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WELL-POSEDNESS AND THE MULTISCALE ALGORITHM FOR HETEROGENEOUS SCATTERING OF MAXWELL'S EQUATIONS IN DISPERSIVE MEDIA

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Abstract. This paper discusses the well-posedness and the multiscale algorithm for the heterogeneous scattering of Maxwell's equations in dispersive media with a periodic microstructure or with many subdivided periodic microstructures. An exact transparent boundary condition is developed to reduce the scattering problem into an initial-boundary value problem in heterogeneous materials. The well-posedness and the stability analysis for the reduced problem are derived. The multiscale asymptotic expansions of the solution for the reduced problem are presented. The convergence results of the multiscale asymptotic method are proved for the dispersive media with a periodic microstructure. A multiscale Crank-Nicolson mixed finite element method (FEM) is proposed where the perfectly matched layer (PML) is utilized to truncate infinite domain problems. Numerical test studies are then carried out to validate the theoretical results.

Key words. Maxwell's equations, dispersive medium, well-posedness, the multiscale asymptotic expansion, finite element method.

1. Introduction

Consider the transient electromagnetic wave incident on a three dimensional dispersive media with a periodic microstructure or many subdivided periodic microstructures, which is called the scatter and is supposed to occupy the bounded domain $\Omega \subset \mathbb{R}^3$. Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz polyhedral convex domain or a bounded smooth domain with a microstructure as shown in Fig. 1(a). The exterior of the volume Ω is denoted by $\Omega_e = \mathbb{R}^3 \setminus \overline{\Omega}$.

Suppose that $(\mathbf{E}^{inc}, \mathbf{H}^{inc})$ is a plane wave incident on the scatter to generate scattered field $(\mathbf{E}^{sc}, \mathbf{H}^{sc})$, which satisfies the following time-domain Maxwell's equations in Ω_e for $t > 0$:

$$(1)\qquad \begin{cases} \eta_0 \partial_t \mathbf{E}^{sc}(\mathbf{x}, t) - \mathbf{curl}\, \mathbf{H}^{sc}(\mathbf{x}, t) = 0, \\ \mu_0 \partial_t \mathbf{H}^{sc}(\mathbf{x}, t) + \mathbf{curl}\, \mathbf{E}^{sc}(\mathbf{x}, t) = 0, \end{cases}$$

where η_0 and μ_0 are the constant permittivity and the constant permeability in the "air region" Ω_e, respectively. It is clear to note that $(\mathbf{E}^{inc}, \mathbf{H}^{inc})$ also satisfies the equation (1). In addition, the scattered field is required to satisfy the Silver-Müller radiation conditions:

$$(2)\qquad \hat{\mathbf{x}} \times (\partial_t \mathbf{E}^{sc} \times \hat{\mathbf{x}}) + \hat{\mathbf{x}} \times \partial_t \mathbf{H}^{sc} = o(|\mathbf{x}|^{-1}),\ \text{as}\, |\mathbf{x}| \to \infty,\ t > 0,$$

where $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$. The total field $(\mathbf{E}^{tot}, \mathbf{H}^{tot})$ in Ω_e consists of the incident field and the scattered field:

$$(3)\quad \mathbf{E}^{tot}(\mathbf{x}, t) = \mathbf{E}^{inc}(\mathbf{x}, t) + \mathbf{E}^{sc}(\mathbf{x}, t),\ \mathbf{H}^{tot}(\mathbf{x}, t) = \mathbf{H}^{inc}(\mathbf{x}, t) + \mathbf{H}^{sc}(\mathbf{x}, t),\ t > 0.$$

In this paper, we investigate the well-posedness and the multiscale algorithm for the heterogeneous scattering of Maxwell's equations in dispersive media with a periodic

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microstructure or with many subdivided periodic microstructures. For the sake of simplicity, we only discuss the corresponding problems in dispersive media with a periodic microstructure in the sequel. Let $\mathbf{E}_\varepsilon(\mathbf{x},t)$ and $\mathbf{H}_\varepsilon(\mathbf{x},t))$ be respectively the electric field and the magnetic field in the scatter, which satisfy Maxwell's equations for the time-domain Lorentz model in Ω, for $t>0$:

$$
\begin{cases}
\boldsymbol{\eta}_\varepsilon(\mathbf{x})\partial_t\mathbf{E}_\varepsilon(\mathbf{x},t) - \mathbf{curl}\ \mathbf{H}_\varepsilon(\mathbf{x},t) + \mathbf{J}_\varepsilon(\mathbf{x},t) = \mathbf{F}(\mathbf{x},t),\\
\boldsymbol{\mu}_\varepsilon(\mathbf{x})\partial_t\mathbf{H}_\varepsilon(\mathbf{x},t) + \mathbf{curl}\ \mathbf{E}_\varepsilon(\mathbf{x},t) = 0,\\
\nabla\cdot(\boldsymbol{\eta}_\varepsilon(\mathbf{x})\mathbf{E}_\varepsilon(\mathbf{x},t)) = \rho(\mathbf{x},t), \quad \nabla\cdot(\boldsymbol{\mu}_\varepsilon(\mathbf{x})\mathbf{H}_\varepsilon(\mathbf{x},t)) = 0,\\
\partial_t\mathbf{J}_\varepsilon(\mathbf{x},t) + \gamma_e\mathbf{J}_\varepsilon(\mathbf{x},t) + \omega_{e0}\displaystyle\int_0^t \mathbf{J}_\varepsilon(\mathbf{x},\tau)d\tau = \eta_\varepsilon(\mathbf{x})\omega_{pe}^2\mathbf{E}_\varepsilon(\mathbf{x},t),
\end{cases}
\tag{4}
$$

where the parameters $\boldsymbol{\eta}_\varepsilon(\mathbf{x})$ and $\boldsymbol{\mu}_\varepsilon(\mathbf{x})$ are permittivity and permeability tensor inside Ω, respectively, ω_{pe} is the electric plasma frequency, ω_{e0} is the electric resonance frequency, γ_e is electric damping frequency, the current $\mathbf{F}(\mathbf{x},t)$ is assumed to be compactly supported in Ω, and $\mathbf{J}_\varepsilon(\mathbf{x},t)$ is the polarization current density. Here $\varepsilon>0$ denotes the relative size of a periodic microstructure of heterogeneous materials, i.e. $0<\varepsilon=l_p/L<1$, where l_p, L are respectively the sizes of a periodic cell and a domain Ω. If we assume that $L=1$, without loss of generality, then the reference periodic cell Q is defined as $Q=\{\boldsymbol{\xi}=(\xi_1,\xi_2,\xi_3): 0<\xi_i<1,\ i=1,2,3\}$ as shown in Fig. 1(b). If let $\boldsymbol{\xi}=\varepsilon^{-1}\mathbf{x}$, then we have $\boldsymbol{\eta}_\varepsilon(\mathbf{x})=\boldsymbol{\eta}(\frac{\mathbf{x}}{\varepsilon})=\boldsymbol{\eta}(\boldsymbol{\xi})$ and $\boldsymbol{\mu}_\varepsilon(\mathbf{x})=\boldsymbol{\mu}(\frac{\mathbf{x}}{\varepsilon})=\boldsymbol{\mu}(\boldsymbol{\xi})$. Here $\boldsymbol{\eta}_\varepsilon^{-1}(\mathbf{x})$ and $\boldsymbol{\mu}_\varepsilon^{-1}(\mathbf{x})$ denote the inverse matrices of $\boldsymbol{\eta}_\varepsilon(\mathbf{x})$ and $\boldsymbol{\mu}_\varepsilon(\mathbf{x})$, respectively.

Remark 1.1. *It should be stated that the interaction of electrons or charged particles with an electric field is often treated classically by the equation of motion named the DLS model*[39] *where the polarization current satisfies the following equation:*

$$\partial_t\mathbf{J}_\varepsilon(\mathbf{x},t) + \gamma_e\mathbf{J}_\varepsilon(\mathbf{x},t) + \omega_{e0}\int_0^t \mathbf{J}_\varepsilon(\mathbf{x},\tau)d\tau = \eta_\varepsilon(\mathbf{x})\omega_{pe}^2\mathbf{E}_\varepsilon(\mathbf{x},t).$$

The model is often called a Lorentz oscillator which gives rise to polarization density, and thus, polarization current. Many models follow from this model. When $\omega_{e0}=0$, the model reduces to the Drude model. Furthermore, if we set $\gamma_e=0$, the cold plasma model follows. Without loss of generality, in the rest of this article, we will assume that all of the physical parameters are positive. By using the above equation, we get

$$\mathbf{J}_\varepsilon(\mathbf{x},t) = \eta_\varepsilon\omega_{pe}^2\int_0^t g(t-\tau)\mathbf{E}_\varepsilon(\mathbf{x},\tau)d\tau = \eta_\varepsilon\omega_{pe}^2 g(t)*\mathbf{E}_\varepsilon(\mathbf{x},t),$$

where $g(t)=\frac{1}{\alpha}e^{-\delta t}\sin(\alpha t)$, $\delta=\frac{\gamma_e}{2}$, $\alpha=\sqrt{\omega_{e0}^2-\delta^2}$, the symbol $*$ denotes a convolution of functions. Notice that $\omega_{e0}>\delta$ in many real applications (see, e.g., [24]*). If assume that $\omega_{e0}=0$, then we have*

$$\mathbf{J}_\varepsilon(\mathbf{x},t) = \eta_\varepsilon\omega_{pe}^2\int_0^t e^{-\gamma_e(t-\tau)}\mathbf{E}_\varepsilon(\mathbf{x},\tau)d\tau = \eta_\varepsilon\omega_{pe}^2 e^{-\gamma_e t}*\mathbf{E}_\varepsilon(\mathbf{x},t).$$

Furthermore, the transmission conditions across the boundary $\partial\Omega$ are imposed for $t>0$:

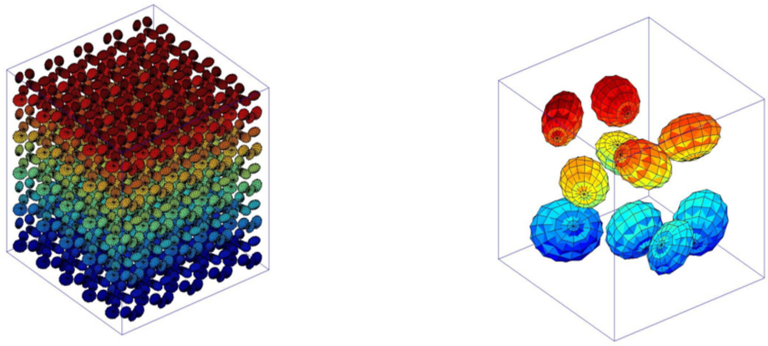
$$\mathbf{n}\times\mathbf{E}_\varepsilon = \mathbf{n}\times\mathbf{E}^{inc} + \mathbf{n}\times\mathbf{E}^{sc}, \qquad \mathbf{n}\times\mathbf{H}_\varepsilon = \mathbf{n}\times\mathbf{H}^{inc} + \mathbf{n}\times\mathbf{H}^{sc}, \tag{5}$$

where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$.

In addition, the initial conditions in Ω or Ω_e are given by

$$\mathbf{E}_\varepsilon(\mathbf{x},0)=\mathbf{U}_0(\mathbf{x}),\ \mathbf{H}_\varepsilon(\mathbf{x},0)=\mathbf{V}_0(\mathbf{x}),\ \mathbf{J}_\varepsilon(\mathbf{x},0)=\mathbf{0}, \tag{6}$$

where $\mathbf{U}_0(\mathbf{x})$ and $\mathbf{V}_0(\mathbf{x})$ are some given functions.



(a) (b)

FIGURE 1. (a) A whole domain Ω of heterogenous materials with a periodic microstructure; (b) the reference cell Q.

For the sake of simplicity, we set

$$\boldsymbol{\mathcal{E}}_\varepsilon = \begin{pmatrix} \mathbf{E}_\varepsilon \\ \mathbf{H}_\varepsilon \end{pmatrix}, \quad \mathcal{A}_\varepsilon = \begin{pmatrix} \boldsymbol{\eta}_\varepsilon & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\mu}_\varepsilon \end{pmatrix}_{6\times 6} = \Big(a_{ij}(\frac{\mathbf{x}}{\varepsilon})\Big)_{6\times 6}. \tag{7}$$

Throughout this paper, the Einstein summation convention on the repeated indices is adopted. Denote by C a generic positive constant independent of ε without distinction. Moreover, the expression $a \lesssim b$ stands for $a \leq Cb$. Denote the Sobolev spaces of the vector-valued functions with boldface letters.

Let B_r be the ball of radius r centered at the origin and choose sufficiently large r such that $\Omega \subset B_r$. We make the following assumptions on the coefficients:

($\mathbf{A}_1$). Let $\boldsymbol{\xi} = \varepsilon^{-1}\boldsymbol{x}$. Suppose that $\boldsymbol{\eta}(\boldsymbol{\xi}) = \eta(\boldsymbol{\xi})I_3$ and $\boldsymbol{\mu}(\boldsymbol{\xi}) = \mu(\boldsymbol{\xi})I_3$, where I_3 is an 3×3 identity matrix. Note that $\eta(\boldsymbol{\xi})$ and $\mu(\boldsymbol{\xi})$ are rapidly oscillating 1-periodic functions with piecewise constants, respectively.

($\mathbf{A_2}$) Let $a_{ij} = a_{ji}$, $a_{ij} \in L^\infty(\Omega)$, and

$$\gamma_0|\boldsymbol{y}|^2 \leq a_{ij}y_iy_j \leq \gamma_1|\boldsymbol{y}|^2, \quad \forall \boldsymbol{y} \in \mathbb{R}^6, \tag{8}$$

where $|\boldsymbol{y}|^2 = y_iy_i$, γ_0 and γ_1 are constants independent of ε.

($\mathbf{A}_3$). Suppose that the incident field $(\mathbf{E}^{inc}(\mathbf{x},t), \mathbf{H}^{inc}(\mathbf{x},t))$ has the traces on ∂B_r belonging to $H^2(0,T;\mathcal{X})$, where $\mathcal{X} = \mathbf{H}^{-\frac{1}{2}}(\mathrm{div};\partial B_r)$, and $\mathbf{F} \in H^1(0,T;\mathbf{L}^2(B_r))$, $\mathbf{F}(\mathbf{x},0) = \mathbf{0}$, $\mathbf{U}_0, \mathbf{V}_0 \in \mathbf{H}(\mathbf{curl}, B_r)$.

The problem (1)-(6) has many applications in electric, communication, materials science and so forth(see, e.g.,[28, 33, 36] and the references therein). We first recall some theoretical results for the well-posedness and stability analysis associated with the problem (1)-(6). The mathematical models for electromagnetic wave propagation in dispersive isotropic media were investigated in [11] by employing energy techniques, spectral theory and dispersion analysis of plane waves. The time-dependent Maxwell's equations with the constitutive relations of linear bianisotropic media were studied in [23] by applying the theory of abstract Volterra equations and strongly continuous semigroups. To our knowledge, there are few theoretical results of the well-posedness and stability analysis for the time-domain scattering problem of Maxwell's equations in heterogeneous dispersive media. In this paper we will use the method of energy, the Lax-Milgram lemma, and the inversion theorem of the Laplace transform to analyze the time-domain scattering problem of Maxwell's equations in heterogeneous dispersive media, where the latter method has also been adopted in [12, 19, 29]. The basic ideas are as follows: we first intend to analyze the scattering problem of a transient electromagnetic plane wave incident in a three-dimensional dispersive media. An exact transparent boundary condition is developed to reduce the the scattering problem into an initial-boundary value problem in heterogeneous materials. The well-posedness and stability analysis for the reduced problem and a priori estimate of the electric field are studied.

It should be mentioned that, if we solve numerically the problem (1)-(6), we will encounter some main difficulties. For example, a direct numerical method such as the finite-difference time-domain (FDTD) method or finite element method cannot produce accurate numerical solutions unless a very fine mesh is required. We recall that the homogenization method gives the overall solution behavior by incorporating the fluctuations due to the heterogeneities. There are a great number of results for the homogenization method of Maxwell's equations in heterogeneous materials (see, e.g., [3, 7, 25, 26, 32, 34, 35]). In particular, Griso et al.[1, 4, 6] used the periodic unfolding method which was introduced in [14] to derive homogenization results of the time-dependent Maxwell's equations in complex materials that are described by constitutive laws involving the time evolution of the electric polarization and magnetization. Barbatis and Stratis[2] studied the periodic homogenization of Maxwell's equations for dissipative bianisotropic media in the time domain, both in $\mathbb{R}^3$ and in a bounded domain with perfect conductor (PEC) boundary condition. However, numerous numerical results have shown that the accuracy of the homogenization method may not be satisfactory if ε is not sufficiently small (see, e.g., [8, 9, 40, 41, 42]). To this end, some multiscale methods were presented, for example, the localized orthogonal decomposition method in [18], the heterogeneous multiscale method (HMM) in [21, 13], the multiscale hybrid-mixed finite element method in [27], and the multiscale asymptotic methods in [9, 26, 40, 41]. In this paper, we present the multiscale asymptotic expansions of the solution for the reduced problem and derive the strong convergence results with an explicit rate for the multiscale asymptotic solutions. Furthermore, a multiscale Crank-Nicolson mixed finite element method is presented while the perfectly matched layer (PML) method is utilized to truncate infinite domain problems.

This paper is outlined as follows. In Section 2, we introduce an exact time-domain transparent boundary condition to reduce the transient electromagnetic scattering from dispersive media into an initial-boundary value problem in heterogeneous materials. The well-posedness and stability analysis for the reduced problem, and a priori estimate of the electric field are derived in Section 3. In Section 4, we first present the formal multiscale asymptotic expansions of the solution for the reduced problem, and then derive the strong convergence results with an explicit rate for the dispersive media with a periodic microstructure. In section 5, a multiscale Crank-Nicolson mixed finite element method for the scattering problem is proposed. Finally, some numerical results are carried out to validate the theoretical results of this paper.

2. Transparent boundary conditions

In this section, we will introduce an exact time-domain transparent boundary condition to reformulate the electromagnetic scattering problem (1)-(6) into an equivalent initial-boundary value problem. For simplicity, without confusion we still use the symbols $\mathbf{E}_\varepsilon$ and $\mathbf{H}_\varepsilon$ in the following equations, for $t > 0$:

$$
\begin{cases}
\tilde{\eta}_\varepsilon \partial_t \mathbf{E}_\varepsilon - \mathbf{curl}\,\mathbf{H}_\varepsilon + \widetilde{\mathbf{J}}_\varepsilon = \mathbf{F}(\mathbf{x},t), & \text{in } B_r,\\
\tilde{\mu}_\varepsilon \partial_t \mathbf{H}_\varepsilon + \mathbf{curl}\,\mathbf{E}_\varepsilon = \mathbf{0}, & \text{in } B_r,\\
\partial_t \mathbf{J}_\varepsilon + \gamma_e \mathbf{J}_\varepsilon + \omega_{e0} \int_0^t \mathbf{J}_\varepsilon dt = \eta_\varepsilon \omega_{pe}^2 \mathbf{E}_\varepsilon, & \text{in } \Omega,\\
\mathbf{E}_\varepsilon|_{t=0} = \mathbf{U}_0,\ \mathbf{H}_\varepsilon|_{t=0} = \mathbf{V}_0, & \text{in } B_r,\\
\widetilde{\mathbf{J}}_\varepsilon|_{t=0} = \mathbf{0}, & \text{in } B_r,\\
\mathscr{G}[\mathbf{n}\times\mathbf{E}_\varepsilon] = \mathbf{n}\times\mathbf{H}_\varepsilon + \mathbf{g}, & \text{on } \partial B_r,
\end{cases}
\tag{9}
$$

where $\mathbf{g} = -\mathbf{n}\times\mathbf{H}^{inc} + \mathscr{G}[\mathbf{n}\times\mathbf{E}^{inc}]$ and $\mathscr{G}$ is the time-domain electric-to-magnetic Calderón operator. Here notice that $\mathbf{E}_\varepsilon = \mathbf{E}^{tot}$, $\mathbf{H}_\varepsilon = \mathbf{H}^{tot}$ in $B_r \setminus \overline{\Omega}$ and

$$
\tilde{\mu}_\varepsilon = \begin{cases} \mu_\varepsilon, & \text{in}\,\Omega \\ \mu_0, & \text{in}\, B_r\setminus\overline{\Omega} \end{cases},\quad
\tilde{\eta}_\varepsilon = \begin{cases} \eta_\varepsilon, & \text{in}\,\Omega \\ \eta_0, & \text{in}\, B_r\setminus\overline{\Omega} \end{cases},\quad
\tilde{\mathbf{J}}_\varepsilon = \begin{cases} \mathbf{J}_\varepsilon, & \text{in}\,\Omega \\ 0, & \text{in}\, B_r\setminus\overline{\Omega} \end{cases},
\tag{10}
$$

where I_3 is an 3×3 identity matrix.

Let $\breve{\mathbf{E}}^{sc}(\mathbf{x},s)$ and $\breve{\mathbf{H}}^{sc}(\mathbf{x},s)$ be the Laplace transform of the scattering fields $\mathbf{E}^{sc}(\mathbf{x},t)$ and $\mathbf{H}^{sc}(\mathbf{x},t)$, respectively. Recall that

$$
\mathscr{L}(\partial_t \mathbf{E}^{sc}) = s\breve{\mathbf{E}}^{sc}(\cdot,s) - \mathbf{E}^{sc}(\cdot,0),\quad \mathscr{L}(\partial_t \mathbf{H}^{sc}) = s\breve{\mathbf{H}}^{sc}(\cdot,s) - \mathbf{H}^{sc}(\cdot,0).
\tag{11}
$$

Given a vector field $\mathbf{u}$, denote by $\mathbf{u}_T = (\mathbf{x}\times\mathbf{u})\times\mathbf{x}$ the tangential component of $\mathbf{u}$ in B_r. By virtue of the frequency domain EtM Calderón operator G_e (see, e.g., [12]), we get the following transparent boundary condition imposed on ∂B_r in the s-domain:

$$
G_e[\hat{\mathbf{x}}\times\breve{\mathbf{E}}^{sc}] = \mathbf{n}\times\breve{\mathbf{H}}^{sc}.
\tag{12}
$$

Lemma 1. (see [12, Lemma 2.5]) *It can be proved that the Calderón operator G_e satisfies the following positivity condition:*

$$
-\mathrm{Re}\int_{\partial B_r} G_e[\hat{\mathbf{x}}\times\breve{\mathbf{E}}^{sc}]\cdot\bar{\breve{\mathbf{E}}}_T^{sc}\,dS \geq 0,
$$

where $\hat{\mathbf{x}}\times\breve{\mathbf{E}}^{sc}\in\mathbf{H}^{-\frac{1}{2}}(\mathrm{div},\partial B_r)$.

We take the inverse Laplace transform of (12), and give the transparent boundary condition in the time domain on ∂B_r

$$
\mathscr{G}[\hat{\mathbf{x}}\times\mathbf{E}^{sc}] = \hat{\mathbf{x}}\times\mathbf{H}^{sc},
\tag{13}
$$

where $\mathscr{G} := \mathscr{L}^{-1}\circ G_e\circ\mathscr{L}$.

