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A POD-BASED FAST ALGORITHM FOR THE NONLOCAL UNSTEADY PROBLEMS

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Abstract. A fast algorithm for the nonlocal unsteady problems was proposed, which can be employed in the numerical simulation of nonlocal diffusion and peridynamic. The surrogate model constructed by the proper orthogonal decomposition (POD) speeds up the process of solving equations by reducing the order of linear equations. Then, the accuracy and efficiency of the proposed algorithm was verified by numerical experiments. The results showed that this approach ensures accuracy while reduces the computational burden of the nonlocal model.

Key words. Nonlocal diffusion, peridynamic, fast algorithm, POD, model reduction.

1. Introduction

The classical theory of continuum mechanics assumes that all internal forces in the body are contact forces, which leads to mathematical models described by partial differential equations. However, the partial differential equation model cannot properly model problems evolving discontinuities, such as damage and fracture. The reason is that it assumes the displacement field is continuously differentiable. Peridynamic (PD) model [1, 2, 3, 4] is a reformulation of continuum theory, which avoids the explicit use of spatial derivatives and the internal force is considered to be a non-contact force. Consequently, PD models are particularly suitable for the representation of discontinuities in displacement fields and crack evolution in materials, its effectiveness in modeling material damages has been shown in numerical simulation of crack nucleation[5], crack propagation and branching[6, 7], phase transformations in solids [8], impact damage[9, 10], polycrystal fracture and so on. Various numerical methods for solving PD problems have been proposed and implemented, including finite difference[11], finite element [12, 13, 14, 15], quadrature, and particle-based[16] methods are successfully applied in numerical simulation of fracture and damage.

However, just like the fractional partial differential equation model[17, 18, 19], due to the nonlocal property of the nonlocal operator, the numerical methods for the PD models yield dense stiffness matrices. Consequently, two main factors restrict the efficiency of numerical simulation, one is the generation of the stiffness matrices, and the other is the computational complexity of the dense linear equations solver. Both two render the numerical simulation of nonlocal models computationally expensive.

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Seeking an efficient and accurate numerical algorithm has been one of the crucial issues in the application of nonlocal models. Extensive research has been developed on the fast algorithm for nonlocal models. A fast finite difference method for the one-dimensional space-fractional diffusion equation was developed in [20], in which the stiffness matrix is full, and it needs a Toeplitz-like matrix expansion. A fast Galerkin finite element method was developed for a one-dimensional PD model in [21], because of the stiffness matrix can be decomposed as a sum of a tridiagonal matrix and a Toeplitz matrix, then the Fast Fourier Transform has been used to accelerate the matrix-vector product, and it was extended to the discontinuous Galerkin method in [22]. The Galerkin method needs to compute more than one layer of integration in the assembly of the stiffness matrix, which constitutes a very large portion of computation time. Therefore, the fast collocation method was developed for nonlocal diffusion problems. In one dimensional case, the stiffness matrix can be decomposed as a sum of a Toeplitz matrix [23, 24] and a low-rank matrix, and the stiffness matrix was proved to be of block-Toeplitz-Toeplitz-block matrices in two-dimensional case [25]. Recently, a fast collocation method is discretized on a uniform partition for two-dimensional peridynamic problems [26], in which the stiffness matrix was also proved to be block-Toeplitz-Toeplitz-block matrices. All of these fast algorithms developed above speed up the solving by exploring the structure of the stiffness matrices, which is discretized on a uniform partition. It can be seen later, the fast algorithm proposed in this paper is independent of the mesh structure.

In this paper, a fast algorithm for the nonlocal unsteady problem was proposed by constructing the reduced-order model (ROM) of the original system. The key idea of ROM is approximating the high-order system with a lower-order system based on the proper orthogonal decomposition (POD) method [27] and the projection coefficients of ROM are obtained by implementing Galerkin projection [28]. In [29], the ROM was firstly used to solve parameterized nonlocal problems. Different from that, this method is utilized in speed up the equation solving at this time. Actually, this algorithm only focus on second factor. However, compared with the existing fast algorithm, our method is independent of the structure of stiffness matrix and it can be easily embedded into existing numerical methods. In the process of solving, it plays a significant role in reducing the computational burden of solving dense linear equations.

The rest of this paper is organized as follows. In section 2, the nonlocal parabolic equation and nonlocal wave equation are introduced. The weak form of these two equations and their finite element discretizations are given in sections 3 and 4, respectively. Section 5 contains POD-based reduced-order models of both two types of equations. Finally, numerical experiments are carried out to verify the efficiency of our fast method.

