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A FINITE-DIFFERENCE SCHEME FOR A LINEAR MULTI-TERM FRACTIONAL-IN-TIME DIFFERENTIAL EQUATION WITH CONCENTRATED CAPACITIES

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Abstract. In this paper, we consider a linear multi-term subdiffusion equation with coefficients which contain Dirac distributions. Also, we consider subdiffusion equations with dynamical boundary conditions. The existence of generalized solutions of these initial-boundary value problems is proved. An implicit finite difference scheme is proposed and its stability and convergence rate are investigated in both cases. The corresponding difference schemes are tested on numerical examples.

Key words. Fractional derivative, fractional PDE, boundary value problem, interface problem, finite differences.

1. Introduction

Fractional calculus has been used as a powerful mathematical tool for the description of many phenomena in applied science. For example, fractional partial differential equations emerge in the modelling of diverse processes such as anomalous diffusion, processes in continuum mechanics as well as processes that occur in viscoelastic media, porous materials, fluids etc. ([7],[13],[14],[11]). In general, fractional derivatives are used for modeling processes with memory effects. Because of the presence of an integral in the definition of fractional derivative, it is clear that they are nonlocal operators.

The analytical solution of differential equations involving fractional derivatives, in some simple cases, can be obtained by using the Laplace transform, the Fourier transform, the Melin transform and some other techniques. Many authors have investigated numerical algorithms including finite difference methods and finite element methods ([8],[18],[12]).

In ([10]) an initial boundary value problem for a generalized multi-term fractional diffusion equation is considered. Solutions of Dirichlet and Robin boundary value problems for multi-term variable distributed order diffusion equations are studied in ([2]). In this article we consider an initial boundary value problem for a multi-term fractional in time equation with an interface. The coefficients of the equation may contain Dirac's delta distribution. It is the so-called problem with concentrated capacity.

The paper is organized as follows. In Section 2 we introduce the Riemann–Liouville and the Caputo derivatives and we mention their basic properties. In Section 3 some new function spaces are defined, especially spaces involving functions with fractional derivatives and anisotropic Sobolev spaces. In Section 4 we formulate the initial-boundary value problem for a linear multi-term fractional in time differential equation with concentrated capacities and define its weak solution. The existence and uniqueness of its (weak) solution are proved. We propose an implicit finite difference scheme and discuss its stability. The analysis of the error and

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the convergence rate of the scheme are presented in this section. One numerical example which is in agreement with the theoretical results is also presented. In Section 5 we consider a subdiffusion equation with a dynamical boundary conditions. In addition to the existence and uniqueness of the solution, a finite difference method is derived, together with its error analysis and convergence rate estimation. At the end, as in Section 4, one numerical example which is in agreement with theoretical results is also presented.

2. Fractional derivatives

Let u be a function defined on a nonempty bounded interval $[a, b]$ and let $k-1 \le \alpha < k$, $k \in \mathbb{N}$. The left Riemann-Liouville fractional derivative of order α is defined as [14]

$$\partial_{a+}^{\alpha} u(t) = \frac{1}{\Gamma(k-\alpha)} \frac{d^k}{dt^k} \int_a^t \frac{u(s)}{(t-s)^{\alpha+1-k}}\, ds, \qquad t \ge a, \tag{1}$$

where the $\Gamma(\cdot)$ is the Gamma function. The right Riemann-Liouville fractional derivative $\partial_{b-}^{\alpha} u(t)$ is defined analogously.

The Caputo fractional derivative is obtained by interchanging the derivative and integral operators in (1)

$$^{C}\partial_{a+}^{\alpha} u(t) = \frac{1}{\Gamma(k-\alpha)} \int_a^t \frac{u^{(k)}(s)}{(t-s)^{\alpha+1-k}}\, ds. \tag{2}$$

These two definitions are not equivalent and are related by the relation

$$\partial_{a+}^{\alpha} u(t) = {}^{C}\partial_{a+}^{\alpha} u(t) + \sum_{j=0}^{k-1} u^{(j)}(a) \frac{(x-a)^{j-\alpha}}{\Gamma(j-\alpha+1)}.$$

In particular, $\partial_{a+}^{\alpha} u(t) = {}^{C}\partial_{a+}^{\alpha} u(t)$ if $u(a) = u'(a) = \cdots = u^{(k-1)}(a) = 0$.

Let us mention two properties of fractional derivatives that will be used hereafter.

For $0 < \alpha < 1$ and continuously differentiable functions $u(t)$ and $v(t)$, the following equality holds:

$$(\partial_{a+}^{\alpha} u,\, v)_{L^2(a,b)} = (u,\, \partial_{b-}^{\alpha} v)_{L^2(a,b)}\,. \tag{3}$$

Also, if $\alpha > 0$ and if u is an infinitely differentiable function in $\mathbb{R}$, with $\operatorname{supp} u \subset (a, b)$, then u satisfies the following relation (see [5]):

$$(\partial_{a+}^{\alpha} u,\, \partial_{b-}^{\alpha} u)_{L^2(a,b)} = \cos \pi\alpha \, \|\partial_{a+}^{\alpha} u\|^2_{L^2(a,+\infty)}\,. \tag{4}$$

For functions of several variables, partial fractional derivatives are defined in an analogous manner, for example,

$$\partial_{t,a+}^{\alpha} u(x,t) = \frac{1}{\Gamma(k-\alpha)} \frac{\partial^k}{\partial t^k} \int_a^t \frac{u(x,s)}{(t-s)^{\alpha+1-k}}\, ds, \quad k-1 < \alpha < k, \quad k \in \mathbb{N}.$$

3. Some function spaces

First, we introduce some notations and define some function spaces, norms and inner products that are used hereafter. Let Ω be an open domain in $\mathbb{R}^n$. As usual, by $C^k(\Omega)$ and $C^k(\overline{\Omega})$ we denote the spaces of k-fold differentiable functions defined on Ω. By $\dot{C}^{\infty}(\Omega) = C_0^{\infty}(\Omega)$ we denote the space of infinitely differentiable functions with compact support in Ω. The space of measurable functions whose square is Lebesgue integrable in Ω is denoted by $L^2(\Omega)$. The inner product and norm of $L^2(\Omega)$ are defined by

$$(u,v)_\Omega = (u,v)_{L^2(\Omega)} = \int_\Omega uv\, d\Omega, \quad \|u\|_\Omega = \|u\|_{L^2(\Omega)} = (u,u)_\Omega^{1/2}.$$

We also use $H^\alpha(\Omega)$ and $\dot{H}^\alpha(\Omega) = H_0^\alpha(\Omega)$ to denote the usual Sobolev spaces [9], whose norms are denoted by $\|u\|_{H^\alpha(\Omega)}$.

For $\alpha > 0$ we further set

$$|u|_{C_+^\alpha[a,b]} = \|\partial_{a+}^\alpha u\|_{C[a,b]}, \qquad |u|_{C_-^\alpha[a,b]} = \|\partial_{b-}^\alpha u\|_{C[a,b]},$$

$$\|u\|^2_{C_\pm^\alpha[a,b]} = \|u\|^2_{C^{[\alpha]^-}[a,b]} + |u|^2_{C_\pm^\alpha[a,b]},$$

$$|u|_{H_+^\alpha(a,b)} = \|\partial_{a+}^\alpha u\|_{L^2(a,b)}, \qquad |u|_{H_-^\alpha(a,b)} = \|\partial_{b-}^\alpha u\|_{L^2(a,b)},$$

$$\|u\|^2_{H_\pm^\alpha(a,b)} = \|u\|^2_{H^{[\alpha]^-}(a,b)} + |u|^2_{H_\pm^\alpha(a,b)},$$

where $[\alpha]^-$ denotes the largest integer $< \alpha$. Then we define $C_\pm^\alpha[a,b]$ as the space of functions $u \in C^{[\alpha]^-}[a,b]$ with finite norm $\|u\|_{C_\pm^\alpha[a,b]}$. The space $H_\pm^\alpha(a,b)$ is defined analogously, while the space $\dot{H}_\pm^\alpha(a,b)$ is defined as the closure of $\dot{C}^\infty(a,b) = C_0^\infty(a,b)$ with respect to the norm $\|\cdot\|_{H_\pm^\alpha(a,b)}$. Since for $\alpha = k \in \mathbb{N}_0$ the fractional derivative reduces to the standard k-th derivative, we have $C_\pm^k[a,b] = C^k[a,b]$ and $H_\pm^k(a,b) = H^k(a,b)$.

The following result holds.

Lemma 1. (see [8]) *For $\alpha > 0$, $\alpha \neq k + 1/2$, $k \in \mathbb{N}_0$, the spaces $\dot{H}_+^\alpha(a,b)$, $\dot{H}_-^\alpha(a,b)$ and $\dot{H}^\alpha(a,b)$ are equal and their norms are equivalent.*

For vector valued functions mapping a real interval $[0,T]$ or $(0,T)$ into a Banach space X we introduce the spaces $C^k([0,T],X)$, $k \in \mathbb{N}_0$ and $H^\alpha((0,T),X)$, $\alpha \geq 0$, in the usual way [9]. In an analogous manner we define the spaces $C_\pm^\alpha([0,T],X)$ and $H_\pm^\alpha((0,T),X)$.

For α, $\beta \geq 0$, we introduce anisotropic Sobolev type spaces:

$$H^{\alpha,\beta}(Q) = L^2((0,T),H^\alpha(0,1)) \cap H^\beta((0,T),L^2(0,1))$$

and

$$H_\pm^{\alpha,\beta}(Q) = L^2((0,T),H^\alpha(0,1)) \cap H_\pm^\beta((0,T),L^2(0,1)).$$

Notice that for $0 \leq \beta < 1/2$: $H_+^{\alpha,\beta}(Q) = H_-^{\alpha,\beta}(Q) = H^{\alpha,\beta}(Q)$.

