INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 17, Number 6, Pages 900–928

AN EXPLICIT FORMULA FOR CORNER SINGULARITY EXPANSION OF THE SOLUTIONS TO THE STOKES EQUATIONS IN A POLYGON

BUYANG LI

Abstract. We present corner singularity expansion for the solutions of the Stokes equations in an arbitrary polygon under the stress boundary condition, with explicit expression of its singular part and quantitative estimate of its regular part. In particular, there is a countably infinite set of angles such that a corner with one of these angles would give the solution precisely one additional logarithmic singularity, whose explicit expression is found.

Key words. Corner singularity, regularity, Stokes equation, stress boundary condition, fractional Sobolev space.

1. Introduction

Consider the Stokes equations with the stress boundary condition in a polygon $\Omega \subset \mathbb{R}^2$, i.e.

$$\left\{\begin{array}{ll} -\nabla \cdot (2\mathbb{D}(\mathbf{u}) - p\mathbb{I}) = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = g & \text{in } \Omega, \\ (2\mathbb{D}(\mathbf{u}) - p\mathbb{I})\mathbf{n} = \mathbf{h} & \text{on } \partial\Omega, \end{array}\right. \tag{1}$$

where $\mathbb{D}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ is the stress tensor, and $\mathbf{n}$ is the unit outward normal vector on the boundary $\partial\Omega$. For any given $(\mathbf{f}, g, \mathbf{h}) \in \mathbf{H}^1(\Omega)' \times L^2(\Omega) \times \mathbf{H}^{1/2}(\partial\Omega)'$, the weak formulation of the problem (1) is to find $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)$ such that

$$\left\{\begin{array}{ll} 2(\mathbb{D}(\mathbf{u}), \mathbb{D}(\mathbf{v}))_\Omega - (p, \nabla \cdot \mathbf{v})_\Omega = (\mathbf{f}, \mathbf{v})_\Omega + (\mathbf{h}, \mathbf{v})_{\partial\Omega}, & \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega), \\ (\nabla \cdot \mathbf{u}, \psi)_\Omega = (g, \psi)_\Omega, & \forall\, \psi \in L^2(\Omega), \end{array}\right. \tag{2}$$

under the compatibility conditions $\int_\Omega \mathbf{f}(x)\mathrm{d}x + \int_{\partial\Omega} \mathbf{h}(x)\mathrm{d}\tau = 0$. Existence and uniqueness of the weak solution can be proved by using the Lax–Milgram lemma (see [16] or appendix A).

For many purposes, higher regularity of the solution is often needed [3, 8, 11]. However, due to the existence of corners of the domain Ω, for a given right-hand side $(\mathbf{f}, g, \mathbf{h}) \in \mathbf{H}^{s-1}(\Omega) \times H^s(\Omega) \times \mathbf{H}^{s-1/2}(\partial\Omega)$ the solution $(\mathbf{u}, p)$ may not be in $\mathbf{H}^{s+1}(\Omega) \times H^s(\Omega)$ for general $s > 0$. It is natural to ask how smooth the solution can be and what specific singularities the solution possesses.

Received by the editors October 5, 2020.
2000 *Mathematics Subject Classification.* 35J25.

Under the Dirichlet boundary conditon $\mathbf{u} = 0$ on $\partial\Omega$, Kellogg and Osborn [13] proved that the solution is in $\mathbf{H}^2(\Omega) \times H^1(\Omega)$ when the polygon is convex and $(\mathbf{f}, g) \in \mathbf{L}^2(\Omega) \times H^1(\Omega)$, with an additional condition $g/r \in L^2(\Omega)$, where r denotes the distance from the corners of the domain. Under the compatibility condition

$$g = 0 \text{ at the corners of the polygon if } s > 1, \tag{3}$$

Dauge [7] proved the $\mathbf{H}^{s+1}(\Omega) \times H^s(\Omega)$ regularity for fractional index $s < \operatorname{Re}\lambda_1(\omega)$ in general polygons, where $\lambda_1(\omega)$ is the first non-integer root of the equation

$$\lambda^2 \sin(\omega)^2 - \sin(\lambda\omega)^2 = 0, \tag{4}$$

and ω is the maximal interior angle of the polygon Ω. In general, $\lambda_1(\omega) > 1$ if Ω is a convex polygon, and $\lambda_1(\omega) > 1/2$ if Ω is a non-convex polygon. With more general boundary conditions, Orlt and Sändig [17] found the range of the index s for which the $\mathbf{H}^{s+1}(\Omega) \times H^s(\Omega)$ regularity holds, Guo and Schwab [10] investigated the analytic regularity of the solution in the framework of L^2-based weighted Sobolev spaces, Maz'ya and Rossmann [15] presented regularity estimates of the solution in terms of L^p-based weighted Sobolev and Hölder spaces.

The basic ideas of [7, 10, 13, 17] are restricting the Stokes equations to a cone centered at a corner of the polygon and transforming the Stokes equation to an ODE system (with a parameter $\lambda \in \mathbb{C}$)

$$\mathcal{L}(\lambda)(\widehat{\mathbf{U}}(\lambda), \widehat{q}(\lambda)) = (\widehat{\mathbf{F}}(\lambda), \widehat{G}(\lambda), \widehat{\mathbf{H}}(\lambda))$$

by using the Mellin transform (or Fourier transform), and then studying the spectrum of the operator $\mathcal{L}$. In particular, if ω denotes the angle of the corner, then $\lambda_1(\omega)$ is the first non-zero eigenvalue of the operator $\mathcal{L}$. In these works, only the regularity of the solution has been studied without much details on the specific expressions of the singularities of the solution. Although the solution is not regular enough when the right-hand side is smooth, explicit expressions of the the singularities of the solution are interesting and helpful in solving the equations numerically [1, 2, 18].

Under general boundary conditions, a decomposition of the solution in the form of

$$(\mathbf{u}, p) = (\mathbf{u}_s, p_s) + \sum_{j,k} C_{\lambda_j,k} (\ln r)^k (r^{\lambda_j} \mathbf{u}_{\lambda_j,\omega,k}(\theta), r^{\lambda_j - 1} p_{\lambda_j,\omega,k}(\theta)) \tag{5}$$

has been mentioned in [17], where the first part is regular and the second part contains the singularities. Under the Dirichlet boundary condition and the condition $g = 0$, the Stokes problem can be converted to a fourth-order bihamornic equation and by this method one can show that only first-order logarithmic singularities appear in the above expression [9], but this method does not apply to the inhomogeneous stress/mixed boundary condition or the case $g \neq 0$. Recently, Choi and

Kweon [4] derived the dominant corner singularity for the solution of the stationary Navier–Stokes equations (with the Dirichlet boundary condition)

$$(\mathbf{u}, p) = (\mathbf{u}_s, p_s) + \sum_j C_{\lambda_j} (r^{\lambda_j} \mathbf{u}_{\lambda_j,\omega}(\theta), r^{\lambda_j - 1} p_{\lambda_j,\omega}(\theta)) \tag{6}$$

when $\omega \neq \omega_*$, where $(\mathbf{u}_s, p_s) \in \mathbf{H}^{s+1}(\Omega) \times H^s(\Omega)$ for $s \in (0, 1]$, and $\omega_* \in (0, 2\pi)$ is the unique root of the equation

$$\tan(\omega_*) - \omega_* = 0. \tag{7}$$

Explicit formulae of $(\mathbf{u}_{\lambda_j,\omega,k}(\theta), p_{\lambda_j,\omega,k}(\theta))$ have not been given for general $\omega \in (0, \pi) \cup (\pi, 2\pi)$ and large fractional s.

In the PhD Thesis of Orlt [16, pp. 63–64], the author have found explicit expressions for the singular terms in the following particular cases:

a) $\lambda = 0$ is an eigenvalue with the geometrical multiplicity 2.

b) $\lambda = 1$, $\omega \neq \pi$, $\omega \neq 2\pi$ and $\omega \neq \tan \omega$ is a simple eigenvalue.

c) $\lambda = 1$ $\omega = \pi$ or $\omega = 2\pi$ is an eigenvalue with the geometrical multiplicity 2.

d) To $\lambda = 1$, $\omega \neq \tan \omega$ belong an eigenfunction and an associate eigenfunction (that means a logarithmic term occurs.)

In the work of Märkl and Sändig [14], explicit singular expansions for all other simple eigenvalues are found.

In this paper, we look for an explicit corner singularity expansion of the solution for any given right-hand side $(\mathbf{f}, g, \mathbf{h}) \in \mathbf{H}^{s-1}_{00}(\Omega) \times H^s_{00}(\Omega) \times \mathbf{H}^{s-1/2}_{00}(\partial\Omega)$ under the stress boundary condition with general $\omega \in (0, \pi) \cup (\pi, 2\pi)$, where $s > 0$ can be fractional and arbitrarily large. Our method can be easily adapted to other boundary conditions. Let $\mathfrak{P}$ be the set of positive roots of the equation

$$\tan(z) - z = 0 \tag{8}$$

and let $\mathfrak{W}$ be the set of $\omega \in (0, \pi) \cup (\pi, 2\pi)$ satisfying

$$z = \sqrt{\frac{\omega^2}{\sin(\omega)^2} - 1} \tag{9}$$

for some $z \in \mathfrak{P}$. Then $\mathfrak{P}$ is a countably infinite set and each $z \in \mathfrak{P}$ corresponds to at most three values of ω satisfying (9), which indicates that $\mathfrak{M}$ is a countably infinite set which contains the ω_* defined in (8). We shall see that when $\omega \notin \mathfrak{M}$ the operator $\mathcal{L}(\lambda)^{-1}$ has a pole of order 1 at each of its spectral point, and the solution has a decomposition in the form of (6). When $\omega \in \mathfrak{M}$, the operator $\mathcal{L}(\lambda)^{-1}$ has a pole of order 2 at

$$\lambda(\omega) := \sqrt{\sin(\omega)^{-2} - \omega^{-2}}, \tag{10}$$

and has a pole of order 1 at other spectral points. The multiple pole at $\lambda(\omega)$ contributes to an additional logarithmic singularity, i.e.

$$(11)\qquad (\mathbf{u},p) = (\mathbf{u}_s,p_s) + \sum_{\lambda_j \neq \lambda(\omega)} C_{\lambda_j}(r^{\lambda_j}\mathbf{u}_{\lambda_j,\omega}(\theta), r^{\lambda_j-1}p_{\lambda_j,\omega}(\theta)) + C^*\Big[(r^{\lambda(\omega)}\mathbf{u}^*_{\lambda(\omega),\omega}(\theta), r^{\lambda(\omega)-1}p^*_{\lambda(\omega),\omega}(\theta)) + (r^{\lambda(\omega)}\mathbf{u}_{\lambda(\omega),\omega}(\theta), r^{\lambda(\omega)-1}p_{\lambda(\omega),\omega}(\theta))\ln r\Big].$$

The explicit expressions of $(\mathbf{u}_{\lambda,\omega}(\theta), p_{\lambda,\omega}(\theta))$ and $(\mathbf{u}^*_{\lambda,\omega}(\theta), p^*_{\lambda,\omega}(\theta))$ will be given. For the reader's convenience, we present a self-contained proof for our results.

2. Notations and main results

Let $\mathbb{R}_+$ denote the set of positive real numbers and let $\mathbb{N}$ be the set of nonnegative integers.

For any given $\omega \in (0,\pi)\cup(\pi,2\pi)$, we let $\mathfrak{Sp}(\mathcal{L};\omega)$ denote the set of all roots λ of the equation

$$(12)\qquad \lambda^2\sin(\omega)^2 - \sin(\lambda\omega)^2 = 0$$

with nonnegative real parts, and define

$$(13)\qquad \operatorname{Re}\mathfrak{Sp}(\mathcal{L};\omega) = \{\eta > 0 : \eta + i\zeta \in \mathfrak{Sp}(\mathcal{L};\omega) \text{ for some } \zeta\in\mathbb{R}\},$$

which is simply the projection of $\mathfrak{Sp}(\mathcal{L};\omega)$ onto the real line. From [7] we know that $\mathfrak{Sp}(\mathcal{L};\omega) = \{0, 1, \lambda_1(\omega), \lambda_2(\omega), \lambda_3(\omega), \cdots\}$ where $\operatorname{Re}\lambda_j(\omega) > 0$ for $j = 1, 2, \cdots$

Let O_m, $m = 1, \cdots, m_0$, be the corners of the polygon Ω in the counter clockwise order, and define $O_0 = O_{m_0}$ and $O_1 = O_{m_0+1}$; let Γ_m denote the open edge between O_m and O_{m+1}, and let $(r_m, \theta_m) = (r_m(x), \theta_m(x))$ be the polar coordinates in the cone centered at O_m spanned by Ω.

We denote by $H^s(\Omega)$ the standard Sobolev space of order $s\in\mathbb{R}$ and define the function spaces

(14)

$$\begin{aligned}
&H^s_{00}(\Omega) := \{v\in H^s(\Omega) : \textstyle\sum_{m=1}^{m_0}\sum_{|\alpha|\le[s]} r_m^{|\alpha|-s}|\partial^\alpha v| \in L^2(\Omega)\} && \text{for } s\ge 0,\\
&H^s_{00}(\Omega) := H^{-s}(\Omega)' \hookrightarrow H^{-s}_{00}(\Omega)' && \text{for } s< 0,\\
&H^0(\partial\Omega) := L^2(\partial\Omega) && \text{for } s = 0,\\
&H^s(\partial\Omega) := \{v\in L^2(\partial\Omega) : v \text{ is the trace of some function in } H^{s+1/2}(\Omega)\} && \text{for } s > 0,\\
&H^s_{\mathrm{pw}}(\partial\Omega) := \{v\in L^2(\partial\Omega) : v|_{\Gamma_m}\in H^s(\Gamma_m),\ m = 1,\dots,m_0\} && \text{for } s\ge 0,\\
&H^s_{\mathrm{pw}}(\partial\Omega) := H^{-s}_{\mathrm{pw}}(\partial\Omega)' && \text{for } s<0,\\
&H^s_{00}(\partial\Omega) := \{v\in H^s_{\mathrm{pw}}(\partial\Omega) : \textstyle\sum_{m=1}^{m_0}\sum_{k=0}^{[s]} r_m^{k-s}|\partial_\tau^k v| \in L^2(\partial\Omega)\} && \text{for } s\ge 0,\\
&H^s_{00}(\partial\Omega) := H^s_{\mathrm{pw}}(\partial\Omega) && \text{for } s<0,
\end{aligned}$$

where $[s]$ denotes the integer part of s. For a non-integer index $s > 0$, $g \in H^s_{00}(\Omega)$ if and only if $g \in H^s(\Omega)$ and (see [6])

$$\partial^\beta g(O_m) = 0 \quad \text{for all multi-indices } \beta \text{ such that } |\beta| \le [s] - 1, \;\; m = 1, \cdots, m_0. \tag{15}$$

Similarly, if $s \ge 0$ and $s - 1/2$ is not an integer, then $h \in H^s_{00}(\partial\Omega)$ if and only if $h \in H^s(\partial\Omega)$ and

$$\partial_\tau^k h(O_m) = 0 \quad \text{for all } k \le [s - 1/2], \;\; m = 1, \cdots, m_0, \tag{16}$$

where ∂_τ denotes the tangential derivative on $\partial\Omega$ along each of the edges. Hence, $H^s_{00}(\Omega) = H^s(\Omega)$ for $s \in [0, 1)$ and $H^s_{00}(\partial\Omega) = H^s_{\mathrm{pw}}(\partial\Omega)$ for $s \in (-\infty, 1/2)$. The vectorial notations $\mathbf{H}^s_{00}(\Omega) = H^s_{00}(\Omega) \times H^s_{00}(\Omega)$ and $\mathbf{H}^s_{00}(\partial\Omega) = H^s_{00}(\partial\Omega) \times H^s_{00}(\partial\Omega)$ will be used in this paper.