2. Nonlocal Unsteady Equation

The problems mainly considered here are nonlocal unsteady problems, Eq. (1) is the nonlocal parabolic equation,

$$
\begin{cases}
\dfrac{\partial u}{\partial t} - \displaystyle\int_{B_\delta(x)} K(x,y)(u(y,t)-u(x,t))dy = f(x,t), & \text{on } \Omega \times (0,T], \\
u(x,t) = g(x,t), & \text{on } \Omega_c \times (0,T], \\
u(x,0) = u_0(x), & \text{on } \Omega \cup \Omega_c.
\end{cases} \tag{1}
$$

and the other is nonlocal wave equation:

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial t^2} - \displaystyle\int_{B_\delta(x)} K(x,y)(u(y,t)-u(x,t))dy = f(x,t), & \text{on } \Omega \times (0,T], \\
u(x,t) = g(x,t), & \text{on } \Omega_c \times (0,T], \\
u(x,0) = u_0(x), & \text{on } \Omega \cup \Omega_c, \\
\dot{u}(x,0) = u_1(x), & \text{on } \Omega \cup \Omega_c,
\end{cases} \tag{2}
$$

where Ω is a bounded open domain and Ω_c is constraint domain, $\Omega_c = \{y \in \mathcal{R}^n \backslash \Omega \mid dist(y,x) \leq \delta, x \in \Omega\}$, i.e., Ω_c is a layer of thickness δ surrounding Ω. The nonlocal operator

$$
Lu(x,t) := \int_{B_\delta(x)} K(x,y)(u(y,t)-u(x,t))dy, \ \forall x \in \Omega, \tag{3}
$$

is corresponding to the classical diffusion operator. For local PDE setting, one must impose appropriate boundary conditions. But for nonlocal models, because the nonlocality of interactions, one must impose volume constraints, that is, constraints acting on the domain, not the boundary of the domain. The Figure 1, 2 show the domain in 1D and 2D. Many kernels have been proposed in literatures. It's

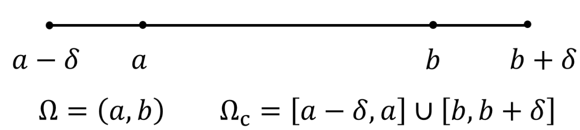

FIGURE 1. One dimensional domain.

worth noticed that the kernel determined the regularity of the solution, which is meaningful in practice. The kernel is symmetric in most cases, so we can define the bilinear form as

$$
A(u,v) = -(Lu,v) = \frac{1}{2}\int_{\Omega\cup\Omega_c}\int_{\Omega\cup\Omega_c} K(x,y)(u(y)-u(x))(v(y)-v(x))dydx \tag{4}
$$

the energy norm $|||u||| = \sqrt{A(u,u)}$, the energy space

$$
V(\Omega \cup \Omega_c) = \{u \in L^2(\Omega \cup \Omega_c) \mid |||u||| < \infty\}. \tag{5}
$$

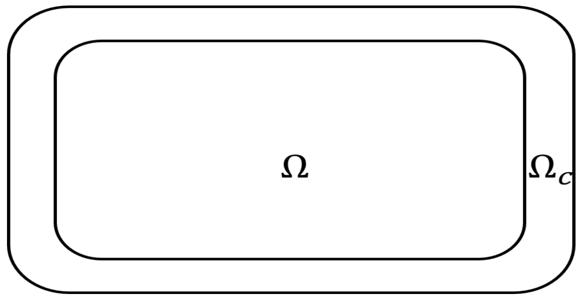

Figure 2. Two dimensional domain.

3. Nonlocal Parabolic Equation

3.1. Weak Formulation. Multiplied by $v \in V = \{v \in L^2(\Omega') \mid v(x) = 0 \text{ on } \Omega_c\}$ in the both side of equation, where $\Omega' = \Omega \cup \Omega_c$, at any $t \in (0, T]$, and integrate over Ω, then

$$\begin{aligned} &\int_\Omega \frac{\partial u(x,t)}{\partial t} v(x)dx \\ &- \int_\Omega \int_{B_\delta(x)} K(x,y)(u(y,t) - u(x,t))v(x)dydx = \int_\Omega f(x,t)v(x)dx, \end{aligned} \tag{6}$$

It can be simplified as,

$$(u_t, v) - (Lu, v) = (f, v). \tag{7}$$

Next select a function space $S(\Omega') \subset L^2(\Omega')$, then define the space $S_0(\Omega') = \{v \in S(\Omega') \mid v = 0 \ on \ \Omega_c\}$ and the affine space $S_g(\Omega') = \{v \in S(\Omega') \mid v = g(x) \ on \ \Omega_c\}$. Further, the weak formulation of the problem (1) can be given as follows:

Problem 1. *(Weak Formulation of Nonlocal Parabolic Equation) Given $f(x,t) \in L^2(\Omega)$, $g(x,t) \in L^2(\Omega_c)$, and $u_0(x) \in L^2(\Omega)$, for any $t \in (0, T)$, find $u(x,t) \in S_g(\Omega')$, such that the initial condition and (7) holds for all $v \in S_0(\Omega')$.*

3.2. Finite Element Discretization. Assumed that there is a finite element space $S^h(\Omega') \subset S(\Omega')$, $S_0^h(\Omega') \subset S_0(\Omega')$, parametrized by a grid-spacing parameter h, and define the finite-dimensional affine space $S_g^h(\Omega') = \{v^h(x,t) \in S^h(\Omega') \mid v^h(x,t) = g^h(x,t) \ on \ \Omega_c\}$, then the semi-discrete problem can be defined. For any $t \in (0, T]$, seek $u^h(x,t) \in S_g^h(\Omega')$ such that

$$(u_t^h, v^h) - (Lu^h, v^h) = (f, v^h), \ \forall v^h \in S_0^h(\Omega'). \tag{8}$$

with $u^h(x, 0) = u_0^h(x)$. Let $\{\phi_i(x)\}_{i=1}^n$ denote the basis function of $S^h(\Omega)$, for example, continuous piecewise linear polynomials with respect to the partition. Then $u^h(x,t), f^h(x,t)$ can be expressed as

$$u^h(x,t) = \sum_{j=1}^n u_j(t)\phi_j(x), \ f^h(x,t) = \sum_{j=1}^n f_j(t)\phi_j(x). \tag{9}$$