Let $\xi \in [0,1]$ and let $\widetilde{L}_\xi^2(0,1)$ be the space of functions defined on the interval [0,1], with the inner product and norm

$$(v,w)_{\widetilde{L}_\xi^2(0,1)} = \int_0^1 v(x)w(x)dx + v(\xi)w(\xi), \quad \|v\|_{\widetilde{L}_\xi^2(0,1)} = (v,v)^{1/2}_{\widetilde{L}_\xi^2(0,1)}.$$

For functions defined in the rectangle $Q = (0,1) \times (0,T)$, we define the space $\widetilde{L}_\xi^2(Q) = L^2((0,T),\widetilde{L}_\xi^2(0,1))$, with inner product and associated norm

$$(v,w)_{\widetilde{L}_\xi^2(Q)} = \iint_Q v(x,t)w(x,t)dxdt + \int_0^T v(\xi,t)w(\xi,t)dt,$$

$$\|v\|_{\widetilde{L}_\xi^2(Q)} = (v,v)^{1/2}_{\widetilde{L}_\xi^2(Q)}.$$

4. Problem with homogeneous Dirichlet boundary conditions

Let $\Omega = (0,1)$ and $I = (0,T)$ be the space and time domain respectively and $Q = \Omega \times I$. We consider the following linear multi-term subdiffusion equation with the presence of the concentrated capacities at the interior points $x = \xi_i$:

$$\sum_{i=1}^{m} [1 + K_i \delta(x - \xi_i)] \partial_{t,0_+}^{\alpha_i} u - \frac{\partial}{\partial x}\left(p \frac{\partial u}{\partial x}\right) = f(x,t), \quad (x,t) \in Q, \tag{5}$$

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t \in \bar{I}, \tag{6}$$

$$u(x,0) = 0, \quad x \in \bar{\Omega}, \tag{7}$$

where K_i, $i = 1, \dots, m$, are positive constants and $\delta(x)$ is Dirac's delta function. The equality in (5) is considered in the sense of the theory of distributions [19]. An analogous problem for $m = 1$ is considered in [4] and for $m = 1$ and $\alpha = 1$ in [6].

For the sake of the simplicity we assume that

$$\xi_1 < \xi_2 < \dots < \xi_m$$

and we set $\xi_0 = 0$ and $\xi_{m+1} = 1$. If the right-hand side in (5) does not contain singular terms, it follows that the solution of this problem for $(x,t) \in (\cup_{i=0}^{m} \Omega_i) \times I$, $\Omega_i = (\xi_i, \xi_{i+1})$, $i = 0, 1, \dots, m$ satisfies the differential equation

$$\sum_{i=1}^{m} \partial_{t,0_+}^{\alpha_i} u - \frac{\partial}{\partial x}\left(p \frac{\partial u}{\partial x}\right) = f(x,t), \tag{8}$$

while for $x = \xi_i$, $i = 1, 2, \dots, m$ the following conjugation conditions are fulfilled

$$[u]_{x=\xi_i} = u(\xi_i + 0, t) - u(\xi_i - 0, t) = 0, \tag{9}$$

$$\left[p \frac{\partial u}{\partial x}\right]_{x=\xi_i} = K_i \partial_{t,0_+}^{\alpha} u(\xi_i, t). \tag{10}$$

We assume that coefficient p satisfies the usual regularity and ellipticity conditions

$$p \in L^\infty(\Omega), \quad 0 < p_0 \le p(x) \le p_1. \tag{11}$$

Taking the inner product of equation (5) with a test function v and formally applying partial integration and relations (3)-(4) one obtains the following weak formulation of the problem (5)-(7): find $u \in \dot{\widetilde{H}}^{1,\boldsymbol{\alpha}/2}(Q)$ such that

$$a(u,v) = l(v)\,, \quad \forall\, v \in \dot{\widetilde{H}}^{1,\boldsymbol{\alpha}/2}(Q), \quad \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m), \tag{12}$$

where

$$\dot{\widetilde{H}}^{1,\boldsymbol{\alpha}/2}(Q) = \bigcap_{i=1}^{m} \dot{H}^{\alpha_i/2}((0,T), \widetilde{L}^2_{\xi_i}(0,1)) \cap L^2((0,T), \dot{H}^1(0,1)).$$

The bilinear form $a(\cdot,\cdot)$ is given by

$$a(u,v) = \sum_{i=1}^{m} \Big[(\partial_{t,0_+}^{\alpha_i/2} u,\, \partial_{t,T_-}^{\alpha_i/2} v)_{L^2(Q)} + K_i (\partial_{t,0_+}^{\alpha_i/2} u(\xi_i,\cdot),\, \partial_{t,T_-}^{\alpha_i/2} v(\xi_i,\cdot))_{L^2(0,T)} \Big] + \left(p \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x} \right)_{L^2(Q)},$$

and the linear functional $l(\cdot)$ is

$$l(v) = (f,v)_{L^2(Q)}\,.$$

In the space $\dot{\widetilde{H}}^{1,\boldsymbol{\alpha}/2}(Q)$ we define a norm $\|\cdot\|_{\widetilde{H}^{1,\boldsymbol{\alpha}/2}(Q)}$ by

$$\|u\|^2_{\widetilde{H}^{1,\boldsymbol{\alpha}/2}(Q)} = \sum_{i=1}^{m} \|u\|^2_{H^{\alpha_i/2}((0,T),\widetilde{L}^2_{\xi_i}(0,1))} + \|u\|^2_{L^2((0,T),H^1(0,1))}.$$

Theorem 1. *Let $\alpha_i \in (0,1)$, $i = 1,\dots,m$, $f \in L^2(Q)$ and let the assumptions* (11) *hold. Then the problem* (5)-(7) *is well posed in $\dot{\widetilde{H}}^{1,\boldsymbol{\alpha}/2}(Q)$ and its weak solution satisfies the a priori estimate*

$$\|u\|_{\widetilde{H}^{1,\boldsymbol{\alpha}/2}(Q)} \le C\|f\|_{L^2(Q)}. \tag{13}$$

The proof follows immediately using the relations (3)-(4), (11) and the Lax-Milgram lemma.

From (13) we immediately deduce the a priori estimate

$$\|u\|_{\widetilde{B}^{1,\boldsymbol{\alpha}/2}(Q)} \le C\|f\|_{L^2(Q)}$$

in the weaker norm [9]

$$\|u\|^2_{\widetilde{B}^{1,\boldsymbol{\alpha}/2}(Q)} = \int_0^T \left[\sum_{i=1}^{m}(T-t)^{-\alpha_i}\|u(\cdot,t)\|^2_{\widetilde{L}^2_{\xi_i}(\Omega)} + \|u(\cdot,t)\|^2_{H^1(\Omega)}\right] dt.$$

4.1. Finite difference approximation. In the rectangle $\bar{Q} = [0,1] \times [0,T]$ we introduce the uniform mesh $\bar{Q}_{h\tau} = \bar{\omega}_h \times \bar{\omega}_\tau$, where $\bar{\omega}_h = \{x_i = ih \,|\, i = 0, 1, \dots, N;\ h = 1/N\}$ and $\bar{\omega}_\tau = \{t_j = j\tau \,|\, j = 0, 1, \dots, M;\ \tau = T/M\}$. We also denote $\omega_h = \bar{\omega}_h \cap (0,1)$, $\omega_h^- = \bar{\omega}_h \cap [0,1)$, $\omega_h^+ = \bar{\omega}_h \cap (0,1]$, $\omega_\tau = \bar{\omega}_\tau \cap (0,T)$, $\omega_\tau^- = \bar{\omega}_\tau \cap [0,T)$ and $\omega_\tau^+ = \bar{\omega}_\tau \cap (0,T]$. For the sake of simplicity, in the sequel we suppose that ξ_i, $i = 1,2,\dots,m$ are rational numbers. Then one can choose the step h, so that $\xi_i \in \omega_h$. We shall use standard notation from the theory of finite difference schemes [16]:

$$v = v(x,t), \quad \hat{v} = v(x,t+\tau),$$

$$v_i = v(x_i,t), \quad t \in \bar{\omega}_\tau\,, \qquad v^j = v(x,t_j), \quad x \in \bar{\omega}_h,$$

$$v_x = \frac{v(x+h,t) - v(x,t)}{h} = v_{\bar{x}}(x+h,t), \quad v_t = \frac{v(h,t+\tau) - v(x,t)}{\tau} = v_{\bar{t}}(x,t+\tau).$$

For a function u which satisfies zero initial condition ($u(x,0) = 0$), the left Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$ on the mesh $\bar{Q}_{h\tau}$ is approximated by the L1-algorithm [13]

$$\partial^{\alpha}_{t,0+}u(x,t_j) \approx \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{j} a_{j-k}u^k_{\bar{t}} =: (\Delta^{\alpha-1}_{t,0+}u_{\bar{t}})^j =: (\Delta^{\alpha}_{t,0+}u)^j, \tag{14}$$

where the coefficients $a_{j-k} = (j-k+1)^{1-\alpha} - (j-k)^{1-\alpha}$, $1 \le k < j \le M$ are strictly decreasing:

$$1 = a_0 > a_1 > \cdots > a_{M-1} > 0. \tag{15}$$

The initial-boundary value problem (5)-(7) is approximated with the following implicit finite difference scheme:

$$\sum_{i=1}^{m}[1 + K_i\delta_{h\xi_i}]\Delta^{\alpha_i}_{t,0+}v - (\bar{p}v_{\bar{x}})_x = \tilde{f}, \quad (x,t) \in Q_{h\tau}, \tag{16}$$

$$v(0,t) = 0, \quad v(1,t) = 0, \quad t \in \bar{\omega}_\tau, \tag{17}$$

$$v(x,0) = 0, \quad x \in \bar{\omega}_h, \tag{18}$$

where

$$\tilde{f}(x,t) = T_x^2 f(x,t), \quad \bar{p}(x) = [p(x) + p(x-h)]/2,$$

$\delta_{h\xi_i}$ is the Dirac mesh function defined by

$$\delta_{h\xi_i} = \delta_h(x-\xi_i) = \begin{cases} 0, & x \in \omega_h \backslash \{\xi_i\}, \\ 1/h, & x = \xi_i, \end{cases}$$

and T_x is the Steklov smoothing operator [17]:

$$T_x f(x,t) = \int_{-1/2}^{1/2} f(x + hx', t)\,\mathrm{d}x',$$

$$T_x^+ v(x,t) = \frac{1}{h}\int_x^{x+h} v(x',t) dx' = T_x^- v(x+h,t),$$

$$T_x^2 f(x,t) = T_x\left(T_x f(x,t)\right) = \int_{-1}^{1} (1-|x'|) f(x+hx',t)\,\mathrm{d}x',$$

We also assume that the coefficient $p(x)$ may have discontinuities of the first kind at the points $x = \xi_i$ and redefine the values $\bar{p}(\xi_i)$ and $\bar{p}(\xi_i + h)$ in the following manner:

$$\bar{p}(\xi_i) = [p(\xi_i - 0) + p(\xi_i - h)]/2\,, \qquad \bar{p}(\xi_i + h) = [p(\xi_i + h) + p(\xi_i + 0)]/2\,.$$

