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NUMERICAL ANALYSIS OF A HISTORY-DEPENDENT VARIATIONAL-HEMIVARIATIONAL INEQUALITY FOR A VISCOPLASTIC CONTACT PROBLEM

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Abstract. In this paper, we consider a mathematical model which describes the quasistatic frictionless contact between a viscoplastic body and a foundation. The contact is modeled with normal compliance and unilateral constraint. We present the variational-hemivariational formulation of the model and prove its unique solvability. Then we introduce a fully discrete scheme to solve the problem and derive an error estimate. Under appropriate regularity assumptions of the exact solution, we obtain the optimal order error estimate. Finally, numerical results are reported to show the performance of the numerical method.

Key words. Variational-hemivariational inequality, viscoplastic material, numerical approximation, optimal order error estimate.

1. Introduction

In this paper, we consider a frictionless contact model for rate-type viscoplastic materials. The constitutive law of such materials can be described in the form of

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t))), \tag{1}$$

where $\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})$ denote the displacement, the stress tensor and the linearized strain tensor, respectively. Operator $\mathcal{E}$ is linear and describes the elastic properties of the material. Operator $\mathcal{G}$ is a nonlinear constitutive function and describes the viscoplastic behavior.

Viscoplastic models are used to describe the behavior of real materials like rubbers, metals, rocks and so on. Concrete examples, experimental background and mechanical interpretation concerning viscoplastic materials can be found in [8]. Mathematical modeling, well-posedness and numerical analysis concerning (1) and its variations can be found in [24, 4, 10, 1, 25] and references therein. For comprehensive studies, we also refer to the book [13]. However, all these monographs are in the framework of variational inequalities.

The notation of hemivariational inequality was first introduced in the 1980's ([23]). It is related to the concept of the generalized gradient of a locally Lipschitz function ([7]). In contrast to variational inequalities with convex structures, hemivariational inequalities are mathematical problems involving nonconvex terms. Particularly, variational-hemivariational inequalities involve both convex and nonconvex terms. During the last three decades, hemivariational inequalities were shown to be a very useful tool, especially in contact mechanics ([21]). Various applications to viscoelastic contact models have been studied in [12, 2, 15, 14]. Moreover, if there is a history-dependent operator in the viscoelastic contact model, the problem leads to the history-dependent hemivariational inequality ([19, 26, 22, 27]).

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Compared with the well-developed studies on viscoelastic contact models, there are relatively few publications devoted to hemivariational inequalities for viscoplastic materials. The difficulty lies in the complex viscoplastic constitutive law. Taking the integral of Equation (1):

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,ds + \boldsymbol{\sigma}(0) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(0)), \tag{2}$$

it naturally contains the history-dependent term. What's more, the constitutive law has an implicit expression of stress field $\boldsymbol{\sigma}$. It means that, when proving the existence result, we need to consider a coupled system which is a history-dependent hemivariational inequality combined with an integral equation, rather than only one hemivariational inequality. When deriving error estimates, since $\boldsymbol{\sigma}$ can not be described by $\boldsymbol{u}$ directly, we have to handle both $\boldsymbol{u}$ and $\boldsymbol{\sigma}$, rather than only $\boldsymbol{u}$.

Related references are in the following. In [5], a quasistatic viscoplastic contact problem is proved to have a unique weak solution. The existence and uniqueness results are obtained for the quasistatic contact model with memory term in [16], moreover with memory and damage terms in [17]. The unique weak solvability for a dynamic contact problem is the topic of [20]. In [18], the dynamic contact problem with damage is proved to have a unique weak solution. To our knowledge, numerical analysis and numerical simulation for hemivariational inequalities for viscoplastic materials have not been investigated in the literature so far and we fill this gap in the present paper. The problem concerned here is a quasistatic contact with normal compliance, unilateral constraint and viscoplastic materials.

The paper is structured as follows. In Section 2, we present some necessary preliminaries. In Section 3, we describe the model of the contact process, derive its variational-hemivariational formulation, state the existence and uniqueness theorem and prove it. Then in Section 4, we introduce a fully discrete scheme and provide the error estimates. Finally, in Section 5, we present some numerical examples which provide numerical evidence of our theoretical results.

2. Preliminaries

In this section, we present some necessary notation and preliminary material which we will use in our paper.

Let X be a Banach space. We first recall the definitions of the generalized directional derivative and the generalized gradient of Clarke for a locally Lipschitz function $\varphi : X \to \mathbb{R}$ ([7]). The generalized directional derivative of φ at $x \in X$ in the direction $v \in X$, denoted by $\varphi^0(x; v)$, is defined by

$$\varphi^0(x; v) = \limsup_{y \to x, t \downarrow 0} \tfrac{\varphi(y+tv) - \varphi(y)}{t}.$$

The generalized gradient of φ at x, denoted by $\partial_{Cl}\varphi(x)$, is a subset of a dual space X^* given by $\partial_{Cl}\varphi(x) = \{\, \zeta \in X^* \mid \varphi^0(x; v) \ge \langle \zeta, v \rangle_{X^* \times X}$ for all $v \in X \}$. In particular, here we present two basic properties provided in [7]:

$$\varphi^0(x; v) = \max \{\, \langle \zeta, v \rangle \mid \zeta \in \partial_{Cl}\varphi(x) \,\}, \tag{3}$$

$$\varphi^0(x; v_1 + v_2) \le \varphi^0(x; v_1) + \varphi^0(x; v_2). \tag{4}$$

Let d be a positive integer. The linear space of second-order symmetric tensors on $\mathbb{R}^d$ is denoted by $\mathbb{S}^d$. The inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \qquad \|\boldsymbol{v}\|_{\mathbb{R}^d} = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2} \qquad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij}\,\tau_{ij}, \qquad \|\boldsymbol{\tau}\|_{\mathbb{S}^d} = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2} \qquad \text{for all } \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d.$$

The convention of summation over repeated indices is used in this paper.