For each $m = 1, \cdots, m_0$, let $\Phi_m(r_m)$ be a fixed smooth cut-off function such that

$$\Phi_m(r_m) = \begin{cases} 1 & \text{in a neighborhood of } O_m, \\ 0 & \text{outside a neighborhood of } O_m. \end{cases} \tag{17}$$

When $\lambda \ne 1$ we define the special functions

$$\begin{aligned} &\big(\mathbf{u}_{\lambda,\omega}(\theta), p_{\lambda,\omega}(\theta)\big)^T \\ &:= \left(\begin{array}{c} \dfrac{\lambda\alpha_{\lambda,\omega}}{\lambda - 1} \cos((\lambda-1)\theta)\sin(\theta) + \lambda \sin[(\lambda-1)\theta]\sin(\theta) - \cos(\lambda\theta) \\ \lambda\cos((\lambda-1)\theta)\sin(\theta) - \alpha_{\lambda,\omega}\left(\dfrac{\cos((\lambda-1)\theta)\cos(\theta)}{\lambda-1} + \sin((\lambda-1)\theta)\sin(\theta)\right) \\ 2\lambda\cos((\lambda-1)\theta) - 2\lambda\alpha_{\lambda,\omega}\sin((\lambda-1)\theta)/(\lambda-1) \end{array} \right), \end{aligned} \tag{18}$$

$(\mathbf{u}^*_{\lambda,\omega}(\theta), p^*_{\lambda,\omega}(\theta))^T$

$$(19)\quad := \begin{pmatrix} \theta\sin(\lambda\theta) + \frac{1}{\lambda+1}\cos(\lambda\theta) + \left(\frac{1}{\lambda+1} - \frac{\lambda\alpha_{\lambda,\omega}\theta}{\lambda-1}\right)\sin((\lambda-1)\theta)\sin(\theta) \\ -\left(\frac{\lambda^2\omega}{(\lambda-1)\sin(\lambda\omega)^2} + \frac{\alpha_{\lambda,\omega}+\lambda\alpha^*_{\lambda,\omega}}{(\lambda-1)^2(\lambda+1)} - \lambda\theta\right)\cos((\lambda-1)\theta)\sin(\theta) \\ \\ \left(\frac{\lambda^2\omega}{(\lambda-1)\sin(\lambda\omega)^2} + \frac{\alpha_{\lambda,\omega}+\lambda\alpha^*_{\lambda,\omega}}{(\lambda-1)^2(\lambda+1)} - \lambda\theta\right)\cos((\lambda-1)\theta)\cos(\theta) + \frac{\alpha_{\lambda,\omega}\theta}{\lambda-1}\sin(\lambda\theta) \\ +\left(\frac{1}{\lambda+1} - \frac{\lambda\alpha_{\lambda,\omega}\theta}{\lambda-1}\right)\cos((\lambda-1)\theta)\sin(\theta) + \left(\cot(\lambda\omega) - \frac{\lambda\omega}{\sin(\lambda\omega)^2} - \frac{\alpha_{\lambda,\omega}}{\lambda+1} + \lambda\theta\right)\cos(\lambda\theta) \\ \\ 2\left(\frac{\lambda^2\omega}{(\lambda-1)\sin(\lambda\omega)^2} + \frac{\alpha_{\lambda,\omega}+\lambda\alpha^*_{\lambda,\omega}}{(\lambda-1)^2(\lambda+1)} - \lambda\theta\right)\sin((\lambda-1)\theta) + 2\left(\frac{1}{\lambda+1} - \frac{\lambda\alpha_{\lambda,\omega}\theta}{\lambda-1}\right)\cos((\lambda-1)\theta) \end{pmatrix},$$

and when $\lambda = 1$ we define

$$(20)\quad (\mathbf{u}_{1,\omega}(\theta), p_{1,\omega}(\theta)) := (-\sin(\theta), \cos(\theta), 0)\,,$$

$$(21)\quad (\mathbf{u}^*_{1,\omega}(\theta), p^*_{1,\omega}(\theta)) := \left(\frac{\omega}{2}\cos(\theta) - \frac{1}{2}\sin(\theta),\ -\frac{\omega}{2}\sin(\theta) - \frac{1}{2}\cos(\theta),\ 2\theta - \omega\right),$$

where we denote

$$(22)\qquad \alpha_{\lambda,\omega} = \cot(\omega) + \lambda\cot(\lambda\omega) \quad \text{and} \quad \alpha^*_{\lambda,\omega} = \lambda\cot(\omega) + \cot(\lambda\omega)$$

to simplify the notations. Our main result is presented in the following theorem.

Theorem 2.1. *Let Ω be a polygon and assume that $(\mathbf{f}, g, \mathbf{h}) \in \mathbf{H}^{s-1}_{00}(\Omega) \times H^s_{00}(\Omega) \times \mathbf{H}^{s-1/2}_{00}(\partial\Omega)$ for some $s \notin \cup_{m=1}^{m_0} \mathrm{Re}\,\mathfrak{Sp}(\mathcal{L};\omega_m)$. Then any a priori $\mathbf{H}^1(\Omega) \times L^2(\Omega)$ weak solution of (1) has the decomposition*

$$(23)\quad \begin{aligned} (\mathbf{u}, p) = (\mathbf{u}_s, p_s) &+ \sum_{m=1}^{m_0} \Phi_m(r_m) \sum_{\substack{\lambda_j \in \mathfrak{Sp}(\mathcal{L};\omega_m)\setminus\mathbb{N} \\ 0<\mathrm{Re}(\lambda_j)<s}} \kappa_{m,\lambda_j}\left(r_m^{\lambda_j}\mathbf{u}_{\lambda_j,\omega_m}(\theta_m), r_m^{\lambda_j-1}p_{\lambda_j,\omega_m}(\theta_m)\right) \\ &+ \sum_{m=1}^{m_0} \kappa_m \Phi_m(r_m)\Big[(r_m^{\lambda(\omega_m)}\mathbf{u}^*_{\lambda(\omega_m),\omega_m}(\theta_m), r_m^{\lambda(\omega_m)-1}p^*_{\lambda(\omega_m),\omega_m}(\theta_m)) \\ &\qquad + (r_m^{\lambda(\omega_m)}\mathbf{u}_{\lambda(\omega_m),\omega_m}(\theta_m), r_m^{\lambda(\omega_m)-1}p_{\lambda(\omega_m),\omega_m}(\theta_m))\ln r_m\Big] \end{aligned}$$

where $(\mathbf{u}_s, p_s) \in \mathbf{H}^{s+1}(\Omega) \times H^s(\Omega)$ is the regular part of the solution, and the other two terms constitute the singular part of the solution, with κ_m and κ_{m,λ_j} being *constants. and*

$$\|\mathbf{u}_s\|_{\mathbf{H}^{s+1}(\Omega)} + \|p_s\|_{H^s(\Omega)} + \sum_{m=1}^{m_0} |\kappa_m| + \sum_{m=1}^{m_0} \sum_{\substack{\lambda_j \in \mathfrak{S}\mathfrak{p}(\mathcal{L};\omega_m)\setminus\mathbb{N} \\ 0<\mathrm{Re}(\lambda_j)<s}} |\kappa_{m,\lambda_j}|$$

$$\leq C(\|\mathbf{f}\|_{\mathbf{H}_{00}^{s-1}(\Omega)} + \|g\|_{H_{00}^s(\Omega)} + \|\mathbf{h}\|_{\mathbf{H}_{00}^{s-1/2}(\partial\Omega)}). \tag{24}$$

If either $\omega_m \notin \mathfrak{M}$ or $s < \lambda(\omega_m)$, then $\kappa_m = 0$. In other words, the third term in (23) *only appears when $s > \lambda(\omega_m)$ and $\omega \in \mathfrak{M}$ simultaneously.*

Remark 2.1. *We have excluded the cases $\omega = \pi$ and $\omega = 2\pi$ in order to use the unified formula* (8)*. Our proof of the quantitative estimate* (24) *relies on the trace theorem which requires $s > 1/2$, which also requires us to exclude the case $\omega = 2\pi$.*

The $[0, s]$-regularity (the shift theorem) below is a special case of Theorem 2.1 when $0 < s < \mathrm{Re}(\lambda_1(\omega))$.

Corollary 2.1. *Let Ω be a polygon and let ω denote its maximal interior angle. If $(\mathbf{f}, g, \mathbf{h}) \in \mathbf{H}_{00}^{s-1}(\Omega) \times H_{00}^s(\Omega) \times \mathbf{H}_{00}^{s-1/2}(\partial\Omega)$ for some $0 < s < \mathrm{Re}(\lambda_1(\omega))$, then the weak solution of* (1) *is in $\mathbf{H}^{s+1}(\Omega) \times H^s(\Omega)$, and*

$$\|\mathbf{u}\|_{\mathbf{H}^{s+1}(\Omega)} + \|p\|_{H^s(\Omega)} \leq C(\|\mathbf{f}\|_{\mathbf{H}_{00}^{s-1}(\Omega)} + \|g\|_{H_{00}^s(\Omega)} + \|\mathbf{h}\|_{\mathbf{H}_{00}^{s-1/2}(\partial\Omega)}). \tag{25}$$

Proof of Theorem 2.1. Since $C^\infty(\overline{\Omega}) \cap H_{00}^s(\Omega)$ is dense in $H_{00}^s(\Omega)$ and $C^\infty(\partial\Omega) \cap H_{00}^{s-1/2}(\partial\Omega)$ is dense in $H_{00}^{s-1/2}(\partial\Omega)$, we can assume that $\mathbf{f}$, g and h are qualitatively C^∞ smooth without loss of generality, provided that our estimates in the proof only depend on the quantitative norm $\|(\mathbf{f}, g, \mathbf{h})\|_{\mathbf{H}^{s+1}(\Omega)\times H^s(\Omega)\times \mathbf{H}^{s-1/2}(\partial\Omega)}$.

Since $\mathrm{Re}\,\lambda_1(\omega) > 1/2$ when $\omega \in [0, \pi) \cup (\pi, 2\pi)$, Theorem 2.1 is equivalent to Corollary 2.1 in the case $s \in (0, 1/2]$, and (25) holds naturally for $s = 0$. In the rest of this paper, we focus on proving Theorem 2.1 for the case $s > 1/2$. Once the case $s > 1/2$ is proved, interpolation between the two cases $s = 0$ and $s = 1/2 + \epsilon$ will give the result for $s \in (0, 1/2]$.

Via a partition of unity, one only need to prove Theorem 2.1 for the variational problem

$$\begin{cases} 2(\mathbb{D}(\mathbf{u}), \mathbb{D}(\mathbf{v}))_{\Omega_0} - (p, \nabla\mathbf{v})_{\Omega_0} = (\mathbf{f}, \mathbf{v})_{\Omega_0} + (\mathbf{h}, \mathbf{v})_{\partial\Omega_0}, & \forall\ \mathbf{v} \in H^1(\Omega_0), \\ (\nabla\cdot\mathbf{u}, \psi)_{\Omega_0} = (g, \psi)_{\Omega_0}, & \forall\ \psi \in L^2(\Omega_0), \end{cases} \tag{26}$$

in the cone Ω_0 centered at one corner, say $O_m = (0, 0)$, and assume that the solution $(\mathbf{u}, p)$ has compact support in the unit ball. In the rest part of the paper, we present estimates for the solution of (26) and, for the reader's convenience, the analysis presented in this paper is self-contained. To simplify the notations, we denote $r = r_m$, $\theta = \theta_m$ and $\omega = \omega_m$.

3. Mellin transform of the Stokes equations

In this section, we formally establish a connection between (26) and a system of ODEs via the Mellin transform, and introduce the function spaces to be used in studying the Mellin transform of the Stokes equations. Strict mathematical analysis is presented in sections 4–5.

3.1. Formal reduction to a system of ODEs. Let $\mathscr{P}$ denote the operator which transforms a given function (either scalar or vector fields) defined on Ω_0 into a function in the polar coordinates defined on $\Omega_0^* = \mathbb{R}_+ \times (0, \omega)$, i.e.

$$\mathscr{P}[v(x_1, x_2)] = v(r\cos(\theta), r\sin(\theta)), \quad \text{if } v \text{ is a scalar function,} \tag{27}$$

$$\mathscr{P}[\mathbf{v}(x_1, x_2)] = \mathbf{v}(r\cos(\theta), r\sin(\theta)), \quad \text{if } \mathbf{v} \text{ is a vector field.} \tag{28}$$

Let

$$\mathbf{r} := (\cos(\theta), \sin(\theta)), \qquad \mathbf{r}^\perp := (-\sin(\theta), \cos(\theta)), \tag{29}$$

and let $\vec{\mathscr{R}}$ denote the following transformation acting on a vector:

$$\vec{\mathscr{R}}\mathbf{v} = (\mathbf{v}\cdot\mathbf{r}, \mathbf{v}\cdot\mathbf{r}^\perp). \tag{30}$$

We define the following notations related to the polar coordinates:

$$\begin{aligned}
&\mathbf{U} = \vec{\mathscr{R}}\mathscr{P}[\mathbf{u}](r,\theta) = (\mathbf{u}(r\cos\theta, r\sin\theta)\cdot\mathbf{r},\ \mathbf{u}(r\cos\theta, r\sin\theta)\cdot\mathbf{r}^\perp),\\
&\mathbf{F} = \vec{\mathscr{R}}\mathscr{P}(\mathbf{f})(r,\theta) = (\mathbf{f}(r\cos\theta, r\sin\theta)\cdot\mathbf{r},\ \mathbf{f}(r\cos\theta, r\sin\theta)),\\
&\mathbf{H} = \vec{\mathscr{R}}\mathscr{P}(\mathbf{h})(r,\theta) = (\mathbf{h}(r\cos\theta, r\sin\theta)\cdot\mathbf{r},\ \mathbf{h}(r\cos\theta, r\sin\theta)),\\
&P = \mathscr{P}(p)(r,\theta) = p(r\cos\theta, r\sin\theta),\\
&G = \mathscr{P}(g)(r,\theta) = g(r\cos\theta, r\sin\theta),
\end{aligned} \tag{31}$$

With these notations, (26) can be written in the polar coordinates as

$$\left\{\begin{array}{l}
\big(2\mathcal{D}(r\partial_r)\mathbf{U}, \mathcal{D}(r\partial_r)\mathbf{V}\big)_{\Omega_0^*} - \big(q,\, r\partial_r V_1 + V_1 + \partial_\theta V_2\big)_{\Omega_0^*} = (r^2\mathbf{F}, \mathbf{V})_{\Omega_0^*} + (r\mathbf{H}, \mathbf{V})_{\partial\Omega_0^*},\\
(r\partial_r U_1 + U_1 + \partial_\theta U_2, \Psi)_{\Omega_0^*} = (rG, \Psi)_{\Omega_0^*},
\end{array}\right. \tag{32}$$

for any $\mathbf{V} \in \vec{\mathscr{R}}\mathscr{P}\mathbf{H}^1(\Omega_0)$ and $\Psi \in \mathscr{P}L^2(\Omega_0)$, where the integrals over Ω_0^* is with respect to the measure $\mathrm{d}r\mathrm{d}\theta/r$ and the integrals over $\partial\Omega_0^* = \mathbb{R}_+ \times \{0, \omega\}$ is with respect to the measure $\mathrm{d}r/r$. The matrix $\mathcal{D}(r\partial_r)$ is given by

$$\mathcal{D}(r\partial_r)\mathbf{U} := \begin{pmatrix} r\partial_r U_1 & (\partial_\theta U_1 + r\partial_r U_2 - U_2)/2 \\ (\partial_\theta U_1 + r\partial_r U_2 - U_2)/2 & U_1 + \partial_\theta U_2 \end{pmatrix}. \tag{33}$$

Via integration by parts, one can see that (32) is the variational formulation of the following equations in the polar coordinates:

(34)
$$\begin{cases} \left[-\left(r\frac{\partial}{\partial r}\right)^2-\frac{\partial^2}{\partial\theta^2}\right]U_1+U_1+2\frac{\partial U_2}{\partial\theta}+r\frac{\partial}{\partial r}(rP)-rP=r^2F_1 & \text{in } \mathbb{R}_+\times(0,\omega),\\ \left[-\left(r\frac{\partial}{\partial r}\right)^2-\frac{\partial^2}{\partial\theta^2}\right]U_2+U_2-2\frac{\partial U_1}{\partial\theta}+\frac{\partial}{\partial\theta}(rP)=r^2F_2 & \text{in } \mathbb{R}_+\times(0,\omega),\\ \left(r\frac{\partial}{\partial r}\right)U_1+U_1+\frac{\partial U_2}{\partial\theta}=rG & \text{in } \mathbb{R}_+\times(0,\omega),\\ \frac{\partial U_1}{\partial\theta}+\left(r\frac{\partial}{\partial r}\right)U_2-U_2=rH_1 & \text{at } \theta=0 \text{ and } \theta=\omega,\\ 2U_1+2\frac{\partial U_2}{\partial\theta}-rP=rH_2 & \text{at } \theta=0 \text{ and } \theta=\omega. \end{cases}$$

To study (32), we use the Mellin transform (with $\lambda=\eta+i\zeta$)

$$\mathscr{M}[v(r)](\lambda)=\int_0^\infty r^{-\lambda-1}v(r)\mathrm{d}r=\int_{-\infty}^\infty e^{-i\zeta t}e^{-\eta t}v(e^t)\mathrm{d}t=\mathscr{F}_t[e^{-\eta t}v(e^t)](\zeta), \tag{35}$$

and its inverse

$$\mathscr{M}^{-1}[\widehat{v}(\lambda)](r)=\frac{1}{2\pi i}\int_{\mathrm{Re}(\lambda)=\eta}r^\lambda\widehat{v}(\lambda)\mathrm{d}\lambda=\frac{1}{2\pi}\int_{\mathbb{R}}r^{\eta+i\zeta}\widehat{v}(\eta+i\zeta)\mathrm{d}\zeta, \tag{36}$$

where $\mathscr{F}_t$ denotes the Fourier transform with respect to t, and by the theory of Fourier transform we have

$$\mathscr{M}[r\partial_r v(r)]=\lambda\mathscr{M}[v(r)](\lambda).$$

By denoting

(37)
$$\begin{aligned} &(\widehat{\mathbf{U}}(\lambda,\theta),\widehat{q}(\lambda,\theta))=\mathscr{M}_*(\mathbf{u},p)(\lambda,\theta):=(\mathscr{M}[\mathbf{U}(r,\theta)](\lambda),\mathscr{M}[rP(r,\theta)](\lambda)),\\ &(\widehat{\mathbf{F}}(\lambda,\theta),\widehat{G}(\lambda,\theta))=\mathscr{M}_*(|x|^2\mathbf{f},g)(\lambda,\theta)=(\mathscr{M}[r^2\mathbf{F}(r,\theta)](\lambda),\mathscr{M}[rG(r,\theta)](\lambda)),\\ &\widehat{G}(\lambda,0)=\mathscr{M}(rG(r,0))(\lambda),\\ &\widehat{G}(\lambda,\omega)=\mathscr{M}(rG(r,\omega)))(\lambda),\\ &\widehat{\mathbf{H}}(\lambda,0)=\mathscr{M}[r\mathbf{H}(r,0)](\lambda),\\ &\widehat{\mathbf{H}}(\lambda,\omega)=\mathscr{M}[r\mathbf{H}(r,\omega)](\lambda), \end{aligned}$$

we call $(\widehat{\mathbf{U}},\widehat{q})$ the *Mellin transform* of $(\mathbf{u},p)$, and call $(\mathbf{u},p)=\mathscr{M}_*^{-1}(\widehat{\mathbf{U}},\widehat{q})$ the *inverse Mellin transform* of $(\widehat{\mathbf{U}},\widehat{q})$. To simplify the notations, we shall often use $\widehat{U}(\lambda)$ to denote the function $\widehat{U}(\lambda,\theta)$.