By eliminating the magnetic field, we get an alternative transparent boundary condition in the s-domain on ∂B_r as follows:

$$
\hat{\mathbf{x}}\times((s\mu_0)^{-1}\mathbf{curl}\,\breve{\mathbf{E}}^{sc}) + G_e[\hat{\mathbf{x}}\times\breve{\mathbf{E}}^{sc}] = 0.
\tag{14}
$$

Taking the inverse Laplace transform of (14) yields an alternative transparent boundary condition in the time-domain:

$$
\hat{\mathbf{x}}\times(\mu_0^{-1}\mathbf{curl}\,\mathbf{E}^{sc}) + \mathscr{C}[\hat{\mathbf{x}}\times\mathbf{E}^{sc}] = 0,
\tag{15}
$$

where $\mathscr{C} = \mathscr{L}^{-1}\circ sG_e\circ\mathscr{L}$. We thus introduce the following lemma:

Lemma 2. see [38, Lemma 4.5-4.6] *Given $\varsigma \geq 0$ and $\mathbf{E}(\cdot,t)\in\mathbf{H}(\mathbf{curl},B_r)$ with $\mathbf{E}(\cdot,0)=\mathbf{0}$, we prove*

$$
\mathrm{Re}\int_0^\varsigma\int_{\partial B_r}\Big(\int_0^t \mathscr{C}[\mathbf{n}\times\mathbf{E}](\tau)d\tau\Big)\cdot\bar{\mathbf{E}}_T(\mathbf{x},t)dSdt \leq 0,
\tag{16}
$$

and

$$\mathrm{Re}\int_0^{\varsigma}\int_{\partial B_r}\Big(\int_0^t \mathscr{C}[\mathbf{n}\times\partial_t\mathbf{E}](\tau)d\tau\Big)\cdot\partial_t\bar{\mathbf{E}}_T(\mathbf{x},t)dSdt\le 0. \tag{17}$$

Lemma 3. (see [38, Lemma 4.2-4.3]) *Given* $\varsigma\ge 0$ *and* $\mathbf{E}(\cdot,t)\in\mathbf{H}(\mathbf{curl},B_r)$ *with* $\mathbf{E}(\cdot,0)=\mathbf{0}$*, we have*

$$\mathrm{Re}\int_0^t\int_{\partial B_r}\mathscr{G}[\mathbf{n}\times\mathbf{E}]\cdot\bar{\mathbf{E}}_T\ dSdt\le 0,\quad \mathrm{Re}\int_0^t\int_{\partial B_r}\mathscr{G}[\mathbf{n}\times\partial_t\mathbf{E}]\cdot\partial_t\bar{\mathbf{E}}_T\ dSdt\le 0.$$

3. The well-posedness and stability analysis

In this section, we will give the proofs of the well-posedness, stability analysis and a priori estimate of the solution for the problem (9).

3.1. The first auxiliary problem. We first discuss the scattering problem of time-harmonic Maxwell's equations with a complex wavenumber, which is a frequency version of the problem (9) under the Laplace transform.

Consider the auxiliary boundary value problem as follows:

$$\begin{cases}\mathbf{curl}\,((s\tilde{\mu}_\varepsilon)^{-1}\mathbf{curl}\,\mathbf{u})+s\tilde{\eta}_\varepsilon\mathbf{u}+\breve{\mathbf{j}}_u=\breve{\mathbf{F}}, & \text{in } B_r,\\ \mathbf{n}\times((s\tilde{\mu}_\varepsilon)^{-1}\mathbf{curl}\,\mathbf{u})+G_e[\mathbf{n}\times\mathbf{u}]=\breve{\mathbf{g}}, & \text{on } \partial B_r,\end{cases} \tag{18}$$

where $\breve{\mathbf{j}}_u=\dfrac{s\tilde{\eta}_\varepsilon\omega_{pe}^2}{s^2+\gamma_e s+\omega_{e0}}\mathbf{u}$, and $s=s_1+is_2$ with $s_1,s_2\in\mathbb{R},\ s_1>0$.

Multiplying the complex conjugate of a test function $\mathbf{v}\in\mathbf{H}(\mathbf{curl},B_r)$, integrating over B_r, and using integration by parts, we get the variational formulation of the problem (18):

$$a(\mathbf{u},\mathbf{v})=\langle\breve{\mathbf{g}},\mathbf{v}\rangle,\quad \mathbf{v}\in\mathbf{H}(\mathbf{curl},B_r), \tag{19}$$

where the sesquilinear form is

$$\begin{aligned}a(\mathbf{u},\mathbf{v})=&\int_{B_r}(s\tilde{\mu}_\varepsilon)^{-1}\mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\bar{\mathbf{v}}d\mathbf{x}+\int_{B_r}s\tilde{\eta}_\varepsilon\mathbf{u}\cdot\bar{\mathbf{v}}d\mathbf{x}\\ &+\int_\Omega\frac{s\eta_\varepsilon\omega_{pe}^2}{s^2+\gamma_e s+\omega_{e0}}\mathbf{u}\cdot\bar{\mathbf{v}}d\mathbf{x}-\langle G_e[\mathbf{n}\times\mathbf{u}],\mathbf{v}\rangle.\end{aligned}$$

For the well-posedness of the variational problem (19) and stability analysis, we have the following theorem:

Theorem 1. *The variational problem* (19) *has a unique solution* $\mathbf{u}\in\mathbf{H}(\mathbf{curl},B_r)$ *which satisfies*

$$\begin{aligned}&\|\mathbf{curl}\,\mathbf{u}\|_{\mathbf{L}^2(B_r)}+\|s\mathbf{u}\|_{\mathbf{L}^2(B_r)}+\|\breve{\mathbf{j}}_u\|_{\mathbf{L}^2(\Omega)}\lesssim s_1^{-1}\|s\mathbf{g}\|_{\mathbf{H}^{-1/2}(div,\partial B_r)}\\ &\quad+s_1^{-1}\||s|^2\mathbf{g}\|_{\mathbf{H}^{-1/2}(div,\partial B_r)}+s_1^{-1}\|s\breve{\mathbf{F}}\|_{\mathbf{L}^2(B_r)},\end{aligned} \tag{20}$$

where $s=s_1+is_2$ *with* $s_1,s_2\in\mathbb{R}$ *and* $s_1>0$.

Proof. Setting $\mathbf{v}=\mathbf{u}$ in (19), we get

$$\begin{aligned}a(\mathbf{u},\mathbf{u})=&\int_{B_r}(s\tilde{\mu}_\varepsilon)^{-1}|\mathbf{curl}\,\mathbf{u}|^2d\mathbf{x}+\int_{B_r}s\tilde{\eta}_\varepsilon|\mathbf{u}|^2d\mathbf{x}-\langle\mathscr{B}[n\times\mathbf{u}],\mathbf{u}\rangle\\ &+\int_\Omega\frac{1}{\eta_\varepsilon\omega_{pe}^2}\bar{s}|\breve{\mathbf{j}}_u|^2+\frac{\gamma_e}{\eta_\varepsilon\omega_{pe}^2}|\breve{\mathbf{j}}_u|^2+\frac{\omega_{e0}}{s\eta_\varepsilon\omega_{pe}^2}|\breve{\mathbf{j}}_u|^2d\mathbf{x}.\end{aligned} \tag{21}$$

Here we use the fact that $\mathbf{u}=\dfrac{1}{\tilde{\eta}_\varepsilon\omega_{pe}^2}(s+\gamma_e+\omega_{e0}/s)\breve{\mathbf{j}}_u$.

Following the lines of the proof of Theorem 2.1 of [12](see also Theorem 3.1 of [19] and Theorem 3.1 of [29]), one can complete the proof of this theorem. □

3.2. The second auxiliary problem. Consider an auxiliary initial-boundary value problem for the time-domain Maxwell's equations with the perfect conductor(PEC) boundary condition:

$$
\begin{cases}
\tilde{\mu}_\varepsilon \partial_t \mathbf{V}(\mathbf{x},t) + \mathbf{curl}\,\mathbf{U}(\mathbf{x},t) = \mathbf{0}, & \text{in } \mathrm{B_r},\\
\tilde{\eta}_\varepsilon \partial_t \mathbf{U}(\mathbf{x},t) - \mathbf{curl}\,\mathbf{V}(\mathbf{x},t) + \widetilde{\mathbf{J}}_u = \mathbf{0}, & \text{in } \mathrm{B_r},\\
\partial_t \mathbf{J}_u + \gamma_e \mathbf{J}_u + \omega_{e0}\int_0^t \mathbf{J}_u dt = \eta_\varepsilon \omega_{pe}^2 \mathbf{U}(\mathbf{x},t), & \text{in } \Omega,\\
\mathbf{n}\times\mathbf{U}(\mathbf{x},t) = \mathbf{0}, & \text{on } \partial\mathrm{B_r},\\
\mathbf{U}(\mathbf{x},0) = \mathbf{U}_0,\ \mathbf{V}(\mathbf{x},0) = \mathbf{V}_0, & \text{in } \mathrm{B_r},\\
\widetilde{\mathbf{J}}_u|_{t=0} = \mathbf{0}, & \text{in } \mathrm{B_r},
\end{cases}
\tag{22}
$$

where $\mathbf{U}_0$ and $\mathbf{V}_0$ are assumed to be compactly supported in B_r, $\widetilde{\mathbf{J}}_u = \mathbf{J}_u$ in Ω and $\widetilde{\mathbf{J}}_u = 0$ in $B_r \setminus \overline{\Omega}$. By the similar arguments as those presented in [19, 29], we can prove the following theorem:

Theorem 2. *The auxiliary problem (22) has a unique solution* $(\mathbf{U},\mathbf{V})$*, which satisfies the stability estimates:*

$$
\begin{aligned}
&\|\mathbf{U}\|_{\mathbf{L}^2(B_r)} + \|\mathbf{V}\|_{\mathbf{L}^2(B_r)} \lesssim \|\mathbf{U}_0\|_{\mathbf{L}^2(B_r)} + \|\mathbf{V}_0\|_{\mathbf{L}^2(B_r)},\\
&\|\partial_t\mathbf{U}\|_{\mathbf{L}^2(B_r)} + \|\partial_t\mathbf{V}\|_{\mathbf{L}^2(B_r)} \lesssim \|\mathbf{U}_0\|_{\mathbf{L}^2(B_r)} + \|\mathbf{curl}\,\mathbf{U}_0\|_{\mathbf{L}^2(B_r)} + \|\mathbf{curl}\,\mathbf{V}_0\|_{\mathbf{L}^2(B_r)},\\
&\|\partial_{tt}\mathbf{U}\|_{\mathbf{L}^2(B_r)} + \|\partial_{tt}\mathbf{V}\|_{\mathbf{L}^2(B_r)} \lesssim \Big\{ \|\mathbf{U}_0\|_{\mathbf{L}^2(B_r)} + \|\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}_0\|_{\mathbf{L}^2(B_r)}\\
&\qquad + \|\mathbf{curl}\,\mathbf{curl}\,\mathbf{V}_0\|_{\mathbf{L}^2(B_r)} + \|\mathbf{curl}\,\mathbf{V}_0\|_{\mathbf{L}^2(B_r)} \Big\}.
\end{aligned}
\tag{23}
$$

3.3. The well-posedness and stability analysis for the problem (9). In this section, we derive the theoretical results of the well-posedness and stability analysis for the problem (9). For the sake of convenience, set $\mathbf{E} = \mathbf{E}_\varepsilon$, $\mathbf{H} = \mathbf{H}_\varepsilon$ and $\widetilde{\mathbf{J}} = \widetilde{\mathbf{J}}_\varepsilon$.

3.3.1. The well-posedness. Let $\mathbf{e} = \mathbf{E} - \mathbf{U}$ and $\mathbf{h} = \mathbf{H} - \mathbf{V}$. It follows from (9) and (22) that $\mathbf{e}$ and $\mathbf{h}$ satisfy the following initial-boundary value problem, for $t > 0$:

$$
\begin{cases}
\mathbf{curl}\,\mathbf{e} + \tilde{\mu}_\varepsilon \partial_t \mathbf{h} = \mathbf{0}, \quad \mathbf{curl}\,\mathbf{h} - \tilde{\eta}_\varepsilon \partial_t \mathbf{e} - \tilde{\mathbf{j}}_e = \mathbf{F}, & \text{in } \mathrm{B_r},\\
\partial_t \mathbf{j}_e + \gamma_e \mathbf{j}_e + \omega_{e0}\displaystyle\int_0^t \mathbf{j}_e(\mathbf{x},\tau)d\tau = \eta_\varepsilon \omega_{pe}^2 \mathbf{e}, & \text{in } \Omega,\\
\mathbf{e}|_{t=0} = \mathbf{0},\ \mathbf{h}|_{t=0} = \mathbf{0}, & \text{in } \mathrm{B_r},\\
\tilde{\mathbf{j}}_e|_{t=0} = \mathbf{0}, & \text{in } \mathrm{B_r},\\
\mathscr{T}[\mathbf{n}\times\mathbf{e}] = \mathbf{n}\times\mathbf{h} - \mathbf{n}\times\mathbf{V} + \mathbf{g}, & \text{on } \partial\mathrm{B_r},
\end{cases}
\tag{24}
$$

where $\tilde{\mathbf{j}}_e = \mathbf{j}_e$ in Ω and $\tilde{\mathbf{j}}_e = 0$ in $B_r \setminus \overline{\Omega}$, $\mathbf{g} = -\mathbf{n}\times\mathbf{H}^{inc} + \mathscr{G}[\mathbf{n}\times\mathbf{E}^{inc}]$.

Let $\breve{\mathbf{e}} = \mathscr{L}\mathbf{e}$ and $\breve{\mathbf{h}} = \mathscr{L}\mathbf{h}$. Taking the Laplace transform of (24) and eliminating $\breve{\mathbf{h}}$, we get the following boundary value problem:

$$
\begin{cases}
\mathbf{curl}\,(s\tilde{\mu}_\varepsilon)^{-1}\mathbf{curl}\,\breve{\mathbf{e}} + s\tilde{\eta}_\varepsilon \breve{\mathbf{e}} + \breve{\mathbf{j}}_e = -\breve{\mathbf{F}}, & \text{in } \mathrm{B_r},\\
\mathbf{n}\times((s\tilde{\mu}_\varepsilon)^{-1}\mathbf{curl}\,\breve{\mathbf{e}}) + \mathscr{B}[\mathbf{n}\times\breve{\mathbf{e}}] = \mathbf{n}\times\breve{\mathbf{V}} - \breve{\mathbf{g}}, & \text{on } \partial\mathrm{B_r},
\end{cases}
\tag{25}
$$

where $\breve{\mathbf{j}}_e = \dfrac{s\eta_\varepsilon\omega_{pe}^2}{s^2+\gamma_e s+\omega_{e0}}\breve{\mathbf{e}}$.

Multiplying the complex conjugate of a test function $\mathbf{v} \in \mathbf{H}(\mathbf{curl}, B_r)$, integrating in B_r, and using integration by parts, we obtain the variational formulation of the problem (25):

$$a(\breve{\mathbf{e}}, \mathbf{v}) = \langle \breve{\mathbf{g}} - \mathbf{n} \times \breve{\mathbf{V}}, \mathbf{v}\rangle + \int_{B_r} \breve{\mathbf{F}} \cdot \bar{\mathbf{v}} d\mathbf{x}, \quad \mathbf{v} \in \mathbf{H}(\mathbf{curl}, B_r), \tag{26}$$

where the sesquilinear form is given by

$$\begin{aligned} a(\breve{\mathbf{e}}, \mathbf{v}) = &\int_{B_r} (s\tilde{\mu}_\varepsilon)^{-1} \mathbf{curl}\, \breve{\mathbf{e}} \cdot \mathbf{curl}\, \bar{\mathbf{v}} d\mathbf{x} + \int_{B_r} s\tilde{\eta}_\varepsilon \breve{\mathbf{e}} \cdot \bar{\mathbf{v}} d\mathbf{x} \\ &+ \int_\Omega \frac{s\eta_\varepsilon\omega_{pe}^2}{s^2+\gamma_e s+\omega_{e0}} \breve{\mathbf{e}} \cdot \bar{\mathbf{v}} d\mathbf{x} - \langle G_e[\mathbf{n} \times \breve{\mathbf{e}}], \mathbf{v}\rangle. \end{aligned} \tag{27}$$

Next we give the proof of the well-posedness for the problem (26).

Lemma 4. *The problem* (26) *has a unique solution* $\breve{\mathbf{e}} \in \mathbf{H}(\mathbf{curl}, B_r)$ *such that*

$$\begin{aligned} &\|\mathbf{curl}\, \breve{\mathbf{e}}\|_{\mathbf{L}^2(B_r)} + \|s\breve{\mathbf{e}}\|_{\mathbf{L}^2(B_r)} + \|\breve{\mathbf{j}}_e\|_{\mathbf{L}^2(\Omega)} \lesssim s_1^{-1} \Big\{ \|s\mathbf{g}\|_{\mathcal{X}} \\ &+ \|s\mathbf{n} \times \breve{\mathbf{V}}\|_{\mathcal{X}} + \||s|^2 \mathbf{g}\|_{\mathcal{X}} + \||s|^2 \mathbf{n} \times \breve{\mathbf{V}}\|_{\mathcal{X}} + \|s\breve{\mathbf{F}}\|_{\mathbf{L}^2(B_r)} \Big\}, \end{aligned} \tag{28}$$

where $s = s_1 + is_2$ *with* $s_1, s_2 \in \mathbb{R}$ *and* $s_1 > 0$.

Proof. Setting $\mathbf{v} = \breve{\mathbf{e}}$ in (27), we obtain

$$\begin{aligned} a(\breve{\mathbf{e}}, \breve{\mathbf{e}}) = &\int_{B_r} (s\tilde{\mu}_\varepsilon)^{-1} |\mathbf{curl}\, \breve{\mathbf{e}}|^2 d\mathbf{x} + \int_{B_r} s\tilde{\eta}_\varepsilon |\breve{\mathbf{e}}|^2 d\mathbf{x} - \langle \mathscr{B}[n \times \breve{\mathbf{e}}], \breve{\mathbf{e}}\rangle \\ &+ \int_\Omega \frac{1}{\eta_\varepsilon\omega_{pe}^2} \bar{s} |\breve{\mathbf{j}}_e|^2 + \frac{\gamma_e}{\eta_\varepsilon\omega_{pe}^2} |\breve{\mathbf{j}}_e|^2 + \frac{\omega_{e0}}{s\eta_\varepsilon\omega_{pe}^2} |\breve{\mathbf{j}}_e|^2 d\mathbf{x}. \end{aligned} \tag{29}$$

Taking the real part of (29) and using Lemma 1, we get

$$\begin{aligned} &\mathrm{Re}\{a(\breve{\mathbf{e}}, \breve{\mathbf{e}})\} \gtrsim \frac{s_1}{|s|^2} (\|\mathbf{curl}\, \breve{\mathbf{e}}\|^2_{\mathbf{L}^2(B_r)} + \|s\breve{\mathbf{e}}\|^2_{\mathbf{L}^2(B_r)} + \|\breve{\mathbf{j}}_e\|^2_{\mathbf{L}^2(\Omega)} \\ &+ \|s\breve{\mathbf{j}}_e\|^2_{\mathbf{L}^2(\Omega)}) + \|\breve{\mathbf{j}}_e\|^2_{\mathbf{L}^2(\Omega)}. \end{aligned} \tag{30}$$

It follows from the Lax-Milgram Lemma that the problem (19) has a unique solution $\breve{\mathbf{e}} \in \mathbf{H}(\mathbf{curl}, B_r)$. Furthermore, from (21) we prove

$$\begin{aligned} \|a(\breve{\mathbf{e}}, \breve{\mathbf{e}})\| \lesssim\ & \|\mathbf{g} - \mathbf{n} \times \breve{\mathbf{V}}\|_{\mathcal{X}} \|\breve{\mathbf{e}}\|_{\mathbf{H}^{-\frac{1}{2}}(curl, \partial B_r)} + \|s^{-1}\breve{\mathbf{F}}\|_{\mathbf{L}^2(B_r)} \|s\breve{\mathbf{e}}\|_{\mathbf{L}^2(B_r)} \\ &\lesssim \|\mathbf{g} - \mathbf{n} \times \breve{\mathbf{V}}\|_{\mathcal{X}} \|\breve{\mathbf{e}}\|_{\mathbf{H}(\mathbf{curl}, B_r)} + \|s^{-1}\breve{\mathbf{F}}\|_{\mathbf{L}^2(B_r)} \|s\breve{\mathbf{e}}\|_{\mathbf{L}^2(B_r)} \\ &\lesssim (\|s^{-1}\mathbf{g}\|_{\mathcal{X}} + \|s^{-1}\mathbf{n} \times \breve{\mathbf{V}}\|_{\mathcal{X}}) \|s\breve{\mathbf{e}}\|_{\mathbf{L}^2(B_r)} \\ &+ (\|\mathbf{g}\|_{\mathcal{X}} + \|\mathbf{n} \times \breve{\mathbf{V}}\|_{\mathcal{X}}) \|\mathbf{curl}\, \breve{\mathbf{e}}\|_{\mathbf{L}^2(B_r)} + \|s^{-1}\breve{\mathbf{F}}\|_{\mathbf{L}^2(B_r)} \|s\breve{\mathbf{e}}\|_{\mathbf{L}^2(B_r)}. \end{aligned} \tag{31}$$

Combining (30)-(31) leads to

$$\begin{aligned} &\|\nabla \times \breve{\mathbf{e}}\|^2_{\mathbf{L}^2(B_r)} + \|s\breve{\mathbf{e}}\|^2_{\mathbf{L}^2(B_r)} + \|s\breve{\mathbf{j}}_e\|^2_{\mathbf{L}^2(\Omega)} + \frac{|s|^2}{s_1} \|\breve{\mathbf{j}}_e\|^2_{\mathbf{L}^2(\Omega)} \\ &\lesssim s_1^{-1} (\|s\mathbf{g}\|_{\mathcal{X}} + \|s\mathbf{n} \times \breve{\mathbf{V}}\|_{\mathcal{X}}) \|s\mathbf{e}\|_{\mathbf{L}^2(B_r)} \\ &+ s_1^{-1} (\||s|^2\mathbf{g}\|_{\mathcal{X}} + \||s|^2\mathbf{n} \times \breve{\mathbf{V}}\|_{\mathcal{X}}) \|\mathbf{curl}\, \mathbf{E}\|_{\mathbf{L}^2(B_r)} \\ &+ s_1^{-1} \|s\breve{\mathbf{F}}\|_{\mathbf{L}^2(B_r)} \|s\breve{\mathbf{e}}\|_{\mathbf{L}^2(B_r)}. \end{aligned} \tag{32}$$