We consider a bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz continuous boundary Γ. Since Γ is Lipschitz continuous, the unit outward normal vector exists a.e. on Γ and is denoted by $\boldsymbol{\nu} = (\nu_i) \in \mathbb{R}^d$. For a vector field $\boldsymbol{v}$, the normal and tangential components of $\boldsymbol{v}$ are $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$. Similarly, for tensor field $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$, the normal and tangential components are $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$. Let Γ_1 denote a measurable part of Γ such that meas$(\Gamma_1) > 0$. Let Γ_3 be a measurable part of Γ .

We introduce the following Hilbert spaces with their inner products:

$$V = \{\boldsymbol{v} = (v_i) \in H^1(\Omega;\mathbb{R}^d) : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\}, \quad (\boldsymbol{u},\boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx,$$

$$Q = \{\boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega;\mathbb{S}^d) : \tau_{ij} = \tau_{ji}\}, \qquad (\boldsymbol{\tau},\boldsymbol{\sigma})_Q = \int_\Omega \boldsymbol{\tau} : \boldsymbol{\sigma}\, dx.$$

We recall the definition of deformation operator $\boldsymbol{\varepsilon} : H^1(\Omega;\mathbb{R}^d) \to L^2(\Omega;\mathbb{S}^d)$:

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \tfrac{1}{2}(u_{i,j} + u_{j,i}),$$

where index following comma indicates a partial derivative. The associated norms in V and Q are denoted respectively by $\|\cdot\|_V$ and $\|\cdot\|_Q$. Completeness of the space $(V, \|\cdot\|_V)$ follows from the use of Korn's inequality, which is allowed under the assumption meas$(\Gamma_1) > 0$. Due to the Sobolev trace theorem, there exists a positive constant c_0 which depends only on Γ_1, Γ_3 and Ω such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3;\mathbb{R}^d)} \le c_0 \|\boldsymbol{v}\|_V \text{ for all } \boldsymbol{v} \in V.$$

We also recall the Green formula

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx + \int_\Omega \mathrm{Div}\boldsymbol{\sigma} \cdot \boldsymbol{v}\, dx = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{v}\, d\Gamma \quad \text{for all} \quad \boldsymbol{v} \in V\ , \boldsymbol{\sigma} \in Q,$$

with the divergence operator defined by $\mathrm{Div}\boldsymbol{\sigma} = (\sigma_{ij,j})$.

Here we present two abstract lemmas which will be used in our paper.

We introduce the following notation first. Consider a normed space Y with its norm $\|\cdot\|_Y$. Let V be a closed subspace of $H^1(\Omega;\mathbb{R}^d)$. We denote the trace operator from V to $L^2(\Gamma_3;\mathbb{R}^d)$ by γ with norm $\|\gamma\| = \|\gamma\|_{\mathcal{L}(V,L^2(\Gamma_3;\mathbb{R}^d))}$ and $\gamma^* : L^2(\Gamma_3;\mathbb{R}^d) \to V^*$ its adjoint. Assume that K is a nonempty, closed and convex subset of V. Next, we consider the operators $A : (0,T) \times V \to V^*, \mathcal{M} : C(0,T;V) \to C(0,T;Y)$, the functional $\varphi : Y \times K \to \mathbb{R}$ and the function $j : \Gamma_3 \times \mathbb{R}^d \to \mathbb{R}$ which satisfy the following hypotheses.

H(A): The operator $A\colon (0,T) \times V \to V^*$ satisfies :

(i) $A(\cdot, v)$ is continuous on $(0,T)$ for all $v \in V$;

(ii) $A(t,\cdot)$ is hemicontinuous and strongly monotone with $m_A > 0$ for all $t \in (0,T)$, i.e. $\langle A(t,v_1) - A(t,v_2), v_1 - v_2\rangle_{V^*\times V} \ge m_A \|v_1 - v_2\|_V^2$ for all $v_1, v_2 \in V$;

(iii) $\|A(t,v)\|_{V^*} \le a_0(t) + a_1\|v\|_V$ for all $v \in V$ and all $t \in (0,T)$ with $a_0 \in L^2(0,T)$, $a_0 \ge 0$ and $a_1 > 0$;

(iv) $A(t,0) = 0$ for all $t \in (0,T)$.