Therefore, the finite element approximation can be written as follows

$$(\sum_{j=1}^{n} \frac{du_j}{dt}\phi_j(x), v^h(x)) - (L(\sum_{j=1}^{n} u_j(t)\phi_j(x)), v^h(x)) = (\sum_{j=1}^{n} f_j(t)\phi_j(x), v^h(x)). \tag{10}$$

Let v^h go through test function space, then we can obtain

$$(\sum_{j=1}^{n} \frac{du_j}{dt}\phi_j(x), \phi_i(x)) - (L(\sum_{j=1}^{n} u_j(t)\phi_j(x)), \phi_i(x)) = (\sum_{j=1}^{n} f_j(t)\phi_j(x), \phi_i(x)). \tag{11}$$

where $i = 1, 2, ..., n$. Because the inner product and the operator are linear, then the formulation above can be written as

$$\sum_{j=1}^{n}(\phi_j(x), \phi_i(x))\frac{du_j}{dt} - \sum_{j=1}^{n}(L(\phi_j(x)), \phi_i(x))u_j(t) = \sum_{j=1}^{n}(\phi_j(x), \phi_i(x))f_j(t). \tag{12}$$

where $i = 1, 2, ..., n$. Now some notation can be introduced for simplicity, set

Mass matrix:	$\mathbf{M} = [(\phi_j(x), \phi_i(x))]_{i,j=1}^n$.
Stifness matrix:	$\mathbf{A}(t) = [(L\phi_j(x), \phi_i(x))]_{i,j=1}^n$.
Load vector:	$\mathbf{F}(t) = [f_j(t)]_{j=1}^n$.
Unknow vector:	$\mathbf{X}(t) = [u_j(t)]_{j=1}^n$.

Then the system can be written as an ordinary differential system.

$$\mathbf{M}\frac{d\mathbf{X}}{dt} - \mathbf{A}\mathbf{X}(t) = \mathbf{M}\mathbf{F}(t).$$

Assumed that we have a uniform partition of $[0, T]$ with mesh size Δt. Then the corresponding θ-scheme is

$$\mathbf{M}\frac{\mathbf{X}_{n+1} - \mathbf{X}_n}{\Delta t} - \theta\mathbf{A}\mathbf{X}_{n+1} - (1-\theta)\mathbf{A}\mathbf{X}_n = \theta\mathbf{M}\mathbf{F}_{n+1} + (1-\theta)\mathbf{M}\mathbf{F}_n.$$

which can be simplified as

$$\tilde{\mathbf{A}}\mathbf{X}_{n+1} = \tilde{\mathbf{b}}$$

where,

$$\begin{aligned}
\tilde{\mathbf{A}} &= \frac{\mathbf{M}}{\Delta t} - \theta\mathbf{A}, \\
\tilde{\mathbf{b}} &= \theta\mathbf{M}\mathbf{F}_{n+1} + (1-\theta)\mathbf{M}\mathbf{F}_n + \frac{\mathbf{M}}{\Delta t}\mathbf{X}_n + (1-\theta)\mathbf{A}\mathbf{X}_n,
\end{aligned}$$

then, if $\theta = 0$, which is well-known as the forward or explicit Euler method. On the other hand, if choosing $\theta = 1$, which corresponds to the backward or implicit Euler method. Both of these methods are first-order accurate methods. The third case would be to choose $\theta = \frac{1}{2}$, which is known as the Crank-Nicolson method and has the advantage that it is second-order accurate.

4. Nonlocal Wave Equation

In order to discretize a second time derivative, we first introduce an additional variable $v = \frac{\partial u}{\partial t}$ and transform the system (2) into a first order system. Then reformulating the original wave equation as follows:

$$\begin{aligned}
\frac{\partial u}{\partial t} - v &= 0 && \text{in } \Omega \times [0,T], && (13)\\
\frac{\partial v}{\partial t} - Lu &= f && \text{in } \Omega \times [0,T], && (14)\\
u(x,t) &= g && \text{on } \Omega_c \times [0,T], && (15)\\
u(x,0) &= u_0(x) && \text{in } \Omega, && (16)\\
\dot{u}(x,0) &= u_1(x) && \text{in } \Omega. && (17)
\end{aligned}$$

Note that we do not have boundary conditions for v at first. So, we enforce $v = \frac{\partial g}{\partial t}$ on the boundary.