Lemma 2. [1] *Let $0 < \alpha < 1$. Then for any function $v(t)$ defined on the grid $\bar{\omega}_\tau$ the following inequality is valid:*

$$v^j\,\Delta_{t,0_+}^{\alpha} v^j \geq \frac{1}{2}\Delta_{t,0_+}^{\alpha} (v^2)^j. \tag{19}$$

Lemma 3. [4] *For every function $v(t)$ defined on the grid $\bar{\omega}_\tau$, which satisfies $v(0) = 0$ and $0 < \alpha < 1$, the following equality is valid:*

$$\tau \sum_{j=1}^{M} (\Delta_{t,0_+}^{\alpha-1}(v_{\bar{t}}^2))^j = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{M} a_{M-j}(v^j)^2.$$

Let us define the following discrete inner products and norms:

$$(u,v)_h = (u,v)_{L^2(\omega_h)} = h \sum_{x\in\omega_h} u(x)v(x), \quad \|v\|_h = \|v\|_{L^2(\omega_h)} = (v,v)_h^{1/2},$$

$$(u,v]_h = (u,v]_{L^2(\omega_h^+)} = h \sum_{x\in\omega_h^+} u(x)v(x), \quad \|v]|_h = \|v]|_{L^2(\omega_h^+)} = (v,v]_h^{1/2},$$

$$[u,v)_h = [u,v)_{L^2(\omega_h^-)} = h \sum_{x\in\omega_h^-} u(x)v(x), \quad |[v\|_h = |[v\|_{L^2(\omega_h^-)} = [v,v)_h^{1/2},$$

$$(u,v)_{\tilde{h}\xi_i} = (u,v)_{\widetilde{L}^2_{\xi_i}(\omega_h)} = h \sum_{x\in\omega_h} u(x)v(x) + u(\xi_i)v(\xi_i),$$

$$\|v\|_{\tilde{h}\xi_i} = \|v\|_{\widetilde{L}^2_{\xi_i}(\omega_h)} = (v,v)_{\tilde{h}\xi_i}^{1/2} = \left\|(1+\delta_{h\xi_i})^{1/2}\, v\right\|_h,$$

$$(u,v)_{\tilde{h}} = (u,v)_{\widetilde{L}_2(\omega_h)} = h \sum_{x\in\omega_h} u(x)v(x) + \sum_{i=1}^{m} u(\xi_i)v(\xi_i),$$

$$\|v\|_{\tilde{h}} = \|v\|_{\widetilde{L}_2(\omega_h)} = (v,v)_{\tilde{h}}^{1/2} = \left\|\left(1 + \sum_{i=1}^{m}\delta_{h\xi_i}\right)^{1/2} v\right\|_h,$$

$$\|v\|_{L^2(Q_{h\tau})} = \left(\tau \sum_{t\in\omega_\tau^+} \|v(\cdot,t)\|_h^2\right)^{1/2}, \quad \|v]|_{L^2(Q_{h\tau})} = \left(\tau \sum_{t\in\omega_\tau^+} \|v(\cdot,t)]|_h^2\right)^{1/2},$$

$$\|v\|_{\widetilde{L}^2(Q_{h\tau})} = \left(\tau \sum_{t\in\omega_\tau^+} \|v(\cdot,t)\|_{\tilde{h}}^2\right)^{1/2},$$

$$\|v\|_{\widetilde{B}^{1,\alpha/2}(Q_{h\tau})} = \left(\|v_{\bar{x}}]|_{L^2(Q_{h\tau})}^2 + \tau \sum_{t\in\omega_\tau^+}\sum_{i=1}^m \Delta_{t,0_+}^{\alpha_i}\|v(\cdot,t)\|_{\tilde{h}\xi_i}^2\right)^{1/2}.$$

Let H_h denote the set of functions defined on the mesh $\bar{\omega}_h$, $B_{h\xi_i}v = (1 + K_i\delta_{h\xi_i})\, v$ and $B_h v = (1 + \sum_{i=1}^m K_i\delta_{h\xi_i})\, v$. Then, for each $v \in H_h$ we have

$$(B_{h\xi_i}v, v)_h = h\sum_{x\in\omega_h} v^2(x) + K_i v^2(\xi_i) \asymp \|v\|_{\tilde{h}\xi_i}^2,$$

$$(B_h v, v)_h = h\sum_{x\in\omega_h} v^2(x) + \sum_{i=1}^m K_i v^2(\xi_i) \asymp \|v\|_{\tilde{h}}^2,$$

and

$$(B_h^{-1}v, v)_h = h\sum_{x\in\omega_h\setminus\{\xi_1,\xi_2,\dots,\xi_m\}} v^2(x) + h^2\sum_{i=1}^m \frac{v^2(\xi_i)}{K_i + h}.$$

Theorem 2. *The finite difference scheme* (16)-(18) *is absolutely stable and its solution satisfies the a priori estimate*

$$\|v\|_{\tilde{B}^{1,\alpha/2}(Q_{h\tau})} \le C\|\tilde{f}\|_{L^2(Q_{h\tau})}. \tag{20}$$

Proof. Let us multiply the equation (16) with hv and sum over the nodes of the mesh ω_h

$$h\sum_{x\in\omega_h}\sum_{i=1}^m v\Delta_{t,0_+}^{\alpha_i} v_{\bar{t}} + \sum_{i=1}^m K_i v(\xi_i,t)\Delta_{t,0_+}^{\alpha_i} v_{\bar{t}}(\xi_i,t) + (\bar{p}v_{\bar{x}}, v_{\bar{x}}]_h = (\tilde{f}, v)_h.$$

Using (19), the ε-inequality and property (11) one can obtain

$$\frac{h}{2}\sum_{x\in\omega_h}\sum_{i=1}^m \Delta_{t,0_+}^{\alpha_i}(v^2)_{\bar{t}} + \frac{1}{2}\sum_{i=1}^m K_i\Delta_{t,0_+}^{\alpha_i}(v^2(\xi_i,t))_{\bar{t}} + p_0\|v_{\bar{x}}]|_h^2 \le \varepsilon\|v\|_h^2 + \frac{1}{4\varepsilon}\|\tilde{f}\|_h^2.$$

From the discrete Friedrichs inequality

$$\|v\|_h \le \frac{1}{\sqrt{8}}\|v_{\bar{x}}]|_h$$

for $0 < \varepsilon < 8p_0$ it follows that

$$\sum_{i=1}^m \Delta_{t,0_+}^{\alpha_i}\|v\|_{\tilde{h}\xi_i}^2 + \|v_{\bar{x}}]|_h^2 \le C\|\tilde{f}\|_h^2.$$

Multiplying the last inequality by τ and summing over the mesh ω_τ^+, one obtains the a priori estimate (20). □

4.2. Convergence of the finite difference scheme. Let u be the solution of the initial-boundary value problem (5)-(7) and v the solution of the finite difference problem (16)-(18). Then the error $z = u - v$ is a solution of the problem

$$\sum_{i=1}^{m} B_{h\xi_i} \Delta_{t,0_+}^{\alpha_i} z - (\bar{p} z_{\bar{x}})_x = \sum_{i=1}^{m} (B_{h\xi_i}\zeta_i + \eta_i) + \chi_x, \quad (x,t) \in Q_{h\tau}, \tag{21}$$

$$z(0,t) = 0, \quad z(1,t) = 0, \quad t \in \bar{\omega}_t, \tag{22}$$

$$z(x,0) = 0, \quad x \in \bar{\omega}_h, \tag{23}$$

where

$$\zeta_i = \Delta_{t,0_+}^{\alpha_i} u - \partial_{t,0_+}^{\alpha_i} u, \quad \eta_i = \partial_{t,0_+}^{\alpha_i}(u - T_x^2 u), \quad \chi = T_x^- \left(p\frac{\partial u}{\partial x}\right) - \bar{p} u_{\bar{x}}.$$

Theorem 3. *The finite difference scheme* (21)-(23) *is absolutely stable and its solution satisfies the a priori estimate*

$$\|z\|_{\tilde{B}^{1,\alpha/2}(Q_{h\tau})} \le C \left[\sum_{i=1}^{m} \left(\left\| B_{h\xi_i}^{1/2} \zeta_i \right\|_h + \left\| B_h^{-1/2} \eta_i \right\|_{L^2(Q_{h\tau})} \right) + \|\chi]|_{L^2(Q_{h\tau})} \right]. \tag{24}$$

The proof is analogous to the proof of Theorem 2, while the right-hand side terms are estimated using summation by part and the inequality

$$\|z\|_{\tilde{h}} \le \sqrt{\frac{1+2m}{8}} \|z_{\bar{x}}]|_h.$$

Thus, in order to obtain an error bound for the finite difference scheme (16)-(18) it is sufficient to estimate the right-hand side terms in (24).

Lemma 4. [18] *Let $0 < \alpha < 1$, $u \in C^2([0,t], C(\bar{\Omega}))$ and $t \in \omega_\tau^+$. Then,*

$$\left| {}^C\partial_{t,0_+}^{\alpha} u - \Delta_{t,0_+}^{\alpha} u \right| \le \tau^{2-\alpha} C \max_{\bar{Q}_t} \left| \frac{\partial^2 u}{\partial t^2} \right|, \tag{25}$$

where $Q_t = (0,t) \times \Omega$.