Using the Cauchy-Schwarz inequality completes the proof of (28). □

Theorem 3. *Under assumptions* (A_1)–(A_3)*, it can be proved that the problem (9) has a unique solution* $(\mathbf{E},\mathbf{H})$ *which satisfies*

$$\mathbf{E}\in L^2(0,T;\mathbf{H}(\mathbf{curl},B_r))\cap H^1(0,T;\mathbf{L}^2(B_r)),$$
$$\mathbf{H}\in L^2(0,T;\mathbf{H}(\mathbf{curl},B_r))\cap H^1(0,T;\mathbf{L}^2(B_r)).$$

Furthermore, we have the following stability estimates:

$$\begin{aligned}&\max_{t\in[0,T]}(\|\partial_t\mathbf{E}\|_{\mathbf{L}^2(B_r)}+\|\partial_t\mathbf{H}\|_{\mathbf{L}^2(B_r)}+\|\mathbf{curl}\,\mathbf{E}\|_{\mathbf{L}^2(B_r)}+\|\mathbf{curl}\,\mathbf{H}\|_{\mathbf{L}^2(B_r)}\\ &+\|\partial_t\tilde{\mathbf{J}}\|_{\mathbf{L}^2(B_r)})\lesssim\max_{t\in[0,T]}\|\partial_t\mathbf{g}\|_{\mathcal{X}}+\|\partial_{tt}\mathbf{g}\|_{L^1(0,T;\mathcal{X})},+\|\mathbf{g}\|_{L^1(0,T;\mathcal{X})}\\ &+\|\mathbf{U}_0\|_{\mathbf{H}(\mathbf{curl},B_r)}+\|\mathbf{V}_0\|_{\mathbf{H}(\mathbf{curl},B_r)}+\|\mathbf{F}\|_{H^1(0,T;\mathbf{L}^2(B_r))}.\end{aligned}\tag{33}$$

Proof. Since

$$\int_0^T\big(\|\mathbf{curl}\,\mathbf{e}\|^2_{\mathbf{L}^2(B_r)}+\|\partial_t\mathbf{e}\|^2_{\mathbf{L}^2(B_r)}\big)dt\lesssim\int_0^\infty e^{-2s_1t}\big(\|\mathbf{curl}\,\mathbf{e}\|^2_{\mathbf{L}^2(B_r)}+\|\partial_t\mathbf{e}\|^2_{\mathbf{L}^2(B_r)}\big)dt,$$

it suffices to estimate the integral

$$\int_0^\infty e^{-2s_1t}\big(\|\mathbf{curl}\,\mathbf{e}\|^2_{\mathbf{L}^2(B_r)}+\|\partial_t\mathbf{e}\|^2_{\mathbf{L}^2(B_r)}\big)dt.$$

We take the Laplace transform on both sides of (24) and obtain

$$\begin{cases}\mathbf{curl}\,\breve{\mathbf{e}}+s\tilde{\mu}_\varepsilon\breve{\mathbf{h}}=\mathbf{0},\quad \mathbf{curl}\,\breve{\mathbf{h}}-s\tilde{\eta}_\varepsilon\breve{\mathbf{e}}-\breve{\mathbf{j}}_e=\breve{\mathbf{F}}, & \text{in } B_r,\\ \mathscr{B}[\mathbf{n}\times\breve{\mathbf{e}}]=\mathbf{n}\times\breve{\mathbf{h}}-\mathbf{n}\times\breve{\mathbf{V}}+\breve{\mathbf{g}}, & \text{on } \partial B_r.\end{cases}\tag{34}$$

It follows from Lemma 4 that

$$\begin{aligned}&\|\mathbf{curl}\,\breve{\mathbf{e}}\|_{\mathbf{L}^2(B_r)}+\|s\breve{\mathbf{e}}\|_{\mathbf{L}^2(B_r)}\lesssim s_1^{-1}(\|s\mathbf{g}\|_{\mathcal{X}}+\|s\mathbf{n}\times\breve{\mathbf{V}}\|_{\mathcal{X}}\\ &+\||s|^2\mathbf{g}\|_{\mathcal{X}}+\||s|^2\mathbf{n}\times\breve{\mathbf{V}}\|_{\mathcal{X}}+\|s\breve{\mathbf{F}}\|_{\mathbf{L}^2(B_r)}).\end{aligned}\tag{35}$$

Combining (34) and (28) implies

$$\begin{aligned}&\|\mathbf{curl}\,\breve{\mathbf{h}}\|_{\mathbf{L}^2(B_r)}+\|s\breve{\mathbf{h}}\|_{\mathbf{L}^2(B_r)}\lesssim s_1^{-1}(\|s\mathbf{g}\|_{\mathcal{X}}+\|s\mathbf{n}\times\breve{\mathbf{V}}\|_{\mathcal{X}}\\ &+\||s|^2\mathbf{g}\|_{\mathcal{X}}+\||s|^2\mathbf{n}\times\breve{\mathbf{V}}\|_{\mathcal{X}}+\|s\breve{\mathbf{F}}\|_{\mathbf{L}^2(B_r)})+\|\breve{\mathbf{F}}\|_{\mathbf{L}^2(B_r)}.\end{aligned}$$

It follows from Lemma 44.1 of [37] that $\breve{\mathbf{e}}$ and $\breve{\mathbf{h}}$ are holomorphic functions of s on the half-plane $s_1>\gamma>0$, where γ is a positive constant. By applying Theorem 43.1 of [37], we can show that there is the inverse Laplace transform of $\breve{\mathbf{e}}$ and $\breve{\mathbf{h}}$, which is supported in $[0,\infty]$.

Let $\mathbf{e}=\mathscr{L}^{-1}(\breve{\mathbf{e}})$, and $\mathbf{h}=\mathscr{L}^{-1}(\breve{\mathbf{h}})$. Since $\breve{\mathbf{e}}=\mathscr{L}(\mathbf{e})=\mathscr{F}(e^{-s_1t}\mathbf{e})$, where $\mathscr{F}$ is the fourier transform with respect to s_2, it follows from the Plancherel identity [15, (2.46)] and (35) that

$$\begin{aligned}&\int_0^\infty e^{-2s_1t}\big(\|\mathbf{curl}\,\mathbf{e}\|^2_{\mathbf{L}^2(B_r)}+\|\partial_t\mathbf{e}\|^2_{\mathbf{L}^2(B_r)}\big)dt\\ &=\frac{1}{2\pi}\int_{-\infty}^\infty\Big\{\|\mathbf{curl}\,\breve{\mathbf{e}}\|^2_{\mathbf{L}^2(B_r)}+\|s\breve{\mathbf{e}}\|^2_{\mathbf{L}^2(B_r)}\Big\}ds_2\\ &\lesssim s_1^{-2}\int_{-\infty}^\infty\Big\{\|s\mathbf{g}\|^2_{\mathcal{X}}+\|s\mathbf{n}\times\breve{\mathbf{V}}\|^2_{\mathcal{X}}+\||s|^2\mathbf{g}\|^2_{\mathcal{X}}\\ &+\||s|^2\mathbf{n}\times\breve{\mathbf{V}}\|^2_{\mathcal{X}}+\|s\breve{\mathbf{F}}\|^2_{\mathbf{L}^2(B_r)}\Big\}ds_2.\end{aligned}$$

Recalling that $\mathbf{F}_{|t=0}=0$ in B_r, $\mathbf{g}|_{t=0}=\partial_t\mathbf{g}|_{t=0}=0$ and $\mathbf{n}\times\mathbf{V}|_{t=0}=\partial_t(\mathbf{n}\times\mathbf{V})|_{t=0}=0$ on ∂B_r, we have $\mathscr{L}(\partial_t\mathbf{F})=s\breve{\mathbf{F}}$ in B_r, $\mathscr{L}(\partial_t\mathbf{g})=s\mathbf{g}$ and $\mathscr{L}(\partial_t(\mathbf{n}\times\mathbf{V}))=$

$s\mathbf{n}\times\check{\mathbf{V}}$ on ∂B_r. It is not difficult to check that

$$|s|^2\check{\mathbf{g}} = (2s_1 - s)s\check{\mathbf{g}} = 2s_1\mathscr{L}(\partial_t\mathbf{g}) - \mathscr{L}(\partial_{tt}\mathbf{g}),$$
$$|s|^2\mathbf{n}\times\check{\mathbf{V}} = (2s_1 - s)s\mathbf{n}\times\check{\mathbf{V}} = 2s_1\mathscr{L}(\partial_t(\mathbf{n}\times\mathbf{V})) - \mathscr{L}(\partial_{tt}(\mathbf{n}\times\mathbf{V})).$$

We thus have

$$\begin{aligned}
&\int_0^\infty e^{-2s_1t}\big(\|\mathbf{curl}\,\mathbf{e}\|^2_{\mathbf{L}^2(B_r)} + \|\partial_t\mathbf{e}\|^2_{\mathbf{L}^2(B_r)}\big)dt\\
&\lesssim(1+s_1^{-2})\int_{-\infty}^{\infty}\|\mathscr{L}(\partial_t\mathbf{g})\|^2_{\mathcal{X}} + s_1^{-2}\int_{-\infty}^{\infty}\|\mathscr{L}(\partial_{tt}\mathbf{g})\|^2_{\mathcal{X}} + \|\mathscr{L}(\partial_t\mathbf{F})\|^2_{\mathbf{L}^2(B_r)}ds_2\\
&\quad+(1+s_1^{-2})\int_{-\infty}^{\infty}\|\mathscr{L}(\partial_t(\mathbf{n}\times\mathbf{V})\|^2_{\mathcal{X}} + s_1^{-2}\int_{-\infty}^{\infty}\|\mathscr{L}(\partial_{tt}(\mathbf{n}\times\mathbf{V}))\|^2_{\mathcal{X}}ds_2\\
&\lesssim(1+s_1^{-2})\int_0^\infty e^{-2s_1t}\|\partial_t\mathbf{g}\|^2_{\mathcal{X}}dt + s_1^{-2}\int_0^\infty e^{-2s_1t}(\|\partial_{tt}\mathbf{g}\|^2_{\mathcal{X}} + \|\partial_t\mathbf{F}\|^2_{\mathbf{L}^2(B_r)})dt\\
&\quad+(1+s_1^{-2})\int_0^\infty e^{-2s_1t}\|\partial_t(\mathbf{n}\times\mathbf{V})\|^2_{\mathcal{X}}dt + s_1^{-2}\int_0^\infty e^{-2s_1t}\|\partial_{tt}(\mathbf{n}\times\mathbf{V})\|^2_{\mathcal{X}}dt,
\end{aligned}$$

and further $\mathbf{e}\in L^2(0,T;\mathbf{H}(\mathbf{curl},B_r))\cap\mathbf{H}^1(0,T;\mathbf{L}^2(B_r))$. Similarly, we show that $\mathbf{h}\in L^2(0,T;\mathbf{H}(\mathbf{curl},B_r))\cap\mathbf{H}^1(0,T;\mathbf{L}^2(B_r))$. Next it turns to the stability analysis. For $t\in(0,T)$, consider an energy function

$$e(t) = \|\tilde{\eta}_\varepsilon^{1/2}\mathbf{E}(\cdot,t)\|^2_{\mathbf{L}^2(B_r)} + \|\tilde{\mu}_\varepsilon^{1/2}\mathbf{H}(\cdot,t)\|^2_{\mathbf{L}^2(B_r)} + \|(\eta_\varepsilon^{1/2}\omega_{pe})^{-1}\mathbf{J}\|^2_{\mathbf{L}^2(\Omega)}.$$

We observe that

$$\begin{aligned}
\int_0^t e'(\tau)d\tau &= \|\tilde{\eta}_\varepsilon^{1/2}\mathbf{E}(\cdot,t)\|^2_{\mathbf{L}^2(B_r)} + \|\tilde{\mu}_\varepsilon^{1/2}\mathbf{H}(\cdot,t)\|^2_{\mathbf{L}^2(B_r)} + \|(\eta_\varepsilon^{1/2}\omega_{pe})^{-1}\mathbf{J}(\cdot,t)\|^2_{\mathbf{L}^2(\Omega)}\\
&\quad-\|\tilde{\eta}_\varepsilon^{1/2}\mathbf{U}_0\|^2_{\mathbf{L}^2(B_r)} + \|\tilde{\mu}_\varepsilon^{1/2}\mathbf{V}_0\|^2_{\mathbf{L}^2(B_r)}.
\end{aligned}$$

Using (9), Lemma 2, and integrating by parts, we have

$$\begin{aligned}
\int_0^t e'(\tau)d\tau &= 2\mathrm{Re}\int_0^t\int_{B_r}\tilde{\eta}_\varepsilon\partial_t\mathbf{E}\cdot\bar{\mathbf{E}} + \tilde{\mu}_\varepsilon\partial_t\mathbf{H}\cdot\bar{\mathbf{H}} + \frac{1}{\tilde{\eta}_\varepsilon\omega_{pe}^2}\partial_t\tilde{\mathbf{J}}\cdot\bar{\tilde{\mathbf{J}}}d\mathbf{x}d\tau\\
&= 2\mathrm{Re}\int_0^t\int_{B_r}\mathbf{curl}\;\mathbf{H}\cdot\bar{\mathbf{E}} - \mathbf{curl}\;\mathbf{E}\cdot\bar{\mathbf{H}}d\mathbf{x} - \gamma_e\tilde{\mathbf{J}}\cdot\bar{\tilde{\mathbf{J}}} - \omega_{e0}\int_0^\tau\tilde{\mathbf{J}}d\zeta\cdot\bar{\tilde{\mathbf{J}}}\\
&\quad+\mathbf{E}\cdot\bar{\tilde{\mathbf{J}}} - \tilde{\mathbf{J}}\cdot\bar{\mathbf{E}} + \mathbf{F}\cdot\bar{\mathbf{E}}d\mathbf{x}d\tau\\
&\le -2\mathrm{Re}\int_0^t\int_{\partial B_r}\mathbf{g}\cdot\bar{\mathbf{E}}_T dSd\tau + 2\mathrm{Re}\int_0^t\int_{B_r}\mathbf{F}\cdot\bar{\mathbf{E}}d\mathbf{x}d\tau\\
&\lesssim\|\mathbf{g}\|_{L^1(0,T;\mathcal{X})}\max_{t\in[0,T]}\|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl},B_r)} + \|\mathbf{F}\|_{L^1(0,T;\mathbf{L}^2(B_r))}\max_{t\in[0,T]}\|\mathbf{E}\|_{\mathbf{L}^2(B_r)}.
\end{aligned} \tag{36}$$

We take the partial derivative with respect to t on both sides of (9), and get the equations of $\partial_t\mathbf{E},\partial_t\mathbf{H}$ and $\partial_t\tilde{\mathbf{J}}$ with the source $\partial_t\mathbf{g} = -\hat{\mathbf{x}}\times\partial_t\mathbf{H}^{inc}+\mathscr{G}[\hat{\mathbf{x}}\times\partial_t\mathbf{E}^{inc}]$ and the initial condition $\partial_t\mathbf{E}|_{t=0} = \tilde{\eta}_\varepsilon^{-1}\mathbf{curl}\,\mathbf{V}_0$, $\partial_t\mathbf{H}|_{t=0} = \tilde{\mu}_\varepsilon^{-1}\mathbf{curl}\,\mathbf{U}_0$, and $\partial_t\tilde{\mathbf{J}}|_{t=0} = \eta_\varepsilon\omega_{pe}^2\mathbf{U}_0$. Repeating the process of the proof of (36) for $\partial_t\mathbf{E},\partial_t\mathbf{H}$ and $\partial_t\tilde{\mathbf{J}}$, we can prove

$$
\begin{aligned}
&\|\tilde{\eta}_\varepsilon^{1/2}\partial_t\mathbf{E}(\cdot,t)\|^2_{\mathbf{L}^2(B_r)} + \|\tilde{\mu}_\varepsilon^{1/2}\partial_t\mathbf{H}(\cdot,t)\|^2_{\mathbf{L}^2(B_r)} + \|(\eta_\varepsilon^{1/2}\omega_{pe})^{-1}\partial_t\mathbf{J}\|^2_{\mathbf{L}^2(\Omega)}\\
&\quad -\Big\{\|\tilde{\eta}_\varepsilon^{-1/2}\mathbf{curl}\,\mathbf{V}_0\|^2_{\mathbf{L}^2(B_r)} + \|\tilde{\mu}_\varepsilon^{-1/2}\mathbf{curl}\,\mathbf{U}_0\|^2_{\mathbf{L}^2(B_r)} + \|\tilde{\eta}_\varepsilon^{1/2}\omega_{pe}\mathbf{U}_0\|^2_{\mathbf{L}^2(B_r)}\Big\}\\
&\le -2Re\int_{\partial B_r}\partial_\tau\mathbf{g}\cdot\bar{\mathbf{E}}_T(t)dS + 2Re\int_0^t\int_{\partial B_r}\partial_{\tau\tau}\mathbf{g}\cdot\bar{\mathbf{E}}_T dSd\tau\\
&\quad +2Re\int_0^t\int_{B_r}\partial_\tau\mathbf{F}\cdot\partial_\tau\bar{\mathbf{E}}d\mathbf{x}d\tau\\
&\lesssim \max_{t\in[0,T]}\|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl},B_r)}\big(\max_{t\in[0,T]}\|\partial_t\mathbf{g}\|_{\mathcal{X}} + \|\partial_{tt}\mathbf{g}\|_{L^1(0,T;\mathcal{X})}\big)\\
&\quad + \max_{t\in[0,T]}\|\partial_t\mathbf{E}\|_{\mathbf{L}^2(B_r)}\|\partial_t\mathbf{F}\|_{L^1(0,T;\mathbf{L}^2(B_r))}.
\end{aligned}
$$

Combining with (36) ends the proof of (33). □

3.3.2. A prior estimate. Next we will derive a prior estimate for the electric field. To demonstrate the impact of the different sources on the final estimate, we consider the following initial-boundary value problems, for $t > 0$:

$$
\begin{cases}
\tilde{\eta}_\varepsilon\partial_{tt}\mathbf{E} + \mathbf{curl}\,(\tilde{\mu}_\varepsilon^{-1}\mathbf{curl}\,\mathbf{E}) + \partial_t\tilde{\mathbf{J}} = \mathbf{0}, & \text{in } B_r,\\
\partial_t\mathbf{J} + \gamma_e\mathbf{J} + \omega_{e0}\int_0^t \mathbf{J}d\tau = \eta_\varepsilon\omega_{pe}^2\mathbf{E}, & \text{in } \Omega,\\
\mathbf{E}|_{t=0} = \mathbf{0},\ \tilde{\eta}_\varepsilon\partial_t\mathbf{E}|_{t=0} = \mathbf{0}, & \text{in } B_r,\\
\tilde{\mathbf{J}}|_{t=0} = \mathbf{0}, & \text{in } B_r,\\
\hat{\mathbf{x}}\times(\tilde{\mu}_\varepsilon^{-1}\mathbf{curl}\ \mathbf{E}) + \mathscr{C}[\hat{\mathbf{x}}\times\mathbf{E}] = \mathbf{g}_2, & \text{on } \partial B_r,
\end{cases}
\tag{37}
$$

and

$$
\begin{cases}
\tilde{\eta}_\varepsilon\partial_{tt}\mathbf{E} + \mathbf{curl}\,(\tilde{\mu}_\varepsilon^{-1}\mathbf{curl}\,\mathbf{E}) + \partial_t\tilde{\mathbf{J}} = \partial_t\mathbf{F}, & \text{in } B_r,\\
\partial_t\mathbf{J} + \gamma_e\mathbf{J} + \omega_{e0}\int_0^t \mathbf{J}d\tau = \eta_\varepsilon\omega_{pe}^2\mathbf{E}, & \text{in } \Omega,\\
\mathbf{E}|_{t=0} = \mathbf{U}_0,\ \tilde{\eta}_\varepsilon\partial_t\mathbf{E}|_{t=0} = \mathbf{curl}\,\mathbf{V}_0, & \text{in } B_r,\\
\tilde{\mathbf{J}}|_{t=0} = \mathbf{0}, & \text{in } B_r,\\
\hat{\mathbf{x}}\times(\tilde{\mu}_\varepsilon^{-1}\mathbf{curl}\ \mathbf{E}) + \mathscr{C}[\hat{\mathbf{x}}\times\mathbf{E}] = \mathbf{0}, & \text{on } \partial B_r,
\end{cases}
\tag{38}
$$

where $\tilde{\eta}_\varepsilon$ and $\tilde{\mu}_\varepsilon$ have been defined in (10), respectively. Here $\tilde{\mathbf{J}} = \mathbf{J}$ in Ω and $\tilde{\mathbf{J}} = \mathbf{0}$ in $B_r \setminus \overline{\Omega}$, $\mathbf{g}_2 = \hat{\mathbf{x}}\times(\mu_\varepsilon^{-1}\mathbf{curl},\ \mathbf{E}^{inc}) + \mathscr{C}[\hat{\mathbf{x}}\times\mathbf{E}^{inc}]$. The variational problem of (37) is to find $\mathbf{E}\in\mathbf{H}(\mathbf{curl}, B_r)$ for $t > 0$, $\forall\mathbf{w}\in\mathbf{H}(\mathbf{curl}, B_r)$ such that

$$
\begin{aligned}
\int_{B_r}\tilde{\eta}_\varepsilon\partial_{tt}\mathbf{E}\cdot\bar{\mathbf{w}}d\mathbf{x} &= -\int_{B_r}\tilde{\mu}_\varepsilon^{-1}\mathbf{curl}\,\mathbf{E}\cdot\mathbf{curl}\,\bar{\mathbf{w}}d\mathbf{x}\\
&\quad -\int_\Omega\partial_t\mathbf{J}\cdot\ \bar{\mathbf{w}}d\mathbf{x} + \langle\mathscr{C}[\mathbf{n}\times\mathbf{E}] - \mathbf{g}_2, \mathbf{w}\rangle.
\end{aligned}
\tag{39}
$$

Theorem 4. *Let* $\mathbf{E}\in\mathbf{H}(\mathbf{curl}, B_r)$ *be solution of* (37). *Under assumptions* (A_1)–(A_3), *for any fixed* $T > 0$, *it holds that*

$$
\begin{aligned}
&\|\partial_t\mathbf{E}\|_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(B_r))} + \|\mathbf{curl}\,\mathbf{E}\|_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(B_r))}\\
&\quad\lesssim T\|\mathbf{g}_2\|_{L^1(0,T;\mathcal{X})} + \|\partial_t\mathbf{g}_2\|_{L^1(0,T;\mathcal{X})},
\end{aligned}
\tag{40}
$$

and

$$\|\mathbf{E}\|_{L^2(0,T;\mathbf{H}(\mathbf{curl},B_r))}+\|\partial_t\mathbf{E}\|_{L^2(0,T;\mathbf{L}^2(B_r))}\lesssim T^{3/2}\|\mathbf{g}_2\|_{L^1(0,T;\mathcal{X})} +(T+T^{1/2})\|\partial_t\mathbf{g}_2\|_{L^1(0,T;\mathcal{X})}. \tag{41}$$

Proof. It should be mentioned that the proof of this theorem is followed the lines of the proof of Theorem 4.4 of [29](see also Theorem 4.2 of [19]). But there are some essential different points. Here we only give these differences.