By using the identity

(38)
$$\int_{\mathbb{R}_+\times(0,\omega)} U(r,\theta)V(r,\theta)\frac{\mathrm{d}r\mathrm{d}\theta}{r} = \int_{\mathbb{R}\times(0,\omega)} \mathscr{M}[U](\eta+i\zeta,\theta)\mathscr{M}[V](-\eta-i\zeta,\theta)\mathrm{d}\zeta\mathrm{d}\theta,$$

(32) is formally transformed to

(39)
$$\begin{cases} \big(2\mathcal{D}(\lambda)\widehat{\mathbf{U}}(\lambda),\mathcal{D}(-\lambda)\widehat{\mathbf{V}}(-\lambda)\big)_{\mathbb{R}\times(0,\omega)} + \big(\widehat{q}(\lambda),\,\lambda\widehat{V}_1(-\lambda)-\widehat{V}_1(-\lambda)-\partial_\theta\widehat{V}_2(-\lambda)\big)_{\mathbb{R}\times(0,\omega)} \\ = (\widehat{\mathbf{F}}(\lambda),\widehat{\mathbf{V}}(-\lambda))_{\mathbb{R}\times(0,\omega)} + (\widehat{\mathbf{H}}(\lambda,0),\widehat{\mathbf{V}}(-\lambda,0))_{\mathbb{R}} + (\widehat{\mathbf{H}}(\lambda,\omega),\widehat{\mathbf{V}}(-\lambda,\omega))_{\mathbb{R}}, \\ (\lambda\widehat{U}_1(\lambda)+\widehat{U}_1(\lambda)+\partial_\theta\widehat{U}_2(\lambda),\widehat{\Psi}(-\lambda))_{\mathbb{R}\times(0,\omega)} = (\widehat{G}(\lambda),\widehat{\Psi}(-\lambda))_{\mathbb{R}\times(0,\omega)}, \end{cases}$$

which holds for any $\widehat{\mathbf{V}}(-\lambda)\in\mathscr{M}\vec{\mathscr{R}}\mathscr{P}[\mathbf{H}^1(\Omega_0)]|_{\mathrm{Re}(\lambda)=\eta}$ and $\widehat{\Psi}(-\lambda)\in\mathscr{M}\mathscr{P}[L^2(\Omega_0)]|_{\mathrm{Re}(\lambda)=\eta}$, where the brackets in (39) denote integrals with respect to $\zeta=\mathrm{Im}(\lambda)$ with fixed $\eta=\mathrm{Re}(\lambda)$. It is easy to check that $\widehat{\mathbf{V}}(-\eta-i\zeta,\theta)=\phi_1(\zeta)\widehat{\mathbf{v}}(\theta)$ and $\widehat{\Psi}(-\eta-i\zeta,\theta)=\phi_2(\zeta)\widehat{\psi}(\theta)$ can be chosen in (39), provided $\mathscr{F}^{-1}\phi_1,\mathscr{F}^{-1}\phi_2\in C_0^\infty(\mathbb{R})$, $\widehat{\mathbf{v}}\in\mathbf{H}^1(0,\omega)$ and $\widehat{\psi}\in L^2(0,\omega)$. Therefore, (39) further implies

$$\begin{cases} \big(2\mathcal{D}(\lambda)\widehat{\mathbf{U}}(\lambda),\mathcal{D}(-\lambda)\widehat{\mathbf{v}}\big)_{(0,\omega)} + \big(\widehat{q}(\lambda),\,\lambda\widehat{v}_1-\widehat{v}_1-\partial_\theta\widehat{v}_2\big)_{(0,\omega)} \\ = (\widehat{\mathbf{F}}(\lambda),\widehat{\mathbf{v}})_{(0,\omega)} + \widehat{\mathbf{H}}(\lambda,0)\widehat{\mathbf{v}}(0) + \widehat{\mathbf{H}}(\lambda,\omega)\widehat{\mathbf{v}}(\omega), \\ (\lambda\widehat{U}_1(\lambda)+\widehat{U}_1(\lambda)+\partial_\theta\widehat{U}_2(\lambda),\widehat{\psi})_{(0,\omega)} = (\widehat{G}(\lambda),\widehat{\psi})_{(0,\omega)}, \end{cases} \tag{40}$$

for any $\widehat{\mathbf{v}}\in\mathbf{H}^1(0,\omega)$ and $\widehat{\psi}\in L^2(0,\omega)$, for almost all $\zeta\in\mathbb{R}$.

For the fixed $\lambda\in\mathbb{C}$, we define $\mathcal{L}(\lambda)(\widehat{\mathbf{U}}(\lambda),\widehat{q}(\lambda))\in\mathbf{H}^1(0,\omega)'\times L^2(0,\omega)$ by

$$\begin{aligned} (\mathcal{L}(\lambda)(\widehat{\mathbf{U}}(\lambda),\widehat{q}(\lambda)),(\widehat{\mathbf{v}},\widehat{\psi})) = {} & \big(2\mathcal{D}(\lambda)\widehat{\mathbf{U}}(\lambda),\mathcal{D}(-\lambda)\widehat{\mathbf{v}}\big)_{(0,\omega)} \\ & + \big(\widehat{q}(\lambda),\,\lambda\widehat{v}_1-\widehat{v}_1-\partial_\theta\widehat{v}_2\big)_{(0,\omega)} \\ & + (\lambda\widehat{U}_1(\lambda)+\widehat{U}_1(\lambda)+\partial_\theta\widehat{U}_2(\lambda),\widehat{\psi})_{(0,\omega)}\,, \end{aligned} \tag{41}$$

for any $(\widehat{\mathbf{v}},\widehat{\psi})\in\mathbf{H}^1(0,\omega)\times L^2(0,\omega)$, and we view $(\widehat{\mathbf{F}},\widehat{G},\widehat{\mathbf{H}})$ as an element of $\mathbf{H}^1(0,\omega)'\times L^2(0,\omega)$ in the following way (via duality):

$$\begin{aligned} \Big((\widehat{\mathbf{F}}(\lambda),\widehat{G}(\lambda),\widehat{\mathbf{H}}(\lambda)),(\widehat{\mathbf{v}},\widehat{\psi})\Big) = {} & \big(\widehat{\mathbf{F}},\widehat{\mathbf{v}}\big)_{(0,\omega)} + \big(\widehat{G},\widehat{\psi}\big)_{(0,\omega)} \\ & + \widehat{\mathbf{H}}(\lambda,0)\widehat{\mathbf{v}}(0) + \widehat{\mathbf{H}}(\lambda,\omega)\widehat{\mathbf{v}}(\omega). \end{aligned} \tag{42}$$

Then (40) can be interpreted as

(43)
$$\mathcal{L}(\lambda)(\widehat{\mathbf{U}}(\lambda),\widehat{q}(\lambda)) = (\widehat{\mathbf{F}}(\lambda),\widehat{G}(\lambda),\widehat{\mathbf{H}}(\lambda)) \ \text{ as an element of } \ \mathbf{H}^1(0,\omega)'\times L^2(0,\omega).$$

One can check that (40) or (43) is the variational formulation of the following equation:

$$
(44)\qquad \begin{cases}
-\dfrac{\mathrm{d}^2\widehat{U}_1}{\mathrm{d}\theta^2}+(1-\lambda^2)\widehat{U}_1+2\dfrac{\mathrm{d}\widehat{U}_2}{\mathrm{d}\theta}+(\lambda-1)\widehat{q}=\widehat{F}_1 & \text{for } \theta\in(0,\omega),\\
-\dfrac{\mathrm{d}^2\widehat{U}_2}{\mathrm{d}\theta^2}+(1-\lambda^2)\widehat{U}_2-2\dfrac{\mathrm{d}\widehat{U}_1}{\mathrm{d}\theta}+\dfrac{\mathrm{d}\widehat{q}}{\mathrm{d}\theta}=\widehat{F}_2 & \text{for } \theta\in(0,\omega),\\
(1+\lambda)\widehat{U}_1+\dfrac{\mathrm{d}\widehat{U}_2}{\mathrm{d}\theta}=\widehat{G} & \text{for } \theta\in(0,\omega),\\
\dfrac{\mathrm{d}\widehat{U}_1}{\mathrm{d}\theta}+(\lambda-1)\widehat{U}_2=\widehat{H}_1 & \text{at } \theta=0 \text{ and } \theta=\omega,\\
2\dfrac{\mathrm{d}\widehat{U}_2}{\mathrm{d}\theta}+2\widehat{U}_1-\widehat{q}=\widehat{H}_2 & \text{at } \theta=0 \text{ and } \theta=\omega.
\end{cases}
$$

3.2. Mapping properties of the Mellin transform. In this subsection, we introduce function spaces related to the mapping properties of the Mellin transform to be used in the rest part of the paper [6]. Readers who are familiar with these properties can pass to the next section.

For any two Banach spaces B_1 and B_2 which are embedded into a common topological space B. The space B_1+B_2 is defined as the set of elements $w\in B$ which admits a decomposition $w=w_1+w_2$ with $w_1\in B_1$ and $w_2\in B_2$ such that

$$\|w\|_{B_1+B_2} := \inf_{w=w_1+w_2}(\|w_1\|_{B_1}+\|w_2\|_{B_2})<\infty,$$

where the infimum extends over all possible decompositions of w. For any Banach space B, we let $L^2(\eta+i\mathbb{R}\mapsto B,(1+|\zeta|)^s\mathrm{d}\zeta)$ denote the space of B-space valued functions defined on the vertical line $\eta+i\mathbb{R}$ of the complex plane such that

$$\|w(\eta+i\zeta)\|_{L^2(\eta+i\mathbb{R}\mapsto B,(1+|\zeta|)^{2s}\mathrm{d}\zeta)} := \left(\int_{\mathbb{R}}(1+|\zeta|)^{2s}\|w(\eta+i\zeta)\|_B^2\mathrm{d}\zeta\right)^{\frac{1}{2}}<\infty,$$

and we also denote $L^2(\eta+i\mathbb{R}\mapsto B):=L^2(\eta+i\mathbb{R}\mapsto B,(1+|\zeta|)^0\mathrm{d}\zeta)$.

If Ω_0 is a cone in $\mathbb{R}^2$ centered at the origin with interior angle ω, we define

(45)

$$
\begin{aligned}
&H^s(\mathbb{R}\times(0,\omega)) = \text{standard Sobolev space of functions on } \mathbb{R}\times(0,\omega) && \text{for } s\geq 0,\\
&H^s(\mathbb{R}\times(0,\omega)) = H^{-s}(\mathbb{R}\times(0,\omega))' && \text{for } s<0,\\
&\widehat{W}^s_\eta(0,\omega) = L^2(\eta+i\mathbb{R}\mapsto H^s(0,\omega))\cap L^2(\eta+i\mathbb{R}\mapsto L^2(0,\omega),(1+|\zeta|)^{2s}\mathrm{d}\zeta),\\
&\widehat{\mathbf{W}}^s_\eta(0,\omega) = L^2(\eta+i\mathbb{R}\mapsto \mathbf{H}^s(0,\omega))\cap L^2(\eta+i\mathbb{R}\mapsto \mathbf{L}^2(0,\omega),(1+|\zeta|)^{2s}\mathrm{d}\zeta),\\
&\widehat{W}^s_\eta = L^2(\eta+i\mathbb{R}\mapsto\mathbb{C},(1+|\zeta|)^{2s}\mathrm{d}\zeta),\\
&\widehat{\mathbf{W}}^s_\eta = L^2(\eta+i\mathbb{R}\mapsto\mathbb{C}^2,(1+|\zeta|)^{2s}\mathrm{d}\zeta).
\end{aligned}
$$

Then by the theory of Fourier transform we have

Lemma 3.1. *For any given $\gamma \in \mathbb{R}$, $w \in H^s_{00}(\Omega_0)$ if and only if its Mellin transform satisfies $\mathscr{M}[r^\gamma \mathscr{P} w] \in \widehat{W}^s_{s-1+\gamma}(0,\omega)$, and $w \in H^s_{00}(\mathbb{R}_+)$ if and only if its Mellin transform satisfies $\mathscr{M}[r^\gamma \mathscr{P} w] \in \widehat{W}^s_{s-1/2+\gamma}$.*

This lemma should be understood in the following way:
(i) if $w \in H^{s+1}_{00}(\Omega_0)$ then its Mellin transform is in $\widehat{W}^s_{s-1+\gamma}(0,\omega)$;
(ii) for any function $\widehat{w} \in \widehat{W}^s_{s-1+\gamma}(0,\omega)$, the inverse Mellin transform $\mathscr{P}^{-1}[r^{-\gamma}\mathscr{M}^{-1}\widehat{w}]$ along the line $\mathrm{Re}(\lambda) = s-1+\gamma$ is in $H^s_{00}(\Omega_0)$.