H($\mathcal{M}$): The operator $\mathcal{M} : C(0,T;V) \to C(0,T;Y)$ satisfies:
$\|(\mathcal{M}u_1)(t) - (\mathcal{M}u_2)(t)\|_Y \le L_\mathcal{M} \int_0^t \|u_1(s) - u_2(s)\|_V ds$ for all $u_1, u_2 \in C(0,T;V)$ and all $t \in (0,T)$ with $L_\mathcal{M} > 0$.

$$\begin{cases} \text{H}(\varphi): \text{ The functional } \varphi : Y \times K \to \mathbb{R} \text{ satisfies:} \\ \text{(i) } \varphi(y,\cdot) \text{ is convex, proper and lower semicontinuous for all } y \in Y; \\ \text{(ii) } 0 \in D(\partial\varphi(y,\cdot)) \text{ for all } y \in Y; \\ \text{(iii) there exists a constant } m_\varphi > 0 \text{ such that } \varphi(u_1,v_2) - \varphi(u_1,v_1) + \varphi(u_2,v_1) \\ \quad -\varphi(u_2,v_2) \le m_\varphi \|u_1 - u_2\|_Y \|v_1 - v_2\|_V \text{ for all } u_i \in Y,\ v_i \in K,\ i = 1,2. \end{cases}$$

$$\begin{cases} \text{H}(j): \text{ The function } j : \Gamma_3 \times \mathbb{R}^d \to \mathbb{R} \text{ satisfies:} \\ \text{(i) } j(\cdot,\xi) \text{ is measurable on } \Gamma_3 \text{ for all } \xi \in \mathbb{R}^d \text{ and there exists } e \in L^2(\Gamma_3;\mathbb{R}^d) \\ \quad \text{such that } j(\cdot, e(\cdot)) \in L^1(\Gamma_3); \\ \text{(ii) } j(x,\cdot) \text{ is locally Lipschitz on } \mathbb{R}^d \text{ for a.e. } x \in \Gamma_3; \\ \text{(iii) } \|\partial j(x,\xi)\|_{\mathbb{R}^d} \le \widetilde{c_0} + \widetilde{c_1}\|\xi\|_{\mathbb{R}^d} \text{ for all } \xi \in \mathbb{R}^d \text{ and a.e. } x \in \Gamma_3 \text{ with } \widetilde{c_0}, \widetilde{c_1} \ge 0; \\ \text{(iv) there exists constant } \widetilde{m_j} \ge 0 \text{ such that } j^0(x,\xi_1;\xi_2-\xi_1) + j^0(x,\xi_2;\xi_1-\xi_2) \\ \quad \le \widetilde{m_j}\|\xi_1 - \xi_2\|^2_{\mathbb{R}^d}, \text{ for all } \xi_1,\xi_2 \in \mathbb{R}^d \text{ and a.e. } x \in \Gamma_3. \end{cases}$$

LEMMA 1. *Assume that H(A), H($\mathcal{M}$), H(φ), H(j) hold, $f \in C(0,T;V^*)$ and*

$$m_A > \max\{\sqrt{3}\widetilde{c_1}, \widetilde{m_j}\}\|\gamma\|^2.$$

Then there exists a unique function $u \in C(0,T;K)$ such that

$$\langle A(t,u(t)), v - u(t)\rangle_{V^*\times V} + \varphi(\mathcal{M}u(t), v) - \varphi(\mathcal{M}u(t), u(t)) \\ + \int_{\Gamma_3} j^0(\gamma u(t); \gamma v - \gamma u(t))d\Gamma \ge \langle f(t), v - u(t)\rangle_{V^*\times V}$$

for all $v \in K$ and all $t \in (0,T)$.

Lemma 1 is a simplified result of Theorem 4.1 in [11]. In our paper, the function j is defined on $\Gamma_3 \times \mathbb{R}^d$ instead of $\Gamma_3 \times \mathbb{R} \times \mathbb{R}^d$. Note that the condition H(j)(iv) is equivalent to the relaxed monotonicity condition of subdifferential

$$(\xi_1^* - \xi_2^*)\cdot(\xi_1 - \xi_2) \ge -\widetilde{m_j}\|\xi_1 - \xi_2\|^2_{\mathbb{R}^d}$$

for all $\xi_i \in \mathbb{R}^d$, $\xi_i^* \in \partial_{Cl} j(x,\xi_i)$, $i = 1,2$, a.e. $x \in \Gamma_3$. Thus, Theorem 4.1 in [11] can be applied here as Lemma 2. Note that the dependence on the spatial variable x is not indicated in order to simplify the notation.

Remark. Assumption H(j) can be satisfied. We take the following function as an example satisfying the assumption H(j).

$$j(r) = \begin{cases} 0, & \text{if } r < 0\ ; \\ -\frac{b}{2a}r^2 + br, & \text{if } 0 \le r \le a\ ; \\ \frac{ab}{2}, & \text{if } r > a. \end{cases} \quad \partial_{Cl} j(r) = \begin{cases} 0, & \text{if } r < 0\ ; \\ [0,b]\,, & \text{if } r = 0\ ; \\ -\frac{b}{a}r + b, & \text{if } 0 < r \le a\ ; \\ 0, & \text{if } r > a. \end{cases}$$

More details can be found in [12, 22].

Then we present the second lemma, which is the well-known Gronwall inequality ([4]).