For $\varsigma\in(t,T)$, define $\boldsymbol{\psi}_1(\mathbf{x},t)=\int_t^\varsigma\mathbf{E}(\mathbf{x},\tau)d\tau,\ \mathbf{x}\in B_r$. Choosing $\mathbf{w}=\boldsymbol{\psi}_1$ in (39), integrating (39) in $[0,\varsigma]$, and taking the real part, we get

$$\begin{aligned}&\int_0^\varsigma\int_{B_r}\tilde{\eta}_\varepsilon\partial_{tt}\mathbf{E}\cdot\bar{\boldsymbol{\psi}}_1d\mathbf{x}dt+\int_0^\varsigma\int_{B_r}\tilde{\mu}_\varepsilon^{-1}\mathbf{curl}\,\mathbf{E}\cdot\mathbf{curl}\,\bar{\boldsymbol{\psi}}_1d\mathbf{x}dt\\&=-\int_0^\varsigma\int_\Omega\partial_t\mathbf{J}\cdot\bar{\boldsymbol{\psi}}_1d\mathbf{x}dt+\int_0^\varsigma\langle\mathscr{C}[\mathbf{n}\times\mathbf{E}]-\mathbf{g}_2,\boldsymbol{\psi}_1\rangle dt.\end{aligned} \tag{42}$$

The terms on the left-hand side of (42) can be treated by virtue of [29, Theorem 4.4] and [19, Theorem 4.2]. Here we only estimate the terms on the right-hand side of (42). It follows from [29, (4.15)] and (37) that

$$\begin{aligned}&\mathrm{Re}\int_0^\varsigma\int_\Omega\partial_t\mathbf{J}\cdot\bar{\boldsymbol{\psi}}_1d\mathbf{x}dt=\mathrm{Re}\int_\Omega\mathbf{J}\cdot\bar{\boldsymbol{\psi}}_1|_0^\varsigma d\mathbf{x}+\mathrm{Re}\int_0^\varsigma\int_\Omega\mathbf{J}\cdot\bar{\mathbf{E}}\ d\mathbf{x}dt\\&=\mathrm{Re}\int_0^\varsigma\int_\Omega\mathbf{J}\cdot\bar{\mathbf{E}}d\mathbf{x}dt=\frac{1}{2}\Big\{\|\frac{1}{\eta_\varepsilon^{1/2}\omega_{pe}}\mathbf{J}(\cdot,\varsigma)\|^2_{\mathbf{L}^2(\Omega)}\\&+\int_0^\varsigma\int_\Omega\gamma_e|\mathbf{J}|^2d\mathbf{x}dt+\|\frac{\omega_{e0}^{1/2}}{\eta_\varepsilon^{1/2}\omega_{pe}}\int_0^\varsigma\mathbf{J}(\cdot,\zeta)d\zeta\|^2_{\mathbf{L}^2(\Omega)}\Big\}.\end{aligned}$$

For $0\le t\le\varsigma\le T$, it follows from [29, (4.16)] that

$$\begin{aligned}&\mathrm{Re}\int_0^\varsigma\int_{\partial B_r}\mathbf{g}(\mathbf{x},t)\cdot\bar{\boldsymbol{\psi}}_1dSdt=\mathrm{Re}\int_0^\varsigma\int_0^t\int_{B_r}\mathbf{g}(\mathbf{x},\tau)\cdot\bar{\mathbf{E}}_T(t)d\mathbf{x}d\tau dt\\&\le\int_0^\varsigma\|\mathbf{g}(\cdot,t)\|_{\mathcal{X}}dt\int_0^\varsigma\|\mathbf{E}(\cdot,t)\|_{\mathbf{H}(\mathbf{curl},B_r)}dt.\end{aligned} \tag{43}$$

Using Lemma 3 and [29, (4.16)] gives

$$\mathrm{Re}\int_0^\varsigma\int_{\partial B_r}\mathscr{C}[\mathbf{n}\times\mathbf{E}]\cdot\bar{\boldsymbol{\psi}}_1dSdt=\mathrm{Re}\int_{\partial B_r}\int_0^\varsigma\big(\int_0^t\mathscr{C}[\mathbf{n}\times\mathbf{E}](\mathbf{x},\tau)d\tau\big)\cdot\bar{\mathbf{E}}_T(t)dtdS\le0. \tag{44}$$

Combining (42)-(44), for any $\varsigma\in[0,T]$, we have

$$\begin{aligned}&\|\tilde{\eta}_\varepsilon^{1/2}\mathbf{E}(\cdot,\varsigma)\|^2_{\mathbf{L}^2(B_r)}+\|\frac{1}{\eta_\varepsilon^{1/2}\omega_{pe}}\mathbf{J}(\cdot,\varsigma)\|^2_{\mathbf{L}^2(\Omega)}+\int_0^\varsigma\int_\Omega\gamma_e|\mathbf{J}|^2\ d\mathbf{x}dt\\&+\int_{B_r}\tilde{\mu}_\varepsilon^{-1}|\int_0^\varsigma\mathbf{curl}\,\mathbf{E}(\mathbf{x},t)dt|^2d\mathbf{x}+\|\frac{\omega_{e0}^{1/2}}{\eta_\varepsilon^{1/2}\omega_{pe}}\int_0^\varsigma\mathbf{J}(\cdot,\zeta)d\zeta\|^2_{\mathbf{L}^2(\Omega)}\\&\le2\int_0^\varsigma\|\mathbf{g}(\cdot,t)\|_{\mathcal{X}}dt\int_0^\varsigma\|\mathbf{E}(\cdot,t)\|_{\mathbf{H}(\mathbf{curl},B_r)}dt.\end{aligned} \tag{45}$$

Since the right-hand side of (45) contains the term $\int_0^\varsigma(\|\mathbf{curl}\,\mathbf{E}\|^2+\|\mathbf{E}\|^2)^{1/2}dt$, one cannot immediately give a prior estimate. To this end, we consider the following equations, for $t > 0$;

$$\begin{cases} \tilde{\eta}_\varepsilon \partial_{tt}(\partial_t \mathbf{E}) + \mathbf{curl}\,(\tilde{\mu}_\varepsilon^{-1}\mathbf{curl}\,\partial_t \mathbf{E}) + \partial_{tt}\tilde{\mathbf{J}} = \mathbf{0}, & \text{in } B_r, \\ \partial_{tt}\mathbf{J} + \gamma_e \partial_t \mathbf{J} + \omega_{e0}\mathbf{J} = \eta_\varepsilon \omega_{pe}^2 \partial_t \mathbf{E}, & \text{in } \Omega, \\ \partial_t \mathbf{E}|_{t=0} = \mathbf{0},\ \partial_{tt}\mathbf{E}|_{t=0} = \mathbf{0}, & \text{in } B_r, \\ \partial_t \tilde{\mathbf{J}}|_{t=0} = \mathbf{0}, & \text{in } B_r, \\ \hat{\mathbf{x}} \times (\tilde{\mu}_\varepsilon^{-1}\mathbf{curl}\,\partial_t \mathbf{E}) + \mathscr{C}[\hat{\mathbf{x}} \times \partial_t \mathbf{E}] = \partial_t \mathbf{g}_2, & \text{on } \partial B_r, \end{cases} \tag{46}$$

where $\partial_t \mathbf{g}_2 = \hat{\mathbf{x}} \times (\tilde{\mu}_\varepsilon^{-1}\mathbf{curl}\ \ \partial_t \mathbf{E}^{inc}) + \mathscr{C}[\hat{\mathbf{x}} \times \partial_t \mathbf{E}^{inc}]$. By introducing a function: $\boldsymbol{\psi}_2(\mathbf{x},t) = \int_t^\varsigma \partial_\tau \mathbf{E}(\mathbf{x},\tau)d\tau,\ \mathbf{x} \in B_r,\ 0 \le t \le \varsigma$, and repeating the process of the proof of (42), we obtain

$$\begin{aligned} &\|\tilde{\eta}_\varepsilon^{1/2}\partial_t \mathbf{E}(\cdot,\varsigma)\|^2_{\mathbf{L}^2(B_r)} + \|\frac{1}{\eta_\varepsilon^{1/2}\omega_{pe}}\partial_t \mathbf{J}(\cdot,\varsigma)\|^2_{\mathbf{L}^2(\Omega)} + \int_0^\varsigma \int_\Omega \gamma_e |\partial_t \mathbf{J}|^2\ d\mathbf{x}dt \\ &+\|\tilde{\mu}_\varepsilon^{-1/2}\mathbf{curl}\,\mathbf{E}(\cdot,t)\|^2_{\mathbf{L}^2(B_r)} + \|\frac{1}{\eta_\varepsilon^{1/2}\omega_{pe}}\mathbf{J}(\cdot,\varepsilon)\|^2_{\mathbf{L}^2(\Omega)} \\ &= 2\mathrm{Re}\int_0^\varsigma \langle \mathscr{C}[\mathbf{n} \times \partial_t \mathbf{E}] - \partial_t \mathbf{g}, \boldsymbol{\psi}_2 \rangle dt. \end{aligned} \tag{47}$$

Using Lemma 3 and [29, (4.16)] implies

$$\begin{aligned} &Re \int_0^\varsigma \int_{\partial B_r} \mathscr{C}[\mathbf{n} \times \partial_t \mathbf{E}] \cdot \bar{\boldsymbol{\psi}}_2 dSdt \\ &= Re \int_{\partial B_r}\int_0^\varsigma \big(\int_0^t \mathscr{C}[\mathbf{n} \times \partial_t \mathbf{E}](\mathbf{x},\tau)d\tau\big) \cdot \partial_t \bar{\mathbf{E}}_T)dtdS \le 0, \end{aligned} \tag{48}$$

and

$$\begin{aligned} \int_0^\varsigma \int_{\partial B_r} \partial_t \mathbf{g} \cdot \bar{\boldsymbol{\psi}}_2 dSdt &= Re \int_{\partial B_r}\int_0^\varsigma (\int_0^t \partial_\tau \mathbf{g}(\mathbf{x},\tau)\, d\tau) \cdot \partial_t \bar{\mathbf{E}}_T(t)dtdS \\ &= Re \int_{\partial B_r} (\int_0^t \partial_\tau \mathbf{g}(\mathbf{x},\tau)\, d\tau) \cdot \bar{\mathbf{E}}_T(\mathbf{x},t)|_0^\varsigma dS \\ &-Re \int_{\partial B_r}\int_0^\varsigma \partial_t \mathbf{g}(\mathbf{x},t) \cdot \bar{\mathbf{E}}_T(\mathbf{x},t)dtdS \\ &\le Re \int_0^\varsigma \|\partial_t \mathbf{g}(\cdot,t)\ \|_{\mathcal{X}} dt\, \|\mathbf{E}(\cdot,\varsigma)\|_{\mathbf{H}(\mathbf{curl},B_r)} \\ &+Re \int_0^\varsigma \|\partial_t \mathbf{g}(\cdot,t)\|_{\mathcal{X}} \|\mathbf{E}(\cdot,t)\|_{\mathbf{H}(\mathbf{curl},B_r)} dt. \end{aligned} \tag{49}$$

For any $\varsigma \in [0,T]$, using (45) and (47)-(49), we get

$$\begin{aligned} &\|\mathbf{E}(\cdot,\varsigma)\|^2_{\mathbf{L}^2(B_r)} + \|\partial_t \mathbf{E}(\cdot,\varsigma)\|^2_{\mathbf{L}^2(B_r)} + \|\mathbf{curl}\,\mathbf{E}(\cdot,\varsigma)\|^2_{\mathbf{L}^2(B_r)} \\ &\quad\lesssim \int_0^\varsigma \|\mathbf{g}(\cdot,t)\|_{\mathcal{X}} dt \int_0^\varsigma \|\mathbf{E}(\cdot,t)\|_{\mathbf{H}(\mathbf{curl},B_r)} dt \\ &\quad+ \int_0^\varsigma \|\partial_t \mathbf{g}(\cdot,t)\|_{\mathcal{X}} dt\, \|\mathbf{E}(\cdot,\varsigma)\|_{\mathbf{H}(\mathbf{curl},B_r)} \\ &\quad+ \int_0^\varsigma \|\partial_t \mathbf{g}(\cdot,t)\|_{\mathcal{X}} \|\mathbf{E}(\cdot,t)\|_{\mathbf{H}(\mathbf{curl},B_r)} dt. \end{aligned} \tag{50}$$

Taking the L^∞-norm with respect to ς on both sides of (45) yields

$$\|\mathbf{E}\|^2_{L^\infty(0,T;\mathbf{L}^2(B_r))} + \|\partial_t\mathbf{E}\|^2_{L^\infty(0,T;\mathbf{L}^2(B_r))} + \|\mathbf{curl}\,\mathbf{E}\|^2_{L^\infty(0,T;\mathbf{L}^2(B_r))}$$
$$\lesssim (T\|\mathbf{g}\|_{L^1(0,T;\mathcal{X})} + \|\partial_t\mathbf{g}\|_{L^1(0,T;\mathcal{X})})\|\mathbf{E}\|_{L^\infty(0,T;\mathbf{H}(\mathbf{curl},B_r))}.$$

By applying the Young inequality, we complete the proof of the estimate (40).

Integrating (50) with respect to ς over $[0,T]$ and using the Cauchy-Schwarz inequality, we have

$$\|\mathbf{E}\|^2_{L^2(0,T;\mathbf{H}(\mathbf{curl},B_r))} + \|\partial_t\mathbf{E}\|^2_{L^2(0,T;\mathbf{L}^2(B_r))} \lesssim (T^{3/2}\|\mathbf{g}\|_{L^1(0,T;\mathcal{X})}$$
$$+(T+T^{1/2})\|\partial_t\mathbf{g}(\cdot,t)\|_{L^1(0,T;\mathcal{X})})\|\mathbf{E}\|_{L^2(0,T;\mathbf{H}(\mathbf{curl},B_r))}.$$

Therefore, using the Young inequality, we complete the proof of (41). □

Theorem 5. *Let $\mathbf{E} \in \mathbf{H}(\mathbf{curl}, B_r)$ be the solution of the problem* (38). *Under assumptions* (A$_1$)–(A$_3$)*, for any fixed $T > 0$, it holds that*

$$\|\mathbf{E}\|_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(B_r))} \lesssim \|\mathbf{U}_0\|_{\mathbf{L}^2(B_r)} + T\|\mathbf{curl}\,\mathbf{V}_0\|_{\mathbf{L}^2(B_r)} + \|\mathbf{F}\|_{L^1(0,T;\mathbf{L}^2(B_r))}, \tag{51}$$

and

$$\begin{aligned}\|\mathbf{E}\|_{L^2(0,T;\mathbf{L}^2(B_r))} \lesssim\ & T^{1/2}\|\mathbf{U}_0\|_{\mathbf{L}^2(B_r)} + T^{3/2}\|\mathbf{curl}\,\mathbf{V}_0\|_{\mathbf{L}^2(B_r)} \\ & + T^{1/2}\|\mathbf{F}\|_{L^1(0,T;\mathbf{L}^2(B_r))}.\end{aligned} \tag{52}$$

Proof. For any $\varsigma \in [0,T]$, similar to (45), we have

$$\begin{aligned}&\|\tilde{\eta}_\varepsilon^{1/2}\mathbf{E}(\cdot,\varsigma)\|^2_{\mathbf{L}^2(B_r)} + \|\frac{1}{\eta_\varepsilon^{1/2}\omega_{pe}}\mathbf{J}(\cdot,\varsigma)\|^2_{\mathbf{L}^2(\Omega)} + \int_0^\varsigma\int_\Omega \gamma_e|\mathbf{J}|^2\, d\mathbf{x}dt \\
&+\int_{B_r}\tilde{\mu}_\varepsilon^{-1}\Big|\int_0^\varsigma \mathbf{curl}\,\mathbf{E}(\mathbf{x},t)dt\Big|^2 d\mathbf{x} + \|\frac{\omega_{e0}^{1/2}}{\eta_\varepsilon^{1/2}\omega_{pe}}\int_0^\varsigma \mathbf{J}(\cdot,\zeta)d\zeta\|^2_{\mathbf{L}^2(\Omega)} \\
&= 2Re\int_0^\varsigma\int_{B_r}\partial_t\mathbf{F}\cdot\bar{\boldsymbol{\psi}}_1 d\mathbf{x}dt = 2Re\int_{B_r}\mathbf{F}\cdot\bar{\boldsymbol{\psi}}_1|_0^\varsigma d\mathbf{x} \\
&+2Re\int_0^\varsigma\int_{B_r}\mathbf{F}\cdot\bar{\mathbf{E}}\, d\mathbf{x}dt = 2Re\int_0^\varsigma\int_{B_r}\mathbf{F}\cdot\bar{\mathbf{E}}d\mathbf{x}dt \\
&\leq 2\int_0^\varsigma \|\mathbf{F}(\cdot,t)\|_{\mathbf{L}^2(B_r)}\|\mathbf{E}(\cdot,t)\|_{\mathbf{L}^2(B_r)}dt.\end{aligned} \tag{53}$$

Taking the L^∞-norm with respect to ς on both sides of (53), and applying the Young inequality, we complete the proof of (51). Integrating (53) with respect to ς over $[0,T]$ and using the Cauchy-Schwarz inequality implies the estimate (52). □

4. Multiscale asymptotic method

In this section, we first present the multiscale asymptotic expansions of the solution for the reduced problem (9) and then we give the convergence results with an explicit rate for the multiscale solutions.

4.1. Definitions of cell functions and the homogenized equations.

In order to facilitate the implementation of the multiscale expansions of the reduced problem problem (9), we first introduce two sets of cells functions. We refer to [8] for the definitions of two sets of cell functions associated with $\boldsymbol{\eta}(\boldsymbol{\xi})$ and $\boldsymbol{\mu}(\boldsymbol{\xi})$ respectively:

$$\theta_k^\eta(\boldsymbol{\xi}),\ \theta_{kl}^\eta(\boldsymbol{\xi}),\ \Theta_1^\eta(\boldsymbol{\xi}),\ \Theta_2^\eta(\boldsymbol{\xi});\ \theta_k^\mu(\boldsymbol{\xi}),\ \theta_{kl}^\mu(\boldsymbol{\xi}),\ \Theta_1^\mu(\boldsymbol{\xi}),\ \Theta_2^\mu(\boldsymbol{\xi}),\ k,l = 1,2,3, \tag{54}$$

where $\theta_k^\eta(\boldsymbol{\xi})$, $\theta_{kl}^\eta(\boldsymbol{\xi})$, $\theta_k^\mu(\boldsymbol{\xi})$, $\theta_{kl}^\mu(\boldsymbol{\xi})$ are scalar cell functions and $\Theta_1^\eta(\boldsymbol{\xi})$, $\Theta_2^\eta(\boldsymbol{\xi})$, $\Theta_1^\mu(\boldsymbol{\xi})$, $\Theta_2^\mu(\boldsymbol{\xi})$ are matrix-valued cell functions defined in the unit cell $Q = (0,1)^3$.

The scalar cells functions $\theta_k^\eta(\boldsymbol{\xi})$, $\theta_{kl}^\eta(\boldsymbol{\xi})$, $\theta_k^\mu(\boldsymbol{\xi})$, $\theta_{kl}^\mu(\boldsymbol{\xi})$ are defined in turn

$$\begin{cases} \dfrac{\partial}{\partial \boldsymbol{\xi}_i}\big(\eta_{ij}(\boldsymbol{\xi})\dfrac{\partial \theta_k^\eta(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}_j}\big) = -\dfrac{\partial \eta_{ik}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}_i}, \quad \boldsymbol{\xi} \in Q, \\ \theta_k^\eta(\boldsymbol{\xi}) = 0, \quad \boldsymbol{\xi} \in \partial Q, \end{cases} \tag{55}$$

$$\begin{cases} \dfrac{\partial}{\partial \boldsymbol{\xi}_i}\big(\eta_{ij}(\boldsymbol{\xi})\dfrac{\partial \theta_{kl}^\eta(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}_j}\big) = -\dfrac{\partial\big(\eta_{ik}(\boldsymbol{\xi})\theta_l^\eta(\boldsymbol{\xi})\big)}{\partial \boldsymbol{\xi}_i} \\ \qquad -\eta_{kj}(\boldsymbol{\xi})\dfrac{\partial \theta_l^\eta(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}_j} - \eta_{kl}(\boldsymbol{\xi}) + \hat{\eta}_{kl}, \quad \boldsymbol{\xi} \in Q, \\ \theta_{kl}^\eta(\boldsymbol{\xi}) = 0, \quad \boldsymbol{\xi} \in \partial Q, \end{cases} \tag{56}$$

$$\begin{cases} \dfrac{\partial}{\partial \boldsymbol{\xi}_i}\big(\mu_{ij}(\boldsymbol{\xi})\dfrac{\partial \theta_k^\mu(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}_j}\big) = -\dfrac{\partial \mu_{ik}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}_i}, \quad \boldsymbol{\xi} \in Q, \\ \theta_k^\mu(\boldsymbol{\xi}) = 0, \quad \boldsymbol{\xi} \in \partial Q, \end{cases} \tag{57}$$

and

$$\begin{cases} \dfrac{\partial}{\partial \boldsymbol{\xi}_i}\big(\mu_{ij}(\boldsymbol{\xi})\dfrac{\partial \theta_{kl}^\mu(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}_j}\big) = -\dfrac{\partial\big(\mu_{ik}(\boldsymbol{\xi})\theta_l^\mu(\boldsymbol{\xi})\big)}{\partial \boldsymbol{\xi}_i} \\ \qquad -\mu_{kj}(\boldsymbol{\xi})\dfrac{\partial \theta_l^\mu(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}_j} - \mu_{kl}(\boldsymbol{\xi}) + \hat{\mu}_{kl}, \quad \boldsymbol{\xi} \in Q, \\ \theta_{kl}^\mu(\boldsymbol{\xi}) = 0, \quad \boldsymbol{\xi} \in \partial Q, \end{cases} \tag{58}$$

where the homogenized coefficient matrices $q(\boldsymbol{\eta}) = (\hat{\eta}_{kl})$ and $q(\boldsymbol{\mu}) = (\hat{\mu}_{kl})$ are calculated by

$$\hat{\eta}_{kl} = \int_Q \big(\eta_{kl}(\boldsymbol{\xi}) + \eta_{kp}(\boldsymbol{\xi})\frac{\partial \theta_l^\eta(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}_p}\big)d\boldsymbol{\xi}, \quad \hat{\mu}_{kl} = \int_Q \big(\mu_{kl}(\boldsymbol{\xi}) + \mu_{kp}(\boldsymbol{\xi})\frac{\partial \theta_l^\mu(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}_p}\big)d\boldsymbol{\xi}. \tag{59}$$

Let

$$\begin{cases} \boldsymbol{\theta}^\eta = (\theta_1^\eta, \theta_2^\eta, \theta_3^\eta) \\ \boldsymbol{\theta}^\mu = (\theta_1^\mu, \theta_2^\mu, \theta_3^\mu), \end{cases} \boldsymbol{\theta}_2^\eta = \begin{pmatrix} \theta_{11}^\eta & \theta_{12}^\eta & \theta_{13}^\eta \\ \theta_{21}^\eta & \theta_{22}^\eta & \theta_{23}^\eta \\ \theta_{31}^\eta & \theta_{32}^\eta & \theta_{33}^\eta \end{pmatrix}, \ \boldsymbol{\theta}_2^\mu = \begin{pmatrix} \theta_{11}^\mu & \theta_{12}^\mu & \theta_{13}^\mu \\ \theta_{21}^\mu & \theta_{22}^\mu & \theta_{23}^\mu \\ \theta_{31}^\mu & \theta_{32}^\mu & \theta_{33}^\mu \end{pmatrix}. \tag{60}$$