If we denote by Γ_1 and Γ_2 the two sides of the cone Ω_0, and denote

$$H^s_{00}(\Gamma_j) := \{v \in H^s(\Gamma_j) : v/r^s \in L^2(\Gamma_j)\}, \tag{46}$$

then

$$\begin{aligned}
&\mathbf{v} \in \mathbf{H}^{s+1}_{00}(\Omega_0) && \text{iff} \quad \mathscr{M}\vec{\mathscr{R}}\mathscr{P}\mathbf{v} \in \widehat{\mathbf{W}}^{s+1}_s(0,\omega),\\
&\phi \in \mathbf{H}^{s}_{00}(\Omega_0) && \text{iff} \quad \mathscr{M}[r\mathscr{P}\phi] \in \widehat{\mathbf{W}}^{s}_s(0,\omega),\\
&\mathbf{f} \in \mathbf{H}^{s-1}_{00}(\Omega_0) && \text{iff} \quad \mathscr{M}[r^2\vec{\mathscr{R}}\mathscr{P}\mathbf{f}] \in \widehat{\mathbf{W}}^{s-1}_s(0,\omega),\\
&g \in H^{s}_{00}(\Omega_0) && \text{iff} \quad \mathscr{M}[r\mathscr{P}g] \in \widehat{W}^{s}_s(0,\omega),\\
&\mathbf{h}|_{\Gamma_j} \in \mathbf{H}^{s-1/2}_{00}(\Gamma_j) && \text{iff} \quad \mathscr{M}[r\vec{\mathscr{R}}\mathscr{P}\mathbf{h}|_{\Gamma_j}] \in \widehat{\mathbf{W}}^{s-1/2}_s, \;\; j=1,2,\\
&g|_{\Gamma_j} \in H^{s-1/2}_{00}(\Gamma_j) && \text{iff} \quad \mathscr{M}[r\mathscr{P}g|_{\Gamma_j}] \in \widehat{W}^{s-1/2}_s, \;\; j=1,2.
\end{aligned} \tag{47}$$

Since

$$\mathbf{H}^{s-1/2}_{00}(\partial\Omega_0) \cong \mathbf{H}^{s-1/2}_{00}(\Gamma_1) \times \mathbf{H}^{s-1/2}_{00}(\Gamma_2),$$

it follows that

$$\begin{aligned}
&\mathbf{h} \in \mathbf{H}^{s-1/2}_{00}(\partial\Omega_0)\\
&\text{iff} \quad \mathscr{M}[r\vec{\mathscr{R}}\mathscr{P}\mathbf{h}] := (\mathscr{M}[r\vec{\mathscr{R}}\mathscr{P}\mathbf{h}|_{\Gamma_j}], \mathscr{M}[r\vec{\mathscr{R}}\mathscr{P}\mathbf{h}|_{\Gamma_j}]) \in \widehat{\mathbf{W}}^{s-1/2}_s \times \widehat{\mathbf{W}}^{s-1/2}_s.
\end{aligned} \tag{48}$$

To simplify the notations, we define the following function spaces:

$$X^s := \mathbf{H}^{s+1}(\Omega_0) \times H^s(\Omega_0), \tag{49}$$

$$Y^s := \mathbf{H}^{s-1}_{00}(\Omega_0) \times H^s_{00}(\Omega_0) \times (\mathbf{H}^{s-1/2}_{00}(\Gamma_1) \times \mathbf{H}^{s-1/2}_{00}(\Gamma_2)), \tag{50}$$

$$\widehat{X}^s := \mathbf{H}^{s+1}(0,\omega) \times H^s(0,\omega), \tag{51}$$

$$\widehat{Y}^s := \mathbf{H}^{s-1}(0,\omega) \times H^s(0,\omega) \times (\mathbb{C}^2 \times \mathbb{C}^2), \tag{52}$$

$$\widehat{X}^s_* := \widehat{\mathbf{W}}^{s+1}_s(0,\omega) \times \widehat{W}^s_s(0,\omega), \tag{53}$$

$$\widehat{Y}^s_* := \widehat{\mathbf{W}}^{s-1}_s(0,\omega) \times \widehat{W}^s_s(0,\omega) \times (\widehat{\mathbf{W}}^{s-1/2}_s \times \widehat{\mathbf{W}}^{s-1/2}_s), \tag{54}$$

The function spaces above have the following properties:

$$(\mathbf{v},\phi)\in \mathbf{H}^{s+1}_{00}(0,\omega)\times H^{s}_{00}(0,\omega) \quad \text{if and only if} \quad \mathscr{M}(\vec{\mathscr{R}}\mathscr{P}[\mathbf{v}], r\mathscr{P}[\phi])\in \widehat{X}^{s}_{*}, \tag{55}$$

$$(\mathbf{f},g,\mathbf{h})\in Y^{s} \quad \text{if and only if} \quad (\mathscr{M}[r^2\vec{\mathscr{R}}\mathscr{P}\mathbf{f}], \mathscr{M}[r\mathscr{P}g], \mathscr{M}[r\vec{\mathscr{R}}\mathscr{P}\mathbf{h}])\in \widehat{Y}^{s}_{*}. \tag{56}$$

For a function $(\mathbf{u},p)$ which is a priori in $\mathbf{H}^1(\Omega_0)\times L^2(\Omega_0)$ and has compact support in the unit ball, initially we only know that its Mellin transform $(\widehat{\mathbf{U}}(\eta+i\zeta),\widehat{q}(\eta+i\zeta))$ is well-defined for $\eta\le 0$, and

$$\|\widehat{\mathbf{U}}(\eta+i\zeta)\|_{H^1(0,\omega)},\ \ \|\widehat{q}(\eta+i\zeta)\|_{L^2(0,\omega)}\in L^2(\mathbb{R}) \ \ \text{as a function of } \zeta \tag{57}$$

$$(1+|\zeta|)\|\widehat{\mathbf{U}}(\eta+i\zeta)\|_{L^2(0,\omega)}\in L^2(\mathbb{R}) \ \ \text{as a function of } \zeta \tag{58}$$

for $\eta\in(-\infty,0]$.

4. Inversion of $\mathcal{L}(\lambda)$

4.1. The spectrum of $\mathcal{L}(\lambda)$. The spectrum of $\mathcal{L}$ is defined as the set of $\lambda\in\mathbb{C}$ with $\mathrm{Re}(\lambda)\ge 0$ such that the operator $\mathcal{L}(\lambda)$ is not invertible. In this subsection, we determine the spectrum of $\mathcal{L}$ and find the solutions of (44) when λ is not in the spectrum.

From [13] we know that when $\lambda\neq 1$ the homogeneous equations corresponding to (44) can be written as

$$\begin{cases} \dfrac{\mathrm{d}^4\widehat{U}_2}{\mathrm{d}\theta^4}+2(1+\lambda^2)\dfrac{\mathrm{d}^2\widehat{U}_2}{\mathrm{d}\theta^2}+(1-\lambda^2)^2\widehat{U}_2=0,\\ \widehat{U}_1=-\dfrac{1}{1+\lambda}\dfrac{\mathrm{d}\widehat{U}_2}{\mathrm{d}\theta},\\ \widehat{q}=\dfrac{1}{1-\lambda^2}\left(\dfrac{\mathrm{d}^3\widehat{U}_2}{\mathrm{d}\theta^3}+(1+\lambda)^2\dfrac{\mathrm{d}\widehat{U}_2}{\mathrm{d}\theta}\right), \end{cases} \tag{59}$$

and the equation of $\widehat{U}_2$ above has four linearly independent solutions

$$\widehat{U}_{21}=\cos(\theta)\frac{\sin(\lambda\theta)}{\lambda}, \tag{60}$$

$$\widehat{U}_{22}=\sin(\theta)\frac{\sin(\lambda\theta)}{\lambda}, \tag{61}$$

$$\widehat{U}_{23}=\cos(\theta)\cos(\lambda\theta), \tag{62}$$

$$\widehat{U}_{24}=\sin(\theta)\cos(\lambda\theta). \tag{63}$$

We define

$$\widehat{U}_{1j}=-\frac{1}{1+\lambda}\frac{\mathrm{d}\widehat{U}_{2j}}{\mathrm{d}\theta} \qquad \text{and} \qquad \widehat{q}_j=\frac{1}{1-\lambda^2}\left(\frac{\mathrm{d}^3\widehat{U}_{2j}}{\mathrm{d}\theta^3}+(1+\lambda)^2\frac{\mathrm{d}\widehat{U}_{2j}}{\mathrm{d}\theta}\right), \tag{64}$$

for $j = 1, 2, 3, 4$, so that $(\widehat{\mathbf{U}}_j, \widehat{q}_j) := (\widehat{U}_{1j}, \widehat{U}_{2j}, \widehat{q}_j)$, $j = 1, 2, 3, 4$, are four linearly independent solutions of the homogeneous equation corresponding to (44). Then direct computation shows that

$$(\widehat{\mathbf{U}}_j, \widehat{q}_j) = (\vec{\mathscr{R}}\widehat{\mathbf{u}}_j, \widehat{q}_j), \tag{65}$$

where $\widehat{\mathbf{u}}_j$ and $\widehat{q}_j$ are given by

$$\begin{aligned}
(\widehat{\mathbf{u}}_1(\theta), \widehat{q}_1(\theta))^T &= \frac{1}{\lambda+1}\begin{pmatrix} -\frac{1}{2}\cos(\lambda\theta) - \frac{1}{2}\cos(2\theta-\lambda\theta) \\ \left(\frac{1}{\lambda}+\frac{1}{2}\right)\sin(\lambda\theta) - \frac{1}{2}\sin(2\theta-\lambda\theta) \\ -2\cos(\theta-\lambda\theta) \end{pmatrix} \\
(\widehat{\mathbf{u}}_2(\theta), \widehat{q}_2(\theta))^T &= \frac{-1}{\lambda+1}\begin{pmatrix} \left(\frac{1}{\lambda}+\frac{1}{2}\right)\sin(\lambda\theta) + \frac{1}{2}\sin(2\theta-\lambda\theta) \\ \sin(\theta-\lambda\theta/2)^2 - \sin(\lambda\theta/2)^2 \\ 2\sin(\theta-\lambda\theta) \end{pmatrix} \\
(\widehat{\mathbf{u}}_3(\theta), \widehat{q}_3(\theta))^T &= \frac{1}{\lambda+1}\begin{pmatrix} -\lambda\sin(\theta-\lambda\theta)\cos(\theta) \\ \left(1+\frac{\lambda}{2}\right)\cos(\lambda\theta) + \frac{1}{2}\lambda\cos(2\theta-\lambda\theta) \\ -2\lambda\sin(\theta-\lambda\theta) \end{pmatrix} \\
(\widehat{\mathbf{u}}_4(\theta), \widehat{q}_4(\theta))^T &= \frac{1}{\lambda+1}\begin{pmatrix} -\left(1+\frac{\lambda}{2}\right)\cos(\lambda\theta) + \frac{\lambda}{2}\cos(2\theta-\lambda\theta) \\ \lambda\cos(\theta-\lambda\theta)\sin(\theta) \\ 2\lambda\cos(\theta-\lambda\theta) \end{pmatrix}.
\end{aligned} \tag{66}$$

We first seek a particular solution of the inhomogeneous equation (44) of the form

$$\widehat{\mathbf{U}}^{\mathrm{p}} = \begin{pmatrix} \widehat{U}_1^{\mathrm{p}} \\ \widehat{U}_2^{\mathrm{p}} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^4 c_j(\lambda,\theta)\widehat{U}_{1j}(\lambda,\theta) \\ \sum_{j=1}^4 c_j(\lambda,\theta)\widehat{U}_{2j}(\lambda,\theta) \end{pmatrix}, \qquad \widehat{q}^{\mathrm{p}} = \sum_{j=1}^4 c_j(\lambda,\theta)\widehat{q}_j(\lambda,\theta) \tag{67}$$

by the method of variation of coefficients, which does not satisfy the boundary conditions in general. Substituting the above three expressions into (44), we obtain

$$\sum_{j=1}^4 (c_j''\widehat{U}_{1j} + 2c_j'\widehat{U}_{1j}' - 2c_j'\widehat{U}_{2j}) = -\widehat{F}_1, \tag{68}$$

$$\sum_{j=1}^4 (c_j''\widehat{U}_{2j} + 2c_j'\widehat{U}_{2j}' + 2c_j'\widehat{U}_{1j} - c_j'\widehat{q}_j) = -\widehat{F}_2, \tag{69}$$

$$\sum_{j=1}^4 c_j'\widehat{U}_{2j} = \widehat{G}, \tag{70}$$

and we impose the following additional equation to determine the coefficients c_j, $j = 1, 2, 3, 4$.

$$\sum_{j=1}^4 c_j'\widehat{U}_{1j} = 0. \tag{71}$$

Then the equations above can be simplified to

$$\sum_{j=1}^{4} c_j' \widehat{U}_{1j} = 0, \tag{72}$$

$$\sum_{j=1}^{4} c_j' \widehat{U}_{1j}' = 2\widehat{G} - \widehat{F}_1, \tag{73}$$

$$\sum_{j=1}^{4} c_j' (\widehat{U}_{2j}' - \widehat{q}_j) = -\widehat{G}' - \widehat{F}_2, \tag{74}$$

$$\sum_{j=1}^{4} c_j' \widehat{U}_{2j} = \widehat{G}. \tag{75}$$

If we denote

$$\mathbb{M} = \begin{pmatrix} \widehat{U}_{11} & \widehat{U}_{12} & \widehat{U}_{13} & \widehat{U}_{14} \\ \widehat{U}_{11}' & \widehat{U}_{12}' & \widehat{U}_{13}' & \widehat{U}_{14}' \\ \widehat{U}_{21}' - q_1 & \widehat{U}_{22}' - q_2 & \widehat{U}_{23}' - q_3 & \widehat{U}_{24}' - q_4 \\ \widehat{U}_{21} & \widehat{U}_{22} & \widehat{U}_{23} & \widehat{U}_{24} \end{pmatrix} \tag{76}$$

whose determinant is $\det \mathbb{M} = 4/(\lambda+1)^2 \neq 0$, then a solution of (72)-(75) is given by

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \Psi(\theta) - \frac{1}{\omega}\int_0^{\omega} \Psi(\theta)\mathrm{d}\theta, \tag{77}$$

with

$$\Psi(\theta) = \int_0^{\theta} \mathbb{M}^{-1} \begin{pmatrix} 0 \\ 2\widehat{G} - \widehat{F}_1 \\ -\widehat{G}' - \widehat{F}_2 \\ \widehat{G} \end{pmatrix} \mathrm{d}\theta.$$

We then look for the solution of (44) in the form of

$$\widehat{\mathbf{U}}(\lambda,\theta) = \widehat{\mathbf{U}}^{\mathrm{p}}(\lambda,\theta) + \sum_{j=1}^{4} b_j(\lambda)\widehat{\mathbf{U}}_j(\lambda,\theta) = \sum_{j=1}^{4}(b_j(\lambda) + c_j(\lambda,\theta))\widehat{\mathbf{U}}_j(\lambda,\theta), \tag{78}$$

$$\widehat{q}(\lambda,\theta) = \widehat{q}^{\mathrm{p}}(\lambda,\theta) + \sum_{j=1}^{4} b_j \widehat{q}_j(\lambda,\theta) = \sum_{j=1}^{4}(b_j(\lambda) + c_j(\lambda,\theta))\widehat{q}_j(\lambda,\theta), \tag{79}$$

by choosing the constants b_j, $j = 1, 2, 3, 4$, to satisfy the boundary conditions in (44). The equations for b_j are

(80)
$$\begin{aligned}
&\sum_{j=1}^{4} b_j\left(\frac{\mathrm{d}\widehat{U}_{1j}}{\mathrm{d}\theta} + (\lambda-1)\widehat{U}_{2j}\right) = \widehat{H}_1 - \left(\frac{\mathrm{d}\widehat{U}_1^{\mathrm{p}}}{\mathrm{d}\theta} + (\lambda-1)\widehat{U}_2^{\mathrm{p}}\right) \quad \text{at } \theta = 0 \text{ and } \omega,\\
&\sum_{j=1}^{4} b_j\left(2\frac{\mathrm{d}\widehat{U}_{2j}}{\mathrm{d}\theta} + 2\widehat{U}_{1j} - \widehat{q}_j\right) = \widehat{H}_2 - \left(2\frac{\mathrm{d}\widehat{U}_2^{\mathrm{p}}}{\mathrm{d}\theta} + 2\widehat{U}_1^{\mathrm{p}} - \widehat{q}^{\mathrm{p}}\right) \quad \text{at } \theta = 0 \text{ and } \omega,
\end{aligned}$$

Substituting $(\widehat{U}_1^{\mathrm{p}}, \widehat{U}_2^{\mathrm{p}}, \widehat{q}^{\mathrm{p}})$ in terms of c_j into the two equations above and using the identities (72)-(75), the two boundary conditions above can be further written as

$$\begin{aligned}
&\sum_{j=1}^{4} b_j\left(\frac{\mathrm{d}\widehat{U}_{1j}}{\mathrm{d}\theta} + (\lambda-1)\widehat{U}_{2j}\right) = \widehat{H}_1 - \sum_{j=1}^{4} c_j\left(\frac{\mathrm{d}\widehat{U}_{1j}}{\mathrm{d}\theta} + (\lambda-1)\widehat{U}_{2j}\right),\\
&\sum_{j=1}^{4} b_j\left(2\frac{\mathrm{d}\widehat{U}_{2j}}{\mathrm{d}\theta} + 2\widehat{U}_{1j} - \widehat{q}_j\right) = \widehat{H}_2 - 2\widehat{G} - \sum_{j=1}^{4} c_j\left(2\frac{\mathrm{d}\widehat{U}_{2j}}{\mathrm{d}\theta} + 2\widehat{U}_{1j} - \widehat{q}_j\right),
\end{aligned} \tag{81}$$

at $\theta = 0$ and ω. Let $\mathbb{A}$ and $\vec{a} = (a_1, a_2, a_3, a_4)^T$ denote the matrix and the right-hand side of the linear system (81), respectively. Then we have

(82)
$$\mathbb{A}^T = \begin{pmatrix}
0 & 2\cos(\lambda\omega)\sin(\omega) + 2\lambda\sin(\lambda\omega)\cos(\omega) & 2 & 2\cos(\lambda\omega)\cos(\omega)\\
\frac{-2}{1+\lambda} & 2\lambda\sin(\lambda\omega)\sin(\omega) - 2\cos(\lambda\omega)\cos(\omega) & 0 & 2\cos(\lambda\omega)\sin(\omega)\\
\frac{2\lambda^2}{1+\lambda} & 2\lambda^2\cos(\lambda\omega)\cos(\omega) - 2\lambda\sin(\lambda\omega)\sin(\omega) & 0 & -2\lambda\sin(\lambda\omega)\cos(\omega)\\
0 & 2\lambda\sin(\lambda\omega)\cos(\omega) + 2\lambda^2\cos(\lambda\omega)\sin(\omega) & 0 & -2\lambda\sin(\lambda\omega)\sin(\omega)
\end{pmatrix},$$

whose determinant is

$$\det(\mathbb{A}) = \frac{\lambda^2\sin(\omega)^2 - \sin(\lambda\omega)^2}{(\lambda+1)^2}16\lambda^2. \tag{83}$$

Thus the matrix $\mathbb{A}$ is invertible if and only if λ is not a root of the equation

$$\lambda^2\sin(\omega)^2 - \sin(\lambda\omega)^2 = 0, \tag{84}$$

which means that the spectrum of $\mathcal{L}$ is

$$\mathfrak{Sp}(\mathcal{L};\omega) = \{0, 1, \lambda_1(\omega), \lambda_2(\omega), \lambda_3(\omega), \cdots\}, \tag{85}$$

with $\mathrm{Re}[\lambda_j(\omega)] > 0$ for $j = 1, 2, \cdots$. When $\lambda \notin \mathfrak{Sp}(\mathcal{L};\omega)$, from (81) one can solve b_j, $j = 1, 2, 3, 4$, and obtain the solution of (44) in the form of (78)-(79).