LEMMA 2. *For a fixed T, let $0 = t_0 < t_1 < \cdots < t_N = T$ and $k_n = t_n - t_{n-1}$ for $n = 1,2,\cdots,N$. Assume $\{g_n\}_{n=0}^N$ and $\{e_n\}_{n=0}^N$ are two sequences of non-negative numbers satisfying*

$$e_n \le cg_n + c\sum_{i=1}^n k_i e_{i-1}, \quad n = 1,\cdots,N,$$

for a constant $c > 0$. Then, for another constant $c > 0$ independent of N,

$$\max_{0\le n\le N} e_n \le c \max_{0\le n\le N} g_n.$$

3. A viscoplastic contact model

In this section, we describe the model of a contact problem, present its variational-hemivariational formulation, state the existence and uniqueness result and finally prove it.

We consider a viscoplastic body which occupies a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) with a Lipschitz continuous boundary Γ. The boundary is divided into three mutually disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $\text{meas}(\Gamma_1) > 0$. The body forces of density $\boldsymbol{f}_0$ act on Ω and surface tractions of density $\boldsymbol{f}_2$ act on Γ_2. The body is clamped on Γ_1, so the displacement field vanishes there. The potential contact surface Γ_3 is a part where the body may come in contact with an obstacle. We also assume that the contact process is quasistatic and we study it in the time interval $[0, T]$. The classical formulation of the contact problem can be written as follows.

Problem P *Find a displacement field* $\boldsymbol{u} : \Omega \times (0, T) \to \mathbb{R}^d$*, a stress field* $\boldsymbol{\sigma} : \Omega \times (0, T) \to \mathbb{S}^d$ *such that*

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t))) \quad \text{in} \quad \Omega \times (0, T), \tag{5}$$

$$\text{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in} \quad \Omega \times (0, T), \tag{6}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_1 \times (0, T), \tag{7}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on} \quad \Gamma_2 \times (0, T), \tag{8}$$

$$\left.\begin{array}{c} \sigma_\nu(t) = \sigma_\nu^1(t) + \sigma_\nu^2(t), \\ -\sigma_\nu^1(t) \in \partial_{Cl}\, j_\nu(u_\nu(t)), \\ u_\nu(t) \le g,\ \sigma_\nu^2(t) \le 0, \\ \sigma_\nu^2(t)(u_\nu(t) - g) = 0, \end{array}\right\} \quad \text{on} \quad \Gamma_3 \times (0, T), \tag{9}$$

$$\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_3 \times (0, T), \tag{10}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0\ ,\ \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \quad \text{in} \quad \Omega. \tag{11}$$

Equation (5) represents the viscoplastic constitutive law introduced in Section 1. Equation (6) is the normalized equilibrium equation for quasistatic process. Conditions (7) and (8) are displacement and traction boundary conditions, respectively. The contact problem is frictionless and is represented by boundary condition (10). The initial conditions are given by (11).

Boundary condition (9) is used to model the contact of the body and a foundation made of a rigid body covered by a layer of an elastic material with thickness $g > 0$, making the normal stress σ_ν on the contact surface to be split into two parts, σ_ν^1 and σ_ν^2. The first part σ_ν^1 describes the deformability with a normal compliance condition, governed by the subdifferential of a nonconvex potential j. The second part σ_ν^2 describes the rigidity of the obstacle with the Signorini unilateral contact condition. Note that the penetration is allowed but is restricted by the relation $u_\nu \le g$. When there is penetration and the normal displacement does not reach the bound g, the contact is under normal compliance condition $-\sigma_\nu \in \partial_{Cl} j_\nu(u_\nu)$. Due to the nonmonotonicity of $\partial_{Cl} j_\nu$, the condition can be used to describe the hardening or the softening phenomena of the foundation. Further examples and interpretations about the nonmonotone normal compliance condition can be found in [21].

In order to study Problem P, we need the following hypotheses on the data.

$$\begin{cases} \text{H}(\mathcal{E}):\ \text{The elasticity tensor } \mathcal{E} = (e_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d \text{ satisfies:} \\ \text{(i) } e_{ijkl} = e_{jikl} = e_{klij} \in L^\infty(\Omega),\ 1 \le i,j,k,l \le d; \\ \text{(ii) There exists } m_{\mathcal{E}} > 0 \text{ such that } \mathcal{E}\boldsymbol{\tau} : \boldsymbol{\tau} \ge m_{\mathcal{E}} \|\boldsymbol{\tau}\|^2 \text{ for all } \boldsymbol{\tau} \in \mathbb{S}^d \text{ and a.e. in } \Omega. \end{cases}$$

$$\begin{cases} \text{H}(\mathcal{G}):\ \text{The constitutive function } \mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d \text{ satisfies:} \\ \text{(i) There exists } L_{\mathcal{G}} > 0 \text{ such that } \|\mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{G}} \\ \quad (\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|) \text{ for all } \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega; \\ \text{(ii) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d; \\ \text{(iii) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0}) \text{ belongs to } Q. \end{cases}$$

H($\boldsymbol{f}$): The densities of body forces and surface tractions satisfy:

$$\boldsymbol{f}_0 \in C(0,T; L^2(\Omega; \mathbb{R}^d)), \quad \boldsymbol{f}_2 \in C(0,T; L^2(\Gamma_2; \mathbb{R}^d)).$$

(H_0): The initial values $\boldsymbol{u}_0 \in U, \boldsymbol{\sigma}_0 \in Q$, where U is the set of admissible displacements, i.e.

$$U = \{\boldsymbol{v} \in V : v_\nu \le g \text{ on } \Gamma_3\}.$$

Finally, the normal superpotential $j_\nu : \Gamma_3 \times \mathbb{R}^d \to \mathbb{R}$ satisfies H(j) given in Section 2.