Remark 4.1. *Under the assumptions* $(\mathbf{A}_1)$*–*$(\mathbf{A}_2)$*, the existence and uniqueness of the solutions for the cell problems (55)-(58) can be established based upon Lax-Milgram lemma. Note that the problems (55)-(58) require the homogeneous Dirichlet's boundary conditions instead of the usual periodic boundary conditions.*

Next we give the definitions of the matrix-valued cell functions $\boldsymbol{\Theta}_1^\eta(\boldsymbol{\xi})$, $\boldsymbol{\Theta}_2^\eta(\boldsymbol{\xi})$, $\boldsymbol{\Theta}_1^\mu(\boldsymbol{\xi})$ and $\boldsymbol{\Theta}_2^\mu(\boldsymbol{\xi})$. We define $\boldsymbol{\Theta}_{1,p}^\eta(\boldsymbol{\xi})$, $\boldsymbol{\Theta}_{1,p}^\mu(\boldsymbol{\xi})$, $p = 1, 2, 3$ in the following ways:

$$\begin{cases} \mathbf{curl}_{\boldsymbol{\xi}}(\boldsymbol{\eta}^{-1}(\boldsymbol{\xi})\mathbf{curl}_{\boldsymbol{\xi}}\Theta_{1,p}^\eta(\boldsymbol{\xi})) = -\mathbf{curl}_{\boldsymbol{\xi}}(\boldsymbol{\eta}^{-1}(\boldsymbol{\xi})\mathbf{e}_p), \quad \boldsymbol{\xi} \in Q, \\ \nabla_{\boldsymbol{\xi}} \cdot \Theta_{1,p}^\eta(\boldsymbol{\xi}) = 0, \quad \boldsymbol{\xi} \in Q, \\ \Theta_{1,p}^\eta(\boldsymbol{\xi}) \times \nu = 0, \quad \boldsymbol{\xi} \in \partial Q, \quad p = 1, 2, 3, \end{cases} \tag{61}$$

$$\begin{cases} \mathbf{curl}_{\boldsymbol{\xi}}(\boldsymbol{\mu}^{-1}(\boldsymbol{\xi})\mathbf{curl}_{\boldsymbol{\xi}}\Theta_{1,p}^\mu(\boldsymbol{\xi})) = -\mathbf{curl}_{\boldsymbol{\xi}}(\boldsymbol{\mu}^{-1}(\boldsymbol{\xi})\mathbf{e}_p), \quad \boldsymbol{\xi} \in Q, \\ \nabla_{\boldsymbol{\xi}} \cdot \Theta_{1,p}^\mu(\boldsymbol{\xi}) = 0, \quad \boldsymbol{\xi} \in Q, \\ \Theta_{1,p}^\mu(\boldsymbol{\xi}) \times \nu = 0, \quad \boldsymbol{\xi} \in \partial Q, \quad p = 1, 2, 3, \end{cases} \tag{62}$$

where $\Theta_{1,p}^{\eta}(\boldsymbol{\xi})$ and $\Theta_{1,p}^{\mu}(\boldsymbol{\xi})$, $p=1,2,3$ are the vector-valued functions, $\nu=(\nu_1,\nu_2,\nu_3)$ is the outward unit normal to ∂Q, $\mathbf{e}_1=\{1,0,0\}^T$, $\mathbf{e}_2=\{0,1,0\}^T$, $\mathbf{e}_3=\{0,0,1\}^T$, $\mathbf{a}^T$ denotes the transpose of a vector $\mathbf{a}$. Let

$$\Theta_1^{\eta}(\boldsymbol{\xi})=(\Theta_{1,1}^{\eta}(\boldsymbol{\xi}),\Theta_{1,2}^{\eta}(\boldsymbol{\xi}),\Theta_{1,3}^{\eta}(\boldsymbol{\xi})),\quad \Theta_1^{\mu}(\boldsymbol{\xi})=(\Theta_{1,1}^{\mu}(\boldsymbol{\xi}),\Theta_{1,2}^{\mu}(\boldsymbol{\xi}),\Theta_{1,3}^{\mu}(\boldsymbol{\xi})).$$

Remark 4.2. *The definitions of $\Theta_{1,p}^{\eta}(\boldsymbol{\xi})$, $\Theta_{1,p}^{\mu}(\boldsymbol{\xi})$, $p=1,2,3$ in* (61) *and* (62) *are similar to* (4.128) *of Ref.* [3]. *However, the essential difference is that we take a perfect conductor boundary condition instead of the periodic boundary condition of Ref.* [3]. *Under assumptions* $(\mathbf{A}_1)$–$(\mathbf{A}_2)$, *the existence and uniqueness of problems* (61) *and* (62) *can be established based upon Lax-Milgram lemma.*

Following the idea of Ref. [8] and [9], the second-order vector-valued cell functions $\Theta_{2,p}^{\eta}(\boldsymbol{\xi})$ and $\Theta_{2,p}^{\mu}(\boldsymbol{\xi})$ are defined as follows:

$$(63)\quad \begin{cases} \mathbf{curl}_{\boldsymbol{\xi}}(\boldsymbol{\eta}^{-1}(\boldsymbol{\xi})\mathbf{curl}_{\boldsymbol{\xi}}\Theta_{2,p}^{\eta}(\boldsymbol{\xi}))=-\mathbf{curl}_{\boldsymbol{\xi}}(\boldsymbol{\eta}^{-1}(\boldsymbol{\xi})\Theta_{1,p}^{\eta}(\boldsymbol{\xi}))\\ \quad -\boldsymbol{\eta}^{-1}(\boldsymbol{\xi})\mathbf{curl}_{\boldsymbol{\xi}}\Theta_{1,p}^{\eta}(\boldsymbol{\xi})-\boldsymbol{\eta}^{-1}(\boldsymbol{\xi})\mathrm{e}_p+q(\boldsymbol{\eta}^{-1})\mathrm{e}_p+\nabla_{\boldsymbol{\xi}}\zeta_{2,p}^{\eta}(\boldsymbol{\xi}),\quad \boldsymbol{\xi}\in Q,\\ \nabla_{\boldsymbol{\xi}}\cdot\Theta_{2,p}^{\eta}(\boldsymbol{\xi})=0,\quad \boldsymbol{\xi}\in Q,\\ \Theta_{2,p}^{\eta}(\boldsymbol{\xi})\times\nu=0,\quad \boldsymbol{\xi}\in\partial Q,\quad p=1,2,3, \end{cases}$$

and

$$(64)\quad \begin{cases} \mathbf{curl}_{\boldsymbol{\xi}}(\boldsymbol{\mu}^{-1}(\boldsymbol{\xi})\mathbf{curl}_{\boldsymbol{\xi}}\Theta_{2,p}^{\mu}(\boldsymbol{\xi}))=-\mathbf{curl}_{\boldsymbol{\xi}}(\boldsymbol{\mu}^{-1}(\boldsymbol{\xi})\Theta_{1,p}^{\mu}(\boldsymbol{\xi}))\\ \quad -\boldsymbol{\mu}^{-1}(\boldsymbol{\xi})\mathbf{curl}_{\boldsymbol{\xi}}\Theta_{1,p}^{\mu}(\boldsymbol{\xi})-\boldsymbol{\mu}^{-1}(\boldsymbol{\xi})\mathrm{e}_p+q(\boldsymbol{\mu}^{-1})\mathrm{e}_p+\nabla_{\boldsymbol{\xi}}\zeta_{2,p}^{\mu}(\boldsymbol{\xi}),\quad \boldsymbol{\xi}\in Q,\\ \nabla_{\boldsymbol{\xi}}\cdot\Theta_{2,p}^{\mu}(\boldsymbol{\xi})=0,\quad \boldsymbol{\xi}\in Q,\\ \Theta_{2,p}^{\mu}(\boldsymbol{\xi})\times\nu=0,\quad \boldsymbol{\xi}\in\partial Q,\quad p=1,2,3, \end{cases}$$

where $\boldsymbol{\eta}^{-1}(\boldsymbol{\xi})$ and $\boldsymbol{\mu}^{-1}(\boldsymbol{\xi})$ denote the inverse matrices of $\boldsymbol{\eta}(\boldsymbol{\xi})$ and $\boldsymbol{\mu}(\boldsymbol{\xi})$, respectively. By using [3, (11.46)], the homogenized coefficient matrices $q(\boldsymbol{\eta}^{-1})$ and $q(\boldsymbol{\mu}^{-1})$ are calculated by

$$(65)\quad \begin{aligned} q(\boldsymbol{\eta}^{-1})&=\mathcal{M}\Big(\boldsymbol{\eta}^{-1}(\boldsymbol{\xi})+\boldsymbol{\eta}^{-1}(\boldsymbol{\xi})\,\mathbf{curl}_{\boldsymbol{\xi}}\,\Theta_1^{\eta}(\boldsymbol{\xi})\Big),\\ q(\boldsymbol{\mu}^{-1})&=\mathcal{M}\Big(\boldsymbol{\mu}^{-1}(\boldsymbol{\xi})+\boldsymbol{\mu}^{-1}(\boldsymbol{\xi})\,\mathbf{curl}_{\boldsymbol{\xi}}\,\Theta_1^{\mu}(\boldsymbol{\xi})\Big), \end{aligned}$$

where the matrix-valued functions $\Theta_1^{\eta}(\boldsymbol{\xi})=(\Theta_{1,1}^{\eta}(\boldsymbol{\xi}),\Theta_{1,2}^{\eta}(\boldsymbol{\xi}),\Theta_{1,3}^{\eta}(\boldsymbol{\xi}))$ and $\Theta_1^{\mu}(\boldsymbol{\xi})=(\Theta_{1,1}^{\mu}(\boldsymbol{\xi}),\Theta_{1,2}^{\mu}(\boldsymbol{\xi}),\Theta_{1,3}^{\mu}(\boldsymbol{\xi}))$ are given in (61) and (62), respectively, $\mathcal{M}v=\int_Q v(\boldsymbol{\xi})d\boldsymbol{\xi}$.

The functions $\zeta_{2,p}^{\eta}(\boldsymbol{\xi})$ and $\zeta_{2,p}^{\mu}(\boldsymbol{\xi})$, $p=1,2,3$ in (63) and (64) are the solutions of the following elliptic equations:

$$(66)\quad \begin{cases} -\Delta_{\boldsymbol{\xi}}\zeta_{2,p}^{\eta}(\boldsymbol{\xi})=\nabla_{\boldsymbol{\xi}}\cdot\widetilde{G}^{\eta}(\boldsymbol{\xi}),\quad \boldsymbol{\xi}\in Q,\\ \zeta_{2,p}^{\eta}(\boldsymbol{\xi})=0,\quad \boldsymbol{\xi}\in\partial Q, \end{cases}$$

and

$$(67)\quad \begin{cases} -\Delta_{\boldsymbol{\xi}}\zeta_{2,p}^{\mu}(\boldsymbol{\xi})=\nabla_{\boldsymbol{\xi}}\cdot\widetilde{G}^{\mu}(\boldsymbol{\xi}),\quad \boldsymbol{\xi}\in Q,\\ \zeta_{2,p}^{\mu}(\boldsymbol{\xi})=0,\quad \boldsymbol{\xi}\in\partial Q, \end{cases}$$

where $\nabla_{\boldsymbol{\xi}}\cdot=div_{\boldsymbol{\xi}}$, and

$$(68)\quad \begin{aligned} \widetilde{G}^{\eta}(\boldsymbol{\xi})&=-\boldsymbol{\eta}^{-1}(\boldsymbol{\xi})\mathbf{curl}_{\boldsymbol{\xi}}\,\Theta_{1,p}^{\eta}(\boldsymbol{\xi})-\boldsymbol{\eta}^{-1}(\boldsymbol{\xi})\mathbf{e}_p+q(\boldsymbol{\eta}^{-1})\mathbf{e}_p,\\ \widetilde{G}^{\mu}(\boldsymbol{\xi})&=-\boldsymbol{\mu}^{-1}(\boldsymbol{\xi})\mathbf{curl}_{\boldsymbol{\xi}}\,\Theta_{1,p}^{\mu}(\boldsymbol{\xi})-\boldsymbol{\mu}^{-1}(\boldsymbol{\xi})\mathbf{e}_p+q(\boldsymbol{\mu}^{-1})\mathbf{e}_p. \end{aligned}$$

It can be verified that

$$(69)\qquad \nabla_{\boldsymbol{\xi}}\cdot(\widetilde{G}^{\eta}(\boldsymbol{\xi})+\nabla_{\boldsymbol{\xi}}\zeta_{2,p}^{\eta}(\boldsymbol{\xi}))=0,\quad \nabla_{\boldsymbol{\xi}}\cdot(\widetilde{G}^{\mu}(\boldsymbol{\xi})+\nabla_{\boldsymbol{\xi}}\zeta_{2,p}^{\mu}(\boldsymbol{\xi}))=0,$$

and

$$\zeta^{\eta}_{2,p}, \quad \zeta^{\mu}_{2,p} \in H^2(Q) \cap H^1_0(Q). \tag{70}$$

Therefore, we define the matrix-valued function $\Theta^{\eta}_2(\boldsymbol{\xi}) = (\Theta^{\eta}_{2,1}(\boldsymbol{\xi}), \Theta^{\eta}_{2,2}(\boldsymbol{\xi}), \Theta^{\eta}_{2,3}(\boldsymbol{\xi}))$ and $\Theta^{\mu}_2(\boldsymbol{\xi}) = (\Theta^{\mu}_{2,1}(\boldsymbol{\xi}), \Theta^{\mu}_{2,2}(\boldsymbol{\xi}), \Theta^{\mu}_{2,3}(\boldsymbol{\xi}))$.

Using the Fourier transform on both sides of (4) with respect to t, we obtain the time-harmonic Maxwell's equations in Ω as follows:

$$\begin{cases} i\omega\boldsymbol{\eta}_\varepsilon(\mathbf{x})\Big[1 + \dfrac{\omega_{pe}^2}{\omega^2 - i\omega\gamma_e - \omega_{e0}}\Big]\mathbf{E}_\varepsilon(\mathbf{x},\omega) - \mathbf{curl}\ \mathbf{H}_\varepsilon(\mathbf{x},\omega) = \mathbf{F}(\mathbf{x},\omega), \\ i\omega\boldsymbol{\mu}_\varepsilon(\mathbf{x})\mathbf{H}_\varepsilon(\mathbf{x},\omega) + \mathbf{curl}\ \mathbf{E}_\varepsilon(\mathbf{x},\omega) = 0, \\ \nabla\cdot(\boldsymbol{\eta}_\varepsilon(\mathbf{x})\mathbf{E}_\varepsilon(\mathbf{x},\omega)) = \rho(\mathbf{x},\omega), \quad \nabla\cdot(\boldsymbol{\mu}_\varepsilon(\mathbf{x})\mathbf{H}_\varepsilon(\mathbf{x},\omega)) = 0. \end{cases} \tag{71}$$

The homogenized Maxwell's equations associated with (71) can be written as

$$\begin{cases} i\omega q(\boldsymbol{\eta})(1 + \dfrac{\omega_{pe}^2}{\omega^2 - i\omega\gamma_e - \omega_{e0}})\mathbf{E}_0(\mathbf{x},\omega) - \mathbf{curl}\ \mathbf{H}_0(\mathbf{x},\omega) = \mathbf{F}(\mathbf{x},\omega), \\ i\omega q(\boldsymbol{\mu})\mathbf{H}_0(\mathbf{x},\omega) + \mathbf{curl}\ \mathbf{E}_0(\mathbf{x},\omega) = 0, \\ \nabla\cdot(q(\boldsymbol{\eta})\mathbf{E}_0(\mathbf{x},\omega)) = \rho(\mathbf{x},\omega), \quad \nabla\cdot(q(\boldsymbol{\mu})\mathbf{H}_0(\mathbf{x},\omega)) = 0, \end{cases} \tag{72}$$

where $q(\boldsymbol{\eta})$ and $q(\boldsymbol{\mu})$ are the homogenized coefficient matrices of $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$, respectively. By virtue of the inverse Fourier transform, we obtain the homogenized Maxwell's equations associated with the original problem (1)-(6) in Ω as follows, for $t > 0$:

$$\begin{cases} q(\boldsymbol{\eta})\partial_t\mathbf{E}_0(\mathbf{x},t) - \mathbf{curl}\ \mathbf{H}_0(\mathbf{x},t) + \mathbf{J}_0(\mathbf{x},t) = \mathbf{F}(\mathbf{x},t), \\ q(\boldsymbol{\mu})\partial_t\mathbf{H}_0(\mathbf{x},t) + \mathbf{curl}\ \mathbf{E}_0(\mathbf{x},t) = 0, \\ \partial_t\mathbf{J}_0(\mathbf{x},t) + \gamma_e\mathbf{J}_0(\mathbf{x},t) + \omega_{e0}\displaystyle\int_0^t \mathbf{J}_0(\mathbf{x},\tau)d\tau = q(\boldsymbol{\eta})\omega_{pe}^2\mathbf{E}_0(\mathbf{x},t), \end{cases} \tag{73}$$

which is coupled to the exterior problem (1)-(2) via the boundary conditions (5). Here the initial conditions (6) are prescribed, the scattering field and the total filed in Ω_e are denoted by $(\mathbf{E}^{sc}_0(\mathbf{x},t), \mathbf{H}^{sc}_0(\mathbf{x},t))$ and $(\mathbf{E}^{tot}_0(\mathbf{x},t), \mathbf{H}^{tot}_0(\mathbf{x},t))$, respectively. The homogenized coefficient matrices $q(\boldsymbol{\eta}) = (\hat{\eta}_{ij})$ and $q(\boldsymbol{\mu}) = (\hat{\mu}_{ij})$ can be calculated by (59).

Using the time-domain transparent boundary condition, we reformulate the scattering problem of the homogenized Maxwell's equations (73) into the following initial-boundary value problem, for $t > 0$:

$$\begin{cases} q(\boldsymbol{\eta})\partial_t\mathbf{E}_0 - \mathbf{curl}\,\mathbf{H}_0 + \tilde{\mathbf{J}}_0 = \mathbf{F}(\mathbf{x},t), & \text{in } B_r, \\ q(\boldsymbol{\mu})\partial_t\mathbf{H}_0 + \mathbf{curl}\,\mathbf{E}_0 = \mathbf{0}, & \text{in } B_r, \\ \partial_t\mathbf{J}_0 + \gamma_e\mathbf{J}_0 + \omega_{e0}\displaystyle\int_0^t \mathbf{J}_0 d\tau = q(\boldsymbol{\eta})\omega_{pe}^2\mathbf{E}_0, & \text{in } \Omega, \\ \mathbf{E}_0|_{t=0} = \mathbf{U}_0, \quad \mathbf{H}_0|_{t=0} = \mathbf{V}_0, & \text{in } B_r, \\ \tilde{\mathbf{J}}_0|_{t=0} = \mathbf{0}, & \text{in } B_r, \\ \mathscr{G}[\mathbf{n}\times\mathbf{E}_0] = \mathbf{n}\times\mathbf{H}_0 + \mathbf{g}, & \text{on } \partial B_r, \end{cases} \tag{74}$$

where $\tilde{\mathbf{J}}_0 = \mathbf{J}_0$ in Ω and $\tilde{\mathbf{J}}_0 = \mathbf{0}$ in $B_r \setminus \overline{\Omega}$. Here we take $\mathbf{E}_0 = \mathbf{E}^{tot}_0$, $\mathbf{H}_0 = \mathbf{H}^{tot}_0$, $q(\boldsymbol{\mu}) = \mu_0 I_3$ and $q(\boldsymbol{\eta}) = \eta_0 I_3$ in $B_r \setminus \overline{\Omega}$.

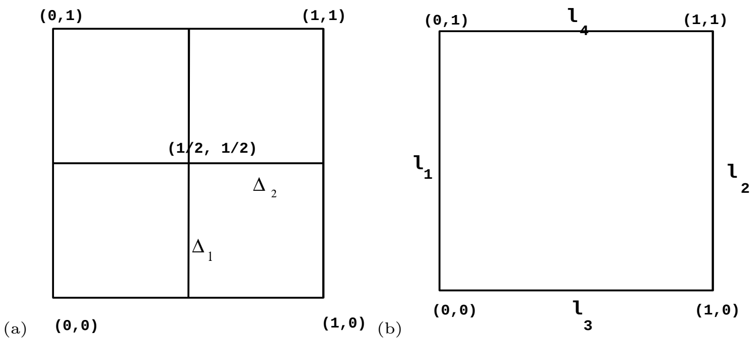


FIGURE 2. (a) the symmetry of Q. (b) the sides of Q.

4.2. Multiscale asymptotic expansions. Now we refer to [8] for the definition of the multiscale asymptotic expansions of the solution for the reduced problem (9) in Ω, for $t > 0$:

$$(75)\quad \begin{aligned} &\mathbf{E}_\varepsilon^{(1)}(\mathbf{x},t) = \mathbf{E}_0(\mathbf{x},t) + \varepsilon\nabla\big(\theta_k^\eta(\boldsymbol{\xi})E_{0k}(\mathbf{x},t)\big) - \varepsilon\boldsymbol{\Theta}_1^\mu(\boldsymbol{\xi})q(\boldsymbol{\mu})\partial_t\mathbf{H}_0(\mathbf{x},t),\\ &\mathbf{E}_\varepsilon^{(2)}(\mathbf{x},t) = \mathbf{E}_0(\mathbf{x},t) + \varepsilon\nabla\Big(\theta_k^\eta(\boldsymbol{\xi})E_{0k}(\mathbf{x},t) + \varepsilon\theta_{kl}^\eta(\boldsymbol{\xi})\partial_{x_l}E_{0k}(\mathbf{x},t)\Big)\\ &\qquad -\varepsilon\boldsymbol{\Theta}_1^\mu(\boldsymbol{\xi})q(\boldsymbol{\mu})\partial_t\mathbf{H}_0(\mathbf{x},t) - \varepsilon^2\boldsymbol{\Theta}_2^\mu(\boldsymbol{\xi})\mathbf{curl}_\mathbf{x}\big(q(\boldsymbol{\mu})\partial_t\mathbf{H}_0(\mathbf{x},t)\big), \end{aligned}$$

$$(76)\quad \begin{aligned} &\mathbf{H}_\varepsilon^{(1)}(\mathbf{x},t) = \mathbf{H}_0(\mathbf{x},t) + \varepsilon\nabla\big(\theta_k^\mu(\boldsymbol{\xi})H_{0k}(\mathbf{x},t)\big) + \varepsilon\boldsymbol{\Theta}_1^\eta(\boldsymbol{\xi})q(\boldsymbol{\eta})\partial_t\mathbf{E}_0(\mathbf{x},t),\\ &\mathbf{H}_\varepsilon^{(2)}(\mathbf{x},t) = \mathbf{H}_0(\mathbf{x},t) + \varepsilon\nabla\Big(\theta_k^\mu(\boldsymbol{\xi})H_{0k}(\mathbf{x},t) + \varepsilon\theta_{kl}^\mu(\boldsymbol{\xi})\partial_{x_l}H_{0k}(\mathbf{x},t)\Big)\\ &\qquad +\varepsilon\boldsymbol{\Theta}_1^\eta(\boldsymbol{\xi})q(\boldsymbol{\eta})\partial_t\mathbf{E}_0(\mathbf{x},t) + \varepsilon^2\boldsymbol{\Theta}_2^\eta(\boldsymbol{\xi})\mathbf{curl}_\mathbf{x}\big(q(\boldsymbol{\eta})\partial_t\mathbf{E}_0(\mathbf{x},t)\big), \end{aligned}$$

where $(\mathbf{E}_0(\mathbf{x},t), \mathbf{H}_0(\mathbf{x},t))$ is the solution of the scattering problem of the homogenized equations (74). The homogenized coefficient matrices $q(\boldsymbol{\eta})$ and $q(\boldsymbol{\mu})$ have been given above. In $B_r \setminus \overline{\Omega}$, we define

$$(77)\quad \mathbf{E}_\varepsilon^{(s)}(\mathbf{x},t) = \mathbf{E}_0^{tot}(\mathbf{x},t), \quad \mathbf{H}_\varepsilon^{(s)}(\mathbf{x},t) = \mathbf{H}_0^{tot}(\mathbf{x},t), \quad s = 1,2,$$

where $\mathbf{E}_\varepsilon^{(s)}(\mathbf{x},t)|_{t=0} = \mathbf{U}_0$, $\mathbf{H}_\varepsilon^{(s)}(\mathbf{x},t)|_{t=0} = \mathbf{V}_0$.