4.1. Weak Formulation. Multiplied by $w \in V = \{v \in L^2(\Omega) \mid v(x) = 0 \text{ on } \ \Omega_c\}$ in the both side of equation, at any $t \in (0,T]$, and integrate over Ω, then

$$(18) \qquad \begin{cases} \displaystyle\int_\Omega \frac{\partial u(x,t)}{\partial t} w(x)dx - \int_\Omega v(x,t)w(x)dx = 0, \\ \displaystyle\int_\Omega \frac{\partial v(x,t)}{\partial t} w(x)dx - \int_\Omega Lu(x,t)w(x)dx = \int_\Omega f(x,t)w(x)dx. \end{cases}$$

It can be simplified as,

$$(19) \qquad \begin{cases} (u_t, w) - (v, w) = 0, \\ (v_t, w) - (Lu, w) = (f, w). \end{cases}$$

Next select a function space $S(\Omega') \in L^2(\Omega')$ and then define the space $S_0(\Omega') = \{v \in S(\Omega^{'}) \mid v = 0 \ on \ \Omega_c\}$ and the affine space $S_g(\Omega') = \{v \in S(\Omega^{'}) \mid v = g(x) \ on \ \Omega_c\}$. Then, the weak formulation of the problem (2) can be given as follows:

Problem 2. *(Weak Formulation of Nonlocal Wave Equation) Given $f(x,t) \in L^2(\Omega)$, $g(x,t) \in L^2(\Omega_c)$, and $u_0(x) \in L^2(\Omega)$, for any $t \in (0,T)$, find $u(x,t) \in S_g(\Omega')$, such that the initial condition and (19) holds for all $v \in S_0(\Omega')$.*

4.2. Finite Element Discretization. Assumed that there is a finite element space $S^h(\Omega') \subset S(\Omega')$, $S_0^h(\Omega') \subset S_0(\Omega')$, parametrized by a grid-spacing parameter h, and define the finite-dimensional affine space $S_g^h(\Omega') = \{v^h(x,t) \in S^h(\Omega^{'}) \mid v^h(x,t) = g^h(x,t) \ on \ \Omega_c\}$, then the semi-discrete problem can be defined. For any $t \in (0,T]$, seek $u^h(x,t) \in S_g^h(\Omega')$ such that

$$(20) \qquad \begin{cases} (u_t^h, w^h) - (v^h, w^h) = 0, \\ (v_t^h, w^h) - (Lu^h, w^h) = (f, w^h). \end{cases}$$

with $u^h(x,0) = u_0^h(x)$. Let $\{\phi_i(x)\}_{i=1}^n$ denote the basis function of $S^h(\Omega)$, for example, continuous piecewise linear polynomials with respect to the partition.

Then $u^h(x,t), f^h(x,t)$ can be expressed as

$$u^h(x,t) = \sum_{j=1}^{n} u_j(t)\phi_j(x),\ v^h(x,t) = \sum_{j=1}^{n} v_j(t)\phi_j(x),\ f^h(x,t) = \sum_{j=1}^{n} f_j(t)\phi_j(x). \tag{21}$$

Therefore, the finite element approximation can be written as follows

$$\begin{cases} (\sum_{j=1}^{n} \frac{du_j}{dt}\phi_j(x), w^h) - (\sum_{j=1}^{n} v_j(t)\phi_j(x), w^h) = 0, \\ (\sum_{j=1}^{n} \frac{dv_j}{dt}\phi_j(x), w^h) - (L(\sum_{j=1}^{n} u_j(t)\phi_j(x)), w^h) = (\sum_{j=1}^{n} f_j(t)\phi_j(x), w^h). \end{cases} \tag{22}$$

Let w^h go through test function space, then we can obtain

$$\begin{cases} (\sum_{j=1}^{n} \frac{du_j}{dt}\phi_j(x), \phi_i(x)) - (\sum_{j=1}^{n} v_j(t)\phi_j(x), \phi_i(x)) = 0, \\ (\sum_{j=1}^{n} \frac{dv_j}{dt}\phi_j(x), \phi_i(x)) - (L(\sum_{j=1}^{n} u_j(t)\phi_j(x)), \phi_i(x)) = (\sum_{j=1}^{n} f_j(t)\phi_j(x), \phi_i(x)). \end{cases} \tag{23}$$

where $i = 1, 2, ..., n$. Because the inner product and the operator are linear, then the formulation above can be written as

$$\begin{cases} \sum_{j=1}^{n}(\phi_j(x), \phi_i(x))\frac{du_j}{dt} - \sum_{j=1}^{n}(\phi_j(x), \phi_i(x))v_j(t) = 0, \\ \sum_{j=1}^{n}(\phi_j(x), \phi_i(x))\frac{dv_j}{dt} - \sum_{j=1}^{n}(L(\phi_j(x)), \phi_i(x))u_j(t) = \sum_{j=1}^{n}(\phi_j(x), \phi_i(x))f_j(t). \end{cases} \tag{24}$$

where $i = 1, 2, ..., n$. Once again, some notation can be introduced for simplicity, set

Mass matrix:	$\mathbf{M} = [(\phi_j(x), \phi_i(x))]_{i,j=1}^n$.
Stifness matrix:	$\mathbf{A}(t) = [(L\phi_j(x), \phi_i(x))]_{i,j=1}^n$.
Load vector:	$\mathbf{F}(t) = [f_j(t)]_{j=1}^n$.
Unknow vector:	$\mathbf{X}(t) = [u_j(t)]_{j=1}^n$.
Unknow vector:	$\mathbf{Y}(t) = [v_j(t)]_{j=1}^n$.