We now set Problem P in its variational form. Take $\boldsymbol{v} \in U$ and $t \in (0,T)$. Using Equation (6) and the Green formula, we deduce that

$$\int_\Omega \boldsymbol{\sigma}(t) : (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))dx = \int_\Omega \boldsymbol{f}_0(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t))dx + \int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t))d\Gamma.$$

Since

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t)) = \sigma_\nu(t)(v_\nu - u_\nu(t)) + \boldsymbol{\sigma}_\tau(t)(\boldsymbol{v}_\tau - \boldsymbol{u}_\tau(t))$$

on $\Gamma_3 \times (0,T)$, split the surface integral over Γ_1, Γ_2 and Γ_3 and take into account condition (8), condition (10) and the fact that $\boldsymbol{v} - \boldsymbol{u}(t) = \boldsymbol{0}$ a.e. on Γ_1. Then we have

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma}(t) : (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))dx &= \int_\Omega \boldsymbol{f}_0(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t))dx \\ &+ \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t))d\Gamma + \int_{\Gamma_3} \sigma_\nu(t)(v_\nu - u_\nu(t))d\Gamma. \end{aligned}$$

From Condition (9), we obtain

$$\begin{aligned} -\sigma_\nu(t)(v_\nu - u_\nu(t)) &= -\sigma_\nu^1(t)(v_\nu - u_\nu(t)) - \sigma_\nu^2(t)(v_\nu - u_\nu(t)) \\ &\le j_\nu^0(u_\nu(t); v_\nu - u_\nu(t)) - \sigma_\nu^2(t)(v_\nu - g) - \sigma_\nu^2(t)(g - u_\nu(t)) \\ &\le j_\nu^0(u_\nu(t); v_\nu - u_\nu(t)). \end{aligned}$$

Therefore,

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma}(t) : (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))dx + \int_{\Gamma_3} j_\nu^0(u_\nu(t); v_\nu - u_\nu(t))d\Gamma \\ \ge \int_\Omega \boldsymbol{f}_0(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t))dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t))d\Gamma. \end{aligned}$$

We further define the function $\boldsymbol{f} : (0,T) \to V$ by

$$(\boldsymbol{f}(t),\boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t)\cdot\boldsymbol{v}dx + \int_{\Gamma_2}\boldsymbol{f}_2(t)\cdot\boldsymbol{v}d\Gamma \quad \text{for all } \boldsymbol{v}\in V,\ t\in(0,T).$$

Together with the initial conditions and the integral of Equation (5), we have the variational formulation of Problem P.

Problem P_V. *Find a displacement field* $\boldsymbol{u}:(0,T)\to U$*, a stress field* $\boldsymbol{\sigma}:(0,T)\to Q$ *such that* $\boldsymbol{u}(0)=\boldsymbol{u}_0$, $\boldsymbol{\sigma}(0)=\boldsymbol{\sigma}_0$ *and*

$$\boldsymbol{\sigma}(t) = \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \tag{12}$$

$$(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + \int_{\Gamma_3} j_\nu^0(u_\nu(t);v_\nu-u_\nu(t))d\Gamma \geq (\boldsymbol{f}(t),v-u(t))_V \tag{13}$$

hold for all $v\in U$ *and all* $t\in(0,T)$.

For Problem P_V, we have the following existence and uniqueness result.

Theorem 3. *Assume* $H(\mathcal{E})$*,* $H(\mathcal{G})$*,* $H(f)$*,* (H_0)*,* $H(j)$ *and*

$$m_{\mathcal{E}} > \max\{\sqrt{3}\widetilde{c_1},\widetilde{m_j}\}\|\gamma\|^2. \tag{14}$$

Then Problem P_V *has a unique solution with the following regularity*

$$\boldsymbol{u}\in C(0,T;U),\quad \boldsymbol{\sigma}\in C(0,T;Q). \tag{15}$$

The proof of Theorem 3 is based on two lemmas shown below. Since the arguments used are similar to those applied in [5, 16, 17] and the modifications are straightforward, we skip the proof of them. We assume that the conditions in Theorem 3 are all satisfied in the following.

Lemma 4. *For each* $\boldsymbol{u}\in C(0,T;V)$*, there exists a unique function* $\mathcal{M}\boldsymbol{u}\in C(0,T;Q)$ *such that*

$$\mathcal{M}\boldsymbol{u}(t) = \int_0^t \mathcal{G}\big(\mathcal{M}\boldsymbol{u}(s)+\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)),\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\big)ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(u_0) \tag{16}$$

for all $t\in(0,T)$*. Moreover, the operator* $\mathcal{M}:C(0,T;V)\to C(0,T;Q)$ *is a history-dependent operator, i.e. there exists* $c>0$ *which only depends on* $d,\mathcal{G},\mathcal{E}$ *such that*

$$\|\mathcal{M}\boldsymbol{u}(t)-\mathcal{M}\boldsymbol{v}(t)\|_Q \leq c\int_0^t \|\boldsymbol{u}(s)-\boldsymbol{v}(s)\|_V ds \tag{17}$$

for all $\boldsymbol{u},\boldsymbol{v}\in C(0,T;V)$ *and all* $t\in(0,T)$.

