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ANALYSIS OF ROTHE METHOD FOR A VARIATIONAL -HEMIVARIATIONAL INEQUALITY IN ADHESIVE CONTACT PROBLEM FOR LOCKING MATERIALS

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Abstract. We study a system of differential variational–hemivariational inequality arising in the modelling of adhesive viscoelastic contact problems for locking materials. The system consists of a variational-hemivariational inequality for the displacement field and an ordinary differential equation for the adhesion field. The contact is described by the unilateral constraint and normal compliance contact condition in which adhesion is taken into account and the friction is modelled by the nonmonotone multivalued subdifferential condition with adhesion. The problem is governed by a linear viscoelastic operator, a nonconvex locally Lipschitz friction potential and the subdifferential of the indicator function of a convex set which describes the locking constraints. The existence and uniqueness of solution to the coupled system are proved. The proof is based on a time-discretization method, known as the Rothe method.

Key words. Variational-hemivariational inequality, Rothe method, adhesion, locking material, unilateral constraint, normal compliance, nonmonotone friction.

1. Introduction

In this paper, we discuss the solvability of a coupled system which consists of an abstract evolution variational-hemivariational inequality and an ordinary differential equation. The system serves as a model for numerous physics and engineering applications. We provide a theoretical illustration of our abstract results and study a quasi-stationary frictional contact problem with adhesion for viscoelastic locking materials. In this problem, the variational-hemivariational inequality describes the displacement field, the ordinary differential equation is for the adhesion field and the subdifferential of the indicator function of a convex set which describes the locking constraints.

Processes of adhesion are important in many industrial settings, such as parts, usually nonmetallic glued together and prevent delamination of composite materials. As a result, in order to obtain more precise models of contact phenomena, it is necessary to add adhesion to the description of contact problems. Here, we adopt the approach model of Frémond [10, 11] and introduce a surface internal variable, the bonding field, which takes values between zero and one, and which describes the fraction of active bonds on the contact surface. The number of literature on adhesive contact problem between a deformable body and a foundation grows rapidly, general models can be found in many contributions, such as [10, 11, 2, 5, 7, 9, 16, 25, 26].

For the locking materials, the strain tensor is constrained to stay in a given convex set. The study of elastic materials with locking effect was first introduced in the pioneering works of Prager [21, 22, 23]. There, the constitutive law of such materials was derived and different mechanical interpretations have been provided.

The main novelties of the paper are described as follows. First, we apply the Rothe method to study a system of a variational-hemivariational inequality and a differential equation. Until now, only few papers devoted to the Rothe method for

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variational-hemivariational inequalities, see [4, 3]. At the same time, they studied only a single variational-hemivariational inequality. The Rothe method to study a system of a hemivariational inequality and a differential equation was first studied in [19]. Here, we promote it to the system of a variational-hemivariational inequality and an ordinary differential equation.

Second, we study a new contact model for locking materials with short memory. Contact problems with locking materials have recently been considered in [1, 18, 27, 28]. For the problem considered in [18] the contact was described by the Signorini unilateral condition and the friction was modeled with a nonmonotone multivalued subdifferential condition. The existence and uniqueness to the problem were proved by using a surjectivity result for pseudomonotone operators as well as the Banach contraction principle. The reference [1] deals with the numerical analysis of the model considered in [18]. The reference [27] considered a model which was frictionless and described with a nonsmooth multivalued interface law which involves unilateral constraints and subdifferential conditions. The existence of a unique weak solution to the problem was proved, and its continuous dependence with respect to the bounds which govern the locking and the normal displacement was established.

We note that all models considered in the above mentioned papers were elliptic. And [28] deals with locking materials with long memory, this leads to a history-dependent inequality. In this paper, we deal with contact problem for locking materials with short memory, which leads to an evolutionary variational-hemivariational inequality.

Third, we show the existence of a unique weak solution to the contact model in this paper. Since the variational formulation of the contact problem consists of a variational-hemivariational inequality and an ordinary differential equation, it is a challenge to derive the existence and uniqueness of the solution for the coupled system.

The rest of the paper is structured as follows. In Section 2 we recall the notation and present some preliminary materials. In Section 3 we provide a classical and variational formulation of the adhesive contact model for locking materials. In section 4 we prove the main existence and uniqueness result, Theorem 3.1, and provide the proof by Rothe method.

2. Preliminaries

In this section, we recall some preliminaries which we will refer to in the sequel. We start with the definitions of Clarke directional derivative and Clarke subdifferential. Let X be a Banach space, X^* its dual. Denote by $\langle\cdot,\cdot\rangle_{X^*\times X}$ the duality pairing between X^* and X.

Definition 2.1. *Let $\psi\colon X\to\mathbb{R}$ be a locally Lipschitz function. The generalized directional derivative, in the sense of Clarke, of ψ at $x\in X$ in the direction $v\in X$, denoted by $\psi^0(x;v)$, is defined by*

$$\psi^0(x;v)=\limsup_{y\to x,\,\lambda\downarrow 0}\frac{\psi(y+\lambda v)-\psi(y)}{\lambda}$$

and the Clarke subdifferential of ψ at x, denoted by $\partial\psi(x)$, is a subset of a dual space X^* given by

$$\partial\psi(x)=\{\,\zeta\in X^*\mid \psi^0(x;v)\ge\langle\zeta,v\rangle_{X^*\times X}\ \forall\, v\in X\,\}.$$

A locally Lipschitz function ψ is called regular (in the sense of Clarke) at $x \in X$ if for all $v \in X$ the one–sided directional derivative $\psi'(x;v)$ exists and satisfies $\psi^0(x;v) = \psi'(x;v)$ for all $v \in X$.

Definition 2.2. *Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a convex functional. Then the convex subdifferntial of φ at $x \in X$ is a subset of X^* given by*

$$\partial\varphi(x) = \{\xi \in X^* : \varphi(x+v) - \varphi(x) \geq \langle \xi, v\rangle_{X^*\times X} \text{ for all } v \in X\}.$$

Now, we pass to the definition of pseudomonotonicity, for both single and multivalued operators.

Definition 2.3. *A single valued operator $A : X \to X^*$ is called pseudomonotone if for any sequence $\{v_n\}_{n=1}^{\infty} \subset X, v_n \to v$ weakly in X and*

$$\limsup_{n\to\infty} \langle Av_n, v_n - v\rangle_{X^*\times X} \leq 0$$

imply that

$$\langle Av, v - y\rangle_{X^*\times X} \leq \liminf_{n\to\infty} \langle Av_n, v_n - y\rangle_{X^*\times X}$$

for every $y \in X$.

Definition 2.4. *A multivalued operator $A : X \to 2^{X^*}$ is called pseudomonotone if the following conditions hold:*

(1) A has values which are nonempty, weakly compact, and convex.

(2) A is upper semicontinuous (usc, for short) from every finite dimensional subspace of X into X^* endowed with the weak topology.

(3) For any sequence $\{v_n\}_{n=1}^{\infty} \subset X$ and any $v_n^* \in A(v_n), v_n \to v$ weakly in X and

$$\limsup_{n\to\infty} \langle v_n^*, v_n - v\rangle_{X^*\times X} \leq 0$$

imply that for any $y \in X$ there exists $u(y) \in A(v)$ such that

$$\langle u(y), v - y\rangle_{X^*\times X} \leq \liminf_{n\to\infty} \langle v_n^*, v_n - y\rangle_{X^*\times X}.$$

Now, let us recall a result which elaborates the pseudomonotone of a multivalued operator proposed in [4, Proposition 2.5].

Proposition 2.5. *Let X and U be two reflexive Banach spaces, let $\gamma : X \to U$ be a linear, continuous and compact operator, and denote by $\gamma^* : U^* \to X^*$ the adjoint operator of γ. Let $J : U \to \mathbb{R}$ be a locally Lipschitz functional, and assume that its Clarke subdifferential satisfies*

$$\|\partial J(v)\|_{U^*} \leq c(1 + \|v\|_U)$$

with $c > 0$. Then the multivalued operator $M : X \to 2^{X^*}$ defined by

$$M(v) = \gamma^* \partial J(\gamma v) \text{ for all } v \in X$$

is pseudomonotone.

Note that, in the statement of Proposition 2.5, U^* represents the dual of U, $\|\cdot\|_U$ and $\|\cdot\|_{U^*}$ denote the norms on the spaces U and U^*, respectively. We now recall a result providing pseudomonotonicity of the sum of two pseudomonotone operators, which corresponds to [6, Proposition 1.3.68].

Proposition 2.6. *Assume that X is a reflexive Banach space and $A_1, A_2 : X \to 2^{X^*}$ are pseudomonotone operators. Then $A_1 + A_2 : X \to 2^{X^*}$ is a pseudomonotone operator.*

The next proposition deals with an existence result for an abstract elliptic inclusion and corresponds to [14, Theorem 2.2].

Proposition 2.7. *Let X be a real reflexive Banach space, let $\tilde{F} : D(\tilde{F}) \subset X \to 2^{X^*}$ be a maximal monotone operator, let $G : D(G) = X \to 2^{X^*}$ be a multivalued pseudomonotone operator, and let $L \in X^*$. Assume that there exist $u_0 \in X$ and $R \geq \|u_0\|_X$ such that $D(\tilde{F}) \cap B_R(0_X) \neq \emptyset$ and*

$$\langle \xi + \eta - L, u - u_0 \rangle_{X^* \times X} > 0$$

for all $u \in D(\tilde{F})$ with $\|u\|_X = R$ and all $\xi \in \tilde{F}(u), \eta \in G(u)$. Then there exists at least an element $u \in D(\tilde{F})$ such that

$$\tilde{F}(u) + G(u) \ni L.$$

In the statement of Proposition 2.7, we denote by $D(\tilde{F})$ and $D(G)$ the effective domains of the operators $\tilde{F}$ and G, respectively, 0_X represents the zero element of X, $B_R(0_X)$ represents the ball of radius R and center 0_X.

We now introduce some spaces of vector-valued function defined on the interval $[0, T]$ where $T > 0$. Let π denote a finite partition of the interval $(0, T)$ by a family of disjoint subintervals $\sigma_i = (a_i, b_i)$ such that $[0, T] = \cup_{i=1}^{n} \bar{\sigma}_i$. Let $\mathcal{N}$ denote the familiy of all such partitions. Then for $1 \leq q < \infty$ we define the seminorm of a function $x : [0, T] \to X$ by equality

$$\|x\|^q_{BV^q(0,T;X)} = \sup_{\pi \in \mathcal{N}} \{ \sum_{\sigma_i \in \pi} \|x(b_i) - x(a_i)\|^q_X \},$$

and the space

$$BV^q(0, T; X) = \{x : [0, T] \to X; \|x\|_{BV^q(0,T;X)} < \infty\}.$$

Assume now that $1 \leq p \leq \infty, 1 \leq q < \infty$, and X, Z are Banach spaces such that $X \subset Z$ with continuous embedding. Denote

$$M^{p,q}(0, T; X, Z) = L^p(0, T; X) \cap BV^q(0, T; Z).$$

Then it is obvious that $M^{p,q}(0, T; X, Z)$ is also a Banach space with the norm $\| \cdot \|_{L^p(0,T;X)} + \| \cdot \|_{BV^q(0,T;Z)}$.

Finally, we end this section by introducing the following compactness result which is proved in [13].

Proposition 2.8. *Let $1 \leq p, q < \infty$. Let $X_1 \subset X_2 \subset X_3$ be real Banach spaces such that X_1 is reflexive, the embedding $X_1 \subset X_2$ is compact, and the embedding $X_2 \subset X_3$ is continuous. Then the embedding $M^{p,q}(0, T; X_1; X_3) \subset L^p(0, T; X_2)$ is compact.*

3. An adhesive viscoelastic contact model

Next, we study a contact problem for an adhesive viscoelastic locking body with unilateral constraints, normal compliance and nonmonotone friction condition.

Assume a viscoelastic body occupies a Lipschitz domain Ω in $\mathbb{R}^d$ with $d = 2, 3$. We use the notation $\boldsymbol{x} = (x_i)_{i=1}^d$ for a generic point in $\overline{\Omega} = \Omega \cup \partial\Omega$ and since the boundary Γ of Ω is Lipschitz continuous, we denote by $\boldsymbol{\nu} = (\nu_i)_{i=1}^d$ the outward unit normal on $\partial\Omega$. We use the notation $\boldsymbol{u} = (u_i)$, $\boldsymbol{\sigma} = (\sigma_{ij})$ and $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ for the displacement vector, the stress tensor, and linearized strain tensor, respectively. Sometimes, we do not indicate explicitly the dependence of the variables on the spatial variable $\boldsymbol{x}$. Recall that the components of the linearized strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u})$ are $\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$, where $u_{i,j} = \partial u_i / \partial x_j$. The indices i, j, k, l run between

1 and d and, unless stated otherwise, the summation convention over repeated indices is used. An index following a comma indicates a partial derivative with respect to the corresponding component of the spatial variable $\boldsymbol{x}$. A superscript prime of a variable stands for the time derivative of the variable. Moreover, we use the notation v_ν and $\boldsymbol{v}_\tau$ for the normal and tangential components of $\boldsymbol{v}$ on $\partial\Omega$ given by $v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}$. The normal and tangential components of the stress field $\boldsymbol{\sigma}$ on the boundary are defined by $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$, respectively. The symbol $\mathbb{S}^d$ represents the space of second order symmetric tensors on $\mathbb{R}^d$.

The boundary $\partial\Omega$ is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 and the measure of Γ_1, denoted $m(\Gamma_1)$, is positive. The body is clamped on Γ_1, thus, the displacement field vanishes there. Time-dependent surface tractions of density $\boldsymbol{f}_2$ act on Γ_2 and time-dependent volume forces of density $\boldsymbol{f}_0$ act in Ω. The part Γ_2 can be empty. We pay attention to the evolutionary process of the mechanical state of the body in the time interval $(0,T)$ with $T > 0$. The mathematical model of the contact problem is stated as follows.