Then the system can be written as an ordinary differential system.

$$\begin{cases} \mathbf{M}\frac{d\mathbf{X}}{dt} - \mathbf{MY} = 0. \\ \mathbf{M}\frac{d\mathbf{Y}}{dt} - \mathbf{AX}(t) = \mathbf{MF}(t). \end{cases} \tag{25}$$

Assumed that we have a uniform partition of $[0, T]$ with mesh size Δt. Then the corresponding θ-scheme is

$$\begin{cases} \mathbf{M}\frac{\mathbf{X}_{n+1} - \mathbf{X}_n}{\Delta t} - \theta\mathbf{MY}_{n+1} - (1-\theta)\mathbf{MY}_n = 0, \\ \mathbf{M}\frac{\mathbf{Y}_{n+1} - \mathbf{Y}_n}{\Delta t} - \theta\mathbf{AX}_{n+1} - (1-\theta)\mathbf{AX}_n = \theta\mathbf{MF}_{n+1} + (1-\theta)\mathbf{MF}_n. \end{cases} \tag{26}$$

The equations above can be simplified a bit by eliminating $\mathbf{Y}_{n+1}$ from the first equation and rearranging terms. We then get

$$\begin{cases} (\mathbf{M}-\theta^2\Delta^2 t\mathbf{A})\mathbf{X}_{n+1} = \mathbf{M}\mathbf{X}_n + \Delta t\mathbf{M}\mathbf{Y}_n + \theta(1-\theta)\Delta^2 t\mathbf{A}\mathbf{X}_n \\ \ + \theta\Delta^2 t[\theta\mathbf{M}\mathbf{F}_{n+1} + (1-\theta)\mathbf{M}\mathbf{F}_n], \\ \mathbf{M}\mathbf{Y}_{n+1} = \mathbf{M}\mathbf{Y}_n + \Delta t[\theta\mathbf{A}\mathbf{X}_{n+1} + (1-\theta)\mathbf{A}\mathbf{X}_n] \\ \ + \Delta t[\theta\mathbf{M}\mathbf{F}_{n+1} + (1-\theta)\mathbf{M}\mathbf{F}_n]. \end{cases} \tag{27}$$

which can be simplified as

$$\begin{cases} \tilde{\mathbf{A}_1}\mathbf{X}_{n+1} = \tilde{\mathbf{b}_1} \\ \tilde{\mathbf{A}_2}\mathbf{Y}_{n+1} = \tilde{\mathbf{b}_2} \end{cases} \tag{28}$$

where,

$$\begin{aligned} &\tilde{\mathbf{A}_1} = (\mathbf{M}-\theta^2\Delta^2 t\mathbf{A}), \\ &\tilde{\mathbf{A}_2} = \mathbf{M}, \\ &\tilde{\mathbf{b}_1} = \mathbf{M}\mathbf{X}_n + \Delta t\mathbf{M}\mathbf{Y}_n + \theta(1-\theta)\Delta^2 t\mathbf{A}\mathbf{X}_n + \theta\Delta^2 t[\theta\mathbf{M}\mathbf{F}_{n+1} + (1-\theta)\mathbf{M}\mathbf{F}_n], \\ &\tilde{\mathbf{b}_2} = \mathbf{M}\mathbf{Y}_n + \Delta t[\theta\mathbf{A}\mathbf{X}_{n+1} + (1-\theta)\mathbf{A}\mathbf{X}_n] + \Delta t[\theta\mathbf{M}\mathbf{F}_{n+1} + (1-\theta)\mathbf{M}\mathbf{F}_n], \end{aligned} \tag{29}$$

then, the same as above section, we can take different θ to get different scheme.

5. Reduce Order Model

5.1. Nonlocal parabolic equation. This section presents a projection-based ROM for the original high-order system. We apply the POD-Galerkin method to project the original system onto a low-dimensional subspace to accerlerate the time process. The general principle of POD method is to construct a m-dimensional basis $\mathbf{\Psi} = [\boldsymbol{\psi_1}, \boldsymbol{\psi_2}, ..., \boldsymbol{\psi_m}] \in \mathcal{R}^{n\times m}(m \ll n)$ such that the solution can be optimally represented by the basis function[29]

$$\mathbf{u}(t) \approx \mathbf{\Psi}\mathbf{a}(t) + \mathbf{u}^{ps}(t) \tag{30}$$

where $\mathbf{a}(t) = [a_1(t), a_2(t), ..., a_m(t)]^T \in \mathcal{R}^m$ is a time-dependent vector, $\mathbf{u}^{ps}(t)$ denotes a particular solution at the time t. In this paper, we take following choice for the particular solution.

$$\mathbf{u}^{ps}(t) = \begin{cases} g^h(x,t), & x \in \Omega_c, \\ 0, & x \in \Omega. \end{cases} \tag{31}$$

In essential, the problem above is a low-rank approximation problem and in the finite dimensional setting can be solved by the singular value decomposition(SVD) of the snapshots matrix $\mathbf{B}_{n\times k} = [\mathbf{u_1}, \mathbf{u_2}, ..., \mathbf{u_k}]$,

$$\mathbf{B}_{n\times k} = \mathbf{U}_{n\times n}\mathbf{\Sigma}_{n\times k}\mathbf{V}^T_{k\times k}. \tag{32}$$

In above equation, $\mathbf{U}$ and $\mathbf{V}^T$ are real orthonormal matrices. The POD basis $\mathbf{\Psi}$ is given by the first m column of $\mathbf{U}$, which corresponds to the m largest singular values of $\mathbf{\Sigma}$. Substituting (30) into the system, that is, $\mathbf{X}(t) = \mathbf{\Psi}\mathbf{a}(t) + \mathbf{u}^{ps}(t)$, and applying the Galerkin projection, then the ROM for nonlocal parabolic equation is obtained.