4.2. Estimates of the solution when $\lambda \notin \mathfrak{S}\mathfrak{p}(\mathcal{L};\omega)$. From (77) we see that

$$\left|\begin{pmatrix} c_1(\lambda,0) \\ c_2(\lambda,0) \\ c_3(\lambda,0) \\ c_4(\lambda,0) \end{pmatrix}\right| = \left|\frac{1}{\omega}\int_0^\omega \int_0^\theta \mathbb{M}^{-1} \begin{pmatrix} 0 \\ 2\widehat{G} - \widehat{F}_1 \\ -\widehat{G}' - \widehat{F}_2 \\ \widehat{G} \end{pmatrix} \mathrm{d}s\mathrm{d}\theta\right|$$

$$= \left|\frac{1}{\omega}\int_0^\omega (\omega-\theta)\mathbb{M}^{-1} \begin{pmatrix} 0 \\ 2\widehat{G} - \widehat{F}_1 \\ -\widehat{G}' - \widehat{F}_2 \\ \widehat{G} \end{pmatrix} \mathrm{d}\theta\right|$$

$$\leq C(\|\widehat{\mathbf{F}}\|_{H^1(0,\omega)'} + \|\widehat{G}\|_{L^2(0,\omega)} + |\widehat{G}(\lambda,0)|), \tag{86}$$

and

$$\left|\begin{pmatrix} c_1(\lambda,\omega) \\ c_2(\lambda,\omega) \\ c_3(\lambda,\omega) \\ c_4(\lambda,\omega) \end{pmatrix}\right| = \left|\begin{pmatrix} c_1(\lambda,0) \\ c_2(\lambda,0) \\ c_3(\lambda,0) \\ c_4(\lambda,0) \end{pmatrix} + \int_0^\omega \mathbb{M}^{-1} \begin{pmatrix} 0 \\ 2\widehat{G} - \widehat{F}_1 \\ -\widehat{G}' - \widehat{F}_2 \\ \widehat{G} \end{pmatrix} \mathrm{d}\theta\right|$$

$$\leq C(\|\widehat{\mathbf{F}}\|_{H^1(0,\omega)'} + \|\widehat{G}\|_{L^2(0,\omega)} + |\widehat{G}(\lambda,0)| + |\widehat{G}(\lambda,\omega)|). \tag{87}$$

It is also easy to see that

$$\sum_{j=1}^4 \left|\frac{\mathrm{d}^k c_j}{\mathrm{d}\theta^k}\right| \leq C_{|\lambda|}\left(\left|\frac{\mathrm{d}^{k-1}\mathbf{F}}{\mathrm{d}\theta^{k-1}}\right| + \left|\frac{\mathrm{d}^k G}{\mathrm{d}\theta^k}\right|\right), \quad k = 1, 2, \cdots, \tag{88}$$

For any function ϕ_j, $j = 1, 2, 3, 4$, whose integral over $(0, \omega)$ is zero, we can choose $\Phi_j(\theta) = \int_0^\theta \phi_j(s)\mathrm{d}s$ and so

$$\sum_{j=1}^4 \int_0^\omega c_j\phi_j \mathrm{d}\theta = \int_0^\omega (\Phi_1'(\theta), \Phi_2'(\theta), \Phi_3'(\theta), \Phi_4'(\theta)) \int_0^\theta \mathbb{M}^{-1} \begin{pmatrix} 0 \\ 2\widehat{G} - \widehat{F}_1 \\ -\widehat{G}' - \widehat{F}_2 \\ \widehat{G} \end{pmatrix} \mathrm{d}s\mathrm{d}\theta$$

$$= -\int_0^\omega (\Phi_1(\theta), \Phi_2(\theta), \Phi_3(\theta), \Phi_4(\theta))\mathbb{M}^{-1} \begin{pmatrix} 0 \\ 2\widehat{G} - \widehat{F}_1 \\ -\widehat{G}' - \widehat{F}_2 \\ \widehat{G} \end{pmatrix} \mathrm{d}\theta$$

$$\leq C_{|\lambda|} \sum_{j=1}^4 \|\Phi_j\|_{H^1(0,\omega)}(\|\widehat{\mathbf{F}}\|_{H^1(0,\omega)'} + \|\widehat{G}\|_{L^2(0,\omega)}) - \int_0^\omega \frac{\mathrm{d}(\Phi_i \mathbb{M}_{i3}^{-1})}{\mathrm{d}\theta} \widehat{G}\mathrm{d}\theta$$

$$\leq C_{|\lambda|} \sum_{j=1}^4 \|\Phi_j\|_{H^1(0,\omega)}(\|\widehat{\mathbf{F}}\|_{H^1(0,\omega)'} + \|\widehat{G}\|_{L^2(0,\omega)})$$

$$\leq C_{|\lambda|} \sum_{j=1}^{4} \|\phi_j\|_{L^2(0,\omega)} (\|\widehat{\mathbf{F}}\|_{H^1(0,\omega)'} + \|\widehat{G}\|_{L^2(0,\omega)}). \tag{89}$$

which implies

$$\sum_{j=1}^{4} \|c_j\|_{L^2(0,\omega)} \leq C_{|\lambda|} (\|\widehat{\mathbf{F}}\|_{H^1(0,\omega)'} + \|\widehat{G}\|_{L^2(0,\omega)}) \tag{90}$$

via duality. Thus by interpolation we have

$$\sum_{j=1}^{4} \|c_j\|_{H^s(0,\omega)} \leq \begin{cases} C_{|\lambda|} (\|\widehat{\mathbf{F}}\|_{H^{1-s}(0,\omega)'} + \|\widehat{G}\|_{H^s(0,\omega)}) & \text{for } s \in [0,1), \\ C_{|\lambda|} (\|\widehat{\mathbf{F}}\|_{H^{s-1}(0,\omega)} + \|\widehat{G}\|_{H^s(0,\omega)}) & \text{for } s \geq 1, \end{cases} \tag{91}$$

By using (72) and (75), we have

$$\frac{\mathrm{d}\widehat{U}_1^{\mathrm{p}}}{\mathrm{d}\theta} = \sum_{j=1}^{4} c_j \widehat{U}_{1j}', \tag{92}$$

$$\frac{\mathrm{d}\widehat{U}_2^{\mathrm{p}}}{\mathrm{d}\theta} = \widehat{G} + \sum_{j=1}^{4} c_j \widehat{U}_{2j}', \tag{93}$$

$$\widehat{q}^{\mathrm{p}} = \sum_{j=1}^{4} c_j . \widehat{q}_j \tag{94}$$

The last four inequalities imply

$$\|\widehat{\mathbf{U}}^{\mathrm{p}}\|_{H^{s+1}(0,\omega)} + \|\widehat{q}^{\mathrm{p}}\|_{H^s(0,\omega)} \leq C_{|\lambda|} (\|\widehat{\mathbf{F}}\|_{H^{s-1}(0,\omega)} + \|\widehat{G}\|_{H^s(0,\omega)}) \quad \text{for } s > 1/2. \tag{95}$$

When $\lambda \notin \mathfrak{Sp}(\mathcal{L};\omega)$, solving the linear system (81) gives

$$\begin{aligned} \sum_{j=1}^{4} |b_j| \leq & C_{|\lambda|} \left(|\widehat{G}(\lambda,0)| + |\widehat{G}(\lambda,\omega)| + |\widehat{\mathbf{H}}(\lambda,0)| + |\widehat{\mathbf{H}}(\lambda,\omega)| + \sum_{j=1}^{4} (|c_j(\lambda,0)| + |c_j(\lambda,\omega)|) \right) \\ \leq & C_{|\lambda|} (\|\widehat{\mathbf{F}}\|_{H^1(0,\omega)'} + \|\widehat{G}\|_{L^2(0,\omega)} + |\widehat{G}(\lambda,0)| + |\widehat{G}(\lambda,\omega)| + |\widehat{\mathbf{H}}(\lambda,0)| + |\widehat{\mathbf{H}}(\lambda,\omega)|), \end{aligned} \tag{96}$$

where we have used (86)-(87).

To conclude, we have

$$\begin{aligned} & \|\widehat{\mathbf{U}}(\lambda)\|_{H^{s+1}(0,\omega)} + \|\widehat{q}(\lambda)\|_{H^s(0,\omega)} \\ & \leq C_D (\|\widehat{\mathbf{F}}(\lambda)\|_{H^{s-1}(0,\omega)} + \|\widehat{G}(\lambda)\|_{H^s(0,\omega)}) + C_D (|\widehat{\mathbf{H}}(\lambda,0)| + |\widehat{\mathbf{H}}(\lambda,\omega)|) \quad \text{for } s > 1/2, \end{aligned} \tag{97}$$

in any compact domain D which does not intersect the spectrum $\mathfrak{Sp}(\mathcal{L};\omega)$. Once this estimate is obtained, it is routine to derive the following Agmon–Nirenberg estimate when $\zeta = \mathrm{Im}(\lambda)$ is sufficiently large [10, 13]:

$$\begin{aligned} & \|\widehat{\mathbf{U}}(\lambda)\|_{H^{s+1}(0,\omega)} + \|\widehat{q}(\lambda)\|_{H^s(0,\omega)} + (1+|\zeta|)^{s+1} \|\widehat{\mathbf{U}}(\lambda)\|_{L^2(0,\omega)} + (1+|\zeta|)^s \|\widehat{q}(\lambda)\|_{L^2(0,\omega)} \\ & \leq C \Big(\|\widehat{\mathbf{F}}(\lambda)\|_{H^{s-1}(0,\omega)} + \|\widehat{G}(\lambda)\|_{H^s(0,\omega)} \Big) + (1+|\zeta|)^{s-1/2} \big(|\widehat{\mathbf{H}}(\lambda,0)| + |\widehat{\mathbf{H}}(\lambda,\omega)| \big). \end{aligned} \tag{98}$$

The last two inequalities imply

(99)
$$\|(\widehat{\mathbf{U}},\widehat{q})\|_{\widehat{X}^s_*}\le C\|(\widehat{\mathbf{F}},\widehat{G},\widehat{\mathbf{H}})\|_{\widehat{Y}^s_*}\quad\text{on the line } \mathrm{Re}(\lambda)=s>1/2 \text{ and } s\notin \mathrm{Re}\,\mathfrak{Sp}(\mathcal{L};\omega).$$

In other words, we have

$$\|\mathcal{L}(\lambda)^{-1}(\widehat{\mathbf{F}}(\lambda),\widehat{G}(\lambda),\widehat{\mathbf{H}}(\lambda))\|_{\widehat{X}^s_*}\le C\|(\widehat{\mathbf{F}}(\lambda),\widehat{G}(\lambda),\widehat{\mathbf{H}}(\lambda))\|_{\widehat{Y}^s_*}\tag{100}$$

on the line $\mathrm{Re}(\lambda)=s>1/2$ with $s\notin \mathrm{Re}\,\mathfrak{Sp}(\mathcal{L};\omega)$.

4.3. Poles of $\mathcal{L}(\lambda)^{-1}$ at $\lambda\in\mathfrak{Sp}(\mathcal{L};\omega)$. From (83) we see that the $\det(\mathbb{A})$ has a second-order or higher-order zero at some point $\lambda\neq 0$ only if the two equations

$$\lambda^2\sin(\omega)^2=\sin(\lambda\omega)^2,\tag{101}$$

$$\tan(\lambda\omega)=\lambda\omega\tag{102}$$

are satisfied simultaneously, which are equivalent to (8)-(9) with $z=\lambda\omega$. Therefore, when $\lambda\notin\mathfrak{M}$, $\mathcal{L}(\lambda)^{-1}:\widehat{Y}_s\to\widehat{X}_s$ has a simple pole at all the spectral points. When $\omega\in\mathfrak{M}$, we have

$$(\mathbb{A}^{-1})_{21}=\frac{(\lambda+1)\sin(\lambda\omega)^2}{2(\lambda^2\sin(\omega)^2-\sin(\lambda\omega)^2)}=\frac{8\lambda^2\sin(\lambda\omega)^2}{(\lambda+1)\det(\mathbb{A})}=O\left(\frac{1}{(\lambda-\lambda(\omega))^2}\right)\tag{103}$$

with $\lambda(\omega):=\sqrt{\sin(\omega)^{-2}-\omega^{-2}}$, thus $\mathcal{L}(\lambda)^{-1}:\widehat{Y}_s\to\widehat{X}_s$ has a pole of order 2 at the spectral point $\lambda(\omega)$, and a simple pole at all the other spectral points with positive real part. (One can check that $\det(\mathbb{A})$ cannot have third-order zeros and therefore $\mathcal{L}(\lambda)^{-1}$ do not have poles of order ≥ 3.)

In section 5, we shall see that the pole of order 2 at $\lambda=\lambda(\omega)$ when $\omega\in\mathfrak{M}$ contributes to a logarithmic singularity of the solution. The pole of $\mathcal{L}(\lambda)^{-1}:\widehat{Y}_s\to\widehat{X}_s$ at $\lambda=0$ does not affect the regularity of a solution which is a priori in $\mathbf{H}^1(\Omega_0)\times L^2(\Omega_0)$ and has compact support.

4.4. Kernel of $\mathcal{L}(\lambda)$ for $\lambda\in\mathfrak{Sp}(\mathcal{L};\omega)$. When $\lambda=0$ and $\omega/\pi\notin\mathbb{N}$, row reduction of $\mathbb{A}$ reduces to

$$\mathbb{A}\Longrightarrow\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\tag{104}$$

which implies that the kernel of $\mathcal{L}(0)$ is a two-dimensional space spanned by $(\cos(\theta),-\sin(\theta),0)$ and $(\sin(\theta),\cos(\theta),0)$.

When $\lambda\omega/\pi\in\mathbb{N}$, $\lambda\neq 0,\pm1$ and $\omega/\pi\notin\mathbb{N}$, row reduction of $\mathbb{A}$ gives

$$\mathbb{A}\Longrightarrow\begin{pmatrix}1&0&0&0\\0&1&-\lambda^2&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\tag{105}$$

which is invertible, thus each nonzero $\lambda\in\mathfrak{Sp}(\mathcal{L};\omega)$ satisfies $\lambda\omega/\pi\notin\mathbb{N}$.