In order to derive the convergence results for the multiscale asymptotic expansions (75)-(77), we need to impose the following conditions on the coefficient matrices and the initial conditions:

($\mathbf{A}_4$). Let $\boldsymbol{\xi} = \varepsilon^{-1}\mathbf{x}$ and the coefficient matrices $\boldsymbol{\eta}(\boldsymbol{\xi}) = \eta(\boldsymbol{\xi})I_3$ and $\boldsymbol{\mu}(\boldsymbol{\xi}) = \mu(\boldsymbol{\xi})I_3$, where I_3 is an 3×3 identity matrix. Assume that $\eta(\boldsymbol{\xi})$ and $\mu(\boldsymbol{\xi})$ are symmetric with respect to the middle plane Δ_k of $Q = (0,1)^3$, where Δ_k, $k = 1,2$, are illustrated in Figure 2(a) in the two dimensional case.

($\mathbf{A}_5$). $\nabla\cdot\mathbf{U} = 0$, $\nabla\cdot\mathbf{V} = 0$.

Remark 4.3. *The condition* (A_4) *indicates that composite materials satisfy geometric symmetric properties in a periodic microstructure, which will be only used for deriving the convergence results for the multiscale asymptotic method.*

Next we give convergence theorems for the multiscale asymptotic solutions defined in (75)-(77).

Theorem 6. *Suppose that $\Omega \subset \mathbb{R}^3$ is the union of entire periodic cells, i.e., $\overline{\Omega} = \bigcup_{\mathbf{z}\in I_\varepsilon} \varepsilon(\mathbf{z} + \overline{Q})$, where $I_\varepsilon = \{\mathbf{z}\in\mathbb{Z}^3,\ \varepsilon(\mathbf{z}+\overline{Q}) \subset \overline{\Omega}\}$ and $\varepsilon > 0$ is any fixed* *small parameter. Let* $(\mathbf{E}_\varepsilon(\mathbf{x},t),\mathbf{H}_\varepsilon(\mathbf{x},t))$ *be the solution of the reduced problem* (9), *and let* $(\mathbf{E}_\varepsilon^{(1)}(\mathbf{x},t),\mathbf{H}_\varepsilon^{(1)}(\mathbf{x},t))$ *and* $(\mathbf{E}_\varepsilon^{(2)}(\mathbf{x},t),\mathbf{H}_\varepsilon^{(2)}(\mathbf{x},t))$ *be the first-order and the second-order multiscale asymptotic solutions defined in* (75)-(77)*, respectively. Under assumptions* (A_1)–(A_5)*, if* $(\mathbf{E}_0,\mathbf{H}_0)\in H^1(0,T;\mathbf{H}^3(\Omega))\cap H^2(0,T;\mathbf{H}^2(\Omega))\cap H^3(0,T;\mathbf{H}^1(\Omega))$*,* $T<\infty$ *and arbitrary, then we obtain the following error estimates:*

(78)
$$\begin{aligned}&\|\mathbf{curl}(\mathbf{E}_\varepsilon-\mathbf{E}_\varepsilon^{(s)})\|_{L^\infty(0,T;\mathbf{L}^2(B_r))}+\|\partial_t(\mathbf{E}_\varepsilon-\mathbf{E}_\varepsilon^{(s)})\|_{L^\infty(0,T;\mathbf{L}^2(B_r))}\\&+\|\mathbf{curl}(\mathbf{H}_\varepsilon-\mathbf{H}_\varepsilon^{(s)})\|_{L^\infty(0,T;\mathbf{L}^2(B_r))}+\|\partial_t(\mathbf{H}_\varepsilon-\mathbf{H}_\varepsilon^{(s)})\|_{L^\infty(0,T;\mathbf{L}^2(B_r))}\le C(T)\varepsilon,\end{aligned}$$

where $s=1,2$*, and* $C(T)$ *is a constant independent of* ε*, but dependent of* T*.*

Proof. We prove Theorem 6 only for the case $s=1$. The case $s=2$ is similar. Due to the limitation of space, the details are omitted.

For the sake of simplicity, we set

$$\mathcal{E}_\varepsilon^{(1)}=\begin{pmatrix}\mathbf{E}_\varepsilon^{(1)}\\ \mathbf{H}_\varepsilon^{(1)}\end{pmatrix},\ \mathbf{R}_1=\begin{pmatrix}\mathbf{R}_1^{(1)}\\ \mathbf{R}_1^{(2)}\end{pmatrix},\ \overrightarrow{\boldsymbol{\nabla}}\times=\begin{pmatrix}0&\mathbf{curl}\\-\mathbf{curl}&0\end{pmatrix},\ \overrightarrow{\boldsymbol{\nabla}}=\begin{pmatrix}\nabla&\mathbf{0}\\ \mathbf{0}&\nabla\end{pmatrix},$$

$$\overrightarrow{\boldsymbol{\nabla}}\cdot=\begin{pmatrix}div&0\\0&div\end{pmatrix},\quad \mathcal{J}=\begin{pmatrix}\omega_{pe}^2g(t)&\mathbf{0}\\ \mathbf{0}&\mathbf{0}\end{pmatrix}_{6\times6}=\Big(a_{ij}(\frac{\mathbf{x}}{\varepsilon})\Big)_{6\times6},$$

$$\boldsymbol{\theta}=\begin{pmatrix}\theta_1^\eta&\theta_2^\eta&\theta_3^\eta\\ \theta_1^\mu&\theta_2^\mu&\theta_3^\mu\end{pmatrix},\quad \boldsymbol{\Theta}_1=\begin{pmatrix}\mathbf{0}&-\Theta_1^\mu\\ \Theta_1^\eta&\mathbf{0}\end{pmatrix}.$$

From (75)-(76), it is not difficult to check that

$$\mathcal{E}_\varepsilon^{(1)}=\mathcal{E}_0+\varepsilon\overrightarrow{\boldsymbol{\nabla}}(\boldsymbol{\theta}\mathcal{E}_0)+\varepsilon\boldsymbol{\Theta}_1\mathcal{A}_0\partial_t\mathcal{E}_0.$$

Using (4), (73) and (75)-(76), taking into account $\mathbf{curl}\to\mathbf{curl}_{\mathbf{x}}+\varepsilon^{-1}\mathbf{curl}_{\boldsymbol{\xi}}$, we get

$$\mathcal{A}_\varepsilon\partial_t(\mathcal{E}_\varepsilon-\mathcal{E}_\varepsilon^{(1)})=\overrightarrow{\boldsymbol{\nabla}}\times(\mathcal{E}_\varepsilon-\mathcal{E}_\varepsilon^{(1)})-\mathcal{A}_\varepsilon\mathcal{J}*(\mathcal{E}_\varepsilon-\mathcal{E}_\varepsilon^{(1)})+\mathbf{R}_1,\tag{79}$$

which holds in the sense of distributions and

$$\begin{aligned}\mathbf{R}_1&=(\mathcal{A}_0-\mathcal{A}_\varepsilon-\mathcal{A}_\varepsilon\overrightarrow{\boldsymbol{\nabla}}_{\boldsymbol{\xi}}\boldsymbol{\theta})\partial_t\mathcal{E}_0-\varepsilon\mathcal{A}_\varepsilon\overrightarrow{\boldsymbol{\nabla}}_{\mathbf{x}}(\boldsymbol{\theta}\partial_t\mathcal{E}_0))-\varepsilon\mathcal{A}_\varepsilon\boldsymbol{\Theta}_1\mathcal{A}_0\partial_{tt}\mathcal{E}_0\\&+\varepsilon\overrightarrow{\boldsymbol{\nabla}}_{\mathbf{x}}\times(\boldsymbol{\Theta}_1\mathcal{A}_0\partial_t\mathcal{E}_0))+\overrightarrow{\boldsymbol{\nabla}}_{\boldsymbol{\xi}}\times(\boldsymbol{\Theta}_1\mathcal{A}_0\partial_t\mathcal{E}_0))-\varepsilon\boldsymbol{\Theta}_1\overrightarrow{\boldsymbol{\nabla}}_{\mathbf{x}}\times\mathcal{A}_0\partial_t\mathcal{E}_0\\&+(\mathcal{A}_0-\mathcal{A}_\varepsilon-\mathcal{A}_\varepsilon\overrightarrow{\boldsymbol{\nabla}}_{\boldsymbol{\xi}}\boldsymbol{\theta})\mathcal{J}*\mathcal{E}_0-\varepsilon\mathcal{A}_\varepsilon\overrightarrow{\boldsymbol{\nabla}}_{\mathbf{x}}(\boldsymbol{\theta}(\mathcal{J}*\mathcal{E}_0))-\varepsilon\mathcal{A}_\varepsilon\boldsymbol{\Theta}_1(\mathcal{J}*\mathcal{A}_0\partial_t\mathcal{E}_0),\end{aligned}\tag{80}$$

where $*$ denotes a convolution of functions with respect to t defined in Introduction. Set

$$\begin{aligned}\mathcal{G}_1(\boldsymbol{\xi},\mathbf{x},t)&=(\mathcal{A}_0-\mathcal{A}_\varepsilon-\mathcal{A}_\varepsilon\overrightarrow{\boldsymbol{\nabla}}_{\boldsymbol{\xi}}\boldsymbol{\theta})\partial_t\mathcal{E}_0,\\ \mathcal{G}_2(\boldsymbol{\xi},\mathbf{x},t)&=(\mathcal{A}_0-\mathcal{A}_\varepsilon-\mathcal{A}_\varepsilon\overrightarrow{\boldsymbol{\nabla}}_{\boldsymbol{\xi}}\boldsymbol{\theta})\mathcal{J}*\mathcal{E}_0,\\ \mathcal{G}_3(\boldsymbol{\xi},\mathbf{x},t)&=\overrightarrow{\boldsymbol{\nabla}}_{\boldsymbol{\xi}}\times(\boldsymbol{\Theta}_1\mathcal{A}_0\partial_t\mathcal{E}_0)).\end{aligned}\tag{81}$$

Under the assumptions of this theorem, for any fixed $t\in(0,T)$, we can check that $\mathcal{G}_i(\boldsymbol{\xi},\mathbf{x},t)$, $i=1,2,3$ are bounded and measurable in $(\boldsymbol{\xi},\mathbf{x})$, 1-periodic in $\boldsymbol{\xi}$, and Lipschitz continuous with respect to $\mathbf{x}$ uniformly in $\boldsymbol{\xi}$. Furthermore, we get

$$\int_Q\mathcal{G}_i(\boldsymbol{\xi},\mathbf{x},t)d\boldsymbol{\xi}=0,\quad i=1,2,3.\tag{82}$$

Given $(\mathbf{E}_0, \mathbf{H}_0) \in H^1(0,T;\mathbf{H}^3(\Omega)) \cap H^2(0,T;\mathbf{H}^2(\Omega)) \cap H^3(0,T;\mathbf{H}^1(\Omega))$, $T < \infty$ and arbitrary. Applying [31, Lemma 1.6], we show that, for $k = 1, 2$,

$$\left|\max_{t\in[0,T]} \int_\Omega \mathbf{R}_1^{(k)} \cdot \bar{\boldsymbol{v}} d\mathbf{x}\right| \leq C(T)\varepsilon \|\boldsymbol{v}\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))},\ \forall \boldsymbol{v} \in L^\infty(0,T;\mathbf{H}^1(\Omega)), \tag{83}$$

where $C(T)$ is a constant independent of ε, but dependent of T.

By taking into account div $\to \mathrm{div}_{\mathbf{x}} + \varepsilon^{-1}\mathrm{div}_{\boldsymbol{\xi}}$, we recall (4), (75), and obtain

$$\begin{aligned}
&\overrightarrow{\boldsymbol{\nabla}} \cdot (\mathcal{A}_\varepsilon(\mathcal{E}_\varepsilon - \mathcal{E}_\varepsilon^{(1)})) = \overrightarrow{\boldsymbol{\nabla}} \cdot (\mathcal{A}_\varepsilon \mathcal{E}_\varepsilon) - \overrightarrow{\boldsymbol{\nabla}} \cdot (\mathcal{A}_\varepsilon \mathcal{E}_\varepsilon^{(1)}) \\
&= \overrightarrow{\boldsymbol{\nabla}} \cdot (\mathcal{A}_0 \mathcal{E}_0) - \overrightarrow{\boldsymbol{\nabla}} \cdot (\mathcal{A}_\varepsilon \mathcal{E}_\varepsilon^{(1)}) = -\varepsilon^{-1} \overrightarrow{\boldsymbol{\nabla}}_{\boldsymbol{\xi}} \cdot (\mathcal{A}_\varepsilon + \mathcal{A}_\varepsilon \overrightarrow{\boldsymbol{\nabla}}_{\boldsymbol{\xi}} \boldsymbol{\theta}) \mathcal{E}_0 \\
&-(\mathcal{A}_\varepsilon + \mathcal{A}_\varepsilon \overrightarrow{\boldsymbol{\nabla}}_{\boldsymbol{\xi}} \boldsymbol{\theta} - \mathcal{A}_0) \overrightarrow{\boldsymbol{\nabla}}_{\mathbf{x}} \cdot \mathcal{E}_0 - \overrightarrow{\boldsymbol{\nabla}}_{\boldsymbol{\xi}} \cdot (\mathcal{A}_\varepsilon \overrightarrow{\boldsymbol{\nabla}}_{\mathbf{x}} (\boldsymbol{\theta} \mathcal{E}_0)) \\
&-\overrightarrow{\boldsymbol{\nabla}}_{\boldsymbol{\xi}} \cdot (\mathcal{A}_\varepsilon \boldsymbol{\Theta}_1 \mathcal{A}_0 \partial_t \mathcal{E}_0) - \varepsilon \overrightarrow{\boldsymbol{\nabla}}_{\mathbf{x}} \cdot (\mathcal{A}_\varepsilon \boldsymbol{\Theta}_1 \mathcal{A}_0 \partial_t \mathcal{E}_0) \\
&= -(\mathcal{A}_\varepsilon + \mathcal{A}_\varepsilon \overrightarrow{\boldsymbol{\nabla}}_{\boldsymbol{\xi}} \boldsymbol{\theta} - \mathcal{A}_0) \overrightarrow{\boldsymbol{\nabla}}_{\mathbf{x}} \cdot \mathcal{E}_0 - \overrightarrow{\boldsymbol{\nabla}}_{\boldsymbol{\xi}} \cdot (\mathcal{A}_\varepsilon \overrightarrow{\boldsymbol{\nabla}}_{\mathbf{x}} (\boldsymbol{\theta} \mathcal{E}_0)) \\
&-\overrightarrow{\boldsymbol{\nabla}}_{\boldsymbol{\xi}} \cdot (\mathcal{A}_\varepsilon \boldsymbol{\Theta}_1 \mathcal{A}_0 \partial_t \mathcal{E}_0) - \varepsilon \overrightarrow{\boldsymbol{\nabla}}_{\mathbf{x}} \cdot (\mathcal{A}_\varepsilon \boldsymbol{\Theta}_1 \mathcal{A}_0 \partial_t \mathcal{E}_0) \equiv \mathcal{H}_1(\boldsymbol{\xi}, \mathbf{x}, t).
\end{aligned} \tag{84}$$

Here we have used the fact that $\overrightarrow{\boldsymbol{\nabla}} \cdot (\mathcal{A}_\varepsilon \mathcal{E}_\varepsilon) = \overrightarrow{\boldsymbol{\nabla}} \cdot (\mathcal{A}_0 \mathcal{E}_0) = \mathbf{0}$, which holds in the sense of distributions under the assumptions $(\mathrm{A}_1) - (\mathrm{A}_5)$.

Set

$$\begin{aligned}
&\mathcal{G}_4(\boldsymbol{\xi}, \mathbf{x}, t) = -(\mathcal{A}_\varepsilon + \mathcal{A}_\varepsilon \overrightarrow{\boldsymbol{\nabla}}_{\boldsymbol{\xi}} \boldsymbol{\theta} - \mathcal{A}_0) \overrightarrow{\boldsymbol{\nabla}}_{\mathbf{x}} \cdot \mathcal{E}_0, \\
&\mathcal{G}_5(\boldsymbol{\xi}, \mathbf{x}, t) = -\overrightarrow{\boldsymbol{\nabla}}_{\boldsymbol{\xi}} \cdot (\mathcal{A}_\varepsilon \overrightarrow{\boldsymbol{\nabla}}_{\mathbf{x}} (\boldsymbol{\theta} \mathcal{E}_0)), \\
&\mathcal{G}_6(\boldsymbol{\xi}, \mathbf{x}, t) = \overrightarrow{\boldsymbol{\nabla}}_{\boldsymbol{\xi}} \cdot (\mathcal{A}_\varepsilon \boldsymbol{\Theta}_1 \mathcal{A}_0 \partial_t \mathcal{E}_0).
\end{aligned} \tag{85}$$

For any fixed $t \in (0,T)$, similarly we can show that the functions $\mathcal{G}_k(\boldsymbol{\xi}, \mathbf{x}, t)$, $k = 4, 5, 6$ are bounded and measurable in $(\boldsymbol{\xi}, \mathbf{x})$, 1-periodic in $\boldsymbol{\xi}$, Lipschitz continuous with respect to $\mathbf{x}$ uniformly in $\boldsymbol{\xi}$, and

$$\int_Q \mathcal{G}_k(\boldsymbol{\xi}, \mathbf{x}, t) d\boldsymbol{\xi} = 0, \quad k = 4, 5, 6. \tag{86}$$

Setting $\mathcal{G}_k = (\mathcal{G}_k^{(1)}, \mathcal{G}_k^{(2)})^T$ and applying [31, Lemma 1.6] again, we have

$$\max_{t\in[0,T]} \left|\int_\Omega \mathcal{G}_k^{(i)}(\frac{\mathbf{x}}{\varepsilon}, \mathbf{x}, t) \cdot \bar{\boldsymbol{v}} d\mathbf{x}\right| \leq C(T)\varepsilon \|\boldsymbol{v}\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))}, \quad i = 1, 2, \tag{87}$$

where $k = 4, 5, 6$, and $C(T)$ is a constant independent of ε, but dependent T.

Set $\mathcal{H}_1 = (\mathcal{H}_1^{(1)}, \mathcal{H}_1^{(2)})^T$. Similarly, for any fixed $t \in (0,T)$, combining (84) and (87) leads to

$$\max_{t\in[0,T]} \left|\int_\Omega \mathcal{H}_1^{(k)}(\frac{\mathbf{x}}{\varepsilon}, \mathbf{x}, t) \cdot \bar{\boldsymbol{v}} d\mathbf{x}\right| \leq C(T)\varepsilon \|\boldsymbol{v}\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))}, \quad k = 1, 2. \tag{88}$$

Set $\boldsymbol{\pi}_\varepsilon(\mathbf{x}, t) = (\pi_\varepsilon^{(1)}, \pi_\varepsilon^{(2)})^T$, where $\pi_\varepsilon^{(1)}$ and $\pi_\varepsilon^{(2)}$ are respectively the solutions of the following elliptic equations:

$$\begin{cases} -\Delta \pi_\varepsilon^{(k)}(\mathbf{x}, t)) = \mathcal{H}_1^{(k)}(\frac{\mathbf{x}}{\varepsilon}, \mathbf{x}, t), & \mathbf{x} \in \Omega, \\ \pi_\varepsilon^{(k)}(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial\Omega, & k = 1, 2, \end{cases} \tag{89}$$

where $t \in (0,T)$ plays the role of a parameter. Combining (88)-(89), for any fixed $t \in (0,T)$, we have

$$\|\pi_\varepsilon^{(k)}\|_{H^1(\Omega)} \leq C\varepsilon, \quad k = 1, 2, \tag{90}$$

where C is a constant independent of ε.