Problem $\mathcal{P}$ Find a displacement field $\boldsymbol{u} : \Omega\times(0,T)\to\mathbb{R}^d$, a stress field $\boldsymbol{\sigma}$: $\Omega\times(0,T)\to\mathbb{S}^d$ and a bonding field $\beta : \Gamma_3\times(0,T)\to[0,1]$ such that

$$(1)\qquad \boldsymbol{\sigma}(t)\in\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}'(t)) + \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \partial I_E(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))) \qquad \text{in } \Omega\times(0,T),$$

$$(2)\qquad \mathrm{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \qquad \text{in } \Omega\times(0,T),$$

$$(3)\qquad \boldsymbol{u}(t) = \boldsymbol{0} \qquad \text{on } \Gamma_1\times(0,T),$$

$$(4)\qquad \boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \qquad \text{on } \Gamma_2\times(0,T),$$

$$(5)\qquad \begin{cases} u_\nu(t)\le g, \quad \sigma_\nu(t)+\xi_\nu(t)\le 0,\\ (u_\nu(t)-g)\,(\sigma_\nu(t)+\xi_\nu(t)) = 0,\\ \xi_\nu(t)\in\partial j_\nu(\beta(t),u_\nu(t)) \end{cases} \qquad \text{on } \Gamma_3\times(0,T),$$

$$(6)\qquad -\boldsymbol{\sigma}_\tau(t)\in\partial j_\tau(\beta(t),\boldsymbol{u}_\tau(t)) \qquad \text{on } \Gamma_3\times(0,T),$$

$$(7)\qquad \beta'(t) = F(t,\boldsymbol{u}(t),\beta(t)) \qquad \text{on } \Gamma_3\times(0,T),$$

$$(8)\qquad \beta(0) = \beta_0 \qquad \text{on } \Gamma_3,$$

$$(9)\qquad \boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \text{in } \Omega.$$

Now, we present a short description of the equations and conditions in Problem $\mathcal{P}$. We refer the reader to [5, 26, 14, 15] for more details on mathematical models in contact mechanics. Eq. (1) represents the constitutive law for viscoelastic materials with locking constraints in which $\mathcal{C}$ is the viscocity operator, $\mathcal{G}$ is the elastic operator, and ∂I_E stands for the convex subdifferential of the indicator function of a set E. Eq. (2) represents the equation of equilibrium, and we use it since we assume that the process is quasi-static. We have the clamped boundary condition (3) on Γ_1 and the surface traction boundary condition (4) on Γ_2.

There are some complex boundary conditions on the boundary Γ_3 in our model which can be one of the traits of novelty in our paper. The function β is a surface internal variable, which is usually called the bonding field or the adhesion field. It describes the pointwise fractional density of active bonds on the contact surface. The evolution of the bounding field considered on Γ_3 is governed by an ordinary differential equation (7) depending on the displacement. If $\beta = 1$ at a point of the contact part, the adhesion is complete and all the bonds are active, and $\beta = 0$ means that all bonds are inactive and there is no adhesion. When $0 < \beta < 1$ then the adhesion is partial and a fracture β of the bonds is active. The function $\beta(0)$ denotes the initial bonding field in (8). The contact condition (5) denotes a model with multivalued normal and unilateral constraint contact boundary condition, which is described by the subgradient of a nonconvex functional j_ν, where j_ν is assumed to be locally Lipschitz in its last variable. On the other hand, the general friction contact condition (6) with adhesion is governed by the subgradient of a nonconvex functional j_τ.

We now focus our interest on the constitutive law (1) which represents the other trait of novelty in our paper. We deduce such a law in the following model: assume that a locking model is connected in parallel with a viscoelastic model with short memory. At each instant t, this stress field is given by the sum

$$\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}^S(t) + \boldsymbol{\sigma}^Q(t)$$

where $\boldsymbol{\sigma}^S$ and $\boldsymbol{\sigma}^Q$ represent the stress of the locking and the viscoelastic model, respectively. From the constitutive law of the locking model we have

$$\boldsymbol{\sigma}^S(t) = \mathcal{G}^S\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \partial I_E(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))) \tag{10}$$

where $\mathcal{G}^S$ is an elasticity operator and E is the set of constraints. $\partial I_E : \mathbb{S}^d \to 2^{\mathbb{S}^d}$ represents the subdifferential of the indicator function of the set E, i.e.,

$$I_E(\boldsymbol{\varepsilon}) = \begin{cases} 0, \ if \ \boldsymbol{\varepsilon} \in \ E \\ +\infty, \ if \ \boldsymbol{\varepsilon} \notin \ E \end{cases} \ for \ all \ \boldsymbol{\varepsilon} \in \ \mathbb{S}^d.$$

For the set E, various examples can be found in the literature, as explained in [8]. A typical example is given by

$$E = \{\boldsymbol{\tau} \in \mathbb{S}^D \mid \mathcal{F}(\boldsymbol{\tau}) \leq k\} \tag{11}$$

where $\mathcal{F} : \mathbb{S}^d \to \mathbb{R}$ is a convex continuous function such that $\mathcal{F}(0) = 0$ and k is a positive constant. It is easy to see that in this case the set E is a nonempty convex closed subset of $\mathbb{S}^d$. Using (11) with the choice

$$\mathcal{F}(\boldsymbol{\tau}) = \frac{1}{2}\|\boldsymbol{\tau}^D\|$$

where $\boldsymbol{\tau}^D$ denotes the deviator of the tensor $\boldsymbol{\tau} \in \mathbb{S}^D$, leads to the Von Mises convex. This convex set was considered in [21, 22] to model the ideal-locking effect.

On the other hand, the viscoelastic constitutive law with short memory is that

$$\boldsymbol{\sigma}^Q(t) = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}'(t)) + \mathcal{G}^Q\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \tag{12}$$

where $\mathcal{C}$ is a viscosity operator and $\mathcal{G}^Q$ is an elasticity operator. Add (10) and (12), and define $\mathcal{G} = \mathcal{G}^S + \mathcal{G}^Q$, we obtain the constitutive law (1).

In the study of Problem $\mathcal{P}$, we use the standard notation for Lebesgue and Sobolev spaces. For $\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d)$, we use the same symbol $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on $\partial\Omega$ and we use the notation v_ν and $\boldsymbol{v}_\tau$ for its normal and tangential traces. In addition, we introduce spaces V and $\mathcal{H}$ as follows:

$$\begin{aligned} &V = \{\boldsymbol{v} = (v_i) \in H^1(\Omega;\mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \ on \ \Gamma_1\}, \\ &\mathcal{H} = L^2(\Omega;\mathbb{S}^d), \\ &H = L^2(\Omega;\mathbb{R}^d). \end{aligned}$$

These are real Hilbert spaces with the canonical inner products in $\mathcal{H}$, and the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}$$

in V. The associated norms are $\|\cdot\|_V$ and $\|\cdot\|_{\mathcal{H}}$. On the other hand, by the trace theorem, we have

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3;\mathbb{R}^d)} \leq c_\gamma\|\gamma\|\|\boldsymbol{v}\|_V \ \forall \ \boldsymbol{v} \in V,$$

where $c_\gamma > 0$ being a constant in the Korn inequality and $\|\gamma\|$ being the norm of the trace operator $\gamma : V \to L^2(\Gamma_3;\mathbb{R}^d)$.

Note that $V \subset H \subset V^*$ form an evolution triple of function spaces, where V is reflexive, and let V^* denotes its dual. The duality pairing between V^* and V and a norm in V are denoted by $\langle\cdot,\cdot\rangle_{V^*\times V}$ and $\|\cdot\|_V$, respectively. Given $0 < T < +\infty$, we introduce spaces $\mathcal{V} = L^2(0,T;V)$ and $\mathcal{W} = \{\boldsymbol{w} \in \mathcal{V} \mid \boldsymbol{w}' \in \mathcal{V}^*\}$, where the time derivative $\boldsymbol{w}' = \partial\boldsymbol{w}/\partial t$ is understood in the sense of vector-valued distributions. The dual of $\mathcal{V}$ is $\mathcal{V}^* = L^2(0,T;V^*)$. It is known that the space $\mathcal{W}$ endowed with the graph norm $\|\boldsymbol{w}\|_{\mathcal{W}} = \|\boldsymbol{w}\|_{\mathcal{V}} + \|\boldsymbol{w}'\|_{\mathcal{V}^*}$ is a separable and reflexive Banach space. We denote $U = L^2(\Gamma_3;\mathbb{R}^d)$, where U is reflexive and let $\langle\cdot,\cdot\rangle_{U^*\times U}$ and $\|\cdot\|_U$ denote the duality between U and U^* and the norm on U, respectively. We also denote $Y = L^2(\Gamma_3)$, $\mathcal{Y} = L^2(0,T;Y)$ and the norm on Y and $\mathcal{Y}$ is denoted by $\|\cdot\|_Y$ and $\|\cdot\|_{\mathcal{Y}}$, respectively.

Moreover, we denote $\mathcal{U} = L^2(0,T;U)$, $\mathcal{U}^* = L^2(0,T;U^*)$ and use the symbols $\langle\cdot,\cdot\rangle_{\mathcal{V}^*\times\mathcal{V}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{U}^*\times\mathcal{U}}$ to denote the duality pairing between $\mathcal{V}$ and $\mathcal{V}^*$, $\mathcal{U}$ and $\mathcal{U}^*$, respectively. We also use the notation $\mathcal{L}(V,V^*)$ for the space of linear continuous operators from V to V^*, and we denote by $\|\cdot\|_{\mathcal{L}(V,V^*)}$ the norm in space $\mathcal{L}(V,V^*)$. Analogously, we introduce the space $\mathcal{L}(V,U)$ and the corresponding norm $\|\cdot\|_{\mathcal{L}(V,U)}$. And then, $C(0,T;V)$ will represent the space of continuous functions defined on $[0,T]$ with values in V. Now, we can consider the following assumptions on the data of Problem $\mathcal{P}$.

For the viscosity tensor $\mathcal{C} : \Omega\times\mathbb{S}^d \to \mathbb{S}^d$, we assume

$$\left\{\begin{array}{l} (a)\ \mathcal{C}_{ijkl} \in L^\infty(\Omega), 1 \le i,j,k,l \le d,\\ (b)\ \mathcal{C}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \boldsymbol{\sigma}\cdot\mathcal{C}\boldsymbol{\tau}\ \textit{for all}\ \boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{S}^d\ \textit{a.e. in}\ \Omega,\\ (c)\ \mathcal{C}\boldsymbol{\tau}\cdot\boldsymbol{\tau} \ge \alpha_1\|\boldsymbol{\tau}\|^2_{\mathbb{S}^d}\ \textit{for all}\ \boldsymbol{\tau}\in\mathbb{S}^d\ \textit{a.e. in}\ \Omega\ \textit{with}\ \alpha_1 > 0.\end{array}\right. \tag{13}$$

For the elasticity tensor $\mathcal{G} : \Omega\times\mathbb{S}^d \to \mathbb{S}^d$, we assume

$$\left\{\begin{array}{l} (a)\ \mathcal{G}_{ijkl} \in L^\infty(\Omega),\ 1 \le i,j,k,l \le d,\\ (b)\ \mathcal{G}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \boldsymbol{\sigma}\cdot\mathcal{G}\boldsymbol{\tau}\ \textit{for all}\ \boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{S}^d\ \textit{a.e. in}\ \Omega,\\ (c)\ \mathcal{G}\boldsymbol{\tau}\cdot\boldsymbol{\tau} \ge \alpha_2\|\boldsymbol{\tau}\|^2_{\mathbb{S}^d}\ \textit{for all}\ \boldsymbol{\tau}\in\mathbb{S}^d\ \textit{a.e. in}\ \Omega\ \textit{with}\ \alpha_2 > 0.\end{array}\right. \tag{14}$$

$$E\ \textit{is a closed, convex subset of}\ \mathbb{S}^d\ \textit{with}\ \mathbf{0}_{\mathbb{S}^d}\in E. \tag{15}$$

j_ν: The normal compliance function $j_\nu : \Gamma_3\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies

$$\left\{\begin{array}{l} (a)\ j_\nu(\cdot,r,s)\ \textit{is measurable on}\ \Gamma_3\ \textit{for all}\ r,s\in\mathbb{R}\ \textit{and}\\ \quad j_\nu(\cdot,0,0)\in L^1(\Gamma_3),\\ (b)\ j_\nu(\boldsymbol{x},r,\cdot)\ \textit{is locally Lipschitz on}\ \mathbb{R}\ \textit{for all}\ r\in\mathbb{R}\ \textit{and a.e.}\\ \quad \boldsymbol{x}\in\Gamma_3,\\ (c)\ |\partial j_\nu(\boldsymbol{x},r,s)| \le c_\nu(1+|s|)\\ \qquad \textit{for all}\ r,s\in\mathbb{R}\ \textit{and a.e.}\ \boldsymbol{x}\in\Gamma_3\ \textit{with}\ c_\nu > 0,\\ (d)\ \textit{either}\ j_\nu(\boldsymbol{x},r,\cdot)\ \textit{or}\ -j_\nu(\boldsymbol{x},r,\cdot)\ \textit{is regular}\\ \quad \textit{for a.e.}\ \boldsymbol{x}\in\Gamma_3\ \textit{and}\ r\in\mathbb{R},\\ (e)\ (r,s)\mapsto j^0_\nu(\boldsymbol{x},r,s;z)\ \textit{is upper semicontinuous}\\ \qquad \textit{for all}\ z\in\mathbb{R}\ \textit{and a.e.}\ \boldsymbol{x}\in\Gamma_3,\ j^0_\nu\ \textit{denotes the}\\ \qquad \textit{Clarke derivative of}\ s\mapsto j_\nu(\boldsymbol{x},r,s)\ \textit{in direction}\ z.\\ (f)\ (\eta_1-\eta_2)(s_1-s_2) \ge -c_2(|r_1-r_2|+|s_1-s_2|)\,|s_1-s_2|\\ \quad \textit{for all}\ \eta_i\in\partial j_\nu(r_i,s_i), r_i,s_i\in\mathbb{R}, i=1,2\ \textit{with}\ c_2\ge 0.\end{array}\right. \tag{16}$$

j_τ: The tangential function $j_\tau : \Gamma_3 \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfies

(17)
$$\begin{cases} (a)\ j_\tau(\cdot, r, \boldsymbol{\xi}) \text{ is measurable on } \Gamma_3 \text{ for all } (r,\boldsymbol{\xi}) \in \mathbb{R}\times\mathbb{R}^d \text{ and} \\ \quad j_\tau(\cdot, 0, \mathbf{0}) \in L^1(\Gamma_3), \\ (b)\ j_\tau(\boldsymbol{x}, r, \cdot) \text{ is locally Lipschitz on } \mathbb{R}^d \text{ for all } r \in \mathbb{R} \text{ and a.e.} \\ \quad \boldsymbol{x} \in \Gamma_3, \\ (c)\ \|\partial j_\tau(\boldsymbol{x}, r, \boldsymbol{\xi})\|_{\mathbb{R}^d} \le c_\tau(1 + \|\boldsymbol{\xi}\|_{\mathbb{R}^d}) \\ \quad\quad \text{for all } r \in \mathbb{R},\ \boldsymbol{\xi} \in \mathbb{R}^d \text{ and a.e. } \boldsymbol{x} \in \Gamma_3 \text{ with } c_\tau > 0, \\ (d)\ \text{either } j_\tau(\boldsymbol{x}, r, \cdot) \text{ or } -j_\tau(\boldsymbol{x}, r, \cdot) \text{ is regular for a.e. } \boldsymbol{x} \in \Gamma_3 \text{ and } r \in \mathbb{R}, \\ (e)\ (r, \boldsymbol{\xi}) \mapsto j_\tau^0(\boldsymbol{x}, r, \boldsymbol{\xi}; \boldsymbol{\eta}) \text{ is upper semicontinuous} \\ \quad\quad \text{for all } \boldsymbol{\eta} \in \mathbb{R}^d \text{ and a.e. } \boldsymbol{x} \in \Gamma_3,\ j_\tau^0 \text{ denotes the} \\ \quad\quad \text{Clarke derivative of } \boldsymbol{\xi} \mapsto j_\tau(\boldsymbol{x}, r, \boldsymbol{\xi}) \text{ in direction } \boldsymbol{\eta}. \\ (f)\ (\zeta_1 - \zeta_2)(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \ge -c_2(|r_1 - r_2| + \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|_{\mathbb{R}^d})\|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|_{\mathbb{R}^d} \\ \quad\quad \text{for all } \zeta_i \in \partial j_\tau(r_i, \boldsymbol{\xi}_i), r_i \in \mathbb{R},\ \boldsymbol{\xi}_i \in \mathbb{R}^d,\ i = 1,2 \text{ with } c_2 \ge 0. \end{cases}$$

Moreover, we assume that the densities of forces and traction satisfy

$$\boldsymbol{f}_0 \in H^1(0,T; L^2(\Omega;\mathbb{R}^d)),\ \boldsymbol{f}_2 \in H^1(0,T; L^2(\Gamma_2;\mathbb{R}^d)),\ \boldsymbol{f}_0(0) \in V. \tag{18}$$

The initial displacement and bonding fields satisfy

$$\boldsymbol{u}_0 \in V, \quad \beta_0 \in Y. \tag{19}$$

The adhesive evolution rate function F satisfies the following assumption.