$$\mathbf{\Psi}^T\mathbf{M}\frac{d(\mathbf{\Psi a}(t)+\mathbf{u}^{ps}(t))}{dt}-\mathbf{\Psi}^T\mathbf{A}(\mathbf{\Psi a}(t)+\mathbf{u}^{ps}(t))=\mathbf{\Psi}^T\mathbf{MF}(t). \tag{33}$$

In this system, the dimension of the ROM is far less than that of the original systems.

$$\begin{cases}\mathbf{\Psi}^T\mathbf{M}\dfrac{d(\mathbf{\Psi a}(t)+\mathbf{u}^{ps}(t))}{dt}-\mathbf{\Psi}^T\mathbf{A}(\mathbf{\Psi a}(t)+\mathbf{u}^{ps}(t))=\mathbf{\Psi}^T\mathbf{MF}(t),\\ \mathbf{a}(0)=\mathbf{\Psi}^T(\mathbf{X}(0)-\mathbf{u}^{ps}(0)).\end{cases} \tag{34}$$

Discrete the above equation in the time direction, the following low-order ordinary differential equation system is obtained.

$$\begin{cases}\mathbf{\Psi}^T\tilde{\mathbf{A}}(\mathbf{\Psi a}_{n+1}+\mathbf{u}^{ps}_{n+1})=\mathbf{\Psi}^T\tilde{\mathbf{b}},\\ \mathbf{a}_0=\mathbf{\Psi}^T(\mathbf{X}_0-\mathbf{u}^{ps}_0).\end{cases} \tag{35}$$

The standard ROM algorithm listed in algorithm 1.

Algorithm 1 ROM of nonlocal parabolic equation

1: Snapshot the data and get snapshots matrix $\mathbf{B}_{n\times k}=[\mathbf{u_1},\mathbf{u_2},...,\mathbf{u_k}]$ in $[0,t_1]$.
2: Get the POD basis by singular value decomposition $\mathbf{B}_{n\times k}=\mathbf{U}_{n\times n}\mathbf{\Sigma}_{n\times k}\mathbf{V}^T_{k\times k}$.
3: Using θ-scheme to solve the ROM of nonlocal parabolic equation in $[0,T]$.

5.2. Nonlocal wave equation. The same as above, we get the snapshots matrix $\mathbf{B}_{n\times k}=[\mathbf{u_1},\mathbf{u_2},...,\mathbf{u_k}]$, $\mathbf{C}_{n\times k}=[\mathbf{v_1},\mathbf{v_2},...,\mathbf{v_k}]$,

$$\begin{aligned}\mathbf{B}_{n\times k}&=\mathbf{U1}_{n\times n}\mathbf{\Sigma}1_{n\times k}\mathbf{V1}^T_{k\times k},\\ \mathbf{C}_{n\times k}&=\mathbf{U2}_{n\times n}\mathbf{\Sigma}2_{n\times k}\mathbf{V2}^T_{k\times k}.\end{aligned} \tag{36}$$

Substituting (30) into the system, that is, $\mathbf{X}(t)=\mathbf{\Psi a}(t)+\mathbf{u}^{ps}(t)$, $\mathbf{Y}(t)=\mathbf{\Phi c}(t)+\mathbf{v}^{ps}(t)$, and applying the Galerkin projection, then the ROM for nonlocal parabolic equation is obtained.

$$\begin{cases}\mathbf{\Psi}^T\mathbf{M}\dfrac{d(\mathbf{\Psi a}(t)+\mathbf{u}^{ps}(t))}{dt}-\mathbf{\Psi}^T\mathbf{M}(\mathbf{\Phi c}(t)+\mathbf{v}^{ps}(t))=0.\\ \mathbf{\Phi}^T\mathbf{M}\dfrac{d(\mathbf{\Phi c}(t)+\mathbf{v}^{ps}(t))}{dt}-\mathbf{\Phi}^T\mathbf{A}(\mathbf{\Psi a}(t)+\mathbf{u}^{ps}(t))(t)=\Phi^T\mathbf{MF}(t).\end{cases} \tag{37}$$

In this system, the dimension of the ROM is far less than that of the original systems.

$$\begin{cases}\mathbf{\Psi}^T\mathbf{M}\dfrac{d(\mathbf{\Psi a}(t)+\mathbf{u}^{ps}(t))}{dt}-\mathbf{\Psi}^T\mathbf{M}(\mathbf{\Phi c}(t)+\mathbf{v}^{ps}(t))=0.\\ \mathbf{\Phi}^T\mathbf{M}\dfrac{d(\mathbf{\Phi c}(t)+\mathbf{v}^{ps}(t))}{dt}-\mathbf{\Phi}^T\mathbf{A}(\mathbf{\Psi a}(t)+\mathbf{u}^{ps}(t))(t)=\mathbf{\Phi}^T\mathbf{MF}(t),\\ \mathbf{a}(0)=\mathbf{\Psi}^T(\mathbf{X}(0)-\mathbf{u}^{ps}(0)),\\ \mathbf{c}(0)=\mathbf{\Phi}^T(\mathbf{Y}(0)-\mathbf{v}^{ps}(0)),\end{cases} \tag{38}$$

Discrete the above equation in the time direction, the following low-order ordinary differential equation system is obtained.