When $\lambda \in \mathfrak{Sp}(\mathcal{L};\omega)$, $\lambda \neq 0, 1$ and $\omega/\pi \notin \mathbb{N}$ (thus $\lambda\omega/\pi \notin \mathbb{N}$), row reduction of $\mathbb{A}$ leads to

$$\mathbb{A} \Longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\lambda^2 & 0 \\ 0 & 0 & 1 & \frac{\sin(\lambda\omega)\cos(\omega)+\lambda\cos(\lambda\omega)\sin(\omega)}{(\lambda^2-1)\sin(\lambda\omega)\sin(\omega)} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{106}$$

which has only one linearly independent nonzero solution

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda^2 \alpha_{\lambda,\omega} \\ \alpha_{\lambda,\omega} \\ 1-\lambda^2 \end{pmatrix}, \tag{107}$$

with

$$\alpha_{\lambda,\omega} = \cot(\omega) + \lambda \cot(\lambda\omega).$$

This implies that the kernel of $\mathcal{L}(\lambda)$ is a one-dimensional space spanned by

$$\begin{aligned} &(\mathbf{U}_{\lambda,\omega}(\theta),\, q_{\lambda,\omega}(\theta))^T \\ &:= \begin{pmatrix} \alpha_{\lambda,\omega}\cos(\lambda\theta)\sin(\theta) + \lambda\sin(\lambda\theta)\sin(\theta) - \cos(\lambda\theta)\cos(\theta) \\ \alpha_{\lambda,\omega}\dfrac{(\lambda-1)\sin(\lambda\theta)\sin(\theta)+\cos((\lambda-1)\theta)}{1-\lambda} + (1+\lambda)\cos(\lambda\theta)\sin(\theta) \\ 2\lambda\cos[(1-\lambda)\theta] - 2\lambda\alpha_{\lambda,\omega}\sin[(1-\lambda)\theta]/(1-\lambda) \end{pmatrix}. \end{aligned} \tag{108}$$

It is straightforward to check that

$$\big(\mathbf{U}_{\lambda,\omega}(\theta), q_{\lambda,\omega}(\theta)\big) = \big(\vec{\mathscr{R}}[\mathbf{u}_{\lambda,\omega}(\theta)], q_{\lambda,\omega}(\theta)\big) \tag{109}$$

with

$$\begin{aligned} &(\mathbf{u}_{\lambda,\omega}(\theta),\, q_{\lambda,\omega}(\theta))^T \\ &:= \begin{pmatrix} \dfrac{\lambda\alpha_{\lambda,\omega}}{\lambda-1}\cos((\lambda-1)\theta)\sin(\theta) + \lambda\sin[(\lambda-1)\theta]\sin(\theta) - \cos(\lambda\theta) \\ \lambda\cos((\lambda-1)\theta)\sin(\theta) - \alpha_{\lambda,\omega}\left[\dfrac{\cos((\lambda-1)\theta)\cos(\theta)}{\lambda-1} + \sin((\lambda-1)\theta)\sin(\theta)\right] \\ 2\lambda\cos((\lambda-1)\theta) - 2\lambda\alpha_{\lambda,\omega}\sin[(\lambda-1)\theta]/(\lambda-1) \end{pmatrix}. \end{aligned} \tag{110}$$

When $\lambda = 1$, one can directly solve (44) with $F_1 = F_2 = G = H_1 = H_2 = 0$ and find that the kernel of $\mathcal{L}(1)$ is a one-dimensional space spanned by $(0,1,0) = (\vec{\mathscr{R}}\mathscr{P}(-\sin(\theta), \cos(\theta)), 0)$.

5. Singularities of the solution

From (57)-(58) we know that $\mathcal{L}(\lambda)(\widehat{\mathbf{U}}(\lambda), \widehat{q}(\lambda)) = (\widehat{\mathbf{F}}(\lambda), \widehat{G}(\lambda), \widehat{\mathbf{H}}(\lambda))$ for $\mathrm{Re}(\lambda) < 0$, and therefore

$$(\widehat{\mathbf{U}}(\lambda), \widehat{q}(\lambda)) = \mathcal{L}(\lambda)^{-1}(\widehat{\mathbf{F}}(\lambda), \widehat{G}(\lambda), \widehat{\mathbf{H}}(\lambda)) \quad \text{for} \quad \mathrm{Re}(\lambda) < 0. \tag{111}$$

Note that $\mathcal{L}(\lambda)^{-1}(\widehat{\mathbf{F}}(\lambda),\widehat{G}(\lambda),\widehat{\mathbf{H}}(\lambda))\in\widehat{X}^s$ is well-defined for $\lambda\notin\mathfrak{Sp}(\mathcal{L};\omega)$ and $\mathrm{Re}(\lambda)\le s$, and obeys (100). When $s\notin\mathrm{Re}\,[\mathfrak{Sp}(\mathcal{L};\omega)]$, by applying the inverse Mellin transform and using the residue theorem, we obtain

$$
\begin{aligned}
(\mathbf{U},q) &= \frac{1}{2\pi i}\int_{\mathrm{Re}(\lambda)=-\epsilon} r^\lambda\mathcal{L}(\lambda)^{-1}(\widehat{\mathbf{F}}(\lambda),\widehat{G}(\lambda),\widehat{\mathbf{H}}(\lambda))\mathrm{d}\lambda\\
&= \frac{1}{2\pi i}\int_{\mathrm{Re}(\lambda)=s} r^\lambda\mathcal{L}(\lambda)^{-1}(\widehat{\mathbf{F}}(\lambda),\widehat{G}(\lambda),\widehat{\mathbf{H}}(\lambda))\mathrm{d}\lambda\\
&\quad+\operatorname*{Res}_{\substack{\lambda\in\mathfrak{Sp}(\mathcal{L};\omega)\\ 0\le\mathrm{Re}(\lambda)<s}}\Big[r^\lambda\mathcal{L}(\lambda)^{-1}(\widehat{\mathbf{F}}(\lambda),\widehat{G}(\lambda),\widehat{\mathbf{H}}(\lambda))\Big]\\
&= (\mathbf{U}_s,q_s)+\operatorname*{Res}_{\substack{\lambda\in\mathfrak{Sp}(\mathcal{L};\omega)\\ 0\le\mathrm{Re}(\lambda)<s}}\Big[r^\lambda\mathcal{L}(\lambda)^{-1}(\widehat{\mathbf{F}}(\lambda),\widehat{G}(\lambda),\widehat{\mathbf{H}}(\lambda))\Big],
\end{aligned}\tag{112}
$$

where $(\mathbf{u}_s,p_s):=((\vec{\mathscr{R}}\mathscr{P})^{-1}\mathbf{U}_s,\mathscr{P}^{-1}q_s)\in X^s$ according to (47), (55) and (100). Upon considering the residue term, we divide the problem into five cases.

(1) $\lambda=0$ is always a spectral point in the region $0\le\mathrm{Re}(\lambda)<s$, and we have

$$
\operatorname*{Res}_{\lambda=0}\Big[r^\lambda\mathcal{L}(\lambda)^{-1}(\widehat{\mathbf{F}}(\lambda),\widehat{G}(\lambda),\widehat{\mathbf{H}}(\lambda))\Big]=\sum_{k=0}^{J-1}(\ln r)^k(\widetilde{\mathbf{U}}_{k,\omega}(\theta),\widetilde{q}_{k,\omega}(\theta))\tag{113}
$$

for some functions $(\widetilde{\mathbf{U}}_{k,\omega}(\theta),\widetilde{q}_{k,\omega}(\theta))$, where J denotes the order of pole at $\lambda=0$. It is easy to see that the solution will not be in $\mathbf{H}^1(\Omega_0)\times L^2(\Omega_0)$ if $J\ne1$. Since our solution is a priori $\mathbf{H}^1(\Omega_0)\times L^2(\Omega_0)$, there must be $J=1$ if this term is not identically zero. Let $\mathscr{L}$ denote the differential operator on the left-hand side of (34). Then, in the case $J=1$, we have

$$
\mathscr{L}(\widetilde{\mathbf{U}}_{0,\omega}(\theta),\widetilde{q}_{0,\omega}(\theta))=\operatorname*{Res}_{\lambda=0}\Big[r^\lambda(\widehat{\mathbf{F}}(\lambda),\widehat{G}(\lambda),\widehat{\mathbf{H}}(\lambda))\Big]=0,\tag{114}
$$

which gives $\mathcal{L}(0)(\widetilde{\mathbf{U}}_{0,\omega}(\theta),\widetilde{q}_{0,\omega}(\theta))=0$ through substituting $(\widetilde{\mathbf{U}}_{0,\omega}(\theta),\widetilde{q}_{0,\omega}(\theta))$ into (34). Since the kernel of $\mathcal{L}(0)$ is a two-dimensional space spanned by $(\cos\theta,-\sin\theta,0)$ and $(\sin\theta,\cos\theta,0)$, it follows that

$$
(\widetilde{\mathbf{U}}_{0,\omega}(\theta),\widetilde{q}_{0,\omega}(\theta))=\kappa_1(\cos\theta,-\sin\theta,0)+\kappa_2(\sin\theta,\cos\theta,0)=(\vec{\mathscr{R}}\mathscr{P}(\kappa_1,\kappa_2),0)\tag{115}
$$

for some constants κ_1 and κ_2. Therefore, (113) contributes to a regular part of the solution.

(2) If $s>1$ then $\lambda=1$ is a spectral point which deserves special treatment. The assumption $(\mathbf{f},g,\mathbf{h})\in\mathbf{H}^{s-1}_{00}(\Omega)\times H^s_{00}(\Omega)\times\mathbf{H}^{s-1/2}_{00}(\partial\Omega)$ guarantees that $(\widehat{\mathbf{F}}(\lambda),\widehat{G}(\lambda),\widehat{\mathbf{H}}(\lambda))$ is analytic up to $\lambda=1$, which further implies that the pole of $\mathcal{L}(\lambda)^{-1}(\widehat{\mathbf{F}}(\lambda),\widehat{G}(\lambda),\widehat{\mathbf{H}}(\lambda))$ at $\lambda=1$ is solely due to the pole of $\mathcal{L}(\lambda)^{-1}$ at $\lambda=1$. We consider two cases below.

Case (2-1): When $\omega\ne\omega_*$, $\mathcal{L}(\lambda)^{-1}$ has a simple pole at $\lambda=1$. In this case, the function $\mathcal{L}(\lambda)^{-1}(\widehat{\mathbf{F}}(\lambda),\widehat{G}(\lambda),\widehat{\mathbf{H}}(\lambda))$ has at most a simple pole at $\lambda=1$ and so

$$
\operatorname*{Res}_{\lambda=1}\Big[r^\lambda\mathcal{L}(\lambda)^{-1}(\widehat{\mathbf{F}}(\lambda),\widehat{G}(\lambda),\widehat{\mathbf{H}}(\lambda))\Big]=r(\widehat{\mathbf{U}}_{1,\omega}(\theta),\widehat{q}_{1,\omega}(\theta))\tag{116}
$$

with $(\widehat{\mathbf{U}}_{1,\omega}(\theta),\widehat{q}_{1,\omega}(\theta))=\operatorname*{Res}_{\lambda=1}\big[\mathcal{L}(\lambda)^{-1}(\widehat{\mathbf{F}}(\lambda),\widehat{G}(\lambda),\widehat{\mathbf{H}}(\lambda))\big]$. Applying the operator $\mathscr{L}$ to (116), we obtain

$$
\mathscr{L}(r(\widehat{\mathbf{U}}_{1,\omega}(\theta),\widehat{q}_{1,\omega}(\theta)))=\operatorname*{Res}_{\lambda=1}\Big[r^\lambda(\widehat{\mathbf{F}}(\lambda),\widehat{G}(\lambda),\widehat{\mathbf{H}}(\lambda))\Big]=0,\tag{117}
$$

which gives $\mathcal{L}(1)(\widehat{\mathbf{U}}_{1,\omega}(\theta), \widehat{q}_{1,\omega}(\theta)) = 0$. Since the kernel of $\mathcal{L}(1)$ is a one-dimensional space spanned by $(0,1,0)$, it follows that

$$r(\widehat{\mathbf{U}}_{1,\omega}(\theta), \widehat{q}_{1,\omega}(\theta)) = (\vec{\mathscr{R}}\mathscr{P}(-\kappa x_2, \kappa x_1), 0) \tag{118}$$

for some constant κ, which is a polynomial in the Cartesian coordinates and so it contributes to a regular part of the solution.

Case (2-2): When $\omega = \omega_*$, the operator $\mathcal{L}(\lambda)^{-1}$ has an order 2 pole at $\lambda = 1$, as we have discussed in section 4.4. In this case, we have

$$\operatorname*{Res}_{\lambda=1}\Big[r^\lambda \mathcal{L}(\lambda)^{-1}(\widehat{\mathbf{F}}(\lambda), \widehat{G}(\lambda), \widehat{\mathbf{H}}(\lambda))\Big] = r(\widehat{\mathbf{U}}^*_{1,\omega}(\theta), \widehat{q}^*_{1,\omega}(\theta)) + r\ln r(\widehat{\mathbf{U}}_{1,\omega}(\theta), \widehat{q}_{1,\omega}(\theta)). \tag{119}$$

By applying the operator $\mathscr{L}$ to (119), we obtain

$$\mathscr{L}\Big(r(\widehat{\mathbf{U}}^*_{1,\omega}(\theta), \widehat{q}^*_{1,\omega}(\theta)) + r\ln r(\widehat{\mathbf{U}}_{1,\omega}(\theta), \widehat{q}_{1a,\omega}(\theta))\Big) = \operatorname*{Res}_{\lambda=1}\Big[r^\lambda(\widehat{\mathbf{F}}(\lambda), \widehat{G}(\lambda), \widehat{\mathbf{H}}(\lambda))\Big] = 0, \tag{120}$$

which gives

$$\mathcal{L}(1)(\widehat{\mathbf{U}}_{1,\omega}(\theta), \widehat{q}_{1,\omega}(\theta)) = 0, \tag{121}$$

$$\mathcal{L}(1)(\widehat{\mathbf{U}}^*_{1,\omega}(\theta), \widehat{q}^*_{1,\omega}(\theta)) = (\mathbf{F}^*, G^*, \mathbf{H}^*) \tag{122}$$

with

$$\mathbf{F}^* = 2\widehat{\mathbf{U}}_{1,\omega} - (\widehat{q}_{1,\omega}, 0), \qquad G^* = -\widehat{\mathbf{U}}_{1,\omega}\cdot\mathbf{e}_1, \qquad \mathbf{H}^* = (-\widehat{\mathbf{U}}_{1,\omega}\cdot\mathbf{e}_2, 0). \tag{123}$$

One can directly solve the ODE system (121)-(138) and see that it has two linearly independent solutions:

$$\begin{cases} (\widehat{\mathbf{U}}_{1,\omega}(\theta), \widehat{q}_{1,\omega}(\theta)) = (0,0,0), \\ (\widehat{\mathbf{U}}^*_{1,\omega}(\theta), \widehat{q}^*_{1,\omega}(\theta)) = (0,1,0) \end{cases} \tag{124}$$

and

$$\begin{cases} (\widehat{\mathbf{U}}_{1,\omega}(\theta), \widehat{q}_{1,\omega}(\theta)) = (0,1,0), \\ (\widehat{\mathbf{U}}^*_{1,\omega}(\theta), \widehat{q}^*_{1,\omega}(\theta)) = \left(\frac{\omega}{2}\cos(2\theta) - \frac{1}{2}\sin(2\theta),\ -\frac{\omega}{2}\sin(2\theta) - \frac{1}{2}\cos(2\theta),\ 2\theta - \omega\right), \end{cases} \tag{125}$$

where (124) contributes to a regular part of the solution and (125) contributes to a singular part of the solution. Also one can check that (125) is a solution of the ODE system (121)-(138) if and only if $\omega = \omega_*$. Substituting (125) into (119) and transforming it to the Cartesian coordinates, we obtain the third term on the right-hand side of (23).

(3) If an integer $k \neq \lambda(\omega)$ and $2 \leq k \leq [s]$ is a spectral point, then $\mathcal{L}(\lambda)^{-1}(\widehat{\mathbf{F}}(\lambda), \widehat{G}(\lambda), \widehat{\mathbf{H}}(\lambda))$ has at most a simple pole at $\lambda = k$. Therefore, we have

$$\operatorname*{Res}_{\lambda=k}\Big[r^\lambda \mathcal{L}(\lambda)^{-1}(\widehat{\mathbf{F}}(\lambda), \widehat{G}(\lambda), \widehat{\mathbf{H}}(\lambda))\Big] = r^k(\widehat{\mathbf{U}}_{k,\omega}(\theta), \widehat{q}_{k,\omega}(\theta)) \tag{126}$$

with $(\widehat{\mathbf{U}}_{k,\omega}(\theta), \widehat{q}_{k,\omega}(\theta)) = \operatorname*{Res}_{\lambda=k}\big[\mathcal{L}(\lambda)^{-1}(\widehat{\mathbf{F}}(\lambda), \widehat{G}(\lambda), \widehat{\mathbf{H}}(\lambda))\big]$, and

$$\mathscr{L}(r^k(\widehat{\mathbf{U}}_{k,\omega}(\theta), \widehat{q}_{k,\omega}(\theta))) = \operatorname*{Res}_{\lambda=k}\Big[r^\lambda(\widehat{\mathbf{F}}(\lambda), \widehat{G}(\lambda), \widehat{\mathbf{H}}(\lambda))\Big] = 0, \tag{127}$$

which gives $\mathcal{L}(k)(\widehat{\mathbf{U}}_{k,\omega}(\theta),\widehat{q}_{k,\omega}(\theta))=0$. Since the kernel of $\mathcal{L}(k)$ is a one-dimensional space spanned by (108) with $\lambda_j=k$, it follows that

$$\big(\widehat{\mathbf{U}}_{k,\omega}(\theta),\widehat{q}_{k,\omega}(\theta)\big)=\kappa\big(\mathbf{U}_{k,\omega}(\theta),q_{k,\omega}(\theta)\big) \tag{128}$$

for some constant κ, which is a polynomial in the Cartesian coordinates in view of the formula (110), and this term contributes to a regular part of the solution.