$F : \Gamma_3 \times (0,T) \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} (a)\ F(\cdot, \cdot, \boldsymbol{\xi}, r) \text{ is measurable on } \Gamma_3 \times (0,T) \text{ for all} \\ \quad (\boldsymbol{\xi}, r) \in \mathbb{R}^d \times \mathbb{R}, \\ (b)\ |F(\boldsymbol{x}, t, \boldsymbol{\xi}_1, r_1) - F(\boldsymbol{x}, t, \boldsymbol{\xi}_2, r_2)| \le L_1(\|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|_{\mathbb{R}^d} + |r_1 - r_2|) \\ \quad \text{for a.e. } (\boldsymbol{x}, t) \in \Gamma_3 \times (0,T) \text{ and all } (\boldsymbol{\xi}_i, r_i) \in \mathbb{R}^d \times \mathbb{R},\ i = 1,2, \\ \quad \text{with } L_1 > 0, \\ (c)\ |F(\boldsymbol{x}, t_1, \boldsymbol{\xi}, r) - F(\boldsymbol{x}, t_2, \boldsymbol{\xi}, r)| \le L_2|t_1 - t_2| \\ \quad \text{for } (\boldsymbol{x}, t_i) \in \Gamma_3 \times (0,T),\ i = 1,2 \text{ and all } (\boldsymbol{\xi}, r) \in \mathbb{R}^d \times \mathbb{R}, \\ \quad \text{with } L_2 > 0, \\ (d)\ F(\boldsymbol{x}, t, \boldsymbol{\xi}, 0) = 0, F(\boldsymbol{x}, t, \boldsymbol{\xi}, r) \ge 0 \text{ for } r \le 0, \text{ and } F(\boldsymbol{x}, t, \boldsymbol{\xi}, r) \\ \quad \le 0 \text{ for } r \ge 1,\ \text{for a.e. } (\boldsymbol{x}, t) \in \Gamma_3 \times (0,T), \text{ and for all } \boldsymbol{\xi} \in \mathbb{R}^d. \end{cases} \tag{20}$$

We now turn to the variational formulation of the contact problem (1)–(9). To this end, we suppose in what follows that $(\boldsymbol{u}, \boldsymbol{\sigma})$ are smooth functions which satisfy (1)–(9). Let $\boldsymbol{v} \in V$. Multiplying the equilibrium equation (2) by $\boldsymbol{v} - \boldsymbol{u}(t)$ and use the Green formula, we deduce that

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}} = \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle_{V^* \times V} + \int_{\Gamma_3} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, d\Gamma \tag{21}$$

for a.e. $t \in (0,T)$, where the elememt $\boldsymbol{f} \in \mathcal{V}^*$ is defined by

$$\langle \boldsymbol{f}(t), \boldsymbol{v}\rangle_{V^* \times V} = (\boldsymbol{f}_0(t), \boldsymbol{v})_H + (\boldsymbol{f}_2(t), \boldsymbol{v})_{L^2(\Gamma_2;\mathbb{R}^d)}\ \forall\, \boldsymbol{v} \in V,\ a.e.\ t \in (0,T). \tag{22}$$

Moreover, from the decomposition formula of $\boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t))$, we have

$$\int_{\Gamma_3} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, d\Gamma = \int_{\Gamma_3} (\sigma_\nu(t)(v_\nu - u_\nu(t)) + \boldsymbol{\sigma}_\tau(t) \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau(t)))\, d\Gamma.$$

Next, we introduce the set of admissible displacement field K_1 defined by

$$K_1 = \{\boldsymbol{v} \in V \mid v_\nu \le g \text{ a.e. on } \Gamma_3\}.$$

Then, for $\boldsymbol{v} \in K_1$, we have

$$\sigma_\nu(t)(v_\nu - u_\nu(t)) = (\sigma_\nu(t)+\xi_\nu(t))(v_\nu - g)+(\sigma_\nu(t)+\xi_\nu(t))(g-u_\nu(t))-\xi_\nu(t)(v_\nu - u_\nu(t)),$$

by the definition of the Clarke subdifferential and the boundary conditions (5), (6), we have

$$\int_{\Gamma_3} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, d\Gamma \geq -\int_{\Gamma_3} j_\nu^0(\boldsymbol{x}, \beta(t), u_\nu(t); v_\nu - u_\nu(t))\, d\Gamma - \int_{\Gamma_3} j_\tau^0(\boldsymbol{x}, \beta(t), \boldsymbol{u}_\tau(t); \boldsymbol{v}_\tau - \boldsymbol{u}_\tau(t))\, d\Gamma.$$

Next, we introduce

$$K_2 = \{\boldsymbol{v} \in V \mid \boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})) \in E \text{ } a.e. \text{ } \boldsymbol{x} \in \Omega\}.$$

From the constitutive law (1), we have

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}') + \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \zeta(\boldsymbol{u}) \text{ and } \zeta(\boldsymbol{u}) \in \partial I_E(\boldsymbol{\varepsilon}(\boldsymbol{u})) \text{ in } \Omega.$$

The latter, for $\boldsymbol{v},\ \boldsymbol{u} \in K_2$, implies

$$\zeta(\boldsymbol{u}) : (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u})) \leq I_E(\boldsymbol{\varepsilon}(\boldsymbol{v})) - I_E(\boldsymbol{\varepsilon}(\boldsymbol{u})) \leq 0 \text{ in } \Omega,$$

thus, we obtain

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}} \leq (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}'(t)) + \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}}.$$

Therefore, from (21), we obtain

$$(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}'(t)) + \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}} + \int_{\Gamma_3} j_\nu^0(\boldsymbol{x}, \beta(t), u_\nu(t); v_\nu - u_\nu(t))\, d\Gamma + \int_{\Gamma_3} j_\tau^0(\boldsymbol{x}, \beta(t), \boldsymbol{u}_\tau(t); \boldsymbol{v}_\tau - \boldsymbol{u}_\tau(t))\, d\Gamma \geq \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle_{V^* \times V}$$

for all $\boldsymbol{v} \in K_1 \cap K_2$, a.e. $t \in (0,T)$. We denote $K = K_1 \cap K_2$, it is clear that K is a closed and convex set with $\mathbf{0}_V \in K$ and we obtain the following variational formulation of Problem $\mathcal{P}$.

Problem $\mathcal{P}^{\mathcal{M}}$ Find a displacement field $\boldsymbol{u} : (0,T) \to V$ and a bonding field $\beta : (0,T) \to Y$ such that $\boldsymbol{u}(t) \in K$ for all $t \in (0,T)$ and

$$\begin{cases} (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}'(t)) + \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}} \\ \quad + \int_{\Gamma_3} j_\nu^0(\boldsymbol{x}, \beta(t), u_\nu(t); v_\nu - u_\nu(t))\, d\Gamma \\ \quad + \int_{\Gamma_3} j_\tau^0(\boldsymbol{x}, \beta(t), \boldsymbol{u}_\tau(t); \boldsymbol{v}_\tau - \boldsymbol{u}_\tau(t))\, d\Gamma \geq \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle_{V^* \times V} \\ \qquad for\ all\ \boldsymbol{v} \in K,\ a.e.\ t \in (0,T), \\ \beta'(t) = F(t, \gamma\boldsymbol{u}(t), \beta(t)) \text{ on } \Gamma_3 \text{ for a.e. } t \in (0,T), \\ \beta(0) = \beta_0 \quad on\ \Gamma_3, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad in\ \Omega. \end{cases} \tag{23}$$

We have the following existence and uniqueness result for Problem $\mathcal{P}^{\mathcal{M}}$. The proof will be shown in the next section.

Theorem 3.1. *Assume the hypotheses on (13)–(20) hold, and moreover,*

$$\alpha_2 > 2c_2. \tag{24}$$

Then Problem $\mathcal{P}^{\mathcal{M}}$ has a unique solution $(\boldsymbol{u}, \beta)$ such that $\boldsymbol{u} \in H^1(0,T;V)$, $\beta \in H^1(0,T;Y)$.

In order to prove Theorem 3.1, in the next section, we first rewrite Problem $\mathcal{P}^{\mathcal{M}}$ into an equivalent form, Problem $\mathcal{Q}^{\mathcal{M}}$. Then we give the proof of the existence and uniqueness result for Problem $\mathcal{Q}^{\mathcal{M}}$, and then Problem $\mathcal{P}^{\mathcal{M}}$ has a unique solution, thus Theorem 3.1 is proved. Moreover, α_2 and c_2 mentioned in Theorem 3.1 represent the constants introduced in (14) and (16).

4. Proof of Theorem 3.1

In order to prove Theorem 3.1, we first define the following operator $A, B : V \to V^*$ by

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle_{V^*\times V} = (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V. \tag{25}$$

$$\langle B\boldsymbol{u}, \boldsymbol{v}\rangle_{V^*\times V} = (\mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V. \tag{26}$$

We also define the function $J : Y \times U \to \mathbb{R}$ by

$$J(\beta, \boldsymbol{u}) = \int_{\Gamma_3} (j_\nu(\boldsymbol{x}, \beta, u_\nu) + j_\tau(\boldsymbol{x}, \beta, \boldsymbol{u}_\tau))\, d\Gamma \quad \forall\, \beta \in Y,\ \boldsymbol{u} \in U. \tag{27}$$

Following the assumption 16(d) and 17(d) and [17, Corollary 4.15(vii)], we obtain that $J(\beta, \cdot)$ or $-J(\beta, \cdot)$ is regular on U for all $\beta \in Y$. And then, applying [17, Corollary 4.15(vi)] and [17, Lemma 3.39(3)] we obtain

$$J^0(\beta, \boldsymbol{u}) = \int_{\Gamma_3} (j^0_\nu(\boldsymbol{x}, \beta, u_\nu) + j^0_\tau(\boldsymbol{x}, \beta, \boldsymbol{u}_\tau))\, d\Gamma, \quad \forall\, \beta \in Y \textit{ and } \boldsymbol{u} \in U. \tag{28}$$

$$\partial J(\beta, \boldsymbol{u}) = \int_{\Gamma_3} (\partial j_\nu(\boldsymbol{x}, \beta, u_\nu) + \partial j_\tau(\boldsymbol{x}, \beta, \boldsymbol{u}_\tau))\, d\Gamma \quad \forall\, \beta \in Y \textit{ and } \boldsymbol{u} \in U. \tag{29}$$

Let $\Phi : V \mapsto \mathbb{R} \cup \{+\infty\}$ be the indicator function of the set K, that is,

$$\Phi(\boldsymbol{v}) = I_K(\boldsymbol{v}) = \begin{cases} 0, & if\ \boldsymbol{v} \in K \\ +\infty, & if\ \boldsymbol{v} \notin K \end{cases} \quad for\ all\ \boldsymbol{v} \in V. \tag{30}$$

Moreover, we give the following operator: $\overline{F} : (0, T) \times U \times Y \to Y$ is defined by

$$\overline{F}(t, \boldsymbol{u}, \beta)(\boldsymbol{x}) = F(\boldsymbol{x}, t, \boldsymbol{u}(\boldsymbol{x}), \beta(\boldsymbol{x}))\ \ for\ all\ \beta \in Y,\ \boldsymbol{u} \in U\ a.e.\ \boldsymbol{x} \in \Gamma_3.$$

Under the above notation, we have

Problem $\mathcal{Q}$. Find $\boldsymbol{u} \in \mathcal{W},\ \beta \in H^1(0, T; Y)$ such that

$$\begin{cases} \langle A\boldsymbol{u}'(t) + B\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle_{V^*\times V} + J^0(\beta(t), \gamma\boldsymbol{u}(t); \gamma\boldsymbol{v} - \gamma\boldsymbol{u}(t)) \\ \quad +\Phi(\boldsymbol{v}) - \Phi(\boldsymbol{u}(t)) \geq \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle_{V^*\times V}\ \ for\ all\ \boldsymbol{v} \in V,\ a.e.\ t \in (0, T), \\ \beta'(t) = \overline{F}(t, \gamma\boldsymbol{u}(t), \beta(t))\ for\ a.e.\ t \in (0, T), \\ \beta(0) = \beta_0,\ \boldsymbol{u}(0) = \boldsymbol{u}_0. \end{cases}$$

Next, we observe that Problem $\mathcal{Q}$ is equivalent to the following problem.

Problem $\mathcal{Q}^{\mathcal{M}}$. Find $\boldsymbol{u} \in \mathcal{W},\ \beta \in H^1(0, T; Y)$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0,\ \beta(0) = \beta_0$ and

$$\begin{aligned} &\langle A\boldsymbol{u}'(t) + B\boldsymbol{u}(t) + \gamma^*\xi(t) - \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle_{V^*\times V} + \Phi(\boldsymbol{v}) - \Phi(\boldsymbol{u}(t)) \\ &\geq 0\ for\ all\ \boldsymbol{v} \in V,\ a.e.\ t \in (0, T) \\ &with\ \xi \in \partial J(\beta(t), \gamma\boldsymbol{u}(t)), \end{aligned} \tag{31}$$

$$\beta'(t) = \overline{F}(t, \gamma\boldsymbol{u}(t), \beta(t))\ for\ a.e.\ t \in (0, T). \tag{32}$$

For the Problem $\mathcal{Q}^{\mathcal{M}}$, we show that under hypotheses (13)–(20), the following properties hold.