Theorem 4. *Let the solution u of the initial-boundary value problem* (5)-(7) *belong to the space $\bigcap_{i=0}^{m}(\bigcap_{j=1}^{m} C_+^{\alpha_j}\left([0,T], H^2(\Omega_i)\right) \cap C\left([0,T], H^3(\Omega_i)\right)) \cap C^2([0,T], C[0,1])$ and $p \in \bigcap_{i=0}^{m} H^2(\Omega_i)$. Then the solution v of the finite difference scheme* (16)-(18) *converges to u and the following convergence rate estimate holds:*

$$\|u - v\|_{\tilde{B}^{1,\alpha/2}(Q_{h\tau})} = O\left(h^2 + \tau^{2-\max_{1\le i\le m}\alpha_i}\right). \tag{26}$$

Proof. Using inequality (25), for $i = 1, 2, \dots, m$ we obtain

$$\|B_{h\xi_i}^{1/2}\zeta_i\|_{L^2(Q_{h\tau})} \le C\tau^{2-\alpha_i} \max_{t\in[0,T]} \max_{x\in[0,1]} \left| \frac{\partial^2 u}{\partial t^2} \right| = C\tau^{2-\alpha_i} \|u\|_{C^2([0,T],C[0,1])}. \tag{27}$$

For $x \neq \xi_i$, $i = 1, \dots, m$, the following integral representation is valid

$$u - T_x^2 u = -\frac{1}{h} \int_{x-h}^{x+h} \int_{x'}^{x} \int_{x''}^{x} \left(1 - \frac{|x'-x|}{h}\right) \frac{\partial^2 u}{\partial x^2}(x''', t)\, \mathrm{d}x''' \mathrm{d}x'' \mathrm{d}x', \tag{28}$$

while for $x = \xi_i, i = 1, 2 \ldots, m$,

$$
\begin{aligned}
u - T_x^2 u = \; & -\frac{1}{h} \int_{\xi_i}^{\xi_i+h} \int_{\xi_i}^{x'} \int_{\xi_i}^{x''} \left(1 - \frac{x' - \xi_i}{h}\right) \frac{\partial^2 u}{\partial x^2}(x''', t)\, \mathrm{d}x''' \mathrm{d}x'' \mathrm{d}x' \\
& -\frac{1}{h} \int_{\xi_i-h}^{\xi_i} \int_{x'}^{\xi_i} \int_{x''}^{\xi_i} \left(1 + \frac{x' - \xi_i}{h}\right) \frac{\partial^2 u}{\partial x^2}(x''', t)\, \mathrm{d}x''' \mathrm{d}x'' \mathrm{d}x' \\
& -\frac{h}{6} \left[\frac{\partial u}{\partial x}(\xi_i + 0, t) - \frac{\partial u}{\partial x}(\xi_i - 0, t)\right].
\end{aligned}
$$

Now, from the previous integral representations and the Sobolev imbedding theorem it immediately follows that

$$
\begin{aligned}
\left\| B_h^{-1/2} \eta_j \right\|_{L^2(Q_{h\tau})} \leq & Ch^2 \sum_{i=0}^{m} \max_{t\in[0,T]} \|\partial_{t,0+}^{\alpha_j} u(\cdot, t)\|_{H^2(\Omega_i)} \\
\leq & Ch^2 \sum_{i=0}^{m} \|u\|_{C_+^{\alpha_j}([0,T], H^2(\Omega_i))}, \quad j = 1, 2, \ldots, m.
\end{aligned} \tag{29}
$$

We decompose the term χ in the following manner: $\chi = \chi_1 + \chi_2 + \chi_3$, where

$$
\begin{aligned}
\chi_1 &= T_x^- \left(p \frac{\partial u}{\partial x}\right) - \left(T_x^- p\right) \left(T_x^- \frac{\partial u}{\partial x}\right), \\
\chi_2 &= \left(T_x^- p - \bar{p}\right) \left(T_x^- \frac{\partial u}{\partial x}\right), \\
\chi_3 &= \bar{p} \left(T_x^- \frac{\partial u}{\partial x} - u_{\bar{x}}\right).
\end{aligned}
$$

The terms χ_i have been estimated in [3]:

$$
\|\chi]|_{L^2(Q_{h\tau})} \leq Ch^2 \sum_{i=0}^{m} \|p\|_{H^2(\Omega_i)} \|u\|_{C([0,T], H^3(\Omega_i))}. \tag{30}
$$

The result (26) then follows from (27)-(30). □

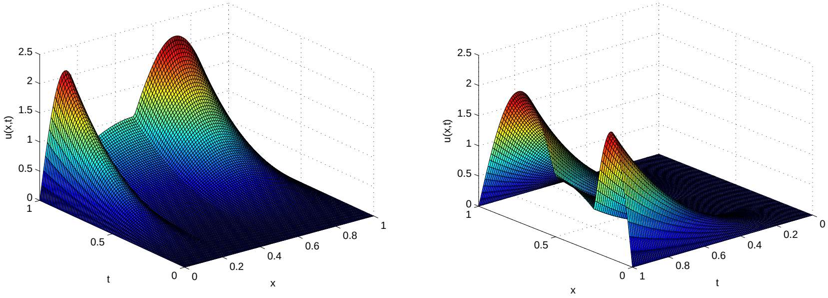

FIGURE 1. The exact solution for $\alpha_1 = 0.45$ and $\alpha_2 = 0.3$.

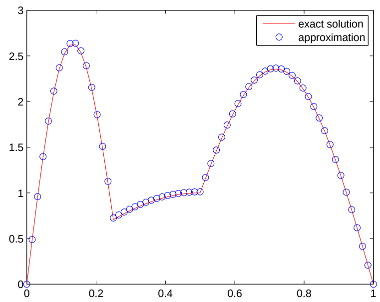

FIGURE 2. The exact solution and its approximation for $\alpha_1 = 0.7$, $\alpha_2 = 0.4$ at the last time level $t = 1$, when $h = 2^{-6}$ and $\tau = 2^{-4}$.

TABLE 1. The experimental error results and the temporal convergence orders for $h = 2^{-11}$ fixed.

α_1	α_2	τ	$\|z\|_{C(Q_{h\tau})}$	CO($\|\cdot\|_C$)	$\|z\|_{\widetilde{B}^{1,\alpha/2}(Q_{h\tau})}$	CO($\|\cdot\|_{\widetilde{B}^{1,\alpha/2}}$)
0.6	0.2	2^{-5}	$5.0926e-03$	1.38	$8.7437e-03$	1.39
		2^{-6}	$1.9630e-03$	1.39	$3.3284e-03$	1.39
		2^{-7}	$7.5081e-04$	1.40	$1.2658e-03$	1.40
		2^{-8}	$2.8521e-04$	1.40	$4.8029e-04$	1.40
		2^{-9}	$1.0732e-04$	1.44	$1.8166e-04$	1.40
		2^{-10}	$3.9570e-05$		$6.8716e-05$	
0.6	0.8	2^{-5}	$1.6534e-02$	1.20	$2.9581e-02$	1.22
		2^{-6}	$7.1905e-03$	1.21	$1.2673e-02$	1.22
		2^{-7}	$3.1174e-03$	1.21	$5.4461e-03$	1.22
		2^{-8}	$1.3493e-03$	1.21	$2.3457e-03$	1.21
		2^{-9}	$5.8344e-04$	1.21	$1.0121e-03$	1.21
		2^{-10}	$2.5201e-04$		$4.3733e-04$	
0.4	0.55	2^{-5}	$5.0595e-03$	1.43	$8.5222e-03$	1.45
		2^{-6}	$1.8770e-03$	1.44	$3.1175e-03$	1.45
		2^{-7}	$6.9076e-04$	1.45	$1.1384e-03$	1.46
		2^{-8}	$2.5266e-04$	1.46	$4.1498e-04$	1.46
		2^{-9}	$9.1800e-05$	1.48	$1.5114e-04$	1.45
		2^{-10}	$3.2940e-05$		$5.5437e-05$	

4.3. Numerical experiment. In this section, we present numerical results to verify the theoretical error estimates stated in Subsection 4.2. We consider (5)-(7) for $m = 2$, $\xi_1 = 1/4$, $\xi_2 = 1/2$, $K_1 = 4\sqrt{2}\pi$, $K_2 = 2\pi$, $p \equiv 1$, and

$$\begin{aligned} f(x,t) =& \sin(\pi x)(\partial_{t,0_+}^{\alpha_1} t^3 + \partial_{t,0_+}^{\alpha_2} t^3 + \pi^2 t^3) \\ &+ \chi_{[0,1/4]}(x) \sin(4\pi x)(\partial_{t,0_+}^{2\alpha_1} t^3 + \partial_{t,0_+}^{\alpha_1+\alpha_2} t^3 + 16\pi^2 \partial_{t,0_+}^{\alpha_1} t^3) \\ &- \chi_{[1/2,1]} \sin(2\pi x)(\partial_{t,0_+}^{\alpha_1+\alpha_2} t^3 + \partial_{t,0_+}^{2\alpha_2} t^3 - 4\pi^2 \partial_{t,0_+}^{\alpha_2} t^3), \end{aligned}$$

where $\chi_{[a,b]}(x)$ is the characteristic function of the interval $[a,b] \subset \mathbb{R}$.

Table 2. The experimental error results and the spatial convergence orders for $\tau = 2^{-12}$ fixed.

α_1	α_2	h	$\|z\|_{C(Q_{h\tau})}$	$\mathrm{CO}(\|\cdot\|_C)$	$\|z\|_{\widetilde{B}^{1,\alpha/2}(Q_{h\tau})}$	$\mathrm{CO}(\|\cdot\|_{\widetilde{B}^{1,\alpha/2}})$
0.6	0.2	2^{-4}	$8.9210e-02$	2.04	$1.9636e-01$	2.01
		2^{-5}	$2.1746e-02$	2.01	$4.8901e-02$	2.00
		2^{-6}	$5.4047e-03$	1.99	$1.2213e-02$	2.00
		2^{-7}	$1.3587e-03$	1.99	$3.0513e-03$	2.00
		2^{-8}	$3.4172e-04$	1.96	$7.6172e-04$	2.01
		2^{-9}	$8.7607e-05$		$1.8955e-04$	
0.6	0.8	2^{-4}	$8.8768e-02$	2.04	$2.0151e-01$	2.01
		2^{-5}	$2.1646e-02$	2.01	$5.0183e-02$	2.00
		2^{-6}	$5.3828e-03$	1.99	$1.2533e-02$	2.00
		2^{-7}	$1.3559e-03$	1.98	$3.1319e-03$	2.00
		2^{-8}	$3.4365e-04$	1.92	$7.8539e-04$	1.90
		2^{-9}	$9.0788e-05$		$2.1036e-04$	
0.4	0.55	2^{-4}	$6.9047e-02$	2.04	$1.5675e-01$	2.01
		2^{-5}	$1.6820e-02$	2.00	$3.9038e-02$	2.00
		2^{-6}	$4.1968e-03$	2.00	$9.7502e-03$	2.00
		2^{-7}	$1.0520e-03$	2.00	$2.4367e-03$	2.00
		2^{-8}	$2.6346e-04$	1.99	$6.0891e-04$	2.00
		2^{-9}	$6.6434e-05$		$1.5211e-04$	

The exact solution is

$$u(x,t) = \sin(\pi x)t^3 + \chi_{[0,1/4]}(x)\sin(4\pi x)\frac{6t^{3-\alpha_1}}{\Gamma(4-\alpha_1)} - \chi_{[1/2,1]}\sin(2\pi x)\frac{6t^{3-\alpha_2}}{\Gamma(4-\alpha_2)}.$$

We solved the problem using the proposed implicit scheme. In Figure 2 we have displayed the exact and numerical solutions on the last time level for comparison. The errors and convergence order in the norms $\|\cdot\|_{C(Q_{h\tau})}$ and $\|\cdot\|_{\widetilde{B}^{1,\alpha/2}(Q_{h\tau})}$ denoted by $\mathrm{CO}(\|\cdot\|_C)$ and $\mathrm{CO}(\|\cdot\|_{\widetilde{B}^{1,\alpha/2}})$, respectively, are given in Table 1 and Table 2. We may conclude that the temporal convergence rate is $2 - \max_{1\le i\le 2}\{\alpha_i\}$ while the spatial convergence rate is 2.