$$
\begin{cases}
\mathbf{\Psi}^T\tilde{\mathbf{A}}_1(\mathbf{\Psi}\mathbf{a}_{n+1}+\mathbf{u}_{n+1}^{ps}) = \mathbf{\Psi}^T\tilde{\mathbf{b_1}}, \\
\mathbf{\Phi}^T\tilde{\mathbf{A}}_2(\mathbf{\Phi}\mathbf{c}_{n+1}+\mathbf{v}_{n+1}^{ps}) = \mathbf{\Phi}^T\tilde{\mathbf{b_2}}, \\
\mathbf{a}_0 = \mathbf{\Psi}^T(\mathbf{X}_0 - \mathbf{u}_0^{ps}), \\
\mathbf{c}_0 = \mathbf{\Phi}^T(\mathbf{Y}_0 - \mathbf{v}_0^{ps})
\end{cases}
\tag{39}
$$

The stardard ROM algorithm listed in algorithm 2.

Algorithm 2 ROM of nonlocal wave equation

1: Snapshot the data and get snapshots matrix $\mathbf{B}_{n\times k} = [\mathbf{u_1}, \mathbf{u_2}, ..., \mathbf{u_k}]$ and $\mathbf{C}_{n\times k} = [\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k}]$ in $[0, t_1]$.
2: Get the POD basis by singular value decomposition $\mathbf{B}_{n\times k} = \mathbf{U1}_{n\times n}\mathbf{\Sigma1}_{n\times k}\mathbf{V1}_{k\times k}^T$, $\mathbf{C}_{n\times k} = \mathbf{U2}_{n\times n}\mathbf{\Sigma2}_{n\times k}\mathbf{V2}_{k\times k}^T$.
3: Using θ-scheme to solve the ROM of nonlocal wave equation in $[0, T]$.

6. Numerical Examples

In this section, we consider the examples of 1D nonlocal parabolic problems and nonlocal wave equation. For all the results reported in this section, 1000 snapshots were used to generate the 4 POD basis. Since this fast algorithm only accelerates the solution of the ordinary differential equation system, we only recorded the time of the time iteration part, and excluded the generation time of the stiffness matrix and the time of sampling. Nevertheless, as can be seen from the numerical experiments below, the efficiency of the fast algorithm is still impressive. All the numerical experiments are performed on a desktop with Intel(R) Core(TM) i7-6700 CPU @ 3.40GHz and Matlab software, *tic* and *toc* functions are used for timing. The following error estimate was proved in [30]: for the piecewise linear finite element method and $u \in H^s(\Omega), s > 0$, then

$$
||u - u_h||_{L^2(\Omega)} \le Ch^{k-\epsilon}\delta^{-1+\epsilon}||u||_{H^s(\Omega)}. \tag{40}
$$

where $k = min\{2, s\}$, $0 < \epsilon \ll 1$, C is independent of h and δ. Numerical experiments show that the ROM method doesn't affect the convergence rate and keep pace with the finite element method.

6.1. Nonlocal unsteady diffusion equation. Consider the following problem:

$$
\begin{cases}
\dfrac{\partial u}{\partial t} - \displaystyle\int_{B_\delta(x)} \frac{2}{\delta^2|x-y|}(u(y)-u(x))dy = f(x,t), & \text{on } (0,1)\times(0,T], \\
u(x,t) = (x^2 - x^4)sin(t), & \text{on } [-\delta,0]\cup[1,1+\delta]\times(0,T], \\
u(x,0) = 0, & \text{on } [-\delta, 1+\delta].
\end{cases}
\tag{41}
$$

this kernel is mostly used in peridynamic model, right hand side $f(x,t) = (\delta^2 + 12x^2 - 2)sin(t) + (x^2(1-x^2))cos(t)$, that has exact solution $u(x,t) = (x^2 - x^4)sin(t)$,

Table 1. Numerical result of nonlocal parabolic equation: $\delta = 0.02$.

h	$L^2(FEM)$	$L^2(ROM)$	Time(FEM)/s	Time(ROM)/s
1/64	$1.1999e-03$	$1.1999e-03$	1.92	0.27
1/128	$3.0142e-04$	$3.0142e-04$	3.07	0.39
1/256	$7.5549e-05$	$7.5549e-05$	8.58	0.67
1/512	$1.8906e-05$	$1.8906e-05$	56.49	1.22
1/1024	$4.7349e-06$	$4.7349e-06$	279.22	1.96
1/2048	$1.1871e-06$	$1.1871e-06$	2576.92	4.14

Table 2. Numerical result of nonlocal parabolic equation: $\delta = 0.2$.

h	$L^2(FEM)$	$L^2(ROM)$	Time(FEM)/s	Time(ROM)/s
1/64	$1.3359e-04$	$1.2363e-04$	3.01	0.68
1/128	$2.6645e-05$	$2.3443e-05$	6.60	0.80
1/256	$5.9914e-06$	$5.0781e-06$	17.40	1.46
1/512	$1.5066e-06$	$1.2776e-06$	117.92	2.53
1/1024	$3.8334e-07$	$3.2589e-07$	923.33	4.60

Table 3. Numerical result of nonlocal parabolic equation: $\delta = 2$.

h	$L^2(FEM)$	$L^2(ROM)$	Time(FEM)/s	Time(ROM)/s
1/64	$1.0416e-03$	$1.0416e-03$	20.37	2.02
1/128	$2.6040e-04$	$2.6039e-04$	110.11	3.97
1/256	$6.5100e-05$	$6.5097e-05$	741.59	7.22
1/512	$1.6275e-05$	$1.6274e-05$	3697.99	15.17
1/1024	$4.0689e-06$	$4.0682e-06$	17796.68	28.88

for $x \in [-\delta, 1+\delta]$ and $t \in [0, T]$. We compare the FEM solution and the ROM solution at $t = T$, $T = 10$, and $\Delta t = 0.001$, $\theta = 0.5$ for different $\delta = 0.02, 0.2, 2.0$.