$H(A)$. The operator $A : V \to V^*$ is linear, bounded, coercive and symmetric, i.e. the following hold:

$$\begin{cases} (i)\ A \in \mathcal{L}(V, V^*). \\ (ii)\ \langle A\boldsymbol{v}, \boldsymbol{v}\rangle_{V^*\times V} \geq \alpha_1 \|\boldsymbol{v}\|_V^2 \ \textit{for all}\ \boldsymbol{v} \in V\ \textit{with}\ \alpha_1 > 0. \\ (iii)\ \langle A\boldsymbol{v}, \boldsymbol{w}\rangle_{V^*\times V} = \langle A\boldsymbol{w}, \boldsymbol{v}\rangle_{V^*\times V}\ \textit{for all}\ \boldsymbol{v}, \boldsymbol{w} \in V. \end{cases}$$

$H(B)$. The operator $B : V \to V^*$ is linear, bounded and coercive, i.e. the following hold:

$$\begin{cases} (i)\ B \in \mathcal{L}(V, V^*). \\ (ii)\ \langle B\boldsymbol{v}, \boldsymbol{v}\rangle_{V^*\times V} \geq \alpha_2 \|\boldsymbol{v}\|_V^2\ \textit{for all}\ \boldsymbol{v} \in V\ \textit{with}\ \alpha_2 > 0. \end{cases}$$

$H(\Phi)$. The functional $\Phi : V \to \mathbb{R} \cup \{+\infty\}$ is convex, proper and lower semicontinuous.

$H(J)$. The functional $J : Y \times U \to \mathbb{R}$ is such that the following hold:

$$\begin{cases} (i)\ \boldsymbol{u} \mapsto J(y, \boldsymbol{u})\ \textit{is locally Lipschitz for all}\ y \in Y. \\ (ii)\ \|\partial J(y, \boldsymbol{u})\|_{U^*} \leq c(1 + \|\boldsymbol{u}\|_U)\ \textit{for all}\ \boldsymbol{u} \in U\ \textit{with}\ c > 0. \\ (iii)\ (y, \boldsymbol{u}) \mapsto J^0(y, \boldsymbol{u}; z)\ \textit{is upper semicontinuous from}\ Y \times U\ \textit{into}\ \mathbb{R} \\ \qquad \textit{for all}\ z \in U. \\ (iiii)\ \textit{There exists}\ m \geq 0\ \textit{such that} \\ \langle \partial J(y, \boldsymbol{u}) - \partial J(y, \boldsymbol{v}), \boldsymbol{u} - \boldsymbol{v}\rangle_{U^*\times U} \geq -m(\|y_1 - y_2\|_Y + \|\boldsymbol{u} - \boldsymbol{v}\|_U)\|\boldsymbol{u} - \boldsymbol{v}\|_U \\ \textit{for all}\ \boldsymbol{u}, \boldsymbol{v} \in U. \end{cases}$$

$H(\gamma)$. The operator $\gamma : V \to U$ is linear, continuous, and compact. Moreover, the associated Nemytskii operator $\bar{\gamma} : M^{2,2}(0, T; V, V^*) \to \mathcal{U}$ defined by $(\bar{\gamma}\boldsymbol{v})(t) = \bar{\gamma}(\boldsymbol{v}(t))$ for all $t \in [0, T]$ is also compact.

$H(0)$. $\boldsymbol{f} \in H^1(0, T; V^*)$, $\boldsymbol{u}_0 \in dom(\Phi)$, $\beta_0 \in Y$ and the following compatiblity condition holds: there exist $\xi_0 \in \partial J(\beta, \gamma\boldsymbol{u}_0)$ and $\eta_0 \in \partial\Phi(\boldsymbol{u}_0)$ such that

$$B\boldsymbol{u}_0 + \gamma^*\xi_0 + \eta_0 - \boldsymbol{f}(0) \in V.$$

$H(\overline{F})$. $\overline{F} : (0, T) \times U \times Y \to Y$ is such that

$$\begin{cases} (i)\ \overline{F}(\cdot, \boldsymbol{u}, y)\ \textit{is measurable on}\ (0, T)\ \textit{for all}\ \boldsymbol{u} \in U\ \textit{and}\ y \in Y. \\ (ii)\ (\boldsymbol{u}, y) \mapsto \overline{F}(t, \boldsymbol{u}, y)\ \textit{is Lipschiz continuous for a.e.}\ t \in (0, T),\ \textit{i.e.,} \\ \textit{there is a constant}\ L_1 > 0\ \textit{such that for all}\ (\boldsymbol{u}_1, y_1),\ (\boldsymbol{u}_2, y_2) \in U \times Y, \\ \quad \|\overline{F}(t, \boldsymbol{u}_1, y_1) - \overline{F}(t, \boldsymbol{u}_2, y_2)\|_Y \leq L_1(\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_U + \|y_1 - y_2\|_Y) \\ \textit{for a.e.}\ t \in (0, T). \\ (iii)\ \textit{there is a constant}\ L_2 > 0\ \textit{such that for all}\ t_1,\ t_2 \in (0, T), \\ \ \|\overline{F}(t_1, \boldsymbol{u}, y) - \overline{F}(t_2, \boldsymbol{u}, y)\|_Y \leq L_2|t_1 - t_2| \\ \textit{for all}\ (\boldsymbol{u}, y) \in U \times Y. \\ (iiii)\ t \mapsto \overline{F}(t, \boldsymbol{0}, 0)\ \textit{belongs to}\ \mathcal{Y}. \end{cases}$$

$H(s)$. Inequality $\alpha_2 > 2m$ holds, where α_2 and m represent the constants introduced in assumptions $H(B)$ and $H(J)(iiii)$, respectively.

First, we note that hypotheses $H(A)$ and $H(B)$ are easy consequences of $\mathcal{C}$ and $\mathcal{G}$, respectively. Also, assumptions j_τ, j_ν imply the assumption $H(J)$. The assumption $H(\Phi)$ satisfies the hypothesis E. Moreover, we have $\boldsymbol{u}_0 \in K = dom(\Phi)$ and it is easy to see that $\boldsymbol{0}_V \in \partial\Phi(\boldsymbol{u}_0)$. In addition, by the properties of the Clarke subdifferential, the set $\partial J(\beta_0, \gamma\boldsymbol{u}_0)$ is nonempty. By (18) and (22) we obtain $\boldsymbol{f}(0) \in V$, and by $\mathcal{G}$ and (26) we have $B\boldsymbol{u}_0 \in V$. This implies that condition $H(0)$ is satisfied with some $\xi_0 \in \partial J(\beta_0, \gamma\boldsymbol{u}_0)$ and $\eta_0 = \boldsymbol{0}_V$. The definition of the operator F implies $H(\overline{F})$. Moreover, it is well known that the trace operator $\gamma : V \to U$ is linear, continuous and compact. We can easily achieve that the Nemytskii operator is also compact by using the method introduced in [4], where Proposition 2.8 is used, and then $H(\gamma)$ also holds. Finally, (24) implies that condition $H(s)$ is satisfied, too.

Now, we give the following existence and uniqueness result of Problem $\mathcal{Q}^{\mathcal{M}}$.

Theorem 4.1. *Assume that $H(A)$, $H(B)$, $H(J)$, $H(\Phi)$, $H(\gamma)$, $H(0)$, $H(\overline{F})$, $H(s)$ hold, then Problem $\mathcal{Q}^{\mathcal{M}}$ has a unique solution $(\boldsymbol{u}, \beta)$ such that* $\boldsymbol{u} \in H^1(0,T;V)$, $\beta \in H^1(0,T;Y)$.

To prove Theorem 4.1, we begin with some auxiliary results. First, we consider the following nonlinear Cauchy problem

$$\begin{cases} \beta'(t) = F(t, u(t), \beta(t)) \text{ for a.e. } t \in (0,T), \\ \beta(0) = \beta_0, \end{cases} \tag{33}$$

where the function $u \in \mathcal{U}$ is fixed and F satisfies hypotheses $H(\overline{F})$. Now, let us recall the following lemma of the above nonlinear Cauchy problem.

Lemma 4.2. *Assume that $H(\overline{F})$ holds, $\beta_0 \in Y$ and $u \in \mathcal{U}$ are given. Then there exists $\beta \in H^1(0,T;Y)$ a unique solution of problem (33). Moreover, given $u_i \in \mathcal{U}$ and denoting by $\beta_i \in H^1(0,T;Y)$ the unique solution corresponding to u_i for i=1, 2, we have*

$$\begin{aligned} &\|\beta_1(t) - \beta_2(t)\|_Y \le c_\beta \int_0^T \|u_1(s) - u_2(s)\|_U \, ds \text{ for a.e. } t \in (0,T) \\ &\text{with } c_\beta > 0. \end{aligned} \tag{34}$$

We refer the reader to [19, Lemma 8] for more details.

Subsequently, we consider an approximation problem based on the discretization of Problem $\mathcal{Q}^{\mathcal{M}}$. For this discrete problem, also known as the Rothe problem, we prove a result of solvability and obtain the estimates of the solution. To this end, let $N \in \mathbb{N}$ be fixed and let $\tau = \frac{T}{N}$ represent the time step. We consider the piecewise constant approximation of $\boldsymbol{f}$ given by

$$\boldsymbol{f}_\tau(t) = \boldsymbol{f}_\tau^k := \frac{1}{\tau} \int_{(k-1)\tau}^{k\tau} \boldsymbol{f}(s) \, ds \text{ for } t \in ((k-1)\tau, k\tau], \text{ for } k = 1, ..., N. \tag{35}$$

Then using [24, Remark 8.15], we obtain that

$$\boldsymbol{f}_\tau \to \boldsymbol{f} \text{ strongly in } \mathcal{V}^* \text{ as } \tau \to 0. \tag{36}$$

We now formulate the following Rothe problem.

Problem $\mathcal{P}_\tau$. Find a sequence $\{\boldsymbol{u}_\tau^k\}_{k=0}^N \subset V$, $\beta_\tau \in H^1(0,T;Y)$ such that $\boldsymbol{u}_\tau^0 = \boldsymbol{u}_0$, $\beta_\tau^0 = \beta_0$ and

(37)

$$\beta_\tau'(t) = \overline{F}(t, \gamma \hat{\boldsymbol{u}}_\tau(t), \beta_\tau(t)) \text{ for a.e. } t \in (0, t_k),$$

$$\begin{aligned} &\left\langle \frac{1}{\tau} A(\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}), \boldsymbol{v} - \boldsymbol{u}_\tau^k \right\rangle_{V^* \times V} + \langle B\boldsymbol{u}_\tau^k, \boldsymbol{v} - \boldsymbol{u}_\tau^k \rangle_{V^* \times V} + \langle \xi_\tau^k, \gamma(\boldsymbol{v} - \boldsymbol{u}_\tau^k) \rangle_{U^* \times U} \\ &\quad + \Phi(\boldsymbol{v}) - \Phi(\boldsymbol{u}_\tau^k) \ge \langle \boldsymbol{f}_\tau^k, \boldsymbol{v} - \boldsymbol{u}_\tau^k \rangle_{V^* \times V} \text{ for all } \boldsymbol{v} \in V, \end{aligned} \tag{38}$$

where

$$\xi_\tau^k \in \partial J(\beta_\tau^k, \gamma \boldsymbol{u}_\tau^k) \text{ for } k = 1, ..., N, \tag{39}$$

and $\hat{\boldsymbol{u}}_\tau(t)$ for $t \in [0, t_k]$ is defined by

$$\hat{\boldsymbol{u}}_\tau(t) = \begin{cases} \sum\limits_{i=1}^{k} \chi_{(t_{i-1},t_i]}(t)\boldsymbol{u}_\tau^{i-1}, \ t > 0, \\ \boldsymbol{u}_0, \ t = 0. \end{cases}$$

Here, $\chi_{(t_{i-1},t_i]}$ is the characteristic function on the interval $(t_{i-1}, t_i]$, that is,

$$\chi_{(t_{i-1},t_i]}(t) = \begin{cases} 1, \ t \in (t_{i-1}, t_i], \\ 0, \ otherwise. \end{cases}$$

Now, we have the following existence result of the solution to Problem $\mathcal{P}_\tau$.

Lemma 4.3. *Assume $H(A)$, $H(B)$, $H(J)$, $H(\Phi)$, $H(\gamma)$, $H(0)$, $H(\overline{F})$, $H(s)$ hold. Then there exists $\tau_1 > 0$ such that Problem $\mathcal{P}_\tau$ has at least one solution for all $\tau \in (0, \tau_1)$.*

Proof. First, we prove the existence of the solution to equation (37), obviously, we can easily obtain that the function $\hat{\boldsymbol{u}}_\tau(t) = \sum\limits_{i=1}^{k} \chi_{(t_{i-1},t_i]}(t)\boldsymbol{u}_\tau^{i-1}$ is well-defined on $[0, t_k]$ and $\hat{\boldsymbol{u}}_\tau \in L^2(0, t_k; V)$. Now we apply Lemma 4.2 to the nonlinear Cauchy problem (37) on the interval $[0, t_k]$, and we deduce that there exists a unique function $\beta_\tau \in H^1(0, t_k; Y)$ such that (37) is satisfied.

It remains to prove the existence of solution to inequality (38). First, we observe that Problem (38) can be formulated in the following equivalent way: given $\boldsymbol{u}_\tau^{k-1} \in V$ with $k = 1, ..., N-1$, find $\boldsymbol{u}_\tau^k \in V$ such that

$$\frac{1}{\tau}A\boldsymbol{u}_\tau^{k-1} + \boldsymbol{f}_\tau^k \in \partial\Phi(\boldsymbol{u}_\tau^k) + \frac{1}{\tau}A\boldsymbol{u}_\tau^k + B\boldsymbol{u}_\tau^k + \gamma^*\partial J(\beta_\tau^k, \gamma\boldsymbol{u}_\tau^k). \tag{40}$$

In order to solve (40), we apply Proposition 2.7 with $\tilde{F}(\boldsymbol{u}) = \partial\Phi(\boldsymbol{u})$ and $G(\boldsymbol{u}) = \frac{1}{\tau}A\boldsymbol{u} + B\boldsymbol{u} + \gamma^*\partial J(\beta_\tau^k, \gamma\boldsymbol{u})$. To this end, since $\tilde{F}$ represents the subdifferential of the function Φ which is proper, convex and semicontinuous, we can easily deduce that it is a maximal monotone operator. Moreover, the operators $\frac{1}{\tau}A$ and B are linear and monotone, they are pseudomonotone. Then, from Propositions 2.5 and 2.6, it follows that G is a pseudomonotone operator.