Next, we use the operator $\mathcal{M}$ found in Lemma 4 to derive the following equivalence.

Lemma 5. *Let* $(\boldsymbol{u},\boldsymbol{\sigma})$ *satisfy (15). Then* $(\boldsymbol{u},\boldsymbol{\sigma})$ *is a solution to Problem* P_V *if and only if*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{M}\boldsymbol{u}(t), \tag{18}$$

$$\begin{aligned}&(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (\mathcal{M}\boldsymbol{u}(t),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q\\ &\quad+\int_{\Gamma_3} j_\nu^0(u_\nu(t);v_\nu-u_\nu(t))d\Gamma \geq (\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}(t))_V \quad \forall \boldsymbol{v}\in U,\end{aligned} \tag{19}$$

for all $t\in(0,T)$.

Now we proceed to prove Theorem 3.

Proof. First, we define operator $A:V\to V^*$ and function $\varphi:Q\times U\to\mathbb{R}$ by

$$\begin{aligned}\langle A\boldsymbol{u},\boldsymbol{v}\rangle_{V^*\times V} &= (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \quad \text{for all } \boldsymbol{u},\boldsymbol{v}\in V,\\ \varphi(\boldsymbol{\sigma},\boldsymbol{v}) &= (\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \qquad \text{for all } \boldsymbol{\sigma}\in Q,\ \boldsymbol{v}\in U.\end{aligned}$$

With this notation, we consider the problem of finding a function $\boldsymbol{u} : (0,T) \to U$ such that the inequality

$$\langle A\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle_{V^*\times V} + \varphi(\mathcal{M}\boldsymbol{u}(t), \boldsymbol{v}) - \varphi(\mathcal{M}\boldsymbol{u}(t), \boldsymbol{u}(t)) + \int_{\Gamma_3} j_\nu^0(u_\nu(t); v_\nu - u_\nu(t))d\Gamma \geq \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle_{V^*\times V} \tag{20}$$

holds for all $\boldsymbol{v} \in U$ and all $t \in (0,T)$.

In order to solve the inequality, we employ Lemma 1 with $K = U$ and $Y = Q$. We use assumption H($\mathcal{E}$)(i) to get that

$$|\langle A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{w}\rangle_{V^*\times V}| \leq d \max_{i,j,k,l} \|e_{ijkl}\|_{L^\infty(\Omega)} \|\boldsymbol{u} - \boldsymbol{v}\|_V \|\boldsymbol{w}\|_V$$

for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$. Hence,

$$\|A\boldsymbol{u} - A\boldsymbol{v}\|_V \leq c\|\boldsymbol{u} - \boldsymbol{v}\|_V.$$

Moreover, from assumption H($\mathcal{E}$)(ii), we have

$$\langle A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v}\rangle \geq m_{\mathcal{E}}\|\boldsymbol{u} - \boldsymbol{v}\|_V^2.$$

Thus, condition H(A) is satisfied.

By definition of operator $\boldsymbol{f}$ and assumption H($\boldsymbol{f}$), we can deduce that $\boldsymbol{f}$ has required regularity. H($\mathcal{M}$), H(j) and H(φ)(i)(ii) can also be satisfied.

Next, since

$$\begin{aligned}\varphi(\boldsymbol{\sigma}_1, \boldsymbol{u}_2) - \varphi(\boldsymbol{\sigma}_1, \boldsymbol{u}_1) + \varphi(\boldsymbol{\sigma}_2, \boldsymbol{u}_1) - \varphi(\boldsymbol{\sigma}_2, \boldsymbol{u}_2) \\ = \big(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}(\boldsymbol{u}_2) - \boldsymbol{\varepsilon}(\boldsymbol{u}_1)\big)_Q \\ \leq \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_Q \, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V\end{aligned}$$

for all $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in Q, \boldsymbol{u}_1, \boldsymbol{u}_2 \in V$, we conclude that H($\varphi$) holds.

It is now a consequence of Lemma 1 that there exists a unique function $\boldsymbol{u} \in C(0,T;U)$ which solves the inequality (20), then solves (19) at the same time. Define $\boldsymbol{\sigma}$ by (18), it follows that $(\boldsymbol{u}, \boldsymbol{\sigma})$ is the unique solution with regularity (15) satisfying (18) and (19). As a result of Lemma 5, Theorem 3 is proved. □

4. A fully discrete scheme and error estimates

In this section, we introduce a fully discrete scheme for Problem P_V and provide a result on error estimates.

Let $V^h \subset V, Q^h \subset Q$ be finite-dimensional spaces which approximate the spaces V and Q. We use $U^h := V^h \bigcap U$ to approximate the convex set U. Here $h > 0$ is a spatial discretization parameter. We assume that

$$\boldsymbol{\varepsilon}(V^h) \subset Q^h, \tag{21}$$

which is very natural and is valid as long as the polynomial degree for space V^h is at most one higher than that for space Q^h.

Let $\mathcal{P}_{Q^h} : Q \to Q^h$ be the orthogonal projection defined through the relation

$$(\mathcal{P}_{Q^h}\boldsymbol{\tau}, \boldsymbol{\tau}^h)_Q = (\boldsymbol{\tau}, \boldsymbol{\tau}^h)_Q \quad \forall \boldsymbol{\tau} \in Q,\ \boldsymbol{\tau}^h \in Q^h.$$

The orthogonal projection has useful non-expansive property: $\|\mathcal{P}_{Q^h}\boldsymbol{\tau}\|_Q \leq \|\boldsymbol{\tau}\|_Q$ for all $\boldsymbol{\tau} \in Q$.