5. Problem with dynamical boundary conditions

Let us consider the following subdiffusion equation:

$$\partial^{\alpha_1}_{t,0_+}u + \partial^{\alpha_2}_{t,0_+}u - \frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right) = f(x,t), \quad (x,t)\in Q, \tag{31}$$

with dynamical boundary conditions:

$$K_1\partial^{\alpha_1}_{t,0_+}u(0,t) = p(0)\frac{\partial u(0,t)}{\partial x}, \quad K_2\partial^{\alpha_2}_{t,0_+}u(1,t) = -p(1)\frac{\partial u(1,t)}{\partial x}, \quad t\in\bar{I}, \tag{32}$$

where K_1 and K_2 are positive constants, and subject to the homogeneous initial condition (7).

The associated bilinear form $a(\cdot,\cdot)$ is given by

$$a(u,v)=\sum_{i=1}^{2}\Big[(\partial_{t,0_+}^{\alpha_i/2}u,\ \partial_{t,T_-}^{\alpha_i/2}v)_{L^2(Q)}+K_i\big(\partial_{t,0_+}^{\alpha_i/2}u(\xi_i,\cdot),\ \partial_{t,T_-}^{\alpha_i/2}v(\xi_i,\cdot)\big)_{L^2(0,T)}\Big]\\ +\left(p\,\frac{\partial u}{\partial x},\frac{\partial v}{\partial x}\right)_{L^2(Q)},$$

where $\xi_1=0$ and $\xi_2=1$ and the linear functional $l(\cdot)$ is defined by

$$l(v)=(f,v)_{L^2(Q)}\,.$$

Lemma 5. *Let $\alpha_i\in(0,1)$ and let the assumptions (11) hold. Then, the bilinear form $a(u,v)$ is bounded on $\widetilde{H}^{1,\boldsymbol{\alpha}/2}(Q)\times\widetilde{H}^{1,\boldsymbol{\alpha}/2}(Q)$. Moreover, this form satisfies Gårding's inequality on $H^{1,\boldsymbol{\alpha}/2}(Q)$: there exist positive constants m and κ such that*

$$a(u,u)+\kappa\|u\|^2_{L^2(Q)}\geq m\|u\|^2_{\widetilde{H}^{1,\boldsymbol{\alpha}/2}(Q)},\qquad\forall\,u\in\widetilde{H}^{1,\boldsymbol{\alpha}/2}(Q).\tag{33}$$

Proof. First, we show that the bilinear form is bounded. Using the Cauchy-Schwarz inequality twice we obtain

$$\begin{aligned}|a(u,v)|=&\bigg|\sum_{i=1}^{2}\Big[(\partial_{t,0_+}^{\alpha_i/2}u,\ \partial_{t,T_-}^{\alpha_i/2}v)_{L^2(Q)}+K_i\big(\partial_{t,0_+}^{\alpha_i/2}u(\xi_i,\cdot),\ \partial_{t,T_-}^{\alpha_i/2}v(\xi_i,\cdot)\big)_{L^2(0,T)}\Big]\\ &+\left(p\,\frac{\partial u}{\partial x},\frac{\partial v}{\partial x}\right)_{L^2(Q)}\bigg|\\ \leq&\sum_{i=1}^{2}\Big[\|\partial_{t,0_+}^{\alpha_i/2}u\|_{L^2(Q)}\|\partial_{t,T_-}^{\alpha_i/2}v\|_{L^2(Q)}\\ &+K_i\|\partial_{t,0_+}^{\alpha_i/2}u(\xi_i,\cdot)\|_{L^2(0,T)}\|\partial_{t,T_-}^{\alpha_i/2}v(\xi_i,\cdot)\|_{L^2(0,T)}\Big]\\ &+p_1\left\|\frac{\partial u}{\partial x}\right\|_{L^2(Q)}\left\|\frac{\partial v}{\partial x}\right\|_{L^2(Q)}\\ \leq&\left\{\sum_{i=1}^{2}\Big[\|\partial_{t,0_+}^{\alpha_i/2}u\|_{L^2(Q)}\|^2+K_i\|\partial_{t,0_+}^{\alpha_i/2}u(\xi_i,\cdot)\|^2_{L^2(0,T)}\Big]+p_1\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(Q)}\right\}^{1/2}\\ &\times\left\{\sum_{i=1}^{2}\Big[\|\partial_{t,T_-}^{\alpha_i/2}v\|^2_{L^2(Q)}+K_i\|\partial_{t,T_-}^{\alpha_i/2}v(\xi_i,\cdot)\|^2_{L^2(0,T)}\Big]+p_1\left\|\frac{\partial v}{\partial x}\right\|^2_{L^2(Q)}\right\}^{1/2}\\ \leq&C\left\{\sum_{i=1}^{2}\|u\|^2_{H^{\alpha_i/2}((0,T),\widetilde{L}_{\xi_i}(\Omega))}+\|u\|^2_{L^2((0,T),H^1(\Omega))}\right\}^{1/2}\\ &\times\left\{\sum_{i=1}^{2}\|v\|^2_{H^{\alpha_i/2}((0,T),\widetilde{L}_{\xi_i}(\Omega))}+\|v\|^2_{L^2((0,T),H^1(\Omega))}\right\}^{1/2}.\end{aligned}$$

Now let us show that Gårding's inequality holds.

$$
\begin{aligned}
&a(u,u)=\sum_{i=1}^{2}\Big[(\partial_{t,0_+}^{\alpha_i/2}u,\ \partial_{t,T_-}^{\alpha_i/2}u)_{L^2(Q)}+K_i\big(\partial_{t,0_+}^{\alpha_i/2}u(\xi_i,\cdot),\ \partial_{t,T_-}^{\alpha_i/2}u(\xi_i,\cdot)\big)_{L^2(0,T)}\Big]\\
&+\left(p\,\frac{\partial u}{\partial x},\frac{\partial u}{\partial x}\right)_{L^2(Q)}\\
&\ge\sum_{i=1}^{2}\cos\frac{\pi\alpha_i}{2}\Big[\|\partial_{t,0_+}^{\alpha_i/2}u\|^2_{L^2((0,\infty),L^2(\Omega))}+K_i\|\partial_{t,0_+}^{\alpha_i/2}u(\xi_i,\cdot)\|^2_{L^2(0,\infty)}\Big]+p_0\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(Q)}\\
&\ge\sum_{i=1}^{2}\cos\frac{\pi\alpha_i}{2}\Big[\|\partial_{t,0_+}^{\alpha_i/2}u\|^2_{L^2((0,T),L^2(\Omega))}+K_i\|\partial_{t,0_+}^{\alpha_i/2}u(\xi_i,\cdot)\|^2_{L^2(0,T)}\Big]+p_0\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(Q)}
\end{aligned}
$$

Using the inequality

$$
\|u\|^2_{L^2(Q)}\le\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(Q)}+\|u(0,t)\|^2_{L^2(0,T)}+\|u(1,t)\|^2_{L^2(0,T)} \tag{34}
$$

we obtain

$$
\begin{aligned}
a(u,u)\ge&\sum_{i=1}^{2}\cos\frac{\pi\alpha_i}{2}\Big[\|\partial_{t,0_+}^{\alpha_i/2}u\|^2_{L^2((0,T),L^2(\Omega))}+K_i\|\partial_{t,0_+}^{\alpha_i/2}u(\xi_i,\cdot)\|^2_{L^2(0,T)}\Big]\\
&+p_0(\|u\|^2_{L^2(Q)}-\|u(0,t)\|^2_{L^2(0,T)}-\|u(1,t)\|^2_{L^2(0,T)})
\end{aligned}
$$

and

$$
\begin{aligned}
2a(u,u)\ge&2\sum_{i=1}^{2}\cos\frac{\pi\alpha_i}{2}\Big[\|\partial_{t,0_+}^{\alpha_i/2}u\|^2_{L^2((0,T),L^2(\Omega))}+K_i\|\partial_{t,0_+}^{\alpha_i/2}u(\xi_i,\cdot)\|^2_{L^2(0,T)}\Big]\\
&+p_0(\|u\|^2_{L^2(Q)}-\|u(0,t)\|^2_{L^2(0,T)}-\|u(1,t)\|^2_{L^2(0,T)})+p_0\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(Q)}.
\end{aligned}
$$

Next, using the ε-inequality

$$
\|u(x,t)\|^2_{L^2(0,T)}\le\varepsilon\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(Q)}+\frac{2}{\varepsilon}\|u\|^2_{L^2(Q)},\quad x=0,1,
$$

where is $\varepsilon>0$, we obtain

$$
\begin{aligned}
2a(u,u)\ge&2\sum_{i=1}^{2}\cos\frac{\pi\alpha_i}{2}\Big[\|\partial_{t,0_+}^{\alpha_i/2}u\|^2_{L^2((0,T),L^2(\Omega))}+K_i\|\partial_{t,0_+}^{\alpha_i/2}u(\xi_i,\cdot)\|^2_{L^2(0,T)}\Big]\\
&+p_0\left(\|u\|^2_{L^2(Q)}-2\varepsilon\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(Q)}-\frac{4}{\varepsilon}\|u\|^2_{L^2(Q)}+\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(Q)}\right)
\end{aligned}
$$

$$
\begin{aligned}
\geq & 2\sum_{i=1}^{2}\cos\frac{\pi\alpha_i}{2}\left[\|\partial_{t,0_+}^{\alpha_i/2}u\|^2_{L^2((0,T),L^2(\Omega))}+K_i\|\partial_{t,0_+}^{\alpha_i/2}u(\xi_i,\cdot)\|^2_{L^2(0,T)}\right]\\
&+p_0\left(\left(1-\frac{4}{\varepsilon}\right)\|u\|^2_{L^2(Q)}+(1-2\varepsilon)\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(Q)}\right)\\
= & 2\sum_{i=1}^{2}\cos\frac{\pi\alpha_i}{2}\left[\|\partial_{t,0_+}^{\alpha_i/2}u\|^2_{L^2((0,T),L^2(\Omega))}+K_i\|\partial_{t,0_+}^{\alpha_i/2}u(\xi_i,\cdot)\|^2_{L^2(0,T)}\right]\\
&+p_0\left(\left(1-\frac{4}{\varepsilon}\right)\|u\|^2_{L^2(Q)}+(1-2\varepsilon)\|u\|^2_{H^1(Q)}-(1-2\varepsilon)\|u\|^2_{L^2(Q)}\right).
\end{aligned}
$$