Let $\boldsymbol{u}_0$ be the element used as the initial condition in Problem P_τ. Let $\boldsymbol{u}$, ξ, η be such that $\boldsymbol{u} \in D(\tilde{F})$, $\xi \in \tilde{F}(\boldsymbol{u})$, $\eta \in G(\boldsymbol{u})$. We will show that, taking $L = \frac{1}{\tau}A\boldsymbol{u}_\tau^{k-1} + \boldsymbol{f}_\tau^k$, the inequality (40) holds for all $\boldsymbol{u}$ such that $\|\boldsymbol{u}\|_V = R$, where $R \geq \|\boldsymbol{u}_0\|_V$. We have

$$\eta = \frac{1}{\tau}A\boldsymbol{u} + B\boldsymbol{u} + \gamma^*\omega \ with \ \omega \in \partial J(\beta_\tau^k, \gamma\boldsymbol{u}). \tag{41}$$

Our aim in what follows is to show that if $\|\boldsymbol{u}\|_V$ is large enough, there holds the following inequality:

$$\Big\langle \frac{1}{\tau}A\boldsymbol{u} + B\boldsymbol{u} + \gamma^*\omega + \xi, \boldsymbol{u} - \boldsymbol{u}_0 \Big\rangle_{V^* \times V} > \Big\langle \frac{1}{\tau}A\boldsymbol{u}_\tau^{k-1} + \boldsymbol{f}_\tau^k, \boldsymbol{u} - \boldsymbol{u}_0 \Big\rangle_{V^* \times V}. \tag{42}$$

To this end, we note that, using our Definition 2.2 and [20, Lemma 2.5], there exist k_1, $k_2 > 0$ such that

$$\langle \xi, \boldsymbol{u} - \boldsymbol{u}_0 \rangle_{V^* \times V} \geq \Phi(\boldsymbol{u}) - \Phi(\boldsymbol{u}_0) \geq -k_1\|\boldsymbol{u}\|_V - k_2 - \Phi(\boldsymbol{u}_0) \ for \ all \ \boldsymbol{u} \in V. \tag{43}$$

Next, from $H(A)$, $H(B)$, $H(J)$, (43), and the inequality $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$, for a, b, $\varepsilon > 0$, we have

$$\begin{aligned}&\Big\langle \frac{1}{\tau}A\boldsymbol{u} + B\boldsymbol{u} + \gamma^*\omega + \xi, \boldsymbol{u} - \boldsymbol{u}_0\Big\rangle_{V^*\times V} \geq \Big(\frac{1}{\tau}\alpha_1 + \alpha_2 - c\|\gamma\|^2_{\mathcal{L}(V,U)} - 3\varepsilon\Big)\|\boldsymbol{u}\|^2_V \\ &\quad - \frac{1}{4\varepsilon}c^2\|\gamma\|^2_{\mathcal{L}(V,U)} - \frac{1}{4\varepsilon}\Big(\frac{1}{\tau}\|A\|_{\mathcal{L}(V,V^*)} + \|B\|_{\mathcal{L}(V,V^*)} + c\|\gamma\|^2_{\mathcal{L}(V,U)}\Big)^2\|\boldsymbol{u}_0\|^2_V \\ &\quad - c\|\gamma\|_{\mathcal{L}(V,U)}\|\boldsymbol{u}_0\|_V - \frac{k_1^2}{4\varepsilon} - k_2 - \Phi(\boldsymbol{u}_0).\end{aligned} \tag{44}$$

On the other hand, we have

$$\begin{aligned}&\Big\langle \frac{1}{\tau}A\boldsymbol{u}_\tau^{k-1} + \boldsymbol{f}_\tau^k, \boldsymbol{u} - \boldsymbol{u}_0\Big\rangle_{V^*\times V} \\ &\leq \varepsilon\|\boldsymbol{u}\|^2_V + \Big(\frac{1}{4\varepsilon} + \frac{1}{2}\Big)\Big\|\frac{1}{\tau}A\boldsymbol{u}_\tau^{k-1} + \boldsymbol{f}_\tau^k\Big\|^2_{V^*} + \frac{1}{2}\|\boldsymbol{u}_0\|^2_V.\end{aligned} \tag{45}$$

We denote $\delta(\tau) = \frac{1}{\tau}\alpha_1 + \alpha_2 - c\|\gamma\|^2_{\mathcal{L}(V,U)}$ and if $\alpha_2 \geq c\|\gamma\|^2_{\mathcal{L}(V,U)}$, we have $\delta(\tau) > 0$ for all $\tau > 0$. Otherwise, if $\alpha_2 < c\|\gamma\|^2_{\mathcal{L}(V,U)}$, we have $\delta(\tau) > 0$ for all $\tau < \tau_1 := \alpha_1(c\|\gamma\|^2_{\mathcal{L}(V,U)} - \alpha_2)^{-1}$, we conclude that for $\varepsilon = \frac{1}{8}\delta(\tau)$, the value $\delta(\tau) - 4\varepsilon$ is positive. Thus, there exists $R_1 > 0$ such that

$$\begin{aligned}&\Big(\frac{1}{\tau}\alpha_1 + \alpha_2 - c\|\gamma\|^2_{\mathcal{L}(V,U)} - 4\varepsilon\Big)\|\boldsymbol{u}\|^2_V \\ &> \frac{1}{4\varepsilon}\Big(\frac{1}{\tau}\|A\|_{\mathcal{L}(V,V^*)} + \|B\|_{\mathcal{L}(V,V^*)} + c\|\gamma\|^2_{\mathcal{L}(V,U)}\Big)^2\|\boldsymbol{u}_0\|^2_V \\ &+ c\|\gamma\|_{\mathcal{L}(V,U)}\|\boldsymbol{u}_0\|_V + \frac{k_1^2}{4\varepsilon} + k_2 + \Phi(\boldsymbol{u}_0) + \frac{1}{4\varepsilon}c^2\|\gamma\|^2_{\mathcal{L}(V,U)} \\ &+ \Big(\frac{1}{4\varepsilon} + \frac{1}{2}\Big)\Big\|\frac{1}{\tau}A\boldsymbol{u}_\tau^{k-1} + \boldsymbol{f}_\tau^k\Big\|^2_{V^*} + \frac{1}{2}\|\boldsymbol{u}_0\|^2_V,\end{aligned} \tag{46}$$

for all $\boldsymbol{u} \in V$ such that $\|\boldsymbol{u}\|_V \geq R_1$. From (44)-(46), we conclude that (42) holds for all $\boldsymbol{u} \in V$ satisfying $\|\boldsymbol{u}\|_V \geq R_1$. Next, by $H(0)$, we have $\partial\Phi(\boldsymbol{u}_0) \neq \emptyset$, so $\boldsymbol{u}_0 \in D(\tilde{F})$, and therefore $D(\tilde{F}) \cap B_{\|\boldsymbol{u}_0\|_V}(\boldsymbol{0}_X) \neq \emptyset$. We denote $R = max\{R_1, \|\boldsymbol{u}_0\|_V\}$, and note that it satisfies the assumption of Proposition 2.7. We can easily conclude that (40) has a solution. Hence, Problem $\mathcal{P}_\tau$ has at least one solution, the proof is completed. □

Now, let us establish the estimate for the solution of the Rothe problem.

Lemma 4.4. *Assume that $H(A)$, $H(B)$, $H(J)$, $H(\Phi)$, $H(\overline{F})$, $H(\gamma)$, $H(0)$ and $H(s)$ hold. Then, there exist $\tau_2 > 0$ and $C > 0$ independent of τ such that for all $\tau \in (0, \tau_2)$ the solution of Problem $\mathcal{P}_\tau$ satisfies that*

$$\max_{k=1,\ldots,N}\|\boldsymbol{u}_\tau^k\|_V \leq C; \tag{47}$$

$$\sum_{k=1}^N\|\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}\|^2_V \leq C; \tag{48}$$

$$\tau\sum_{k=1}^N\Big\|\frac{\beta_\tau^k - \beta_\tau^{k-1}}{\tau}\Big\|^2_Y \leq C; \tag{49}$$

$$\tau\sum_{k=1}^N\Big\|\frac{\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}}{\tau}\Big\|^2_V \leq C. \tag{50}$$

Proof. We take $\boldsymbol{v}_0 \in dom(\Phi)$ as the test function in (38) and obtain

$$\begin{aligned}
&\Big\langle \frac{1}{\tau}A(\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}), \boldsymbol{u}_\tau^k \Big\rangle_{V^*\times V} + \langle B\boldsymbol{u}_\tau^k, \boldsymbol{u}_\tau^k\rangle_{V^*\times V} + \langle \xi_\tau^k, \gamma \boldsymbol{u}_\tau^k\rangle_{U^*\times U} + \Phi(\boldsymbol{u}_\tau^k)\\
&\quad \le \Big\langle \frac{1}{\tau}A(\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}, \boldsymbol{v}_0)\Big\rangle_{V^*\times V} + \langle B\boldsymbol{u}_\tau^k, \boldsymbol{v}_0\rangle_{V^*\times V}\\
&\quad + \langle \xi_\tau^k, \gamma \boldsymbol{v}_0\rangle_{U^*\times U} + \Phi(\boldsymbol{v}_0) + \langle \boldsymbol{f}_\tau^k, \boldsymbol{u}_\tau^k - \boldsymbol{v}_0\rangle_{V^*\times V}
\end{aligned} \tag{51}$$

with $\xi_\tau^k \in \partial J(\beta_\tau^k, \gamma \boldsymbol{u}_\tau^k)$ and $k = 1, ..., N$.

From $H(A)$ we obtain

$$\begin{aligned}
\langle A(\boldsymbol{u}-\boldsymbol{v}), \boldsymbol{u}\rangle_{V^*\times V} &= \frac{1}{2}\langle A\boldsymbol{u}, \boldsymbol{u}\rangle_{V^*\times V} - \frac{1}{2}\langle A\boldsymbol{v}, \boldsymbol{v}\rangle_{V^*\times V}\\
&+ \frac{1}{2}\langle A(\boldsymbol{u}-\boldsymbol{v}), (\boldsymbol{u}-\boldsymbol{v})\rangle_{V^*\times V} \quad \textit{for all } \boldsymbol{u}, \boldsymbol{v} \in V.
\end{aligned} \tag{52}$$

Then, using this identity as well as assumptions $H(B)$, $H(J)$ and [20, Lemma 2.5], we obtain

$$\begin{aligned}
&\Big\langle \frac{1}{\tau}A(\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}), \boldsymbol{u}_\tau^k \Big\rangle_{V^*\times V} + \langle B\boldsymbol{u}_\tau^k, \boldsymbol{u}_\tau^k\rangle_{V^*\times V} + \langle \xi_\tau^k, \gamma \boldsymbol{u}_\tau^k\rangle_{U^*\times U} + \Phi(\boldsymbol{u}_\tau^k)\\
&\quad \ge \frac{1}{2\tau}\langle A\boldsymbol{u}_\tau^k, \boldsymbol{u}_\tau^k\rangle_{V^*\times V} - \frac{1}{2\tau}\langle A\boldsymbol{u}_\tau^{k-1}, A\boldsymbol{u}_\tau^{k-1}\rangle_{V^*\times V}\\
&\quad + \frac{1}{2\tau}\langle A(\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}), (\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1})\rangle_{V^*\times V} + (\alpha_2 - c\|\gamma\|^2_{\mathcal{L}(V,U)} - 2\varepsilon)\|\boldsymbol{u}_\tau^k\|_V^2\\
&\quad - \frac{1}{4\varepsilon}(c^2\|\gamma\|^2_{\mathcal{L}(V,U)} + k_1^2) - k_2
\end{aligned} \tag{53}$$

where k_1, k_2 are positive constants which do not depend on τ. Moreover, we have the estimate

$$\begin{aligned}
&\Big\langle \frac{1}{\tau}A(\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}), \boldsymbol{v}_0 \Big\rangle_{V^*\times V} + \langle B\boldsymbol{u}_\tau^k, \boldsymbol{v}_0\rangle_{V^*\times V} + \langle \xi_\tau^k, \gamma \boldsymbol{v}_0\rangle_{U^*\times U} + \Phi(\boldsymbol{v}_0)\\
&\quad + \langle \boldsymbol{f}_\tau^k, \boldsymbol{u}_\tau^k - \boldsymbol{v}_0\rangle_{V^*\times V} \le \Big\langle \frac{1}{\tau}A(\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}), \boldsymbol{v}_0 \Big\rangle_{V^*\times V} + 3\varepsilon\|\boldsymbol{u}_\tau^k\|_V^2\\
&\quad + \frac{1}{4\varepsilon}(\|B\|^2_{\mathcal{L}(V,V^*)}\|\boldsymbol{v}_0\|_V^2 + c^2\|\gamma\|^4_{\mathcal{L}(V,U)}\|\boldsymbol{v}_0\ \|_V^2) + c\|\gamma\|_{\mathcal{L}(V,U)}\|\boldsymbol{v}_0\|_V\\
&\quad + \Big(\frac{1}{4\varepsilon} + \frac{1}{2}\Big)\|\boldsymbol{f}_\tau^k\|^2_{V^*} + \Phi(\boldsymbol{v}_0) + \frac{1}{2}\|\boldsymbol{v}_0\|_V^2.
\end{aligned} \tag{54}$$

Next, by combining (51)–(54), we find that

$$\begin{aligned}
&\frac{1}{2}\langle A\boldsymbol{u}_\tau^k, \boldsymbol{u}_\tau^k\rangle_{V^*\times V} + \frac{1}{2}\langle A(\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}), \boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}\rangle_{V^*\times V} + \tau\alpha_2\|\boldsymbol{u}_\tau^k\|_V^2\\
&\quad \le \langle A(\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}), \boldsymbol{v}_0\rangle_{V^*\times V} + \frac{1}{2}\langle A\boldsymbol{u}_\tau^{k-1}, \boldsymbol{u}_\tau^{k-1}\rangle_{V^*\times V}\\
&\quad + \tau(c\|\gamma\|^2_{\mathcal{L}(V,U)} + 5\varepsilon)\|\boldsymbol{u}_\tau^k\|_V^2 + \Big(\frac{1}{4\varepsilon} + \frac{1}{2}\Big)\tau\|\boldsymbol{f}_\tau^k\|^2_{V^*}\\
&\quad + \frac{\tau}{4\varepsilon}(\|B\|^2_{\mathcal{L}(V,V^*)}\|\boldsymbol{v}_0\|_V^2 + c^2\|\gamma\|^4_{\mathcal{L}(V,U)}\|\boldsymbol{v}_0\|_V^2 + c^2\|\gamma\|^2_{\mathcal{L}(V,U)} + k_1^2)\\
&\quad + \tau(\Phi(\boldsymbol{v}_0) + \frac{1}{2}\|\boldsymbol{v}_0\|_V^2 + k_2 + c\|\gamma\|_{\mathcal{L}(V,U)}\|\boldsymbol{v}_0\|_V).
\end{aligned} \tag{55}$$