We use a possibly non-uniform partition of the time interval $[0,T] : 0 = t_0 < t_1 < t_2 < \cdots < t_N = T$. We denote the time step size $k_n = t_n - t_{n-1}$ for $n = 1, \cdots, N$ and the maximal time step size $k = \max_n k_n$. For a continuous function $g = g(t)$, we write $g_n = g(t_n)$. Everywhere in the sequel, c will denote a general positive constant independent of discretization parameters h and k. The values of c may change in different inequalities.

Choose $\boldsymbol{u}_0^h \in U^h$ and $\boldsymbol{\sigma}_0^h \in Q^h$ to be the approximation of initial values $\boldsymbol{u}_0 \in U$ and $\boldsymbol{\sigma}_0 \in Q$. We construct the following fully discrete approximation scheme for Problem P_V.

Problem P_V^{hk} *Find* $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset U^h$ *and* $\boldsymbol{\sigma}^{hk} = \{\boldsymbol{\sigma}_n^{hk}\}_{n=0}^N \subset Q^h$ *such that* $\boldsymbol{u}_0^{hk} = \boldsymbol{u}_0^h, \boldsymbol{\sigma}_0^{hk} = \boldsymbol{\sigma}_0^h$ *and for* $n = 1, 2, \cdots, N$,

$$\boldsymbol{\sigma}_n^{hk} = \mathcal{P}_{Q^h}\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}) + \sum_{j=1}^{n} k_j \mathcal{P}_{Q^h}\mathcal{G}(\boldsymbol{\sigma}_{j-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk})) + \boldsymbol{\sigma}_0^{hk} - \mathcal{P}_{Q^h}\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^{hk}), \tag{22}$$

$$(\boldsymbol{\sigma}_n^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \boldsymbol{u}_n^{hk}))_Q + \int_{\Gamma_3} j_\nu^0(u_{n\nu}^{hk}; v_\nu^h - u_{n\nu}^{hk})d\Gamma \geq (\boldsymbol{f}_n, \boldsymbol{v}^h - \boldsymbol{u}_n^{hk})_V \, \forall \boldsymbol{v}^h \in U^h. \tag{23}$$

For the existence and uniqueness result of the fully-discrete scheme P_V^{hk}, we need to prove that, with $\{\boldsymbol{u}_j^{hk}\}_{j\leq n-1}$ known, $\boldsymbol{u}_n^{hk}$ is uniquely determined by (22) and (23). In fact, we only need to consider an elliptic variational-hemivariational inequality: find $\boldsymbol{u}_n^{hk} \in U^h$ such that for all $\boldsymbol{v}^h \in U^h$, there holds

$$\begin{aligned}&(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \boldsymbol{u}_n^{hk}))_Q + \textstyle\int_{\Gamma_3} j_\nu^0(u_{n\nu}^{hk}; v_\nu^h - u_{n\nu}^{hk})d\Gamma \geq (\boldsymbol{f}_n, \boldsymbol{v}^h - \boldsymbol{u}_n^{hk})_V \\ &-\Big(\sum_{j=1}^{n} k_j \mathcal{G}(\boldsymbol{\sigma}_{j-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \boldsymbol{u}_n^{hk})\Big)_Q - (\boldsymbol{\sigma}_0^{hk} - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \boldsymbol{u}_n^{hk}))_Q.\end{aligned}$$

From [15], we conclude that this inequality has a unique weak solution, which implies the unique solvability of our Problem P_V^{hk}.

In order to derive error estimates, we first prove the following theorem.

Theorem 6. *Let* $\{\boldsymbol{u}_n^{hk}\}_{n=0}^N$ *and* $\{\boldsymbol{\sigma}_n^{hk}\}_{n=0}^N$ *be the unique solution of Problem* P_V^{hk}. *There exists a constant* $c > 0$ *such that*

$$\|\boldsymbol{\sigma}_n^{hk}\|_Q + \|\boldsymbol{u}_n^{hk}\|_V \leq c, \quad 0 \leq n \leq N.$$

Proof. From H($\mathcal{G}$), we have

$$\begin{aligned}&\|\mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}) - \mathcal{G}(\mathbf{0}, \mathbf{0})\|_Q \leq L_{\mathcal{G}}(\|\boldsymbol{\sigma}\|_Q + \|\boldsymbol{\varepsilon}\|_Q) \\ \Rightarrow &\|\mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon})\|_Q \leq \|\mathcal{G}(\mathbf{0}, \mathbf{0})\|_Q + L_{\mathcal{G}}(\|\boldsymbol{\sigma}\|_Q + \|\boldsymbol{\varepsilon}\|_Q).\end{aligned}$$