Further,

$$
\begin{aligned}
a(u,u)+\frac{p_0}{2}\left(\frac{4}{\varepsilon}-2\varepsilon\right)\|u\|^2_{L^2(Q)}\geq &\sum_{i=1}^{2}\cos\frac{\pi\alpha_i}{2}\left[\|\partial_{t,0_+}^{\alpha_i/2}u\|^2_{L^2((0,T),L^2(\Omega))}\right.\\
&\left.+K_i\|\partial_{t,0_+}^{\alpha_i/2}u(\xi_i,\cdot)\|^2_{L^2(0,T)}\right]+\frac{p_0}{2}(1-2\varepsilon)\|u\|^2_{H^1(Q)}.
\end{aligned}
$$

Taking $0<\varepsilon<\frac{1}{2}$ we obtain Gårding's inequality:

$$
a(u,u)+\kappa\|u\|^2_{L^2(Q)}\geq m\|u\|^2_{\widetilde{H}^{1,\alpha/2}(Q)},
$$

where $\kappa=\frac{p_0}{2}\left(\frac{4}{\varepsilon}-2\varepsilon\right)$ and

$$
m=\left(\frac{1}{\sum_{i=1}^{2}\cos\frac{\pi\alpha_i}{2}}+\frac{1}{\sum_{i=1}^{2}K_i\cos\frac{\pi\alpha_i}{2}}+\frac{1}{\frac{p_0}{2}(1-2\varepsilon)}\right)^{-1}.
$$

□

Theorem 5. *Let $\alpha_i\in(0,1)$, $f\in L^2(Q)$ and let the assumptions* (11) *hold. Then, the problem* (31)-(32) *is well posed in $\dot{\widetilde{H}}^{1,\boldsymbol{\alpha}/2}(Q)$ and its weak solution satisfies the a priori estimate*

$$
\|u\|_{\widetilde{H}^{1,\boldsymbol{\alpha}/2}(Q)}\leq C\|f\|_{L^2(Q)}. \tag{35}
$$

Proof. From the proof of the previous lemma it is obvious that

$$
a(u,u)\geq C\left[\sum_{i=1}^{2}\|u\|^2_{H^{\alpha_i/2}((0,T),\tilde{L}^2_{\xi_i}(0,1))}+\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(Q)}\right]. \tag{36}
$$

It can be shown that the following fractional Poincaré-Friedrichs inequality is valid ([5]):

$$
\|u(x,\cdot)\|^2_{L^2(0,T)}\leq C_0\|\partial_{t,0_+}^{\beta}u(x,\cdot)\|^2_{L^2(0,T)}, \tag{37}
$$

where $\beta\in(0,1/2)$. Using (34), (36) and (37) it follows that

$$
\|u\|^2_{L^2(Q)}\leq\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(Q)}+\sum_{i=1}^{2}C_i\partial_{t,0+}^{\alpha_i/2}\|u(\xi_i,t)\|^2_{L^2(0,T)}\leq C_3a(u,u). \tag{38}
$$

Finally, from (36) and (38) we obtain

$$
\begin{aligned}
a(u,u)\geq & C_4\left[\sum_{i=1}^{2}\|u\|^2_{H^{\alpha_i/2}((0,T),\tilde{L}^2_{\xi_i}(0,1))}+\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(Q)}+\|u\|^2_{L^2(Q)}\right]\\
&=C_4\|u\|^2_{\widetilde{H}^{1,\alpha/2}(Q)}.
\end{aligned} \tag{39}
$$

The last inequality means that the bilinear functional $a(u, v)$ is coercive. Now, from the Lax-Milgram lemma it follows that we have a unique weak solution. Finally, from

$$C_4\|u\|^2_{\tilde{H}^{1,\alpha/2}(Q)} \le a(u,u) \le \|f\|_{L^2(Q)}\|u\|_{L^2(Q)} \le \|f\|_{L^2(Q)}\|u\|_{\tilde{H}^{1,\alpha/2}(Q)},$$

the a priori estimate (35) directly follows. □

5.1. Finite difference approximation. We approximate the problem (31) with the following finite difference scheme

$$[1+K_1\delta_{h\xi_1}]\Delta^{\alpha_1}_{t,0+}v + [1+K_2\delta_{h\xi_2}]\Delta^{\alpha_2}_{t,0+}v - (\bar{p}v_{\bar{x}})_{\hat{x}} = \tilde{f}, \quad x \in \bar{\omega}_h, \quad t \in \omega_\tau^+, \tag{40}$$

where

$$\delta_{h\xi_1} = \begin{cases} 0, & x \in \omega_h^+ \\ 2/h, & x = 0 \end{cases}, \qquad \delta_{h\xi_2} = \begin{cases} 0, & x \in \omega_h^- \\ 2/h, & x = 1 \end{cases},$$

$$v_{\hat{x}} = \begin{cases} \frac{2}{h}v(h), & x = 0 \\ v_x, & x \in \omega_h \\ -\frac{2}{h}v(1), & x = 1 \end{cases}$$

and

$$\bar{p}\,(x) = [p(x) + p(x-h)]/2.$$

An the right-hand side we have used Steklov smoothing operators

$$\tilde{f} := \tilde{T}_x f := \begin{cases} T_x^{2+}f, & x = 0 \\ T_x^2 f, & x \in \omega_h \\ T_x^{2-}f, & x = 1 \end{cases},$$

where for $x = 0$ and $x = 1$ we act with the asymmetric Steklov averaging operators

$$T_x^{2+}f(x,t) = \frac{2}{h}\int_0^h \left(1 - \frac{x'}{h}\right) f(x',t)dx', \quad x = 0,$$

$$T_x^{2-}f(x,t) = \frac{2}{h}\int_{1-h}^1 \left(1 + \frac{x'-1}{h}\right) f(x',t)dx', \quad x = 1.$$

Let us define the following inner products and norms

$$[u,v]_h = h\sum_{x\in\omega_h} u(x)v(x) + \frac{h}{2}u(0)v(0) + \frac{h}{2}u(1)v(1), \quad |[v]|_h = [v,v]_h^{1/2},$$

$$\|v\|^2_{H^1(\bar{\omega}_h)} = |[v]|_h^2 + \|v_{\bar{x}}]|_h^2,$$

$$[u,v]_{\tilde{h}\xi_1} = [u,v]_h + u(0)v(0), \quad |[v]|_{\tilde{h}\xi_1} = [v,v]_{\tilde{h}\xi_1}^{1/2},$$

$$[u,v]_{\tilde{h}\xi_2} = [u,v]_h + u(1)v(1), \quad |[v]|_{\tilde{h}\xi_2} = [v,v]_{\tilde{h}\xi_2}^{1/2}$$

and

$$|[v]|_{\widetilde{B}^{1,\alpha/2}(Q_{h\tau})} = \left(\|v_{\bar{x}}]|^2_{L^2(Q_{h\tau})} + \tau\sum_{t\in\omega_\tau^+}\sum_{i=1}^2 \Delta^{\alpha_i}_{t,0+}|[v(\cdot,t)]|^2_{\tilde{h}\xi_i} \right)^{1/2}.$$

The following relations are satisfied

$$[-(\bar{p}v_{\bar{x}})_{\hat{x}}, v]_h = h\sum_{i=1}^N \bar{p}_i v^2_{\bar{x},i} \ge p_0(v_{\bar{x}}, v_{\bar{x}}],$$

$$[B_{h\xi_1}v,v]_h = [(1+K_1\delta_{h\xi_1})v,v]_h = [v,v]_h + K_1v^2(0) \asymp |[v]|^2_{\tilde{h}\xi_1},$$
$$[B_{h\xi_2}v,v]_h = [(1+K_2\delta_{h\xi_2})v,v]_h = [v,v]_h + K_2v^2(1) \asymp |[v]|^2_{\tilde{h}\xi_2},$$
$$[B_hv,v]_h = [v,v]_h + K_1v^2(0) + K_2v^2(1) \asymp |[v]|^2_{\tilde{h}},$$

and

$$[B_h^{-1}v,v]_h = h\sum_{x\in\omega_h} v^2(x) + h^2\frac{v^2(0)}{4K_1+2h} + h^2\frac{v^2(1)}{4K_2+2h}\,.$$

The bilinear form $a(u,v)$ is approximated with the discrete bilinear form:

$$a_h(u,v) = \sum_{i=1}^{2}[B_{h\xi_i}\Delta^{\alpha_i}_{t,0+}u_{\bar{t}},v]_h + \tau\sum_{t\in\omega_\tau^+}(\bar{p}u_{\bar{x}},v_{\bar{x}}]_h.$$

Lemma 6. *Let $\alpha_i \in (0,1)$ and let the assumptions* (11) *hold. Then, for sufficiently small mesh sizes h and τ, the bilinear form $a_h(u,v)$ is bounded on the space of discrete functions $\widetilde{B}^{1,\boldsymbol{\alpha}/2}(Q_{h\tau})\times\widetilde{B}^{1,\boldsymbol{\alpha}/2}(Q_{h\tau})$. Moreover, this form satisfies Gårding's inequality on $\widetilde{B}^{1,\boldsymbol{\alpha}/2}(Q_{h\tau})$: there exist positive constants $\widetilde{m}$ and $\widetilde{\kappa}$ such that*

$$a_h(v,v) + \widetilde{\kappa}\|v\|^2_{L^2(Q_{h\tau})} \geq \widetilde{m}\|v\|^2_{\widetilde{B}^{1,\boldsymbol{\alpha}/2}(Q_{h\tau})}. \tag{41}$$

The proof is analogous to the proof of Lemma 5.