We define that

$$C(\varepsilon) = \Big(\frac{1}{4\varepsilon} + \frac{1}{2}\Big),$$
$$D(\varepsilon) = \frac{1}{4\varepsilon}(\|B\|^2_{\mathcal{L}(V,V^*)}\|\boldsymbol{v}_0\|^2_V + c^2\|\gamma\|^4_{\mathcal{L}(V,U)}\|\boldsymbol{v}_0\|^2_V + c^2\|\gamma\|^2_{\mathcal{L}(V,U)} + k_1^2)$$
$$+ (\Phi(\boldsymbol{v}_0) + \frac{1}{2}\|\boldsymbol{v}_0\|^2_V + k_2 + c\|\gamma\|_{\mathcal{L}(V,U)}\|\boldsymbol{v}_0\|_V),$$

write (55) for $k = 1, ..., n \leq N$, add the resulting inequalities, and combine assumption $H(A)$, we obtain

$$\frac{1}{2}\alpha_1\|\boldsymbol{u}^n_\tau\|^2_V + \frac{1}{2}\alpha_1\sum_{k=1}^n\|\boldsymbol{u}^k_\tau - \boldsymbol{u}^{k-1}_\tau\|^2_V + \tau\sum_{k=1}^n\alpha_2\|\boldsymbol{u}^k_\tau\|^2_V$$
$$\leq \epsilon\|\boldsymbol{u}^n_\tau\|^2_V + \frac{1}{4\epsilon}\|A\|^2_{\mathcal{L}(V,V^*)}\|\boldsymbol{v}_0\|^2_V + \frac{1}{2}\|A\|_{\mathcal{L}(V,V^*)}\|\boldsymbol{u}^0_\tau\|^2_V$$
$$+ \tau\sum_{k=1}^n(c\|\gamma\|^2_{\mathcal{L}(V,U)} + 5\varepsilon)\|\boldsymbol{u}^k_\tau\|^2_V + C(\varepsilon)\|\boldsymbol{f}_\tau\|^2_{\mathcal{V}^*} + TD(\varepsilon).$$

This inequality implies that

$$\Big(\frac{1}{2}\alpha_1 - \tau c\|\gamma\|^2_{\mathcal{L}(V,U)} - \epsilon - 5\tau\varepsilon\Big)\|\boldsymbol{u}^n_\tau\|^2_V \leq \tau\sum_{k=1}^{n-1}(c\|\gamma\|^2_{\mathcal{L}(V,U)} + 5\varepsilon)\|\boldsymbol{u}^k_\tau\|^2_V$$
$$+ \frac{1}{4\epsilon}\|A\|^2_{\mathcal{L}(V,V^*)}\|\boldsymbol{v}_0\|^2_V + \frac{1}{2}\|A\|_{\mathcal{L}(V,V^*)}\|\boldsymbol{u}^0_\tau\|^2_V + C(\varepsilon)\|\boldsymbol{f}_\tau\|^2_{\mathcal{V}^*} + TD(\varepsilon). \tag{56}$$

Let $\tau_2 = \alpha_1(2c\|\gamma\|^2_{\mathcal{L}(V,U)})^{-1}$, and assume that $\tau < \tau_2$. Then there exist $\epsilon,\ \varepsilon > 0$ such that $(\frac{1}{2}\alpha_1 - \tau c\|\gamma\|^2_{\mathcal{L}(V,U)} - \epsilon - 5\tau\varepsilon) > 0$. On the other hand, from (36) we know that the sequence $\{\boldsymbol{f}_\tau\}$ is bounded in $\mathcal{V}^*$ as $\tau \to 0$. Therefore, we can apply the discrete Gronwall lemma, i.e., [12, Lemma 7.25]. As a result, from (56) we obtain (47). And then, (48) follows from (47) and (56).

We now give the proof of inequalities (49) and (50). To this end, we first give the following Lemma and denote $\delta\boldsymbol{u}^k_\tau = \frac{\boldsymbol{u}^k_\tau - \boldsymbol{u}^{k-1}_\tau}{\tau}$, $\delta\beta^k_\tau = \frac{\beta^k_\tau - \beta^{k-1}_\tau}{\tau}$.

Lemma 4.5. *Assume that $H(\overline{F}), H(\gamma)$ hold. Then, there exist $C_i > 0,\ i = 1, 2, 3$ independent of τ such that the following inequality holds:*

$$\|\delta\beta^k_\tau\|^2_Y \leq C_1\tau\Big(\sum_{i=1}^{k-1}\|\delta\beta^i_\tau\|^2_Y + \sum_{i=1}^{k-1}\|\gamma\delta\boldsymbol{u}^i_\tau\|^2_U\Big) + C_2 + C_3(\|\gamma\boldsymbol{u}^0_\tau\|^2_U + \|\beta^0_\tau\|^2_Y).$$

Proof. Discretize the differential equation (32), we have

$$\frac{\beta^k_\tau - \beta^{k-1}_\tau}{\tau} = \overline{F}(t_{k-1}, \gamma\boldsymbol{u}^{k-1}_\tau, \beta^{k-1}_\tau),$$

that is

$$\beta^k_\tau - \beta^{k-1}_\tau = \tau\sum_{i=1}^{k-1}(\overline{F}(t_i, \gamma\boldsymbol{u}^i_\tau, \beta^i_\tau) - \overline{F}(t_{i-1}, \gamma\boldsymbol{u}^{i-1}_\tau, \beta^{i-1}_\tau)) + \tau\overline{F}(t_0, \gamma\boldsymbol{u}^0_\tau, \beta^0_\tau).$$

From $H(\overline{F})$ and $H(\gamma)$, we have

$$\begin{aligned}
&\overline{F}(t_i,\gamma \boldsymbol{u}_\tau^i,\beta_\tau^i)-\overline{F}(t_{i-1},\gamma \boldsymbol{u}_\tau^{i-1},\beta_\tau^{i-1})\\
&\quad=\overline{F}(t_i,\gamma \boldsymbol{u}_\tau^i,\beta_\tau^i)-\overline{F}(t_i,\gamma \boldsymbol{u}_\tau^{i-1},\beta_\tau^{i-1})\\
&\quad+\overline{F}(t_i,\gamma \boldsymbol{u}_\tau^{i-1},\beta_\tau^{i-1})-\overline{F}(t_{i-1},\gamma \boldsymbol{u}_\tau^{i-1},\beta_\tau^{i-1})\\
&\quad\le L_1(\|\gamma \boldsymbol{u}_\tau^i-\gamma \boldsymbol{u}_\tau^{i-1}\|_U+\|\beta_\tau^i-\beta_\tau^{i-1}\|_Y)+L_2\tau.
\end{aligned}$$

And then

$$\begin{aligned}
\|\beta_\tau^k-\beta_\tau^{k-1}\|_Y\le{}&\tau L_1\sum_{i=1}^{k-1}(\|\gamma \boldsymbol{u}_\tau^i-\gamma \boldsymbol{u}_\tau^{i-1}\|_U+\|\beta_\tau^i-\beta_\tau^{i-1}\|_Y)\\
&+L_2\sum_{i=1}^{k-1}\tau^2+\tau L_1(\|\gamma \boldsymbol{u}_\tau^0\|_U+\|\beta_\tau^0\|_Y).
\end{aligned}$$

We obtain that

$$\|\delta\beta_\tau^k\|_Y^2\le C_1\tau\Big(\sum_{i=1}^{k-1}\|\delta\beta_\tau^i\|_Y^2+\sum_{i=1}^{k-1}\|\gamma\delta \boldsymbol{u}_\tau^i\|_U^2\Big)+C_2+C_3(\|\gamma \boldsymbol{u}_\tau^0\|_U^2+\|\beta_\tau^0\|_Y^2).$$

Here, $C_1=6TL_1^2$, $C_2=3L_2^2T^2$, $C_3=6L_1^2$. □

Next, we take $\xi_0\in\partial J(\beta_0,\gamma \boldsymbol{u}_0),\eta_0\in\partial\Phi(\boldsymbol{u}_0)$ and define $\boldsymbol{u}_\tau^{-1}=\boldsymbol{u}_0+\tau(B\boldsymbol{u}_0+\gamma^*\xi_0+\eta_0-\boldsymbol{f}(0))$. Then it follows that

$$\delta \boldsymbol{u}_\tau^0=\boldsymbol{f}(0)-B\boldsymbol{u}_0-\gamma^*\xi_0-\eta_0. \tag{57}$$

Taking $\boldsymbol{v}=\boldsymbol{u}_\tau^{k-1}$ in (38) we obtain

$$\begin{aligned}
&\tau\langle A\delta \boldsymbol{u}_\tau^k,\delta \boldsymbol{u}_\tau^k\rangle_{V^*\times V}+\tau\langle B\boldsymbol{u}_\tau^k,\delta \boldsymbol{u}_\tau^k\rangle_{V^*\times V}+\tau\langle\xi_\tau^k,\gamma\delta \boldsymbol{u}_\tau^k\rangle_{U^*\times U}\\
&+\Phi(\boldsymbol{u}_\tau^k)-\Phi(\boldsymbol{u}_\tau^{k-1})\le\tau\langle \boldsymbol{f}_\tau^k,\delta \boldsymbol{u}_\tau^k\rangle_{V^*\times V}
\end{aligned}$$

with

$$\xi_\tau^k\in\partial J(\beta_\tau^k,\gamma \boldsymbol{u}_\tau^k).$$

Moreover, replacing k with $k-1$ in (38) and taking $\boldsymbol{v}=\boldsymbol{u}_\tau^k$ to obtain

$$\begin{aligned}
&-\tau\langle A\delta \boldsymbol{u}_\tau^{k-1},\delta \boldsymbol{u}_\tau^k\rangle_{V^*\times V}-\tau\langle B\boldsymbol{u}_\tau^{k-1},\delta \boldsymbol{u}_\tau^k\rangle_{V^*\times V}\\
&-\tau\langle\xi_\tau^{k-1},\gamma\delta \boldsymbol{u}_\tau^k\rangle_{U^*\times U}+\Phi(\boldsymbol{u}_\tau^{k-1})-\Phi(\boldsymbol{u}_\tau^k)\le-\tau\langle \boldsymbol{f}_\tau^{k-1},\delta \boldsymbol{u}_\tau^k\rangle_{V^*\times V}
\end{aligned}$$

with

$$\xi_\tau^{k-1}\in\partial J(\beta_\tau^{k-1},\gamma \boldsymbol{u}_\tau^{k-1}).$$

Now, we add the last two inequalities and use (52), $H(B)$, $H(J)(iiii)$ to obtain

$$\begin{aligned}
&\frac{1}{2}\langle A\delta \boldsymbol{u}_\tau^k,\delta \boldsymbol{u}_\tau^k\rangle_{V^*\times V}+\frac{1}{2}\langle A(\delta \boldsymbol{u}_\tau^k-\delta \boldsymbol{u}_\tau^{k-1}),\delta \boldsymbol{u}_\tau^k-\delta \boldsymbol{u}_\tau^{k-1}\rangle_{V^*\times V}\\
&+\tau(\alpha_2-m-\varepsilon)\|\delta \boldsymbol{u}_\tau^k\|_V^2\le\frac{1}{2}\langle A\delta \boldsymbol{u}_\tau^{k-1},\delta \boldsymbol{u}_\tau^{k-1}\rangle_{V^*\times V}\\
&+\frac{1}{4\varepsilon}\frac{1}{\tau}\|\boldsymbol{f}_\tau^k-\boldsymbol{f}_\tau^{k-1}\|_{V^*}^2+m\tau\|\delta \boldsymbol{u}_\tau^k\|_V\|\delta\beta_\tau^k\|_Y.
\end{aligned} \tag{58}$$

We now write (58) for $k = 1, ..., n \leq N$; Moreover, we add the resulting inequalities, use Lemma 4.5 to obtain

$$
\begin{aligned}
&\frac{1}{2}\alpha_1\|\delta \boldsymbol{u}_\tau^n\|_V^2 + \sum_{k=1}^{n} \tau(\alpha_2 - m - \varepsilon - m\varepsilon)\|\delta \boldsymbol{u}_\tau^k\|_V^2 + \tau \sum_{k=1}^{n} \|\delta\beta_\tau^k\|_Y^2 \\
&\leq \Big(C_1\tau^2 + \frac{C_1 m\tau^2}{4\varepsilon}\Big)\sum_{k=1}^{n}\Big(\sum_{i=1}^{k-1}\|\delta \boldsymbol{u}_\tau^i\|_V^2 + \sum_{i=1}^{k-1}\|\delta\beta_\tau^i\|_Y^2\Big) + \frac{1}{4\varepsilon}\|\boldsymbol{f}'\|_{\mathcal{V}^*}^2 \\
&+ \frac{mC_2T}{4\varepsilon} + \Big(C_3T + \frac{C_3 mT}{4\varepsilon}\Big)(\|\boldsymbol{u}_\tau^0\|_V^2 + \|\beta_\tau^0\|_Y^2) + C_2T \\
&+ 2\|A\|_{\mathcal{L}(V,V^*)}(\|\boldsymbol{f}(0)\|^2 + \|B\|_{\mathcal{L}(V,V^*)}^2\|\boldsymbol{u}_0\|_V^2 + \|\gamma\|_{\mathcal{L}(V,U)}^2\|\xi_0\|_V^2 + \|\eta_0\|_V^2).
\end{aligned} \tag{59}
$$

Estimates (49) and (50) are now direct consequences of the discrete Gronwall lemma, i.e., [12, Lemma 7.25], (59) and assumption $H(s)$. □

Now, let us give the proof of Theorem 4.1. To this end, we use the estimates obtained above in the study of the Rothe problem. For a given index $\tau > 0$, We first define the piecewise linear and piecewise constant interpolant functions $\boldsymbol{u}_\tau : (0, T) \to V$ and $\bar{\boldsymbol{u}}_\tau : (0, T) \to V$, respectively,

$$
\bar{\boldsymbol{u}}_\tau(t) = \begin{cases} \boldsymbol{u}_\tau^k, \ t \in ((k-1)\tau, k\tau], \ k = 1, ..., N, \\ \boldsymbol{u}_\tau^0, \ t = 0, \end{cases} \tag{60}
$$

and

$$
\boldsymbol{u}_\tau(t) = \boldsymbol{u}_\tau^k + \Big(\frac{t}{\tau} - k\Big)(\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}), \ t \in ((k-1)\tau, k\tau], \ k = 1, ..., N. \tag{61}
$$