In this numerical example, we use the piecewise linear finite element method to simulate the model problem. we observe the numerical performance of the scheme for various δ, where δ denote the horizon parameter in the nonlocal model. In Table 1, 2, 3, we present the L^2 errors and the CPU time of the finite element method and the reduced order model developed in section 5 for $\delta = 0.02, 0.2, 2.0$, respectively. The error indicates the reliability of our fast algorithm and keeps the optimal convergence rate. we observe that the fast algorithm has greatly improved computational efficiency. Besides, with increasing the δ, the stiffness matrix is denser than before, and the performance of ROM is enhanced and consumes much less CPU time.

TABLE 4. Numerical result of nonlocal wave equation: $\delta = 0.02$.

h	$L^2(FEM)$	$L^2(ROM)$	Time(FEM)/s	Time(ROM)/s
1/64	$2.2085e-03$	$2.2112e-03$	1.92	0.54
1/128	$4.7529e-04$	$4.7727e-04$	4.88	0.62
1/256	$1.1242e-04$	$1.1305e-04$	13.31	0.95
1/512	$2.7516e-05$	$2.7686e-05$	117.32	1.70
1/1024	$6.8125e-06$	$6.8558e-06$	550.38	2.92
1/2048	$1.6951e-06$	$1.7060e-06$	3772.69	6.28

TABLE 5. Numerical result of nonlocal wave equation: $\delta = 0.2$.

h	$L^2(FEM)$	$L^2(ROM)$	Time(FEM)/s	Time(ROM)/s
1/64	$1.3403e-04$	$1.3397e-04$	4.43	1.10
1/128	$2.6843e-05$	$2.6491e-05$	11.53	1.57
1/256	$6.0566e-06$	$5.9521e-06$	32.68	2.45
1/512	$1.5240e-06$	$1.4963e-06$	249.122	5.03
1/1024	$3.8742e-07$	$3.8005e-07$	1423.22	8.05

TABLE 6. Numerical result of nonlocal wave equation: $\delta = 2$.

h	$L^2(FEM)$	$L^2(ROM)$	Time(FEM)/s	Time(ROM)/s
1/64	$1.0417e-03$	$1.0417e-03$	37.38	4.21
1/128	$2.6042e-04$	$2.6043e-04$	227.98	8.26
1/256	$6.5105e-05$	$6.5108e-05$	1227.66	16.23
1/512	$1.6276e-05$	$1.6306e-05$	5988.93	33.28
1/1024	$4.0687e-06$	$4.1888e-06$	26470.54	76.77

6.2. Nonlocal wave equation. Consider the following problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \displaystyle\int_{B_\delta(x)} \frac{2}{\delta^2|x-y|}(u(y)-u(x))dy = f(x,t), & \text{on}(0,1)\times(0,T], \\ u(x,t) = (x^2-x^4)sin(t), & \text{on}[-\delta,0]\cup[1,1+\delta]\times[0,T], \\ u(x,0) = 0, & \text{on}[-\delta,1+\delta], \\ \dot{u}(x,0) = x^2-x^4, & \text{on}[-\delta,1+\delta]. \end{cases} \tag{42}$$

this kernel is mostly used in peridynamic model, where $\delta = 0.125$, right hand side $f(x,t) = (\delta^2 + 12x^2 - 2)sin(t) + (x^2(1-x^2))cos(t)$, that has exact solution $u(x,t) = (x^2-x^4)sin(t)$, for $x \in [-\delta, 1+\delta]$ and $t \in [0,T]$. We compare the FEM solution and the ROM solution at $t = T$, $T = 10$, and $\Delta t = 0.001$, $\theta = 0.5$ for different $\delta = 0.02, 0.2, 2$.

we now turn to a more interesting problem for which it is a nonlocal wave equation, is mostly used in peridynamic. The volume constraints are given by the exact solution. For these choices, we solve both the full finite element method and the ROM method, and the numerical result as described in the table. Results analogous to those in Table 4, 5, 6 for different δ, this further confirms our assertion that when the stiffness matrices that are much denser compared with those for the small δ the ROM method is performed well.

7. Conclusions

In this paper, we applied reduce order model for nonlocal unsteady problems. In particular, we have developed the ROM of a finite element scheme, which is suited for numerical simulation a nonlocal model. The fast method has significantly reduced computational complexity. However, although this ROM method we used with the finite element method, it can see this method is independent of the numerical scheme and can be easily used in other numerical schemes to improve the computational efficiency. Numerical experiments confirm the results, the methods' accuracy has been shown numerically. Motivate by these results, the reduced-order model will be widely used in numerical simulation of the peridynamic model.

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