Theorem 6. *The finite difference scheme* (40) *is absolutely stable and its solution satisfies the a priori estimate*

$$\tau\sum_{i=1,2}\sum_{t\in\omega_\tau^+}\Delta^{\alpha_i}_{t,0+}|[v(\cdot,t)]|^2_{\tilde{h}\xi_i} + \|v_{\bar{x}}]|^2_{L^2(Q_{h\tau})} \leq C|[\tilde{f}]|^2_{L^2(Q_{h\tau})}. \tag{42}$$

Proof. We multiply (40) with hv and sum over the mesh nodes $\bar{\omega}_h$. Using the properties (19), (11) and the ε-inequality we obtain

$$\frac{1}{2}[1,\Delta^{\alpha_1}_{t,0+}(v^2)_{\bar{t}}]_{\tilde{h}\xi_1} + \frac{1}{2}[1,\Delta^{\alpha_2}_{t,0+}(v^2)_{\bar{t}}]_{\tilde{h}\xi_2} + p_0\|v_{\bar{x}}]|^2_h \leq \varepsilon|[v]|^2_h + \frac{1}{4\varepsilon}|[\tilde{f}]|^2_h.$$

Using the inequality

$$|[v]|^2_h \leq \frac{1}{2}\|v_{\bar{x}}]|^2 + 2(v_0^2+v_N^2)$$

we obtain

$$\frac{1}{2}[1,\Delta^{\alpha_1}_{t,0+}(v^2)_{\bar{t}}]_{\tilde{h}\xi_1} + \frac{1}{2}[1,\Delta^{\alpha_2}_{t,0+}(v^2)_{\bar{t}}]_{\tilde{h}\xi_2} + \left(p_0-\frac{\varepsilon}{2}\right)\|v_{\bar{x}}]|^2_h \leq 2\varepsilon\left(v_0^2+v_N^2\right) + \frac{1}{4\varepsilon}|[\tilde{f}]|^2_h.$$

Multiplying the last inequality by τ and summing over the mesh ω_τ^+, we get

$$\begin{aligned}&\tau\sum_{t\in\omega_\tau^+}\Delta^{\alpha_1}_{t,0+}|[v(\cdot,t)]|^2_{\tilde{h}\xi_1} + \tau\sum_{t\in\omega_\tau^+}\Delta^{\alpha_2}_{t,0+}|[v(\cdot,t)]|^2_{\tilde{h}\xi_2} + \|v_{\bar{x}}]|^2_{L^2(Q_{h\tau})}\\ &\leq \left(2\varepsilon+\frac{4\varepsilon}{2p_0-\varepsilon}\right)\tau\sum_{t\in\omega_\tau^+}(v_0^2+v_N^2) + \left(\frac{1}{4\varepsilon}+\frac{2}{4\varepsilon(2p_0-\varepsilon)}\right)|[\tilde{f}]|^2_{L^2(Q_{h\tau})}.\end{aligned} \tag{43}$$

In order to estimate the sum

$$\tau\sum_{t\in\omega_\tau^+}v_0^2 = \tau\sum_{t\in\omega_\tau^+}v^2(0,t) \tag{44}$$

we will use Lemma 3. First, notice that the first summand on the left-hand side of the inequality (43) contains the sum

$$\tau\sum_{t\in\omega_\tau^+}\Delta^{\alpha_1}_{t,0+}\left(v^2(0,t)\right) = \frac{\tau^{1-\alpha_1}}{\Gamma(2-\alpha_1)}\sum_{j=1}^{M}a^{[1-\alpha_1]}_{M-j}(v_0^j)^2. \tag{45}$$

We will show that the sum (44) can be estimated by the sum (45).

Applying the mean value theorem for $\theta \in (0,1)$ we get the following estimate for the coefficient $a_{M-j}^{[1-\alpha_1]}$:

$$a_{M-j}^{[1-\alpha_1]} \geq a_{M-1}^{[1-\alpha_1]} = M^{1-\alpha_1} - (M-1)^{1-\alpha_1} = \frac{1-\alpha_1}{(M-\theta)^{\alpha_1}} > \frac{1-\alpha_1}{M^{\alpha_1}} = (1-\alpha_1)\tau^{\alpha_1}.$$

By using this in (45) we get

$$\frac{\tau^{1-\alpha_1}}{\Gamma(2-\alpha_1)} \sum_{j=1}^{M} a_{M-j}^{[1-\alpha_1]} (v_0^j)^2 > \frac{1-\alpha_1}{\Gamma(2-\alpha_1)} \tau \sum_{j=1}^{M} (v_0^j)^2 = C_{\alpha_1} \tau \sum_{t\in\omega_\tau^+} v_0^2.$$

An analogous estimate can be obtained for

$$\tau \sum_{t\in\omega_\tau^+} v_N^2 = \tau \sum_{t\in\omega_\tau^+} v^2(1,t).$$

So, by choosing a small enough ε we get the a priori estimate (42), where

$$C = \max\left\{ \left(\frac{1}{C_{\alpha_1}} + \frac{1}{C_{\alpha_2}}\right)\left(2\varepsilon + \frac{4\varepsilon}{2p_0-\varepsilon}\right), \frac{1}{4\varepsilon} + \frac{2}{4\varepsilon(2p_0-\varepsilon)} \right\}$$

and $0 \leq \varepsilon \leq 2p_0$. □

5.2. Error analysis. Let u be the solution of the initial-boundary value problem (31)-(32) with the initial condition (7) and let v be the solution of the difference problem (40). Then, the error $z = u - v$ satisfies

$$\sum_{i=1}^{2} B_{h\xi_i} \Delta_{t,0_+}^{\alpha_i} z - (\bar{p} z_{\bar{x}})_{\hat{x}} = \sum_{i=1}^{2} (B_{h\xi_i}\phi_i + \mu_i) + \nu, \quad (x,t) \in \bar{Q}_{h\tau}, \tag{46}$$

$$z(x,0) = 0, \quad x \in \bar{\omega}_h, \tag{47}$$

where

$$\phi_i = \Delta_{t,0_+}^{\alpha_i} u - \partial_{t,0_+}^{\alpha_i} u, \quad \mu_i = \partial_{t,0_+}^{\alpha_i}(u - \tilde{T}_x u), \quad i = 1,2,$$

and

$$\nu = \tilde{T}_x \frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right) - (\bar{p} u_{\bar{x}})_{\hat{x}}.$$

Theorem 7. *Let the solution u of the initial-boundary value problem* (31)-(32) *and* (7) *belong to the space* $\bigcap_{i=1}^{2} (C_+^{\alpha_i}(\bar{I}, H^2(Q_i)) \cap C(\bar{I}, H^3(Q_i))) \cap C^2(\bar{I}, C[0,1])$ *and* $p \in H^2(Q)$. *Then, the solution v of the finite difference scheme* (40) *converges to u and the following convergence rate estimate holds:*

$$|[u-v]|_{\tilde{B}^{1,\alpha/2}(Q_{h\tau})} = O\left(h^2 + \tau^{2-\max_{1\leq i\leq 2}\alpha_i}\right). \tag{48}$$

Proof. From the a priori estimate (42) we directly deduce the inequality

$$|[z]|_{\widetilde{B}^{1,\alpha/2}(Q_{h\tau})} \leq C_1 \left(\sum_{i=1}^{2} \left(|[\phi_i]|_{\widetilde{L}^2(Q_{h\tau})} + |[B_h^{-1/2}\mu_i]|_{L^2(Q_{h\tau})} \right) + |[\nu]|_{L^2(Q_{h\tau})} \right). \tag{49}$$

Thus, the problem of deriving a convergence rate estimate for the finite difference scheme (40) is reduced to estimating the right-hand side terms in the inequality (49). Using inequality (25), for $i = 1,2$ we obtain

$$\|\phi_i\|_{\widetilde{L}^2(Q_{h\tau})} \leq C\tau^{2-\alpha_i} \max_{t\in[0,T]} \max_{x\in[0,1]} \left|\frac{\partial^2 u}{\partial t^2}\right| \leq C\tau^{2-\max\alpha_i} \|u\|_{C^2([0,T],C[0,1])}. \tag{50}$$

From integral representations (28) we have that

$$\begin{aligned}
&u - T_x^{2+}u = u - \frac{2}{h}\int_0^h \left(1 - \frac{x'}{h}\right) u(x',t)dx' \\
&= \frac{2}{h}\int_0^h \left(1 - \frac{x'}{h}\right)(u(0,t) - u(x',t))dx' \\
&= \frac{2}{h}\int_0^h \left(1 - \frac{x'}{h}\right)\int_0^{x'} \frac{\partial u}{\partial x}(x'',t)dx'dx'' \\
&= \frac{2}{h}\int_0^h \int_0^{x'} \int_0^{x''} \left(1 - \frac{x'}{h}\right)\frac{\partial^2 u}{\partial x^2}(x''',t)dx'''dx''dx' + \frac{h}{3}\frac{\partial u}{\partial x}(0,t), \quad x = 0,
\end{aligned}$$

and

$$\begin{aligned}
u - T_x^{2-}u = {}& \frac{2}{h}\int_{1-h}^1 \int_{x'}^1 \int_{x''}^1 \left(1 + \frac{x'-1}{h}\right)\frac{\partial^2 u}{\partial x^2}(x''',t)dx'''dx''dx' \\
&+ \frac{h}{3}\frac{\partial u}{\partial x}(1,t), \quad x = 1,
\end{aligned}$$

using the Sobolev imbedding theorem it immediately follows that

$$\begin{aligned}
|[B_h^{-1/2}\mu_i]|_{L^2(Q_{h\tau})} \le{}& Ch^2 \max_{t\in[0,T]} \|\partial_{t,0_+}^{\alpha_i}u(\cdot,t)\|_{H^2(\Omega)} \\
\le{}& Ch^2\,\|u\|_{C_+^{\alpha_i}([0,T],H^2(\Omega))}, \quad i = 1,2.
\end{aligned} \tag{51}$$

The term ν is decomposed in the following manner: $\nu = \sum_{k=1}^6 \nu_k$, where

$$\begin{aligned}
\nu_1 &= T_x^2\left(p\frac{\partial^2 u}{\partial x^2}\right) - \left(T_x^2 p\right)\left(T_x^2\frac{\partial^2 u}{\partial x^2}\right), \\
\nu_2 &= \left(T_x^2 p - p\right)\left(T_x^2\frac{\partial^2 u}{\partial x^2}\right), \\
\nu_3 &= T_x^2\left(p'\frac{\partial u}{\partial x}\right) - \left(T_x^2 p'\right)\left(T_x^2\frac{\partial u}{\partial x}\right), \\
\nu_4 &= \left(T_x^2 p' - \frac{1}{2}\left(p_x + p_{\bar{x}}\right)\right)\left(T_x^2\frac{\partial u}{\partial x}\right), \\
\nu_5 &= \frac{1}{2}\left(p_x + p_{\bar{x}}\right)\left(T_x^2\frac{\partial u}{\partial x} - \frac{1}{2}(u_{\bar{x}} + u_x)\right), \\
\nu_6 &= -\frac{1}{4}\left(p_x - p_{\bar{x}}\right)\left(u_x - u_{\bar{x}}\right),
\end{aligned}$$

