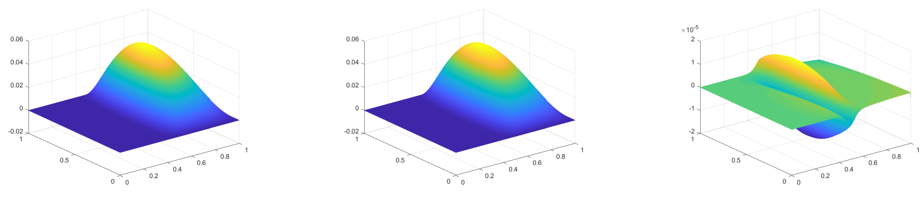

FIGURE 1. WG solution (Left), exact solutions (Middle) and the error (Right) for $(P_1(K); P_1(e); [P_2(K)]^2)$ element and $\alpha = 0.1I_2$.

$|||\cdot|||$ norm. The SFWG scheme with P_k elements has convergence rate of $\mathcal{O}(h^{k+1})$ and $\mathcal{O}(h^k)$ in L^2-norm and H^1 norm, respectively.

Example 4.4. *As the final example, we use a L-shaped domain $\Omega = [-1,1]^2 \setminus (0,1) \times (-1,0)$ with the following data: $\alpha = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \boldsymbol{\beta} = (1,2)^T, c = 1$, and the exact solution is $u(x,y) = \sin(\pi x)\sin(\pi y) + x^2$. Table 5 shows the performance of the SFWG scheme (9) for the problem (1)-(2) on polynomial of degrees $k = 1, 2$. The results indicate that the SFWG method with P_k elements has convergence rate*

TABLE 5. Error analysis and convergence rates for the SFWG scheme (9) on L-shaped domain $\Omega = [-1,1]^2 \setminus (0,1) \times (-1,0)$.

k	h	$\|Q_h u - u_h$	*Rate*	$\|Q_0 u - u_0\|$	*Rate*
	1/2	*1.8632E-00*	-	*1.2581E-01*	-
	1/4	*9.3121E-01*	*1.00*	*4.0281E-02*	*1.64*
1	*1/8*	*4.6375E-01*	*1.00*	*1.0780E-02*	*1.90*
	1/16	*2.3156E-01*	*1.00*	*2.7431E-03*	*1.97*
	1/32	*1.1574E-01*	*1.00*	*6.8887E-04*	*1.99*
	1/64	*5.7863E-02*	*1.00*	*1.7241E-04*	*2.00*
	1/2	*5.6650E-01*	-	*1.9072E-02*	-
	1/4	*1.4311E-01*	*1.99*	*2.3484E-03*	*3.02*
2	*1/8*	*3.5605E-02*	*2.00*	*2.9224E-04*	*3.00*
	1/16	*8.8820E-03*	*2.00*	*3.6469E-05*	*3.00*
	1/32	*2.2192E-03*	*2.00*	*4.5531E-06*	*3.00*
	1/64	*5.5474E-04*	*2.00*	*5.6868E-07*	*3.00*

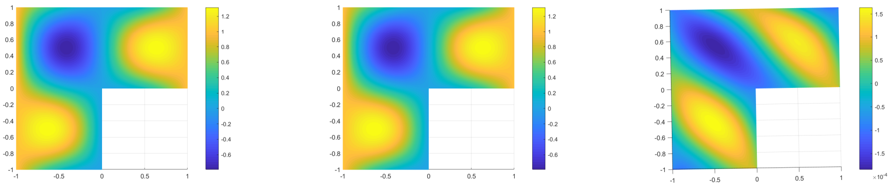

FIGURE 2. WG solution (Left), exact solutions (Middle) and the error (Right) for $(P_1(K); P_1(e); [P_2(K)]^2)$ element on a L-shaped domain.

of $\mathcal{O}(h^{k+1})$ and $\mathcal{O}(h^k)$ in L^2-norm and H^1 norm, respectively. The numerical solution, the exact solution and their error are shown in Fig 2.

5. Conclusion

The stabilizer free weak Galerkin methods for the convection-diffusion-reaction problems studied in this paper. The error estimates and convergence of the SFWG scheme is derived. Numerical results show that while the same rate of convergence can be obtained using standard weak Galerkin scheme (4) and SFWG scheme (9), the SFWG method is more efficient and easier to implement for the convection-diffusion-reaction (1)-(2).

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