(4) If λ_j is a non-integer spectral point with $0<\mathrm{Re}(\lambda_j)<s$ such that either $\omega\notin\mathfrak{M}$ or $\lambda_j\neq\lambda_j(\omega)$, then $\mathcal{L}(\lambda)^{-1}$ has a simple pole at $\lambda=\lambda_j$ and so

$$\operatorname*{Res}_{\lambda=\lambda_j}\Big[r^\lambda\mathcal{L}(\lambda)^{-1}(\widehat{\mathbf{F}}(\lambda),\widehat{G}(\lambda),\widehat{\mathbf{H}}(\lambda))\Big]=r^{\lambda_j}(\widehat{\mathbf{U}}_{\lambda_j,\omega}(\theta),\widehat{q}_{\lambda_j,\omega}(\theta)) \tag{129}$$

with $(\widehat{\mathbf{U}}_{\lambda_j,\omega}(\theta),\widehat{q}_{\lambda_j,\omega}(\theta))=\operatorname*{Res}_{\lambda=\lambda_j}\big[\mathcal{L}(\lambda)^{-1}(\widehat{\mathbf{F}}(\lambda),\widehat{G}(\lambda),\widehat{\mathbf{H}}(\lambda))\big]$, and

$$\mathscr{L}(r^{\lambda_j}(\widehat{\mathbf{U}}_{\lambda_j,\omega}(\theta),\widehat{q}_{\lambda_j,\omega}(\theta)))=\operatorname*{Res}_{\lambda=\lambda_j}\Big[r^\lambda(\widehat{\mathbf{F}}(\lambda),\widehat{G}(\lambda),\widehat{\mathbf{H}}(\lambda))\Big]=0, \tag{130}$$

which gives $\mathcal{L}(\lambda_j)(\widehat{\mathbf{U}}_{\lambda_j,\omega}(\theta),\widehat{q}_{\lambda_j,\omega}(\theta))=0$. Since the kernel of $\mathcal{L}(\lambda_j)$ is a one-dimensional space spanned by (108), it follows that

$$\big(\widehat{\mathbf{U}}_{\lambda_j,\omega}(\theta),\widehat{q}_{\lambda_j,\omega}(\theta)\big)=\kappa\big(\mathbf{U}_{\lambda_j,\omega}(\theta),q_{\lambda_j,\omega}(\theta)\big), \tag{131}$$

for some constant κ, which belongs to the singular part of the solution. One can check that the function given by (18) satisfies

$$(\widehat{\mathbf{U}}_{\lambda(\omega),\omega}(\theta),\widehat{q}_{\lambda(\omega),\omega}(\theta))=(\vec{\mathscr{R}}\mathscr{P}[\mathbf{u}_{\lambda(\omega),\omega}(\theta)],q_{\lambda(\omega),\omega}(\theta)). \tag{132}$$

Transformed to the Cartesian coordinates, (129) can be absorbed into the second term on the right-hand side of (23).

(5) When $\omega\in\mathfrak{M}\backslash\{\omega_*\}$, the operator $\mathcal{L}(\lambda)^{-1}$ has a pole of order 2 at $\lambda=\lambda(\omega):=\sqrt{\sin(\omega)^{-2}-\omega^{-2}}\neq 1$, as we have discussed in section 4.4. This pole of order 2 indicates that

$$\begin{aligned}&\operatorname*{Res}_{\lambda=\lambda(\omega)}\Big[r^\lambda\mathcal{L}(\lambda)^{-1}(\widehat{\mathbf{F}}(\lambda),\widehat{G}(\lambda),\widehat{\mathbf{H}}(\lambda))\Big]\\&=r^{\lambda(\omega)}(\widehat{\mathbf{U}}^*_{\lambda(\omega),\omega}(\theta),\widehat{q}^*_{\lambda(\omega),\omega}(\theta))+r^{\lambda(\omega)}\ln r(\widehat{\mathbf{U}}_{\lambda(\omega),\omega}(\theta),\widehat{q}_{\lambda(\omega),\omega}(\theta)).\end{aligned} \tag{133}$$

By applying the operator $\mathscr{L}$ to (119), we obtain

$$\begin{aligned}&\mathscr{L}\big(r^{\lambda(\omega)}(\widehat{\mathbf{U}}^*_{\lambda(\omega),\omega}(\theta),\widehat{q}^*_{\lambda(\omega),\omega}(\theta))+r^{\lambda(\omega)}\ln r(\widehat{\mathbf{U}}_{\lambda(\omega),\omega}(\theta),\widehat{q}_{\lambda(\omega),\omega}(\theta))\big)\\&=\operatorname*{Res}_{\lambda=\lambda(\omega)}\Big[r^\lambda(\widehat{\mathbf{F}}(\lambda),\widehat{G}(\lambda),\widehat{\mathbf{H}}(\lambda))\Big]=0,\end{aligned} \tag{134}$$

which gives

$$\mathcal{L}(\lambda(\omega))(\widehat{\mathbf{U}}_{\lambda(\omega),\omega}(\theta),\widehat{q}_{\lambda(\omega),\omega}(\theta))=0, \tag{135}$$

$$\mathcal{L}(\lambda(\omega))(\widehat{\mathbf{U}}^*_{\lambda(\omega),\omega}(\theta),\widehat{q}^*_{\lambda(\omega),\omega}(\theta))=(\widehat{\mathbf{F}}^*,\widehat{G}^*,\widehat{\mathbf{H}}^*) \tag{136}$$

with

$$\widehat{\mathbf{F}}^*=2\lambda(\omega)\widehat{\mathbf{U}}_{\lambda(\omega),\omega}-(\widehat{q}_{\lambda(\omega),\omega},0), \tag{137}$$

$$\widehat{G}^*=-\widehat{\mathbf{U}}_{\lambda(\omega),\omega}\cdot\mathbf{e}_1,\qquad \widehat{\mathbf{H}}^*=(-\widehat{\mathbf{U}}_{\lambda(\omega),\omega}\cdot\mathbf{e}_2,0). \tag{138}$$

We shall see that the ODE system (135)-(136) has two linearly independent solutions. In fact, from section 4.4 we know that if $(\widehat{\mathbf{U}}_{\lambda(\omega),\omega}(\theta), \widehat{q}_{\lambda(\omega),\omega}(\theta)) = (0,0,0)$ then (136) has only one linearly independent solution

$$\left\{ \begin{array}{l} (\widehat{\mathbf{U}}_{\lambda(\omega),\omega}(\theta), \widehat{q}_{\lambda(\omega),\omega}(\theta)) = (0,0,0), \\ (\widehat{\mathbf{U}}^*_{\lambda(\omega),\omega}(\theta), \widehat{q}^*_{\lambda(\omega),\omega}(\theta)) = \big(\mathbf{U}_{\lambda(\omega),\omega}(\theta), q_{\lambda(\omega),\omega}(\theta)\big), \end{array} \right. \tag{139}$$

where $(\mathbf{U}_{\lambda(\omega),\omega}(\theta), q_{\lambda(\omega),\omega}(\theta))$ is given by (108). In this case, (133) can be absorbed into the second term of (23) (after transformation to the Cartesian coordinates).

If $(\widehat{\mathbf{U}}_{\lambda(\omega),\omega}(\theta), \widehat{q}_{\lambda(\omega),\omega}(\theta)) \neq (0,0,0)$, then (135) has only one linearly independent solution

$$(\widehat{\mathbf{U}}_{\lambda(\omega),\omega}(\theta), \widehat{q}_{\lambda(\omega),\omega}(\theta)) = \big(\mathbf{U}_{\lambda(\omega),\omega}(\theta), q_{\lambda(\omega),\omega}(\theta)\big),$$

and the solution of (136) is given by (78)-(79), where c_j is given by

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \int_0^\theta \mathbb{M}^{-1} \begin{pmatrix} 0 \\ 2\widehat{G}^* - \widehat{F}_1^* \\ -\dfrac{\mathrm{d}\widehat{G}^*}{\mathrm{d}\theta} - \widehat{F}_2^* \\ \widehat{G}^* \end{pmatrix} \mathrm{d}\theta. \tag{140}$$

Direct calculations (with the help of Matlab's symbolic computation toolbox) show that

(141)

$$\begin{aligned} c_1 = {} & \Big(\tfrac{1}{4}(\lambda-1)(\lambda+3) - \tfrac{1}{4}(\lambda+1)^2\cos(2\theta) - \tfrac{\lambda+1}{4(\lambda-1)}\alpha_{\lambda,\omega}\sin(2\theta)\Big)\cos(\lambda\theta)^2 \\ & +\alpha_{\lambda,\omega}\Big(\tfrac{\lambda(\lambda+1)}{2(\lambda-1)}\cos(\theta)^2 - \tfrac{(\lambda-1)(\lambda+2)}{2(\lambda-1)}\Big)\sin(\lambda\theta)\cos(\lambda\theta) + \tfrac{\lambda}{\lambda-1}\alpha_{\lambda,\omega}\theta + 1 \\ c_2 = {} & \Big(\alpha_{\lambda,\omega}\big(\tfrac{\lambda+1}{2(\lambda-1)}\cos(\theta)^2 - \tfrac{1}{\lambda-1}\big) - \tfrac{(\lambda+1)^2}{2}\sin(\theta)\cos(\theta)\Big)\cos(\lambda\theta)^2 \\ & +\Big(\tfrac{\lambda(\lambda+1)}{2(\lambda-1)}\alpha_{\lambda,\omega}\sin(\theta)\cos(\theta) - 1\Big)\sin(\lambda\theta)\cos(\lambda\theta) - \tfrac{1}{2}\alpha_{\lambda,\omega} - \lambda(\lambda+1)\theta \\ c_3 = {} & \tfrac{1}{2\lambda(\lambda-1)}\alpha_{\lambda,\omega}(\lambda^2\sin(\theta)^2 + \lambda\sin(\theta)^2 - 2)\sin(\lambda\theta)^2 \\ & +\Big(\tfrac{(\lambda+1)^2}{2\lambda}\cos(\theta)^2 + \tfrac{(\lambda+1)}{2\lambda(\lambda-1)}\alpha_{\lambda,\omega}\sin(\theta)\cos(\theta) - \tfrac{\lambda^2+2\lambda-1}{2\lambda}\Big)\sin(\lambda\theta)\cos(\lambda\theta) \\ c_4 = {} & \Big(\tfrac{1}{\lambda} - \tfrac{(\lambda+1)}{2(\lambda-1)}\alpha_{\lambda,\omega}\sin(\theta)\cos(\theta)\Big)\sin(\lambda\theta)^2 - \tfrac{\lambda}{\lambda-1}\alpha_{\lambda,\omega}\theta \\ & +\Big(\tfrac{(\lambda+1)^2}{2\lambda}\sin(\theta)\cos(\theta) - \tfrac{1}{2\lambda(\lambda-1)}\alpha_{\lambda,\omega}((\lambda+1)\cos(\theta)^2 - 2)\Big)\sin(\lambda\theta)\cos(\lambda\theta), \end{aligned}$$

and the constants b_j are the solution of (81). Although (140) defers from (77) by a constant (in order to simplify the expression of c_j), this will not affect the final solution. Via elementary row operations, the augmented matrix $[\mathbb{A}\,|\,\vec{a}]$ reduces to

$$[\mathbb{A}\,|\,\vec{a}] \Longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a_1^*(\omega) \\ 0 & 1 & -\lambda(\omega)^2 & 0 & a_2^*(\omega) \\ 0 & 0 & 1 & \frac{\sin(\lambda(\omega)\omega)\cos(\omega)+\lambda(\omega)\cos(\lambda(\omega)\omega)\sin(\omega)}{(\lambda(\omega)^2-1)\sin(\lambda(\omega)\omega)\sin(\omega)} & a_3^*(\omega) \\ 0 & 0 & 0 & 0 & a_4^*(\omega) \end{array}\right) \tag{142}$$

for some constants $a_j^*(\omega)$, $j = 1, 2, 3, 4$. Clearly, the linear system (81) has only one linearly independent solution if $a_4^*(\omega) = 0$, and has no non-zero solution if $a_4^*(\omega) \neq 0$. We claim that $a_4^*(\omega) = 0$ when $\omega \in \mathfrak{M}\backslash\{\omega_*\}$ and the linear system (81) has only one linearly independent solution. To verify this claim, one can simply check that the vector (b_1, b_2, b_3, b_4) given by

$$\begin{aligned} & b_1 = -1, \\ & b_2 = -\frac{\alpha_{\lambda,\omega}(\lambda+1)}{2(\lambda-1)} - \frac{\mathbf{c}(\lambda,\omega)\cdot\mathbf{d}_{\lambda,\omega} - \frac{2\lambda^2}{\lambda-1}\cot(\omega) - (\frac{(\lambda+1)^2}{2} + \frac{2\lambda}{\lambda-1})\cot(\lambda\omega)}{(\lambda^2-1)/\lambda}, \\ & b_3 = -\frac{\mathbf{c}(\lambda,\omega)\cdot\mathbf{d}_{\lambda,\omega} - \frac{2\lambda^2}{\lambda-1}\cot(\omega) - (\frac{(\lambda+1)^2}{2} + \frac{2\lambda}{\lambda-1})\cot(\lambda\omega)}{\lambda(\lambda^2-1)}, \\ & b_4 = 0, \end{aligned} \tag{143}$$

is a solution of (81) (see appendix B), where $\mathbf{c}(\lambda, \omega) := (c_1, c_2, c_3, c_4)^T$ is given by (141) and

$$\mathbf{d}_{\lambda,\omega} = \begin{pmatrix} \cot(\lambda\omega) + \lambda\cot(\omega) \\ \lambda - \cot(\lambda\omega)\cot(\omega) \\ -\lambda + \lambda^2\cot(\lambda\omega)\cot(\omega) \\ \lambda^2\cot(\lambda\omega) + \lambda\cot(\omega) \end{pmatrix}. \tag{144}$$

Therefore, the second linearly independent solution of (135)-(136) is given by

$$\begin{cases} (\widehat{\mathbf{U}}_{\lambda(\omega),\omega}(\theta), \widehat{q}_{\lambda(\omega),\omega}(\theta)) = \big(\mathbf{U}_{\lambda(\omega),\omega}(\theta), q_{\lambda(\omega),\omega}(\theta)\big), \\ (\widehat{\mathbf{U}}^*_{\lambda(\omega),\omega}(\theta), \widehat{q}^*_{\lambda(\omega),\omega}(\theta)) = \sum_{j=1}^4 (c_k(\theta) + b_k)(\widehat{\mathbf{U}}_k(\theta), \widehat{q}_k(\theta)), \end{cases} \tag{145}$$

where $(\widehat{\mathbf{U}}_k(\theta), \widehat{q}_k(\theta))$ is given by (65), c_k and b_k are given by (141)-(143). Using the symbolic computation of Matlab, one can check that the function given by (21) satisfies

$$(\widehat{\mathbf{U}}^*_{\lambda(\omega),\omega}(\theta), \widehat{q}^*_{\lambda(\omega),\omega}(\theta)) = (\vec{\mathscr{R}}\mathscr{P}[\mathbf{u}^*_{\lambda(\omega),\omega}(\theta)], q^*_{\lambda(\omega),\omega}(\theta)). \tag{146}$$

Substituting (145) into (133) and transforming it to the Cartesian coordinates, we obtain the third term on the right-hand side of (23).

To conclude, the solution has a decomposition in the form of (23) in the cone Ω_0. The proof of Theorem 2.1 is complete. □

Acknowledgements

This work was supported in part by an internal grant of The Hong Kong Polytechnic University (project code: ZZKQ).