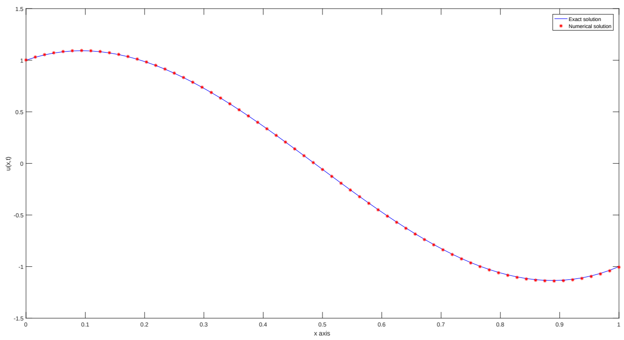

FIGURE 3. The exact solution and its approximation for $\alpha_1 = 0.6$, $\alpha_2 = 0.8$ at the last time level $t = 1$, when $h = 2^{-6}$ and $\tau = 2^{-6}$.

for $x \in \omega_h$, while for $x = 0$ we have

$$\begin{aligned}
\nu_1 &= T_x^{2+}\left(p\frac{\partial^2 u}{\partial x^2}\right) - \left(T_x^{2+}p\right)\left(T_x^{2+}\frac{\partial^2 u}{\partial x^2}\right),\\
\nu_2 &= \left(T_x^{2+}p - p\right)\left(T_x^{2+}\frac{\partial^2 u}{\partial x^2}\right),\\
\nu_3 &= T_x^{2+}\left(p'\frac{\partial u}{\partial x}\right) - \left(T_x^{2+}p'\right)\left(T_x^{2+}\frac{\partial u}{\partial x}\right),\\
\nu_4 &= \left(T_x^{2+}p' - p_x\right)\left(T_x^{2+}\frac{\partial u}{\partial x}\right),\\
\nu_5 &= p_x\left(T_x^{2+}\frac{\partial u}{\partial x} - u_x\right),\\
\nu_6 &= 0
\end{aligned}$$

and, for $x = 1$,

$$\begin{aligned}
\nu_1 &= T_x^{2-}\left(p\frac{\partial^2 u}{\partial x^2}\right) - \left(T_x^{2-}p\right)\left(T_x^{2-}\frac{\partial^2 u}{\partial x^2}\right),\\
\nu_2 &= \left(T_x^{2-}p - p\right)\left(T_x^{2-}\frac{\partial^2 u}{\partial x^2}\right),\\
\nu_3 &= T_x^{2-}\left(p'\frac{\partial u}{\partial x}\right) - \left(T_x^{2-}p'\right)\left(T_x^{2-}\frac{\partial u}{\partial x}\right),\\
\nu_4 &= \left(T_x^{2-}p' - p_{\bar{x}}\right)\left(T_x^{2-}\frac{\partial u}{\partial x}\right),\\
\nu_5 &= p_{\bar{x}}\left(T_x^{2-}\frac{\partial u}{\partial x} - u_{\bar{x}}\right),\\
\nu_6 &= 0.
\end{aligned}$$

The terms ν_i are estimated as in [4]:

$$|[\nu]|_{L^2(Q_{h\tau})} \leq Ch^2\|p\|_{H^2(\Omega)}\|u\|_{C([0,T],H^3(\Omega))}. \tag{52}$$

The result (48) follows from (50)-(52). $\square$

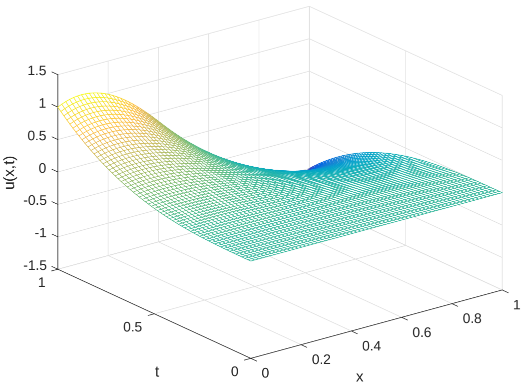

FIGURE 4. The exact solution for $\alpha_1 = 0.6$ and $\alpha_2 = 0.8$.

TABLE 3. The experimental error results and the temporal convergence orders for $h = 2^{-11}$ fixed.

α_1	α_2	τ	$\|z\|_{C(Q_{h\tau})}$	CO($\|\cdot\|_C$)	$\|z\|_{\tilde{B}^{1,\alpha/2}(Q_{h\tau})}$	CO($\|\cdot\|_{\tilde{B}^{1,\alpha/2}}$)
0.6	0.2	2^{-5}	$4.9749e-03$	1.37	$6.3275e-03$	1.39
		2^{-6}	$1.9240e-03$	1.38	$2.4173e-03$	1.39
		2^{-7}	$7.3880e-04$	1.39	$9.2194e-04$	1.39
		2^{-8}	$2.8249e-04$	1.39	$3.5114e-04$	1.39
		2^{-9}	$1.0779e-04$	1.39	$1.3366e-04$	1.39
		2^{-10}	$4.1145e-05$		$5.0927e-05$	
0.6	0.8	2^{-5}	$1.2679e-02$	1.19	$2.0282e-02$	1.22
		2^{-6}	$5.5663e-03$	1.19	$8.6849e-03$	1.22
		2^{-7}	$2.4342e-03$	1.20	$3.7277e-03$	1.22
		2^{-8}	$1.0622e-03$	1.20	$1.6036e-03$	1.21
		2^{-9}	$4.6297e-04$	1.20	$6.9123e-04$	1.21
		2^{-10}	$2.0175e-04$		$2.9858e-04$	
0.4	0.55	2^{-5}	$3.7364e-03$	1.42	$5.5162e-03$	1.45
		2^{-6}	$1.4008e-03$	1.43	$2.0178e-03$	1.45
		2^{-7}	$5.2112e-04$	1.43	$7.3689e-04$	1.45
		2^{-8}	$1.9299e-04$	1.44	$2.6890e-04$	1.45
		2^{-9}	$7.1335e-05$	1.43	$9.8191e-05$	1.45
		2^{-10}	$2.6416e-05$		$3.5986e-05$	

5.3. Numerical experiment. In order to verify the theoretical error estimates from Subsection 5.2 we consider (5)-(7) for $K_1 = K_2 = 1$, $p \equiv 1$ and

$$\begin{aligned} f(x,t) = &\cos(\pi x)(\partial_{t,0_+}^{\alpha_1} t^3 + \partial_{t,0_+}^{\alpha_2} t^3 + \pi^2 t^3) + x(1-x^2)(\partial_{t,0_+}^{2\alpha_1} t^3 + \partial_{t,0_+}^{\alpha_1+\alpha_2} t^3) \\ &+ x^2(x-1)(\partial_{t,0_+}^{\alpha_1+\alpha_2} t^3 + \partial_{t,0_+}^{2\alpha_2} t^3) - 2(3x+2)\partial_{t,0_+}^{\alpha_1} t^3 + 2(3x-1)\partial_{t,0_+}^{\alpha_2} t^3. \end{aligned}$$

TABLE 4. The experimental error results and the spatial convergence orders for $\tau = 2^{-13}$fixed.

α_1	α_2	h	$\|z\|_{C(Q_{h\tau})}$	$CO(\|\cdot\|_C)$	$\|z\|_{\widetilde{B}^{1,\alpha/2}(Q_{h\tau})}$	$CO(\|\cdot\|_{\widetilde{B}^{1,\alpha/2}})$
0.6	0.2	2^{-4}	$5.2243e-03$	2.00	$7.0540e-03$	2.01
		2^{-5}	$1.3063e-03$	2.00	$1.7566e-03$	2.00
		2^{-6}	$3.2652e-04$	2.00	$4.3938e-04$	1.98
		2^{-7}	$8.1575e-05$	2.00	$1.1101e-04$	1.93
		2^{-8}	$2.0337e-05$	1.73	$2.9086e-05$	1.73
		2^{-9}	$6.1287e-06$		$8.7973e-06$	
0.6	0.8	2^{-4}	$5.1533e-03$	2.00	$8.9737e-03$	2.00
		2^{-5}	$1.2908e-03$	1.99	$2.2461e-03$	1.97
		2^{-6}	$3.2462e-04$	1.90	$5.7423e-04$	1.86
		2^{-7}	$8.6749e-05$	1.35	$1.5805e-04$	1.51
		2^{-8}	$3.4144e-05$	0.70	$5.5454e-05$	0.82
		2^{-9}	$2.0992e-05$		$3.1324e-05$	
0.4	0.55	2^{-4}	$4.8541e-03$	2.00	$7.2254e-03$	2.01
		2^{-5}	$1.2140e-03$	2.00	$1.7988e-03$	2.00
		2^{-6}	$3.0368e-04$	2.00	$4.4965e-04$	1.99
		2^{-7}	$7.6088e-05$	1.99	$1.1331e-04$	1.95
		2^{-8}	$1.9190e-05$	1.78	$2.9351e-05$	1.80
		2^{-9}	$5.5716e-06$		$8.4205e-06$	

The exact solution is

$$u(x,t) = \cos(\pi x)t^3 + x(1-x)^2 \frac{6t^{3-\alpha_1}}{\Gamma(4-\alpha_1)} - x^2(1-x)\frac{6t^{3-\alpha_2}}{\Gamma(4-\alpha_2)}.$$

The problem (5)-(7) is approximated by the finite difference scheme (40). In Figure 3 we have displayed the exact and numerical solutions at the last time level, for comparison. In Figure 4 we have displayed the exact solution with $\alpha_1 = 0.6$ and $\alpha_2 = 0.8$. The errors and convergence orders were considered in the norms $\|\cdot\|_{C(Q_{h\tau})}$ and $\|\cdot\|_{\widetilde{B}^{1,\alpha/2}(Q_{h\tau})}$. Using the same labels for the error and convergence orders as in Table 1 and Table 2 from Subsection 4.3., in Table 3 and Table 4 we present the results obtained for this numerical example. We may conclude, as in the previous section, that the temporal convergence rate is $2 - \max_{1\le i\le 2}\{\alpha_i\}$ while the spatial convergence rate is 2.

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