If we take

$$\widetilde{\mathcal{E}}_\varepsilon^{(1)}(\mathbf{x},t) = \mathcal{E}_\varepsilon^{(1)}(\mathbf{x},t) - \mathcal{A}_\varepsilon^{-1}\overrightarrow{\boldsymbol{\nabla}}\,\boldsymbol{\pi}_\varepsilon(\mathbf{x},t), \tag{91}$$

then it is obvious that

$$\overrightarrow{\boldsymbol{\nabla}} \cdot \big(\mathcal{A}_\varepsilon(\mathcal{E}^\varepsilon(\mathbf{x},t) - \widetilde{\mathcal{E}}_1^\varepsilon(\mathbf{x},t))\big) = 0. \tag{92}$$

Assume that Ω is the union of entire cells. We recall the boundary conditions of cell functions θ_k^η, θ_k^μ, Θ_1^η and Θ_1^μ, and obtain

$$\begin{aligned}(\mathbf{E}_\varepsilon(\mathbf{x},t) - \mathbf{E}_\varepsilon^{(1)}(\mathbf{x},t)) \times \mathbf{n} &= (\mathbf{E}^{sc}(\mathbf{x},t) - \mathbf{E}_0^{sc}(\mathbf{x},t)) \times \mathbf{n}, \quad \mathbf{x} \in \partial\Omega,\\ (\mathbf{H}_\varepsilon(\mathbf{x},t) - \mathbf{H}_\varepsilon^{(1)}(\mathbf{x},t)) \times \mathbf{n} &= (\mathbf{H}^{sc}(\mathbf{x},t) - \mathbf{H}_0^{sc}(\mathbf{x},t)) \times \mathbf{n}, \quad \mathbf{x} \in \partial\Omega,\end{aligned} \tag{93}$$

where $(\mathbf{E}^{sc}, \mathbf{H}^{sc})$ and $(\mathbf{E}_0^{sc}, \mathbf{H}_0^{sc})$ satisfy (1)-(2), and $\mathbf{n}$ is the outward unit normal to $\partial\Omega$. Furthermore, for any fixed $t \in (0,T)$, we know that $\pi_\varepsilon^{(k)} \in H^2(\Omega) \cap H_0^1(\Omega)$, and obtain (see [3, p. 144])

$$\nabla\, \pi_\varepsilon^{(k)} \times \mathbf{n} = 0. \tag{94}$$

For $\mathbf{x} \in \partial\Omega$, we thus get

$$\begin{aligned}(\mathbf{E}_\varepsilon(\mathbf{x},t) - \widetilde{\mathbf{E}}_\varepsilon^{(1)}(\mathbf{x},t)) \times \mathbf{n} &= (\mathbf{E}^{sc}(\mathbf{x},t) - \mathbf{E}_0^{sc}(\mathbf{x},t)) \times \mathbf{n},\\ (\mathbf{H}_\varepsilon(\mathbf{x},t) - \widetilde{\mathbf{H}}_\varepsilon^{(1)}(\mathbf{x},t)) \times \mathbf{n} &= (\mathbf{H}^{sc}(\mathbf{x},t) - \mathbf{H}_0^{sc}(\mathbf{x},t)) \times \mathbf{n}.\end{aligned} \tag{95}$$

From (79) and (91), we have

$$\mathcal{A}_\varepsilon\, \partial_t(\mathcal{E}_\varepsilon - \widetilde{\mathcal{E}}_\varepsilon^{(1)}) = \overrightarrow{\boldsymbol{\nabla}} \times (\mathcal{E}_\varepsilon - \widetilde{\mathcal{E}}_\varepsilon^{(1)}) - \mathcal{A}_\varepsilon \mathcal{J} * (\mathcal{E}_\varepsilon - \widetilde{\mathcal{E}}_\varepsilon^{(1)}) + \mathcal{R}_1, \tag{96}$$

where

$$\mathcal{R}_1 = \mathbf{R}_1 - \overrightarrow{\boldsymbol{\nabla}}(\partial_t \boldsymbol{\pi}_\varepsilon) + \overrightarrow{\boldsymbol{\nabla}} \times (\mathcal{A}_\varepsilon^{-1}\overrightarrow{\boldsymbol{\nabla}}\,\boldsymbol{\pi}_\varepsilon) + \mathcal{J} * \overrightarrow{\boldsymbol{\nabla}}\,\boldsymbol{\pi}_\varepsilon(\mathbf{x},t) \tag{97}$$

where $\mathbf{R}_1$ has been given in (80) and $\mathcal{R}_1 = (\mathcal{R}_1^{(1)}, \mathcal{R}_1^{(2)})^T$.

Denote by $\mathbf{e} = \mathbf{E}_\varepsilon - \widetilde{\mathbf{E}}_\varepsilon^{(1)}$, $\mathbf{h} = \mathbf{H}_\varepsilon - \widetilde{\mathbf{H}}_\varepsilon^{(1)}$, $\mathbf{e}_s = \mathbf{E}^{sc} - \mathbf{E}_0^{sc}$, and $\mathbf{h}_s = \mathbf{H}^{sc} - \mathbf{H}_0^{sc}$, where $(\mathbf{e}_s, \mathbf{h}_s)$ satisfies (1)-(2). It follows from (79) and (93) that $(\mathbf{e}, \mathbf{h})$ is the solution of the following initial-boundary value problem, for $t > 0$

$$\begin{cases}\tilde{\eta}_\varepsilon \partial_t \mathbf{e} - \mathbf{curl}\,\mathbf{h} = -\tilde{\eta}_\varepsilon \omega_{pe}^2 g(t) * \mathbf{e} + \mathcal{R}_1^{(1)}, & \text{in } B_r,\\ \tilde{\mu}_\varepsilon \partial_t \mathbf{h} + \mathbf{curl}\,\mathbf{e} = \mathcal{R}_1^{(2)}, & \text{in } B_r,\\ \mathbf{e}|_{t=0} = \mathbf{0},\ \mathbf{h}|_{t=0} = \mathbf{0}, & \text{in } B_r,\\ \mathscr{G}[\mathbf{n} \times \mathbf{e}] = \mathbf{n} \times \mathbf{h}, & \text{on } \partial B_r,\end{cases} \tag{98}$$

where $\mathcal{R}_1^{(1)} = \mathcal{R}_1^{(2)} \equiv \mathbf{0}$ in $B_r \setminus \Omega$.

Since η_ε and μ_ε are respectively rapidly oscillating 1-periodic functions with piecewise constants thanks to $(A_1) - (A_2)$, using (92), we have

$$\begin{aligned}\|\mathbf{div}\,\mathbf{e}\|_{\mathbf{L}^2(\Omega)}^2 &\lesssim \|\eta_\varepsilon^2 \mathbf{div}\,\mathbf{e}\|_{\mathbf{L}^2(\Omega)}^2 = \|\mathbf{div}\,(\eta_\varepsilon \mathbf{e})\|_{\mathbf{L}^2(\Omega)}^2 = 0,\\ \|\mathbf{div}\,\mathbf{h}\|_{\mathbf{L}^2(\Omega)}^2 &\lesssim \|\mu_\varepsilon^2 \mathbf{div}\,\mathbf{h}\|_{\mathbf{L}^2(\Omega)}^2 = \|\mathbf{div}\,(\mu_\varepsilon \mathbf{h})\|_{\mathbf{L}^2(\Omega)}^2 = 0.\end{aligned} \tag{99}$$

For any fixed $t \in (0,T)$, set $\mathbf{j}(\cdot,t) = \eta_\varepsilon \omega_{pe}^2 g(t) * \mathbf{e}$ and define an energy function as follows:

$$e(t) = \|\tilde{\eta}_\varepsilon^{1/2}\mathbf{e}(\cdot,t)\|_{\mathbf{L}^2(B_r)}^2 + \|\tilde{\mu}_\varepsilon^{1/2}\mathbf{h}(\cdot,t)\|_{\mathbf{L}^2(B_r)}^2 + \|(\eta_\varepsilon^{1/2}\omega_{pe})^{-1}\mathbf{j}(\cdot,t)\|_{\mathbf{L}^2(\Omega)}^2.$$

We observe that

$$\int_0^t e'(\tau)d\tau = \|\tilde{\eta}_\varepsilon^{1/2}\mathbf{e}(\cdot,t)\|_{\mathbf{L}^2(B_r)}^2 + \|\tilde{\mu}_\varepsilon^{1/2}\mathbf{h}(\cdot,t)\|_{\mathbf{L}^2(B_r)}^2 + \|(\eta_\varepsilon^{1/2}\omega_{pe})^{-1}\mathbf{j}(\cdot,t)\|_{\mathbf{L}^2(\Omega)}^2.$$

From (98)-(99), using Lemma 3, Corollary 3.6 of [20, p. 55] and integrating by parts, we get

$$
\begin{aligned}
&\int_0^t e'(t)dt = 2\mathrm{Re}\int_0^t\int_{B_r}\Big\{\tilde{\eta}_\varepsilon\partial_t\mathbf{e}\cdot\bar{\mathbf{e}}+\tilde{\mu}_\varepsilon\partial_t\mathbf{h}\cdot\bar{\mathbf{h}}+\frac{1}{\tilde{\eta}_\varepsilon\omega_{pe}^2}\partial_t\mathbf{j}\cdot\bar{\mathbf{j}}\Big\}d\mathbf{x}d\tau\\
&\le 2\mathrm{Re}\int_0^t\int_{\partial B_r}\mathscr{G}[\mathbf{n}\times\mathbf{e}]\cdot\bar{\mathbf{e}}_T\,dSd\tau+2\mathrm{Re}\int_0^t\int_\Omega\mathcal{R}_1^{(1)}\cdot\bar{\mathbf{e}}+\mathcal{R}_1^{(2)}\cdot\bar{\mathbf{h}}\,d\mathbf{x}d\tau\\
&\le 2\mathrm{Re}\int_0^t\int_\Omega\Big\{\mathcal{R}_1^{(1)}\cdot\bar{\mathbf{e}}+\mathcal{R}_1^{(2)}\cdot\bar{\mathbf{h}}\Big\}d\mathbf{x}d\tau\\
&\le C(t)\varepsilon\Big\{\max_{t\in[0,T]}\|\mathbf{e}\|_{\mathbf{H}(\mathbf{curl},\Omega)}+\max_{t\in[0,T]}\|\mathbf{h}\|_{\mathbf{H}(\mathbf{curl},\Omega)}\Big\}.
\end{aligned}
\tag{100}
$$

By taking the partial derivative with respect to t on both sides of (98), we can follow (100) for $\partial_t\mathbf{e}$ and $\partial_t\mathbf{h}$ and get the following estimate:

$$
\begin{aligned}
&\|\tilde{\eta}_\varepsilon^{1/2}\partial_t\mathbf{e}(\cdot,t)\|^2_{\mathbf{L}^2(B_r)}+\|\tilde{\mu}_\varepsilon^{1/2}\partial_t\mathbf{h}(\cdot,t)\|^2_{\mathbf{L}^2(B_r)}+\|(\eta_\varepsilon^{1/2}\omega_{pe})^{-1}\partial_t\mathbf{j}(\cdot,t)\|^2_{\mathbf{L}^2(\Omega)}\\
&\le 2\mathrm{Re}\textstyle\int_0^t\int_\Omega\Big\{\partial_\tau\mathcal{R}_1^{(1)}\cdot\partial_\tau\bar{\mathbf{e}}+\partial_\tau\mathcal{R}_1^{(2)}\cdot\partial_\tau\bar{\mathbf{h}}\Big\}d\mathbf{x}d\tau\\
&\le 2\mathrm{Re}\textstyle\int_\Omega\Big\{\partial_t\mathcal{R}_1^{(1)}\cdot\bar{\mathbf{e}}+\partial_t\mathcal{R}_1^{(2)}\cdot\bar{\mathbf{h}}\Big\}d\mathbf{x}-2\mathrm{Re}\int_0^t\int_\Omega\Big\{\partial_{\tau\tau}\mathcal{R}_1^{(1)}\cdot\bar{\mathbf{e}}+\partial_{\tau\tau}\mathcal{R}_1^{(2)}\cdot\bar{\mathbf{h}}\Big\}d\mathbf{x}d\tau\\
&\le C(t)\varepsilon\Big\{\max_{t\in[0,T]}\|\mathbf{e}\|_{\mathbf{H}(\mathbf{curl},\Omega)}+\max_{t\in[0,T]}\|\mathbf{h}\|_{\mathbf{H}(\mathbf{curl},\Omega)}\Big\}.
\end{aligned}
$$

Therefore, using (100), we complete the proof of (78). □

Theorem 7. *Suppose that $\Omega\subset\mathbb{R}^3$ is the union of entire periodic cells, i.e., $\overline{\Omega}=\bigcup_{\mathbf{z}\in I_\varepsilon}\varepsilon(\mathbf{z}+\overline{Q})$, where $I_\varepsilon=\{\mathbf{z}\in\mathbb{Z}^3,\ \varepsilon(\mathbf{z}+\overline{Q})\subset\overline{\Omega}\}$ and $\varepsilon>0$ is any fixed small parameter. Let $(\mathbf{E}_\varepsilon(\mathbf{x},t),\mathbf{H}_\varepsilon(\mathbf{x},t))$ be the solution of the reduced problem* (9)*, and let $(\mathbf{E}_\varepsilon^{(1)}(\mathbf{x},t),\mathbf{H}_\varepsilon^{(1)}(\mathbf{x},t))$ and $(\mathbf{E}_\varepsilon^{(2)}(\mathbf{x},t),\mathbf{H}_\varepsilon^{(2)}(\mathbf{x},t))$ be the first-order and the second-order multiscale asymptotic solutions defined in* (75)-(77)*, respectively. Under assumptions* (A_1)–(A_5)*, if $(\mathbf{E}_0,\mathbf{H}_0)\in H^1(0,T;\mathbf{H}^3(\Omega))\cap H^2(0,T;\mathbf{H}^2(\Omega))\cap H^3(0,T;\mathbf{H}^1(\Omega))$, then we have the following error estimates:*

$$
\|\mathbf{curl}(\mathbf{E}_\varepsilon-\mathbf{E}_\varepsilon^{(s)})\|_{L^2(0,T;\mathbf{L}^2(B_r))}+\|\partial_t(\mathbf{E}_\varepsilon-\mathbf{E}_\varepsilon^{(s)})\|_{L^2(0,T;\mathbf{L}^2(B_r))}\le C(T)\varepsilon,
\tag{101}
$$

where $s=1,2$ and $C(T)$ is a constant independent of ε, but dependent of T.

Proof. Due to the limitation of space, we only give the proof of (101) for the case $s=1$. The case $s=2$ is similar.

Set

$$
\mathcal{E}_\varepsilon-\tilde{\mathcal{E}}_\varepsilon^{(1)}=\begin{pmatrix}\mathbf{e}\\ \mathbf{h}\end{pmatrix}.
$$

From (98), by eliminating $\mathbf{h}$, we get the following initial-boundary value problem:

$$
\begin{cases}
\tilde{\eta}_\varepsilon\partial_{tt}\mathbf{e}=\mathbf{curl}\,\tilde{\mu}_\varepsilon^{-1}\mathbf{curl}\,\mathbf{e}-\partial_t\widetilde{\mathbf{J}}_e+\partial_t\mathcal{R}_1^{(1)}+\mathbf{curl}\,(\tilde{\mu}_\varepsilon^{-1}\mathcal{R}_1^{(2)}), & \text{in } B_r\\
\mathbf{e}|_{t=0}=\mathbf{0},\quad \partial_t\mathbf{e}|_{t=0}=\mathbf{0}, & \text{in } B_r\\
\mathbf{n}\times(\tilde{\mu}_\varepsilon^{-1}\mathbf{curl}\,\mathbf{e})+\mathscr{C}[\mathbf{n}\times\mathbf{e}]=\mathbf{0}, & \text{on } \partial B_r,
\end{cases}
\tag{102}
$$

where $\mathscr{C}=\mathscr{L}^{-1}\circ sG_e\circ\mathscr{L}$ and $\mathbf{J}_e=\eta_\varepsilon\omega_{pe}^2\int_0^t g(t-\tau)\mathbf{e}(\mathbf{x},\tau)d\tau$, $\widetilde{\mathbf{J}}_e=\mathbf{J}_e$ in Ω and $\widetilde{\mathbf{J}}_e=\mathbf{0}$ in $B_r\setminus\overline{\Omega}$.

For $\varsigma \in (t, T)$, we introduce an auxiliary function $\boldsymbol{\psi}_1(\mathbf{x}, t) = \int_t^{\varsigma} \mathbf{e}(\mathbf{x}, \tau) d\tau,\ \mathbf{x} \in B_r$. Following the lines of the proof of Theorem 4, we prove

$$
\begin{aligned}
&\|\tilde{\eta}_\varepsilon^{1/2}\mathbf{e}(\cdot,\varsigma)\|^2_{\mathbf{L}^2(B_r)} + \|\frac{1}{\tilde{\eta}_\varepsilon^{1/2}\omega_{pe}}\tilde{\mathbf{J}}_e(\cdot,\varsigma)\|^2_{\mathbf{L}^2(B_r)} + \int_0^{\varsigma}\int_{B_r}\gamma_e|\tilde{\mathbf{J}}_e|^2\, d\mathbf{x}dt \\
&+ \int_{B_r}\tilde{\mu}_\varepsilon^{-1}|\int_0^{\varsigma}\mathbf{curl}\,\mathbf{e}(\mathbf{x},t)dt|^2 d\mathbf{x} + \|\frac{\omega_{e0}^{1/2}}{\tilde{\eta}_\varepsilon^{1/2}\omega_{pe}}\int_0^{\varsigma}\tilde{\mathbf{J}}_e(\cdot,\zeta)d\zeta\|^2_{\mathbf{L}^2(B_r)} \\
&\le 2\mathrm{Re}\int_0^{\varsigma}\int_{B_r}\left\{\partial_t\mathcal{R}_1^{(1)} + \mathbf{curl}\,(\mu_\varepsilon^{-1}\mathcal{R}_1^{(2)})\right\}\cdot\bar{\boldsymbol{\psi}}_1\, d\mathbf{x}dt.
\end{aligned}
\tag{103}
$$

Furthermore, we have

$$
\begin{aligned}
&\int_0^{\varsigma}\int_{B_r}\partial_t\mathcal{R}_1^{(1)}\cdot\bar{\boldsymbol{\psi}}_1 d\mathbf{x}dt = \int_{B_r}\mathcal{R}_1^{(1)}\cdot\bar{\boldsymbol{\psi}}_1|_0^{\varsigma}d\mathbf{x} + \int_0^{\varsigma}\int_{B_r}\mathcal{R}_1^{(1)}\cdot\bar{\mathbf{e}}\, d\mathbf{x}dt \\
&= \int_0^{\varsigma}\int_{B_r}\mathcal{R}_1^{(1)}\cdot\bar{\mathbf{e}}\, d\mathbf{x}dt \le C(T)\varepsilon\|\,\mathbf{e}\|_{L^2(0,T;\mathbf{H}(\mathbf{curl},\Omega))},
\end{aligned}
\tag{104}
$$

and

(105)

$$
\begin{aligned}
&\int_0^{\varsigma}\int_{B_r}\mathbf{curl}\big(\tilde{\mu}^{-1}(\tfrac{\mathbf{x}}{\varepsilon})\mathcal{R}_1^{(2)}\big)\cdot\bar{\boldsymbol{\psi}}_1 d\mathbf{x}dt = \int_{B_r}\int_0^{\varsigma}\int_0^t\mathbf{curl}\big(\tilde{\mu}^{-1}(\tfrac{\mathbf{x}}{\varepsilon})\mathcal{R}_1^{(2)}\big)d\tau\cdot\bar{\mathbf{e}}dtd\mathbf{x} \\
&= \int_{B_r}\int_0^{\varsigma}\Big(\mathbf{curl}_{\mathbf{x}}\Big(\{\big(\tilde{\mu}^{-1}(\boldsymbol{\xi}) + \tilde{\mu}^{-1}(\boldsymbol{\xi})\mathbf{curl}_{\xi}\Theta_1^{\mu}(\boldsymbol{\xi})\big)q(\mu) - I_3 - \nabla_{\boldsymbol{\xi}}\theta^{\mu}(\xi)\}\mathbf{H}_0(\mathbf{x},t)\Big) \\
&\quad +\varepsilon^{-1}\{\mathbf{curl}_{\xi}\big(\tilde{\mu}^{-1}(\boldsymbol{\xi}) + \tilde{\mu}^{-1}(\boldsymbol{\xi})\mathbf{curl}_{\xi}\Theta_1^{\mu}(\boldsymbol{\xi})\big)q(\mu)\mathbf{H}_0(\mathbf{x},t)\} \\
&\quad -\mathbf{curl}_{\xi}\big(\nabla_{\mathbf{x}}\partial_t\mathbf{H}_0(\mathbf{x},t)\theta^{\mu}(\boldsymbol{\xi})\big) - \mathbf{curl}_{\xi}\big(\Theta_1^{\eta}(\boldsymbol{\xi})q(\boldsymbol{\eta})\partial_t\mathbf{E}_0(\mathbf{x},t)\big) \\
&\quad +\mathbf{curl}_{\xi}\Big(\tilde{\mu}^{-1}(\boldsymbol{\xi})\mathbf{curl}_{\mathbf{x}}\big(\Theta_1^{\mu}(\boldsymbol{\xi})q(\mu)\mathbf{H}_0(\mathbf{x},t)\big)\Big) \\
&\quad -\varepsilon\mathbf{curl}_{\mathbf{x}}\big(\nabla_{\mathbf{x}}\partial_t\mathbf{H}_0(\mathbf{x},t)\theta^{\mu}\big) - \varepsilon\mathbf{curl}_{\mathbf{x}}\big(\Theta_1^{\eta}(\boldsymbol{\xi})q(\boldsymbol{\eta})\partial_t\mathbf{E}_0(\mathbf{x},t)\big) \\
&\quad +\varepsilon\mathbf{curl}_{\mathbf{x}}\Big(\tilde{\mu}^{-1}(\boldsymbol{\xi})\mathbf{curl}_{\mathbf{x}}\big(\Theta_1^{\mu}(\boldsymbol{\xi})q(\mu)\mathbf{H}_0(\mathbf{x},t)\big)\Big)\Big)\cdot\bar{\mathbf{e}}\, dtd\mathbf{x}.
\end{aligned}
$$

Using (2.4) of [8] and (11.46) of [3, p. 145], it can be shown that

$$
\int_Q\big((\mu^{-1}(\boldsymbol{\xi}) + \mu^{-1}(\boldsymbol{\xi})\mathbf{curl}_{\boldsymbol{\xi}}\,\boldsymbol{\Theta}_1)q(\mu) - I_3 - \nabla_{\boldsymbol{\xi}}\theta^{\mu}\big)d\boldsymbol{\xi} = q(\mu^{-1})q(\mu) - I_3 = 0.
$$

Here we have used the fact $q(\mu^{-1}) = q(\mu)^{-1}$ (see (5.61)-(5.62) of [3, p. 64]). On the other hand, we recall the cell problem (2.7)-(2.8) defined in [8] and have

$$
\mathbf{curl}_{\boldsymbol{\xi}}\big(\mu^{-1}(\boldsymbol{\xi}) + \mu^{-1}(\boldsymbol{\xi})\mathbf{curl}_{\boldsymbol{\xi}}\Theta_1^{\mu}\big) = \mathbf{0}.
$$

Similarly, we use [31, Lemma 1.6] and obtain

$$
|\int_0^{\varsigma}\int_{\Omega}\mathbf{curl}\,\big(\mu^{-1}(\frac{\mathbf{x}}{\varepsilon})\partial_t\mathcal{R}_1^{(2)}\big)\cdot\bar{\boldsymbol{\psi}}_1 d\mathbf{x}dt| \le C(T)\varepsilon\|\mathbf{e}\|_{L^2(0,T;\mathbf{H}(\mathbf{curl};\Omega))},
\tag{106}
$$

where $C(T)$ is a constant independent of ε, but dependent of T.