Here the sequence $\{\boldsymbol{u}_\tau^k\}_{k=0}^N$ is a solution of Problem P_τ, obtained under the assumptions of Lemma 4.3. In addition, we consider the piecewise constant interpolant $\bar{\xi}_\tau : (0, T) \to U^*$ given by

$$
\bar{\xi}_\tau(t) = \xi_\tau^k, \ t \in ((k-1)\tau, k\tau], \ k = 1, ..., N, \tag{62}
$$

where the sequence $\{\xi_\tau^k\}_{k=0}^N$ satisfies (39). Then we note that (38) can be written equivalently, as

$$
\begin{aligned}
&\langle A\boldsymbol{u}_\tau'(t) + B\bar{\boldsymbol{u}}_\tau(t) - \boldsymbol{f}_\tau(t), \boldsymbol{v}(t) - \bar{\boldsymbol{u}}_\tau(t)\rangle_{V^*\times V} + \langle \bar{\xi}_\tau(t), \bar{\gamma}(\boldsymbol{v}(t) - \bar{\boldsymbol{u}}_\tau(t))\rangle_{U^*\times U} \\
&+ \Phi(\boldsymbol{v}(t)) - \Phi(\bar{\boldsymbol{u}}_\tau(t)) \geq 0 \quad \textit{for all } \boldsymbol{v} \in \mathcal{V} \textit{ a.e. } t \in (0, T).
\end{aligned} \tag{63}
$$

We now define the Nemytskii operators $\mathcal{A}, \ \mathcal{B} : \mathcal{V} \to \mathcal{V}^*$ as $(\mathcal{A}\boldsymbol{w})(t) = A(\boldsymbol{w}(t))$ and $(\mathcal{B}\boldsymbol{w})(t) = B(\boldsymbol{w}(t))$ for all $\boldsymbol{w} \in \mathcal{V}$ and all $t \in (0, T)$. Thus, from (63) we get

$$
\begin{aligned}
&\langle \mathcal{A}\boldsymbol{u}_\tau' + \mathcal{B}\bar{\boldsymbol{u}}_\tau - \boldsymbol{f}_\tau, \boldsymbol{v} - \bar{\boldsymbol{u}}_\tau\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle \bar{\xi}_\tau, \bar{\gamma}(\boldsymbol{v} - \bar{\boldsymbol{u}}_\tau)\rangle_{\mathcal{U}^*\times\mathcal{U}} \\
&+ \int_0^T \Phi(\boldsymbol{v}(t)) - \Phi(\bar{\boldsymbol{u}}_\tau(t))\, dt \geq 0 \ \ \textit{for all } \boldsymbol{v} \in \mathcal{V},
\end{aligned} \tag{64}
$$

where $\bar{\gamma}$ is the Nemytskii operator introduced in assumption $H(\gamma)$. Let τ_2 be the constant obtained in the proof of Lemma 4.4 and then we obtain the following result.

Lemma 4.6. *Assume that $H(A)$, $H(B)$, $H(J)$, $H(\Phi)$, $H(\overline{F})$, $H(0)$, $H(\gamma)$ and $H(s)$ hold. Then for all $\tau \in (0,\tau_2)$, the functions defined by (60)–(62) satisfy*

$$\|\bar{\boldsymbol{u}}_\tau\|_{L^\infty(0,T;V)} \le C, \tag{65}$$

$$\|\bar{\boldsymbol{u}}_\tau\|_{M^{2,2}(0,T;V,V^*)} \le C, \tag{66}$$

$$\|\boldsymbol{u}_\tau\|_{C(0,T;V)} \le C, \tag{67}$$

$$\|\boldsymbol{u}'_\tau\|_{\mathcal{V}} \le C, \tag{68}$$

$$\|\bar{\xi}_\tau\|_{\mathcal{U}^*} \le C, \tag{69}$$

with positive constant C which is not dependent on τ.

Proof. Since the estimate (47) holds for all $\tau \in (0,\tau_2)$, the estimates (65) and (67) follow directly from (47). Next, from (65), we see that the sequence $\{\bar{\boldsymbol{u}}_\tau\}$ remains bounded in $\mathcal{V}$. Thus, in order to establish the estimate for (66), we only need to estimate $\|\bar{\boldsymbol{u}}_\tau\|_{BV^2(0,T;V^*)}$. To do this, we consider a division $0 = a_0 < a_1 <, ..., < a_n = T$, where $a_i \in ((m_i-1)\tau, m_i\tau]$ is such that $\bar{\boldsymbol{u}}_\tau(a_i) = \boldsymbol{u}_\tau^{m_i}$ with $m_0 = 0,\ m_n = N$, and $m_{i+1} > m_i$ for $i = 1, ..., N-1$. Then

$$\begin{aligned}\|\bar{\boldsymbol{u}}_\tau\|^2_{BV^2(0,T;V^*)} &= \sum_{i=1}^n \|\boldsymbol{u}_\tau^{m_i} - \boldsymbol{u}_\tau^{m_{i-1}}\|^2_{V^*}\\ &\le \sum_{i=1}^n \Big((m_i - m_{i-1}) \sum_{k=m_{i-1}+1}^{m_i} \|\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}\|^2_{V^*}\Big)\\ &\le \Big(\sum_{i=1}^n (m_i - m_{i-1})\Big)\Big(\sum_{i=1}^n \sum_{k=m_{i-1}+1}^{m_i} \|\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}\|^2_{V^*}\Big)\\ &= N\sum_{k=1}^N \|\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}\|^2_{V^*} = T\tau \sum_{k=1}^N \Big\|\frac{\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}}{\tau}\Big\|^2_{V^*}\\ &\le C\tau \sum_{k=1}^N \Big\|\frac{\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}}{\tau}\Big\|^2_{V}.\end{aligned} \tag{70}$$

It is enough to see that $\|\bar{\boldsymbol{u}}_\tau\|^2_{BV^2(0,T;V^*)}$ is bounded from (50) and (70), moreover, we conclude from here that (66) holds. To prove (68), we observe that

$$\|\boldsymbol{u}'_\tau\|^2_{\mathcal{V}} = \tau \sum_{k=1}^N \Big\|\frac{\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}}{\tau}\Big\|^2_V,$$

then we use (50) to obtain the estimates. Finally, for the proof of (69) we note that

$$\begin{aligned}\|\bar{\xi}_\tau\|^2_{\mathcal{U}^*} &= \sum_{k=1}^N \tau\|\xi_\tau^k\|^2_{U^*} \le \tau \sum_{k=1}^N c^2(1 + \|\gamma \boldsymbol{u}_\tau^k\|_U)^2\\ &\le 2Tc^2 + 2c^2\tau\|\gamma\|^2_{\mathcal{L}(V,U)} \sum_{k=1}^N \|\boldsymbol{u}_\tau^k\|^2_V \le C.\end{aligned}$$

We complete the proof of the lemma. □

Now, we have all the ingredients to prove Theorem 4.1.

Proof. First, from Lemma 4.3 we can see that, for $\tau > 0$ small enough, there exists a solution to Problem P_τ. We consider such a solution, and we use it to construct the functions $\boldsymbol{u}_\tau$, $\bar{\boldsymbol{u}}_\tau$ *and* $\bar{\xi}_\tau$ denoted by (60)-(62). For a subsequence which we still denote by τ, we claim that there exists $(\boldsymbol{u}, \xi, \beta) \in \mathcal{V} \times \mathcal{U}^* \times H^1(0,T;Y)$ such that, the following results hold:

$$\bar{\boldsymbol{u}}_\tau \to \boldsymbol{u} \quad \textit{weakly in } \mathcal{V}, \tag{71}$$

$$\bar{\gamma}\bar{\boldsymbol{u}}_\tau \to \bar{\gamma}\boldsymbol{u} \quad \textit{strongly in } \mathcal{U}, \tag{72}$$

$$\bar{\xi}_\tau \to \xi \quad \textit{weakly in } \mathcal{U}^*, \tag{73}$$

$$\boldsymbol{u}_\tau \to \boldsymbol{u} \quad \textit{weakly in } \mathcal{V}, \tag{74}$$

$$\boldsymbol{u}'_\tau \to \boldsymbol{u}' \quad \textit{weakly in } \mathcal{V}, \tag{75}$$

$$\beta_\tau \to \beta \quad \textit{in } H^1(0,T;Y). \tag{76}$$

First, from the estimate (65), the continuous embedding $L^\infty(0,T;V) \subset \mathcal{V}$, and the reflexivity of the space $\mathcal{V}$, the convergence (71) holds. The convergence (72) follows from the estimate (66) and assumption $H(\gamma)$. The bound (69) and the reflexivity of the space $\mathcal{U}^*$ imply the convergence (73). For the proof of (74), we recall that (67) implies that the sequence $\{\boldsymbol{u}_\tau\}$ is bounded in $\mathcal{V}$, and therefore we can assume that there exists $\boldsymbol{u}_1 \in \mathcal{V}$ such that $\boldsymbol{u}_\tau \to \boldsymbol{u}_1$ weakly in $\mathcal{V}$ as $\tau \to 0$. Thus, from (71) it follows that $\bar{\boldsymbol{u}}_\tau - \boldsymbol{u}_\tau \to \boldsymbol{u} - \boldsymbol{u}_1$ weakly in $\mathcal{V}$ as $\tau \to 0$. On the other hand, we note that

$$\|\bar{\boldsymbol{u}}_\tau - \boldsymbol{u}_\tau\|^2_{\mathcal{V}} = \sum_{k=1}^{N} \int_{(k-1)\tau}^{k\tau} (k\tau - t)^2 \left\| \frac{\boldsymbol{u}^k_\tau - \boldsymbol{u}^{k-1}_\tau}{\tau} \right\|^2_V dt = \frac{\tau^2}{3} \|\boldsymbol{u}'_\tau\|^2_{\mathcal{V}},$$

and therefore, follows from the bound (68) of $\{\boldsymbol{u}'_\tau\}$, we have $\boldsymbol{u} = \boldsymbol{u}_1$. From the above, we can conclude that (74) holds. The convergence (75) holds following from the same bound (68). Finally, thanks to [19], the convergence (76) holds.

Now, we show that $(\boldsymbol{u}, \xi, \beta) \in \mathcal{V} \times \mathcal{U}^* \times H^1(0,T;Y)$ is a solution of Problem $\mathcal{Q}^{\mathcal{M}}$. To this end, we start with passing to the limit in the initial condition. Since the embedding $\{\boldsymbol{v} \in \mathcal{V},\ \boldsymbol{v}' \in \mathcal{V}\} \subset C(0,T;V)$ is continuous, we have $\boldsymbol{u}_\tau \to \boldsymbol{u}$ weakly in $C(0,T;V)$ as $\tau \to 0$ following from (74) and (75). In particular, we have

$$\boldsymbol{u}_\tau(t) \to \boldsymbol{u}(t) \ \ \textit{weakly in } V, \ \ \textit{as } \tau \to 0 \ \ \textit{for all } t \in [0,T]. \tag{77}$$

Similarly, we have

$$\beta_\tau(t) \to \beta(t) \ \ \textit{in } Y, \ \ \textit{as } \tau \to 0 \ \ \textit{for all } t \in [0,T]. \tag{78}$$

Hence, since $\boldsymbol{u}_\tau(0) = \boldsymbol{u}_0,\ \beta_\tau(0) = \beta_0$ for all $\tau > 0$, we get $\boldsymbol{u}(0) = \boldsymbol{u}_0,\ \beta(0) = \beta_0$. Now as $\tau \to 0$, we pass to the limit in (64). For $\boldsymbol{v} \in \mathcal{V}$, we calculate

$$\langle \mathcal{A}\boldsymbol{u}'_\tau, \boldsymbol{v} - \bar{\boldsymbol{u}}_\tau \rangle_{\mathcal{V}^* \times \mathcal{V}} = \langle \mathcal{A}\boldsymbol{u}'_\tau, \boldsymbol{v} \rangle_{\mathcal{V}^* \times \mathcal{V}} - \langle \mathcal{A}\boldsymbol{u}'_\tau, \boldsymbol{u}_\tau \rangle_{\mathcal{V}^* \times \mathcal{V}} + \langle \mathcal{A}\boldsymbol{u}'_\tau, \boldsymbol{u}_\tau - \bar{\boldsymbol{u}}_\tau \rangle_{\mathcal{V}^* \times \mathcal{V}}. \tag{79}$$

Since $\mathcal{A}$ is linear and continuous, it is weakly continuous. So from (75) we have $\mathcal{A}\boldsymbol{u}'_\tau \to \mathcal{A}\boldsymbol{u}_\tau$ weakly in $\mathcal{V}^*$, i.e.,

$$\lim_{\tau \to 0} \langle \mathcal{A}\boldsymbol{u}'_\tau, \boldsymbol{v} \rangle_{\mathcal{V}^* \times \mathcal{V}} = \langle \mathcal{A}\boldsymbol{u}', \boldsymbol{v} \rangle_{\mathcal{V}^* \times \mathcal{V}}. \tag{80}$$

On the other hand, we have

$$\begin{aligned} &\limsup_{\tau \to 0} (-\langle \mathcal{A}\boldsymbol{u}'_\tau, \boldsymbol{u}_\tau \rangle_{\mathcal{V}^* \times \mathcal{V}}) \\ &= \limsup_{\tau \to 0} \Big(\frac{1}{2} \langle A\boldsymbol{u}_\tau(0), \boldsymbol{u}_\tau(0) \rangle_{V^* \times V} - \frac{1}{2} \langle A\boldsymbol{u}_\tau(T), \boldsymbol{u}_\tau(T) \rangle_{V^* \times V} \Big) \\ &= \frac{1}{2} \langle A\boldsymbol{u}(0), \boldsymbol{u}(0) \rangle_{V^* \times V} - \liminf_{\tau \to 0} \frac{1}{2} \langle A\boldsymbol{u}_\tau(T), \boldsymbol{u}_\tau(T) \rangle_{V^* \times V}. \end{aligned} \tag{81}$$