References

[1] H. Blum and M. Dobrowolski: On finite element methods for elliptic equations on domains with corners. Computing, 28 (1982), pp. 53–63.

[2] Z. Cai and S. Kim: A finite element method using singular functions for the poisson equation: corner singularities. SIAM J. Numer. Anal., 39 (2001), pp. 286–299.

[3] Y. Cao, M. Gunzburger, X. He, and X. Wang: Parallel, non-iterative, multi-physics domain decomposition methods for time-dependent Stokes–Darcy systems. Math. Comp., 83 (2014), pp. 1617–1644.

[4] H.J. Choi and J.R. Kweon: The stationary Navier–Stokes system with no-slip boundary condition on polygons: corner singularity and regularity. Comm. Partial Differential Eq., 38 (2013), pp. 1235–1255.
[5] C. Bernardi, M. Dauge, and Y. Maday: Polynomials in the Sobolev World. Preprint, 2007. <hal-00153795v1>
[6] M. Dauge: Elliptic Boundary Value Problems in Corner Domains, Springer-Verlag Berlin Heidelberg, 1988.
[7] M. Dauge: Stationary Stokes and Navier–Stokes systems on two- or three-dimensional domains with corners. part i: linearized equations. SIAM J. Math. Anal., 20 (1989), pp. 74–97.
[8] R. Durán, R.H. Nochetto, and J.P. Wang: Sharp maximum norm error estimates for finite element approximations of the Stokes problem in 2-D. Math. Comp., 51 (1988), pp. 491–506.
[9] P. Grisvard: Elliptic Problems in Nonsmooth Domains. SIAM 2011.
[10] B. Guo and C. Schwab: Analytic regularity of Stokes flow on polygonal domains in countably weighted Sobolev spaces. J. Comput. Appl. Math., 190 (2006), pp. 487–519.
[11] J. Guzmán and D. Leykekhman: Pointwise error estimates of finite element approximations to the Stokes problem on convex polyhedra. Math. Comp., 81 (2012), pp. 1879–1902.
[12] V. John and J. Layton: Analysis of numerical errors in large eddy simulation. SIAM J. Numer. Anal., 40 (2002), pp. 995–1020.
[13] R.B. Kellogg and J.E. Osborn: A regularity for the Stokes problem in a convex polygon. J. Funct. Anal., 21 (1976), pp. 397–431.
[14] P. Märkl and A.M. Sändig: Singularities of the Stokes System in Polygons. Preprint. Bericht 2008/009 des Instituts für Angewandte Analysis und Numerische Simulation der Universität Stuttgart, 2008.
[15] V. Mazya and J. Rossmann: A mixed boundary value problems for the stationary Navier–Stokes system in polyhedral domains. Arch. Rational Mech. Anal., 194 (2009), pp. 669–712.
[16] M. Orlt: Regularitätsuntersuchungen und FEM-FehlerabschÄtzungen für allgemeine Randwertprobleme der Navier-Stokes-Gleichungen. PhD Thesis, Universität Rostock, 1998.
[17] M. Orlt and A.M. Sändig: Regularity of viscous Navier–Stokes flows in nonsmooth domains. Boundary Value Problems and Integral Equations in Nonsmooth Domains, edited by M. Costabel and M. Dauge, Copyright 1995 by Marcel Dekker, Inc.
[18] M. Sarkis and X. Tu. Singular Function Mortar Finite Element Methods. Comput. Methods Appl. Math., 3 (2003), pp. 202–218.

Appendix A: Existence and uniqueness of the weak solution

It suffices to prove the existence and uniqueness of the weak solution $(\mathbf{w}, p) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)$ for the problem

(A.1)
$$\begin{cases} 2(\mathbb{D}(\mathbf{w}), \mathbb{D}(\mathbf{v}))_\Omega - (p, \nabla\cdot\mathbf{v})_\Omega = (\mathbf{f}, \mathbf{v})_\Omega + (\mathbb{F}, \mathbb{D}(\mathbf{v}))_\Omega + (\mathbf{h}, \mathbf{v})_{\partial\Omega}, & \forall\ \mathbf{v} \in \mathbf{H}^1(\Omega), \\ (\nabla\cdot\mathbf{w}, \psi)_\Omega = 0, & \forall\ \psi \in L^2(\Omega), \end{cases}$$

with $\mathbf{f} \in \mathbf{H}^1(\Omega)'$, $\mathbf{h} \in \mathbf{H}^{1/2}(\partial\Omega)'$ and $\mathbb{F} \in L^2(\Omega)^{2\times 2}$. This problem corresponds to $g = 0$ in (1). In general, if $g \in L^2(\Omega)$ and $g \neq 0$, then there exists $\mathbf{a} \in \mathbf{H}^1(\Omega)$ such that $\nabla\cdot\mathbf{a} = g$. In this case, (1) is equivalent to (A.1) with $\mathbf{w} := \mathbf{u} - \mathbf{a}$ and $\mathbb{F} = -2\mathbb{D}(\mathbf{a})$.

In the following, we prove the existence of a weak solution for (A.1) under the compatibility condition

$$\int_\Omega \mathbf{f}(x)\mathrm{d}x + \int_{\partial\Omega} \mathbf{h}(x)\mathrm{d}\tau = 0. \tag{A.2}$$

Consider the divergence-free closed subspaces of $\mathbf{H}^1(\Omega)$,

$$\mathbf{H}^1_{\rm div}(\Omega) := \{\, \mathbf{w} \in \mathbf{H}^1(\Omega) : \nabla\cdot\mathbf{w} = 0 \,\}, \tag{A.3}$$

$$\mathring{\mathbf{H}}^1_{\rm div}(\Omega) := \{\, \mathbf{w} \in \mathbf{H}^1(\Omega) : \nabla\cdot\mathbf{w} = 0,\ \textstyle\int_\Omega \mathbf{w}\,{\rm d}x = 0 \,\}, \tag{A.4}$$

which are equipped with the norm of $\mathbf{H}^1(\Omega)$. Under the compatibility condition (A.2), one can define the following continuous linear functionals on $\mathbf{H}^1_{\rm div}(\Omega)$ and $\mathring{\mathbf{H}}^1_{\rm div}(\Omega)$, respectively, by

$$\widetilde{\ell}(\mathbf{v}) := (\mathbf{f},\mathbf{v})_\Omega + (\mathbb{F},\mathbb{D}(\mathbf{v}))_\Omega + (\mathbf{h},\mathbf{v})_{\partial\Omega}, \quad \forall\,\mathbf{v}\in\mathbf{H}^1_{\rm div}(\Omega), \tag{A.5}$$

$$\ell(\mathbf{v}) := (\mathbf{f},\mathbf{v})_\Omega + (\mathbb{F},\mathbb{D}(\mathbf{v}))_\Omega + (\mathbf{h},\mathbf{v})_{\partial\Omega}, \quad \forall\,\mathbf{v}\in\mathring{\mathbf{H}}^1_{\rm div}(\Omega). \tag{A.6}$$

which satisfy

$$\widetilde{\ell}(\mathbf{v}) = \ell\big(\mathbf{v} - \tfrac{1}{|\Omega|}\textstyle\int_\Omega \mathbf{v}(x){\rm d}x\big), \quad \forall\,\mathbf{v}\in\mathbf{H}^1_{\rm div}(\Omega). \tag{A.7}$$

By Korn's inequality [12], we have

$$2(\mathbb{D}(\mathbf{w}),\mathbb{D}(\mathbf{w}))_\Omega \ge \kappa_1\|\nabla\mathbf{w}\|^2_{L^2(\Omega)} \ge \kappa_2\|\mathbf{w}\|^2_{H^1(\Omega)}, \quad \forall\,\mathbf{w}\in\mathring{\mathbf{H}}^1_{\rm div}(\Omega),$$

for some positive constants κ_1 and κ_2. Hence, the Lax–Milgram lemma implies that there exists a unique weak solution $\mathbf{w}\in\mathring{\mathbf{H}}^1_{\rm div}(\Omega)$ of the following variational problem:

$$2(\mathbb{D}(\mathbf{w}),\mathbb{D}(\mathbf{v}))_\Omega = \ell(\mathbf{v}), \quad \forall\,\mathbf{v}\in\mathring{\mathbf{H}}^1_{\rm div}(\Omega). \tag{A.8}$$

In view of (A.7), we have

$$\begin{aligned} 2\big(\mathbb{D}(\mathbf{w}),\mathbb{D}(\mathbf{v})\big)_\Omega &= 2\big(\mathbb{D}(\mathbf{w}),\mathbb{D}\big(\mathbf{v} - \tfrac{1}{|\Omega|}\textstyle\int_\Omega \mathbf{v}(x){\rm d}x\big)\big)_\Omega \\ &= \ell(\mathbf{v} - \tfrac{1}{|\Omega|}\textstyle\int_\Omega \mathbf{v}(x){\rm d}x) \\ &= \widetilde{\ell}(\mathbf{v}) \\ &= (\mathbf{f},\mathbf{v})_\Omega + (\mathbb{F},\mathbb{D}(\mathbf{v}))_\Omega + (\mathbf{h},\mathbf{v})_{\partial\Omega}, \qquad \forall\,\mathbf{v}\in\mathbf{H}^1_{\rm div}(\Omega). \end{aligned} \tag{A.9}$$

If we define a linear functional Λ on $\mathbf{H}^1(\mathbb{R}^2)$ by

$$\Lambda(\mathbf{v}) := 2\big(\mathbb{D}(\mathbf{w}),\mathbb{D}(\mathbf{v})\big)_\Omega - (\mathbf{f},\mathbf{v})_\Omega - (\mathbb{F},\mathbb{D}(\mathbf{v}))_\Omega - (\mathbf{h},\mathbf{v})_{\partial\Omega}, \quad \forall\,\mathbf{v}\in\mathbf{H}^1(\mathbb{R}^2), \tag{A.10}$$

then (A.9) yields

$$\Lambda(\mathbf{v}) = 0, \quad \forall\,\mathbf{v}\in\mathbf{H}^1(\mathbb{R}^2) \text{ such that } \nabla\cdot\mathbf{v} = 0. \tag{A.11}$$

Hence, there exists $p\in L^2(\mathbb{R}^2)$ such that

$$\Lambda(\mathbf{v}) = (p,\nabla\cdot\mathbf{v})_{\mathbb{R}^2}, \quad \forall\,\mathbf{v}\in\mathbf{H}^1(\mathbb{R}^2). \tag{A.12}$$

The identity (A.10) implies that $\Lambda(\mathbf{v})$ is independent of the values of $\mathbf{v}$ outside the domain Ω, which means that $p = 0$ outside Ω. Hence, we have

$$\Lambda(\mathbf{v}) = (p,\nabla\cdot\mathbf{v})_\Omega, \quad \forall\,\mathbf{v}\in\mathbf{H}^1(\mathbb{R}^2), \tag{A.13}$$

which together with (A.10) yields

$$2(\mathbb{D}(\mathbf{w}),\mathbb{D}(\mathbf{v}))_\Omega - (\mathbf{f},\mathbf{v})_\Omega - (\mathbb{F},\mathbb{D}(\mathbf{v}))_\Omega - (\mathbf{h},\mathbf{v})_{\partial\Omega} = (p,\nabla\cdot\mathbf{v})_\Omega, \quad \forall\,\mathbf{v}\in\mathbf{H}^1(\Omega). \tag{A.14}$$

This proves the existence of a weak solution $(\mathbf{w},p)\in\mathbf{H}^1(\Omega)\times L^2(\Omega)$ for the problem (A.1). The uniqueness of the weak solution, up to a constant addition in $\mathbf{w}$, can be proved in the routine way.

Appendix B: Verification of $a_4^*(\omega) = 0$ when $\omega \in \mathfrak{M} \backslash \{\omega_*\}$

In this appendix, we use Matlab's symbolic computation toolbox to verify that the vector (b_1, b_2, b_3, b_4) given by (143) is a solution of (81) when $\omega \in \mathfrak{M} \backslash \{\omega_*\}$, where (c_1, c_2, c_3, c_4) is given by (141) and $(\mathbf{F}, G, \mathbf{H})$ is given by (138). Therefore, the augmented matrix (142) is of rank 3 and $a_4^*(\omega) = 0$ when $\omega \in \mathfrak{M} \backslash \{\omega_*\}$.

We first write the following commands in the file StokesTest.M, and then run the file in the command window of Matlab R2012b.

```
syms L t w z % define L t w z as symbols
load('c.mat') % the column vector c is defined by (141)
a=cot(w)+L*cot(L*w); % the parameter $\alpha_{\lambda,\omega}$
cw=subs(c,t,w); % the value of c at $\theta=\omega$
cw=conj(cw'); % the transpose of the vector c
d=[cot(L*w)+L*cot(w), L-cot(L*w)*cot(w), -L+L^2*cot(L*w)*cot(w), L*a];
b1=-1;
b3=-(sum(cw.*d)-2*L^2/(L-1)*cot(w)-((L+1)^2/2+2*L/(L-1))*cot(L*w))
/(L*(L^2-1));
b2=-a*(L+1)/(2*(L-1))+b3*L^2;
b4=0;
b=[b1;b2;b3;b4];
% The following three lines define $(\mathbf{U}_{\lambda,\omega}, q_{\lambda,\omega})$; see (108)
V1=a*cos(L*t)*sin(t)+L*sin(L*t)*sin(t)-cos(L*t)*cos(t);
V2=a*((L-1)*sin(L*t)*sin(t)+cos((L-1)*t))/(1-L)+(1+L)*cos(L*t)*sin(t);
Q=2*L*cos((1-L)*t)-2*a*L*sin((1-L)*t)/(1-L);
% The following 5 lines define $\mathbf{F}^*$, $G^*$ and $\mathbf{H}^*$ in (137)-(138)
F1=2*L*V1-Q;
F2=2*L*V2;
G=-V1;
H1=-V2;
H2=0;
% The following three lines define $(\mathbf{U}_j, q_j)$, $j=1,2,3,4$
UU2=[cos(t)*sin(L*t)/L, sin(t)*sin(L*t)/L, cos(t)*cos(L*t),
sin(t)*cos(L*t)];
UU1=simplify(-diff(UU2,t)/(1+L));
qq=simplify((diff(UU2,t,3)+(1+L)^2*diff(UU2,t))/(1-L^2));
% The following 8 lines define the matrix $\mathbb{A}$ and the right-hand
side of (81)
E1=diff(UU1,t)+(L-1)*UU2;
E2=2*diff(UU2,t)+2*UU1-qq;
d10=H1;
d1w=H1-sum(cw.*E1);
d20=H2-2*G;
d2w=H2-2*G-sum(cw.*E2);
dd=[subs(d10,t,0); subs(d1w,t,w); subs(d20,t,0); subs(d2w,t,w)];
A=[subs(E1,t,0); subs(E1,t,w); subs(E2,t,0); subs(E2,t,w)];
```

```
A=simplify(A);
E=A*b-dd; %%%%%% We want to verify that E=0 %%%%%%
E13=simplify(E(1:3)) % show the expressions of E(1), E(2), E(3)
% The following 12 lines simplify the expression of E(4)
% by using (101)-(102) and assuming sin(λω) > 0
aa=simplify(E(4));
bb=expand(aa);
aa=subs(bb,cos(L*w),sin(w)/w);
bb=subs(aa,cos(w)^2,1-sin(w)^2);
aa=subs(bb,sin(L*w),L*sin(w));
aa=simplify(aa);
bb=subs(aa,sin(w),w*sin(z)/z);
aa=subs(bb,w,z/L);
aa=simplify(aa);
bb=subs(aa,sin(z)^2,z^2/(1+z^2));
aa=subs(bb,sin(z)^4,z^4/(1+z^2)^2);
E4=simplify(aa) % show the simplified expression of E(4)
% The following 12 lines simplify the expression of E(4)
% by using (101)-(102) and assuming sin(λω) < 0
aa=simplify(E(4));
bb=expand(aa);
aa=subs(bb,cos(L*w),-sin(w)/w);
bb=subs(aa,cos(w)^2,1-sin(w)^2);
aa=subs(bb,sin(L*w),-L*sin(w));
aa=simplify(aa);
bb=subs(aa,sin(w),-w*sin(z)/z);
aa=subs(bb,w,z/L);
aa=simplify(aa);
bb=subs(aa,sin(z)^2,z^2/(1+z^2));
aa=subs(bb,sin(z)^4,z^4/(1+z^2)^2);
E4=simplify(aa) % show the simplified expression of E(4)
```

```
>> StokesTest % run this in the command window of Matlab R2012b
E13 =
0
0
0
E4 =
0
E4 =
0
```

Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong.
E-mail: buyang.li@polyu.edu.hk
URL: https://www.polyu.edu.hk/ama/profile/byli/