By taking the partial derivative with respect to t on both sides of (102), we follow (103) for $\partial_t \mathbf{e}$ and $\partial_t \mathbf{h}$, and then get the following estimate:

$$
\begin{aligned}
&\|\tilde{\eta}_\varepsilon^{1/2}\partial_t \mathbf{e}(\cdot,\varsigma)\|^2_{\mathbf{L}^2(B_r)} + \|\tfrac{1}{\eta_\varepsilon^{1/2}\omega_{pe}}\partial_t \mathbf{J}_e(\cdot,\varsigma)\|^2_{\mathbf{L}^2(\Omega)} + \textstyle\int_0^\varsigma\int_\Omega \gamma_e|\partial_t\mathbf{J}_e|^2\, d\mathbf{x}dt \\
&+\|\tilde{\mu}_\varepsilon^{-1/2}\mathbf{curl}\,\mathbf{e}(\cdot,\varsigma)\|^2_{\mathbf{L}^2(B_r)} + \|\frac{1}{\eta_\varepsilon^{1/2}\omega_{pe}}\mathbf{J}_e(\cdot,\varepsilon)\|^2_{\mathbf{L}^2(\Omega)} \\
&\le 2\mathrm{Re}\textstyle\int_0^\varsigma\int_{B_r}(\partial_{tt}\mathcal{R}_1^{(1)} + \mathbf{curl}\,(\tilde{\mu}_\varepsilon^{-1}\partial_t\mathcal{R}_1^{(2)}))\cdot\bar{\boldsymbol{\psi}}_2\, d\mathbf{x}dt \\
&\le 2\mathrm{Re}\textstyle\int_{B_r}\int_0^\varsigma\int_0^t(\partial_{\tau\tau}\mathcal{R}_1^{(1)} + \mathbf{curl}\,(\tilde{\mu}_\varepsilon^{-1}\partial_\tau\mathcal{R}_1^{(2)}))d\tau\cdot\partial_t\bar{\mathbf{e}}\, dtd\mathbf{x} \\
&\le 2\mathrm{Re}\textstyle\int_{B_r}\int_0^\varsigma(\partial_{tt}\mathcal{R}_1^{(1)} + \mathbf{curl}\,(\tilde{\mu}_\varepsilon^{-1}\partial_t\mathcal{R}_1^{(2)}))\cdot\bar{\mathbf{e}}\, dtd\mathbf{x} \\
&\le C(T)\varepsilon\|\mathbf{e}\|_{L^2(0,T;\mathbf{H}(\mathbf{curl};\Omega))}.
\end{aligned}
\tag{107}
$$

Therefore, by combining (103), we complete the proof of (101). □

5. A multiscale Crank-Nicolson mixed finite element method and numerical examples

5.1. A multiscale Crank-Nicolson mixed finite element method. Based on the theoretical results presented in Section 3, we first introduce a multiscale Crank-Nicolson mixed finite element method for heterogeneous scattering problems of Maxwell's equations (1)-(6). The method can be described as follows:

Step 1. Compute the scalar cell functions $\theta_k^\eta(\boldsymbol{\xi})$, $\theta_k^\mu(\boldsymbol{\xi})$, $\theta_{kl}^\eta(\boldsymbol{\xi})$, $\theta_{kl}^\mu(\boldsymbol{\xi})$, $k,l=1,2,3$ and the matrix-valued cell functions $\Theta_1^\eta(\boldsymbol{\xi})$, $\Theta_1^\mu(\boldsymbol{\xi})$, $\Theta_2^\eta(\boldsymbol{\xi})$, $\Theta_2^\mu(\boldsymbol{\xi})$ in the unit cell $Q=(0,1)^3$, respectively. Then compute the homogenized coefficient matrices $q(\eta)$ and $q(\mu)$, respectively.

Step 2. Solve numerically the scattering problems of the homogenized Maxwell's equations (73).

Step 3. Use the difference quotients to compute the derivatives of the solution $(\mathbf{E}_0(\mathbf{x},t),\mathbf{H}_0(\mathbf{x},t))$ for the homogenized time-dependent Maxwell's equations (73). The detailed formulas can be found in [8, 9, 10].

At **Step 1**, we refer to [8, 9] for computing the scalar cell functions $\theta_k^\eta(\boldsymbol{\xi})$, $\theta_k^\mu(\boldsymbol{\xi})$, $\theta_{kl}^\eta(\boldsymbol{\xi})$, $\theta_{kl}^\mu(\boldsymbol{\xi})$, $k,l=1,2,3$ and the matrix-valued cell functions $\Theta_1^\eta(\boldsymbol{\xi})$, $\Theta_2^\eta(\boldsymbol{\xi})$, $\Theta_1^\mu(\boldsymbol{\xi})$, $\Theta_2^\mu(\boldsymbol{\xi})$. For the numerical algorithms and the convergence, we refer the reader to [9, 10, 30, 40, 42]. Once the cell functions $\theta_k^\eta(\boldsymbol{\xi})$, $\theta_k^\mu(\boldsymbol{\xi})$, $k=1,2,3$ are calculated numerically, we can get the numerical solutions of the homogenized coefficient matrices $q(\eta)$ and $q(\mu)$ by using the formulas (59).

Next we focus on discussing the numerical computation for the scattering problem of the homogenized Maxwell's equations (74). In the real computation, we replace the homogenized coefficient matrices $q(\eta)$ and $q(\mu)$ by their numerical solutions $\widehat{\eta}^{h_0}$ and $\widehat{\mu}^{h_0}$, where h_0 is the mesh size for solving the cell problems. Therefore, the modified homogenized Maxwell's equations associated with (74) can be written as follows, for $t>0$

$$
\begin{cases}
\widehat{\eta}^{h_0}\partial_t\overline{\mathbf{E}}_0 - \mathbf{curl}\,\overline{\mathbf{H}}_0 + \widetilde{\overline{\mathbf{J}}}_0 = \mathbf{F}, & \text{in } B_r,\\
\widehat{\mu}^{h_0}\partial_t\overline{\mathbf{H}}_0 + \mathbf{curl}\,\overline{\mathbf{E}}_0 = \mathbf{0}, & \text{in } B_r,\\
\partial_t\overline{\mathbf{J}}_0 + \gamma_e\overline{\mathbf{J}}_0 + \omega_{e0}\int_0^t\overline{\mathbf{J}}_0 d\tau = \widehat{\eta}^{h_0}\omega_{pe}^2\overline{\mathbf{E}}_0, & \text{in } \Omega,\\
\overline{\mathbf{E}}_0(\mathbf{x},0) = \mathbf{U}_0,\ \overline{\mathbf{H}}_0(\mathbf{x},0) = \mathbf{V}_0, & \text{in } B_r\\
\widetilde{\overline{\mathbf{J}}}_0(\mathbf{x},0) = \mathbf{0}, & \text{in } B_r,\\
\mathscr{G}[\mathbf{n}\times\overline{\mathbf{E}}_0] = \mathbf{n}\times\overline{\mathbf{H}}_0 + \mathbf{g}, & \text{on } \partial B_r.
\end{cases}
\tag{108}
$$

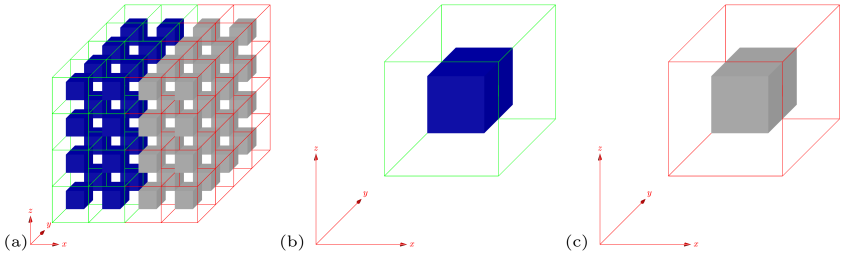


FIGURE 3. (a) A whole domain Ω of composite materials with two subdivided periodic microstructures: blue and grey; (b) the reference cell $Q = (0, 1)^3$; (c) the reference cell $Q^* = (0, 1)^3$.

Remark 5.1. *Using Proposition 3.3 of* [9]*, we can estimate the difference between the solution of the modified homogenized Maxwell's equations (108) and that of the homogenized Maxwell's equations (74).*

At **Step 2**, we employ the lowest order divergence and curl conforming (called Raviart-Thomas-Nédélec) tetrahedral element, utilize PML to truncate infinite domain, and use the implicit Crank-Nicolson scheme to solve the scattering problems of the modified homogenized Maxwell's equations in a coarse mesh and at the larger time step.

Remark 5.2. *The error analysis of the Crank-Nicolson mixed finite element method for solving the modified homogenized Maxwell's equations with constant coefficients and the PML equations can be found in the paper* [22, 28]*. Based on the theoretical results of the proposed multiscale method in this paper (see Theorems 6 and 7), it is not difficult to obtain the error estimates for the multiscale Crank-Nicolson mixed finite element method. Due to the limitation of space, we omit the convergence results and their proofs.*

5.2. Numerical examples. To validate the multiscale method and the convergence results presented in this paper, next we give some numerical examples.

Example 5.1. *We consider the time-dependent scattering problems* (1)-(6) *of Maxwell's equations in dispersive media with two subdivided periodic microstructures, which are displayed with different colors such as blue and grey. The different reference cells Q and Q^* are shown in fig. 3:(b) and (c), respectively. The whole domain Ω of composite materials is shown in fig. 3:(a). Here we take $\varepsilon = \dfrac{1}{4}$, and the excitation is a plane wave from the direction $(\theta^{inc}, \phi^{inc})$. The incident electric field is given by*

$$\mathbf{E}^{inc} = (\cos\alpha\hat{\theta} + \sin\alpha\hat{\phi})\hat{\mathbf{E}}_0 f\big(t - \hat{k}^{inc} \cdot (\mathbf{r} - \mathbf{r}_0)/c_0\big),$$

where α is the polarization angle, $\hat{\mathbf{E}}_0$ is the peak field strength, $\mathbf{r}_0$, is a reference position vector, $c_0 = 1$ is the speed of light, $\hat{k}^{inc}$ is the unite vector along the incident direction, and $f\big(t - \hat{k}^{inc} \cdot (\mathbf{r} - \mathbf{r}_0)/c_0\big)$ is the temporal profile of the incident field. Here a tapered sinusoidal temporal profile is given by $f(t) = (1 - exp(-t/\tau_p))\sin(\omega_0 t)$,

TABLE 1. Comparison of computational costs in Examples 5.1.

	Original eqs.	Scalar cell eqs.	Matrix-valued cell eqs.	Homogenized eqs.
Elements	769250	92090	92090	233631
Dof	5539923	17406	112228	1691124
The time step	$\Delta t = 0.001$			$\Delta t' = 0.005$

TABLE 2. Comparison of the computational errors.

	$\frac{\|\mathbf{e}_0\|_{(0)}}{\|\mathbf{E}_\varepsilon\|_{(0)}}$	$\frac{\|\mathbf{e}_1\|_{(0)}}{\|\mathbf{E}_\varepsilon\|_{(0)}}$	$\frac{\|\mathbf{e}_0\|_{(1)}}{\|\mathbf{E}_\varepsilon\|_{(1)}}$	$\frac{\|\mathbf{e}_1\|_{(1)}}{\|\mathbf{E}_\varepsilon\|_{(1)}}$	$\frac{\|\mathbf{h}_0\|_{(0)}}{\|\mathbf{H}_\varepsilon\|_{(0)}}$	$\frac{\|\mathbf{h}_1\|_{(0)}}{\|\mathbf{H}_\varepsilon\|_{(0)}}$	$\frac{\|\mathbf{h}_0\|_{(1)}}{\|\mathbf{H}_\varepsilon\|_{(1)}}$	$\frac{\|\mathbf{h}_1\|_{(1)}}{\|\mathbf{H}_\varepsilon\|_{(1)}}$
Case 4.1	0.0283	0.0174	0.2706	0.1572	0.0809	0.0404	0.5835	0.3933
Case 4.2	0.0191	0.0118	0.1487	0.0796	0.1094	0.0650	0.6060	0.4087

where $\tau_p = 4T_0$ *and* $T_0 = 2\pi/\omega_0$.

Case 4.1. $$\mu(\frac{\mathbf{x}}{\varepsilon}) = \begin{cases} 5.0, \text{ in a cube of Q}, \\ 1.0, \text{ others of Q}, \\ 0.1, \text{ in a cube of Q}^*, \\ 1.0, \text{ others of Q}^*, \end{cases} \quad \eta(\frac{\mathbf{x}}{\varepsilon}) = \begin{cases} 2.5, \text{ in a cube of Q}, \\ 1.0, \text{ others of Q}, \\ 0.01, \text{ in a cube of Q}^*, \\ 1.0, \text{ others of Q}^*, \end{cases}$$

Case 4.2. $$\mu(\frac{\mathbf{x}}{\varepsilon}) = \begin{cases} 1.0, \text{ in a cube of Q}, \\ 4.0, \text{ others of Q}, \\ 0.01, \text{ in a cube of Q}^*, \\ 1.0, \text{ others of Q}^*, \end{cases} \quad \eta(\frac{\mathbf{x}}{\varepsilon}) = \begin{cases} 2.5, \text{ in a cube of Q}, \\ 8.0, \text{ others of Q}, \\ 0.1, \text{ in a cube of Q}^*, \\ 1.0, \text{ others of Q}^*. \end{cases}$$

In order to demonstrate the numerical accuracy of the proposed method, the exact solution $(\mathbf{E}_\varepsilon(\mathbf{x},t), \mathbf{H}_\varepsilon(\mathbf{x},t))$ of the time-dependent Maxwell's equations (1)-(6) with rapidly oscillating coefficients must be available. Since the elements of the coefficient matrices $\eta(\frac{\mathbf{x}}{\varepsilon})$ and $\mu(\frac{\mathbf{x}}{\varepsilon})$ maybe are discontinuous, generally speaking, it is an extremely difficult task or even impossible to give the exact solution of the above Maxwell's equations. To this end, we replace the exact solution $(\mathbf{E}_\varepsilon(\mathbf{x},t), \mathbf{H}_\varepsilon(\mathbf{x},t))$ by the numerical solution in a very fine mesh and at a small time step. It should be emphasized that this step is not necessary in the real applications.

Without confusion we denote by $(\mathbf{E}_\varepsilon(\mathbf{x},t), \mathbf{H}_\varepsilon(\mathbf{x},t))$ the numerical solution of the time-dependent Maxwell's equations (1)-(6) in a fine mesh and at a small time step, and let $(\mathbf{E}_{0,h}(\mathbf{x},t), \mathbf{H}_{0,h}(\mathbf{x},t))$ be the numerical solution of the modified homogenized Maxwell's equations in a coarse mesh and at the larger time step. $(\mathbf{E}^{(1)}_{\varepsilon,h}(\mathbf{x},t), \mathbf{H}^{(1)}_{\varepsilon,h}(\mathbf{x},t))$ are the first-order multiscale numerical solutions of the problems (1)-(6) based on (75)-(76), respectively. Set $\mathbf{e}_0 = \mathbf{E}_\varepsilon - \mathbf{E}_{0,h}$, $\mathbf{e}_1 = \mathbf{E}_\varepsilon - \mathbf{E}^{(1)}_{\varepsilon,h}$, $\mathbf{h}_0 = \mathbf{H}_\varepsilon - \mathbf{H}_{0,h}$, and $\mathbf{h}_1 = \mathbf{H}_\varepsilon - \mathbf{H}^{(1)}_{\varepsilon,h}$. For convenience, We introduce the notation $\|\mathbf{u}\|_{(0)} = \|\mathbf{u}(\cdot,T)\|_{\mathbf{L}^2(\Omega)}$ and $\|\mathbf{u}\|_{(1)} = \|\mathbf{u}(\cdot,T)\|_{\mathbf{H}(\mathbf{curl};\Omega)}$.

Comparison of computational costs for solving the related problems is listed in Table 1. The relative numerical errors of the homogenization method and the first-order multiscale method in the $\mathbf{L}^2(\Omega)$-norm and in the $\mathbf{H}(\mathbf{curl};\Omega)$-norm at the time $T = 8.0$ for Cases 4.1 and 4.2 are shown in Table 2. The evolution of the relative errors of the homogenization method and the first-order multiscale method for computing $(\mathbf{E}_\varepsilon(\mathbf{x},t), \mathbf{H}_\varepsilon(\mathbf{x},t))$ in $\mathbf{L}^2(\Omega)$ norm and in $\mathbf{H}(\mathbf{curl};\Omega)$ norm for Cases 4.1 and 4.2 are displayed in Fig. 4-5. Comparison of the computational results for the related component of the solutions is displayed in Fig. 6.

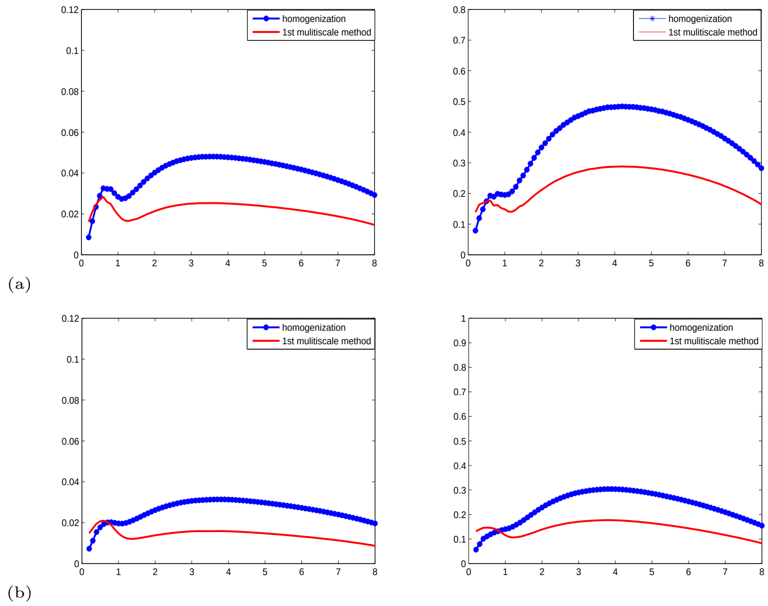


FIGURE 4. The evolution of the relative errors for computing $\mathbf{E}_\varepsilon(\mathbf{x}, t)$ in $\mathbf{L}^2(\Omega)$ norm and in $\mathbf{H}(\mathbf{curl};\Omega)$ norm in the following cases: (a)Case 4.1; (b)Case 4.2 .

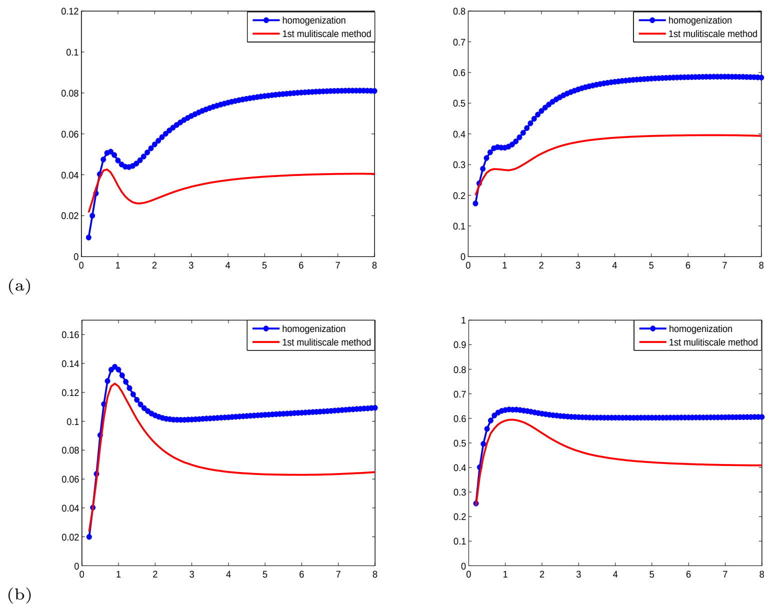


FIGURE 5. The evolution of the relative errors for computing $\mathbf{H}_\varepsilon(\mathbf{x}, t)$ in $\mathbf{L}^2(\Omega)$ norm and in $\mathbf{H}(\mathbf{curl};\Omega)$ norm in the following cases: (a)Case 4.1; (b)Case 4.2 .

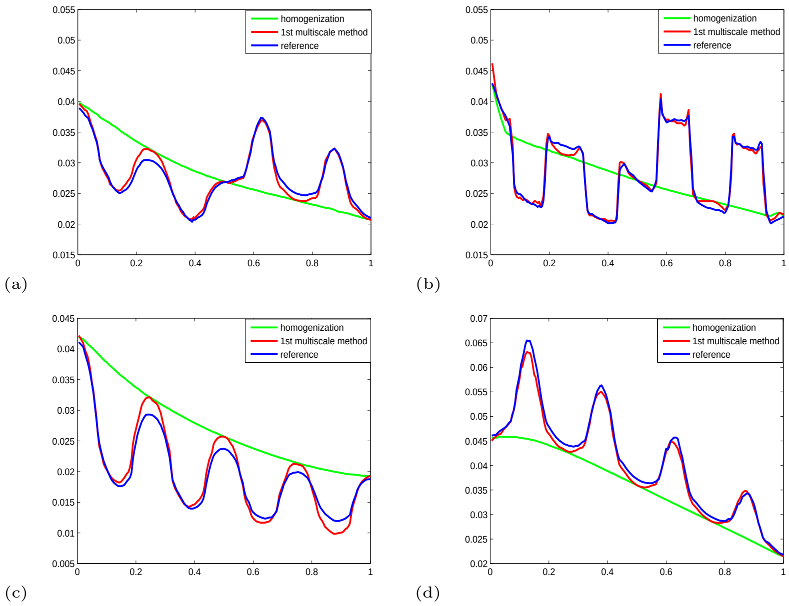


FIGURE 6. comparison of the computational results for the third component of the reference solution $\mathbf{E}_\varepsilon$,the homogenized solution $\mathbf{E}_0$, and the first-order multiscale solution $\mathbf{E}_\varepsilon^{(1)}(\mathbf{x},t)$ at time t =3.0s in the case 4.1 : (a) in the line $z = 0.625, x = y$; (b) in the line $x = y = z$; comparison of the computational results for the first component of the reference solution $\mathbf{H}_\varepsilon$,the homogenized solution $\mathbf{H}^0$, and the first-order multiscale solution $\mathbf{H}_\varepsilon^{(1)}(\mathbf{x},t)$ at time t =3.0s in the case 4.1 : (c) in the line $x = 0.375, y = z$; (d) in the line $x = 0.625, y = z$.

Remark 5.3. *By observing the above numerical results, we note that it fails to provide satisfactory results for the homogenization method. The multiscale approach, however, results in more accurate numerical solutions for the scattering problem of time-dependent Maxwell's equations in dispersive media with a periodic microstructure or with many subdivided periodic microstructures. Therefore, the numerical results confirm the convergence results presented in this paper.*

5.3. Concluding remarks. Finally, we give some remarks and present some unsolved problems.

Remark 5.4. *Provided that the solution* $(\mathbf{E}_0(\mathbf{x},t), \mathbf{H}_0(\mathbf{x},t))$ *of the homogenized Maxwell's equations* (73) *with constant coefficients is smooth enough, the error estimates are obtained. Rigorous regularity analysis for the solutions of 3D time-dependent Maxwell's equations is very challenging and still open. We refer the reader to these references* [6, 16, 17]. *However, the formal multiscale asymptotic expansions are useful in developing the efficient numerical method for 3D time-dependent Maxwell's equations in heterogeneous dispersive media.*

Remark 5.5. *We assume that a domain* Ω *is the union of entire periodic cells in Theorems* 6 *and* 7. *The convergence for the multiscale method of this paper in a general domain* Ω *is still an unsolved problem to authors.*

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