Moreover, we observe that the functional $V \ni \boldsymbol{v} \to \langle A\boldsymbol{v}, \boldsymbol{v}\rangle_{V^*\times V}$ is continuous and convex, therefore, it is weakly lower semicontinuous. Thus, (77) yields

$$\langle A\boldsymbol{u}(T), \boldsymbol{u}(T)\rangle_{V^*\times V} \leq \liminf_{\tau\to 0}\langle A\boldsymbol{u}_\tau(T), \boldsymbol{u}_\tau(T)\rangle_{V^*\times V},$$

we combine this inequality with (81), and obtain

$$\begin{aligned}&\limsup_{\tau\to 0}(-\langle \mathcal{A}\boldsymbol{u}'_\tau, \boldsymbol{u}_\tau\rangle_{\mathcal{V}^*\times\mathcal{V}}) \\ &\leq \frac{1}{2}\langle A\boldsymbol{u}(0), \boldsymbol{u}(0)\rangle_{V^*\times V} - \frac{1}{2}\langle A\boldsymbol{u}(T), \boldsymbol{u}(T)\rangle_{V^*\times V} = -\langle \mathcal{A}\boldsymbol{u}', \boldsymbol{u}\rangle_{\mathcal{V}^*\times\mathcal{V}}.\end{aligned} \tag{82}$$

Next, by using an calculus, we obtain

$$\begin{aligned}&\langle \mathcal{A}\boldsymbol{u}'_\tau, \boldsymbol{u}_\tau - \bar{\boldsymbol{u}}_\tau\rangle_{\mathcal{V}^*\times\mathcal{V}} = \sum_{k=1}^{N}\int_{\tau(k-1)}^{\tau k}\langle \mathcal{A}\boldsymbol{u}'_\tau(t), \boldsymbol{u}_\tau(t) - \bar{\boldsymbol{u}}_\tau(t)\rangle_{V^*\times V}\; dt \\ &= \sum_{k=1}^{N}\int_{\tau(k-1)}^{\tau k}\left\langle A\frac{1}{\tau}(\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}), \Big(\frac{t}{\tau} - k\Big)(\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1})\right\rangle_{V^*\times V} dt \\ &= \sum_{k=1}^{N}\frac{1}{\tau}\langle A(\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}), \boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}\rangle_{V^*\times V}\int_{\tau(k-1)}^{\tau k}\Big(\frac{t}{\tau} - k\Big)\; dt. \\ &= -\sum_{k=1}^{N}\frac{1}{2}\langle A(\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}), (\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1})\rangle_{V^*\times V} \leq 0\end{aligned} \tag{83}$$

Combining this inequality with (79), (80), and (82), we obtain that

$$\limsup_{\tau\to 0}\langle \mathcal{A}\boldsymbol{u}'_\tau, \boldsymbol{v} - \bar{\boldsymbol{u}}_\tau\rangle_{\mathcal{V}^*\times\mathcal{V}} \leq \langle \mathcal{A}\boldsymbol{u}', \boldsymbol{v} - \boldsymbol{u}\rangle_{\mathcal{V}^*\times\mathcal{V}}. \tag{84}$$

Next, let us estimate

$$\limsup_{\tau\to 0}\langle \mathcal{B}\bar{\boldsymbol{u}}_\tau, \boldsymbol{v} - \bar{\boldsymbol{u}}_\tau\rangle_{\mathcal{V}^*\times\mathcal{V}} \leq \limsup_{\tau\to 0}\langle \mathcal{B}\bar{\boldsymbol{u}}_\tau, \boldsymbol{v}\rangle_{\mathcal{V}^*\times\mathcal{V}} - \liminf_{\tau\to 0}\langle \mathcal{B}\bar{\boldsymbol{u}}_\tau, \bar{\boldsymbol{u}}_\tau\rangle_{\mathcal{V}^*\times\mathcal{V}}. \tag{85}$$

Since the operator $\mathcal{B}$ is weakly continuous, from (75) we have $\mathcal{B}\boldsymbol{u}'_\tau \to \mathcal{B}\boldsymbol{u}_\tau$ *weakly in* $\mathcal{V}^*$, i.e.,

$$\limsup_{\tau\to 0}\langle \mathcal{B}\bar{\boldsymbol{u}}_\tau, \boldsymbol{v}\rangle_{\mathcal{V}^*\times\mathcal{V}} = \lim_{\tau\to 0}\langle \mathcal{B}\bar{\boldsymbol{u}}_\tau, \boldsymbol{v}\rangle_{\mathcal{V}^*\times\mathcal{V}} = \langle \mathcal{B}\boldsymbol{u}, \boldsymbol{v}\rangle_{\mathcal{V}^*\times\mathcal{V}}. \tag{86}$$

By using a lower semicontinuity argument, it follows that

$$\liminf_{\tau\to 0}\langle \mathcal{B}\bar{\boldsymbol{u}}_\tau, \bar{\boldsymbol{u}}_\tau\rangle_{\mathcal{V}^*\times\mathcal{V}} \geq \langle \mathcal{B}\boldsymbol{u}, \boldsymbol{u}\rangle_{\mathcal{V}^*\times\mathcal{V}}, \tag{87}$$

and, combining this inequality with (85) and (86), we see that

$$\limsup_{\tau\to 0}\langle \mathcal{B}\bar{\boldsymbol{u}}_\tau, \boldsymbol{v} - \bar{\boldsymbol{u}}_\tau\rangle_{\mathcal{V}^*\times\mathcal{V}} \leq \langle \mathcal{B}\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}\rangle_{\mathcal{V}^*\times\mathcal{V}}. \tag{88}$$

The next, from (36) and (71) we have

$$\langle \boldsymbol{f}_\tau, \boldsymbol{v} - \bar{\boldsymbol{u}}_\tau\rangle_{\mathcal{V}^*\times\mathcal{V}} \to \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}\rangle_{\mathcal{V}^*\times\mathcal{V}} \;\; as\; \tau \to 0. \tag{89}$$

Moreover, combining (72) and (73), we obtain that

$$\langle \bar{\xi}_\tau, \bar{\gamma}(\boldsymbol{v} - \bar{\boldsymbol{u}}_\tau)\rangle_{\mathcal{U}^*\times\mathcal{U}} \to \langle \xi, \bar{\gamma}(\boldsymbol{v} - \boldsymbol{u})\rangle_{\mathcal{U}^*\times\mathcal{U}} \;\; as\; \tau \to 0. \tag{90}$$

Finally, we define such a functional $\Psi : \mathcal{V} \to \mathbb{R} \cup \{+\infty\}$ as

$$\Psi(\boldsymbol{v}) = \int_0^T \Phi(\boldsymbol{v}(t))\; dt \quad for\; all\; \boldsymbol{v} \in \mathcal{V}.$$

We now show that Ψ is lower semicontinuous. Let $\boldsymbol{u}_n \to \boldsymbol{u}$ strongly in $\mathcal{V}$, by using [20, Lemma 2.5], there exist $k_1, k_2 \in \mathbb{R}$ such that for $\boldsymbol{v} \in V$ we have $\Phi(\boldsymbol{v}) \geq k_1 \|\boldsymbol{v}\|_V + k_2$, which implies that

$$\int_0^T \Phi(\boldsymbol{u}_n(t))\, dt \geq k_2 T + k_1 \int_0^T \|\boldsymbol{u}_n(t)\|_V\, dt \geq k_2 T - |k_1| \sqrt{T} \|\boldsymbol{u}_n\|_{\mathcal{V}} \geq \bar{k}, \tag{91}$$

where $\bar{k}$ is a positive constant which does not depend on n. Now consider a convergent subsequence of $\Psi(\boldsymbol{u}_n)$, i.e., $\Psi(\boldsymbol{u}_n) \to M$ as $n \to \infty$. And then, for a subsequence, we have $\boldsymbol{u}_{n_k}(t) \to \boldsymbol{u}(t)$ strongly in V for a.e. $t \in (0,T)$. We have

$$\Phi(\boldsymbol{u}(t)) \leq \liminf_{n\to\infty} \Phi(\boldsymbol{u}_{n_k}(t))\ a.e.\ t \in (0,T)$$

from the lower semicontinuity of Φ. As for (91), we use the Fatou lemma to obtain

$$\int_0^T \Phi(\boldsymbol{u}(t))dt \leq \int_0^T \liminf_{n_k\to\infty} \Phi(\boldsymbol{u}_{n_k}(t))dt \leq \liminf_{n_k\to\infty} \int_0^T \Phi(\boldsymbol{u}_{n_k}(t))\, dt = M,$$

which shows that Ψ is a lower semicontinuous function. And then, since Φ is convex, Ψ is obviously convex, as a result, it is weakly sequentially lower semicontinuous. From (71) we have

$$\Psi(\boldsymbol{u}) \leq \liminf_{\tau\to 0} \Psi(\bar{\boldsymbol{u}}_\tau),$$

and

$$\limsup_{\tau\to 0} \int_0^T (\Phi(\boldsymbol{v}(t)) - \Phi(\bar{\boldsymbol{u}}_\tau(t)))\, dt \leq \int_0^T (\Phi(\boldsymbol{v}(t)) - \Phi(\boldsymbol{u}(t)))\, dt. \tag{92}$$

To conclude the proof, we now show that $\xi(t) \in \partial J(\beta(t), \gamma\boldsymbol{u}(t))\ for\ a.e.\ t \in (0,T)$. Since $\bar{\xi}_\tau(t) \in \partial J(\beta_\tau(t), \gamma\bar{\boldsymbol{u}}_\tau(t))\ for\ a.e.\ t \in (0,T)$, from (72)–(73), (76), the upper semicontinuity of ∂J (cf. [19, Lemma 7]) and the convergence theorem of Aubin and Cellina (cf. [15]), we have

$$\xi(t) \in \partial J(\beta(t), \gamma\boldsymbol{u}(t))\ for\ a.e.\ t \in (0,T).$$

We deduce that $(\boldsymbol{u}, \beta)$ is a solution of Problem $\mathcal{Q}^{\mathcal{M}}$, moreover, $\boldsymbol{u} \in H^1(0,T;V)$, $\beta \in H^1(0,T;Y)$ which completes the proof of the existence part.

Finally, let us focus on the proof of the uniqueness.

Assume that $\boldsymbol{u}_1, \boldsymbol{u}_2$ are two distinct solutions to the Problem $\mathcal{Q}^{\mathcal{M}}$. We have for $\boldsymbol{v} \in \mathcal{V}$ and a.e. $t \in (0,T)$

$$\begin{aligned} &\langle A\boldsymbol{u}_1'(t) + B\boldsymbol{u}_1(t) + \gamma^*\xi_1(t) - \boldsymbol{f}(t), \boldsymbol{v}(t) - \boldsymbol{u}_1(t)\rangle_{V^*\times V} \\ &\quad + \Phi(\boldsymbol{v}(t)) - \Phi(\boldsymbol{u}_1(t)) \geq 0\ \ with\ \xi_1(t) \in \partial J(\beta_1(t), \gamma\boldsymbol{u}_1(t)). \end{aligned} \tag{93}$$

$$\begin{aligned} &\langle A\boldsymbol{u}_2'(t) + B\boldsymbol{u}_2(t) + \gamma^*\xi_2(t) - \boldsymbol{f}(t), \boldsymbol{v}(t) - \boldsymbol{u}_2(t)\rangle_{V^*\times V} \\ &\quad + \Phi(\boldsymbol{v}(t)) - \Phi(\boldsymbol{u}_2(t)) \geq 0\ \ with\ \xi_2(t) \in \partial J(\beta_2(t), \gamma\boldsymbol{u}_2(t)). \end{aligned} \tag{94}$$

In (93) and (94), we take $\boldsymbol{v} = \boldsymbol{u}_2$, $\boldsymbol{v} = \boldsymbol{u}_1$ respectively and then add two inequalities, we have

$$\begin{aligned} &\langle A\boldsymbol{u}_1'(t) - A\boldsymbol{u}_2'(t), \boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\rangle_{V^*\times V} + \langle B\boldsymbol{u}_1(t) - B\boldsymbol{u}_2(t), \boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\rangle_{V^*\times V} \\ &\quad + \langle \xi_1(t) - \xi_2(t), \gamma\boldsymbol{u}_1(t) - \gamma\boldsymbol{u}_2(t)\rangle_{U^*\times U} \leq 0 \\ &\quad with\ \xi_i(t) \in \partial J(\beta_i(t), \gamma\boldsymbol{u}_i(t)),\ i = 1, 2. \end{aligned} \tag{95}$$

Using now $H(A)$, $H(B)$, $H(J)$ and (95) yields

$$\begin{aligned}&\alpha_1\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2 \le \langle A(\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)), \boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\rangle_{V^*\times V}\\ &+\int_0^t \alpha_2\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2\,dt \le \|\gamma\|^2_{\mathcal{L}(V,U)}\int_0^t m\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2\,ds\\ &+\|\gamma\|_{\mathcal{L}(V,U)}\int_0^t m\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V\|\beta_1(s)-\beta_2(s)\|_Y\,ds\\ &+\langle A(\boldsymbol{u}_1(0)-\boldsymbol{u}_2(0)), \boldsymbol{u}_1(0)-\boldsymbol{u}_2(0)\rangle_{V^*\times V}.\end{aligned} \tag{96}$$

From Lemma 4.2, we have

$$\|\beta_1(t)-\beta_2(t)\|_Y \le c_\beta\int_0^t \|\gamma\|\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V\ ds. \tag{97}$$

Moreover, we have $\boldsymbol{u}_1(0)=\boldsymbol{u}_2(0)=\boldsymbol{u}_0$, and therefore (96), (97) imply that

$$\begin{aligned}&\alpha_1\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2+\|\beta_1(t)-\beta_2(t)\|_Y^2\\ &\le \|\gamma\|^2_{\mathcal{L}(V,U)}(m+\frac{m}{4\varepsilon}+c_\beta)\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2\ ds\\ &+\varepsilon m\int_0^t\|\beta_1(s)-\beta_2(s)\|_Y^2\ ds.\end{aligned} \tag{98}$$

We now use the Gronwall lemma, i.e. [12, Lemma 7.24] to conclude that $\boldsymbol{u}_1(t)=\boldsymbol{u}_2(t)$, $\beta_1(t)=\beta_2(t)$ for a.e. $t\in(0,T)$, this completes the proof. □

A triple of functions $(\boldsymbol{u},\boldsymbol{\sigma},\beta)$ such that $(\boldsymbol{u},\beta)$ is a solution of Problem $\mathcal{P}^{\mathcal{M}}$ and $\boldsymbol{\sigma}$ is given by the constitutive law (1) is called a weak solution to Problem $\mathcal{P}$. From the proof of Theorem 4.1, we obtain that Problem $\mathcal{Q}^{\mathcal{M}}$ has a unique solution such that $\boldsymbol{u}\in H^1(0,T;V)$ and $\beta\in H^1(0,T;Y)$. And then, we can conclude that Problem $\mathcal{P}$ has a unique weak solution under the Theorem 3.1, and the solution satisfies the regularity $\boldsymbol{u}\in H^1(0,T;V)$, $\boldsymbol{\sigma}\in L^2(0,T;\mathcal{H})$ and $\beta\in H^1(0,T;Y)$.

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