So we now combine (22), H($\mathcal{E}$)(i) and the definition of the norm in space V to obtain

$$\begin{aligned}\|\boldsymbol{\sigma}_n^{hk}\|_Q &\leq \|\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk})\|_Q + \sum_{j=1}^{n} k_j \|\mathcal{G}(\boldsymbol{\sigma}_{j-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk}))\|_Q + \|\boldsymbol{\sigma}_0^{hk}\|_Q + \|\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^{hk})\|_Q \\ &\leq c\|\boldsymbol{u}_n^{hk}\|_V + c\sum_{j=1}^{n} k_j(\|\boldsymbol{\sigma}_{j-1}^{hk}\|_Q + \|\boldsymbol{u}_{j-1}^{hk}\|_V) + c\|\mathcal{G}(\mathbf{0}, \mathbf{0})\|_Q + \|\boldsymbol{\sigma}_0^{hk}\|_Q + c\|\boldsymbol{u}_0^{hk}\|_V \\ &\leq c\|\boldsymbol{u}_n^{hk}\|_V + c\sum_{j=1}^{n} k_j(\|\boldsymbol{\sigma}_{j-1}^{hk}\|_Q + \|\boldsymbol{u}_{j-1}^{hk}\|_V) + c.\end{aligned}$$

Take $\boldsymbol{v}^h = \mathbf{0} \in U^h$ and use (22) in (23):

$$\begin{aligned}(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}))_Q &\leq \int_{\Gamma_3} j_\nu^0(u_{n\nu}^{hk}; -u_{n\nu}^{hk})d\Gamma + (\boldsymbol{f}_n, \boldsymbol{u}_n^{hk})_V \\ &-(\boldsymbol{\sigma}_0^{hk} - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}))_Q - (\sum_{j=1}^{n} k_j \mathcal{G}(\boldsymbol{\sigma}_{j-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}))_Q.\end{aligned}$$

Note that, from H(j)(iii)(iv) and (3), we can obtain

$$\begin{aligned}\int_{\Gamma_3} j_\nu^0(u_{n\nu}^{hk}; -u_{n\nu}^{hk})d\Gamma &\leq \widetilde{m}_j\|\gamma\|^2\|u_{n\nu}^{hk}\|_V^2 - \int_{\Gamma_3} j_\nu^0(0; u_{n\nu}^{hk})d\Gamma \\ &\leq \widetilde{m}_j\|\gamma\|^2\|\boldsymbol{u}_n^{hk}\|_V^2 + \widetilde{c}_0\sqrt{\text{meas}(\Gamma_3)}\|\gamma\|\|\boldsymbol{u}_n^{hk}\|_V.\end{aligned}$$

Further combined with H($\mathcal{G}$)(i), we have

$$\begin{aligned}m_{\mathcal{G}}\|\boldsymbol{u}_n^{hk}\|_V^2 \leq \widetilde{m}_j\|\gamma\|^2\|\boldsymbol{u}_n^{hk}\|_V^2 + \widetilde{c}_0\sqrt{\text{meas}(\Gamma_3)}\|\gamma\|\|\boldsymbol{u}_n^{hk}\|_V + \|\boldsymbol{f}_n\|_V\|\boldsymbol{u}_n^{hk}\|_V \\ +c\sum_{j=1}^n k_j(\|\boldsymbol{\sigma}_{j-1}^{hk}\|_Q + \|\boldsymbol{u}_{j-1}^{hk}\|_V)\|\boldsymbol{u}_n^{hk}\|_V + c\|\mathcal{G}(\mathbf{0},\mathbf{0})\|_Q\|\boldsymbol{u}_n^{hk}\|_V \\ +\|\boldsymbol{\sigma}_0^{hk} - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^{hk})\|_Q\|\boldsymbol{u}_n^{hk}\|_V.\end{aligned}$$

Taking into account (14), we can find the expression

$$\|\boldsymbol{u}_n^{hk}\|_V \leq c + c\sum_{j=1}^n k_j(\|\boldsymbol{\sigma}_{j-1}^{hk}\|_Q + \|\boldsymbol{u}_{j-1}^{hk}\|_V).$$

Thus,

$$\|\boldsymbol{\sigma}_n^{hk}\|_Q + \|\boldsymbol{u}_n^{hk}\|_V \leq c + c\sum_{j=1}^n k_j(\|\boldsymbol{\sigma}_{j-1}^{hk}\|_Q + \|\boldsymbol{u}_{j-1}^{hk}\|_V).$$

From Lemma 2, we conclude that there is a constant $c > 0$ such that

$$\|\boldsymbol{\sigma}_n^{hk}\|_Q + \|\boldsymbol{u}_n^{hk}\|_V \leq c. \qquad \square$$

Now we proceed to derive error estimates. Taking the unique solution of Problem P_V at time $t = t_n$, we have

$$\boldsymbol{\sigma}_n = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) + \int_0^{t_n}\mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \tag{24}$$

$$(\boldsymbol{\sigma}_n, \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}_n))_Q + \int_{\Gamma_3} j_\nu^0(u_{n\nu}; v_\nu - u_{n\nu})d\Gamma \geq (\boldsymbol{f}_n, \boldsymbol{v} - \boldsymbol{u}_n)_V \quad \forall \boldsymbol{v} \in U. \tag{25}$$

Subtracting (22) from (24), we obtain

$$\begin{aligned}&\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk} \\ =\ &\boldsymbol{\sigma}_n - \mathcal{P}_{Q^h}\boldsymbol{\sigma}_n + \mathcal{P}_{Q^h}\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk} \\ =\ &(I_Q - \mathcal{P}_{Q^h})(\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_0) + \mathcal{P}_{Q^h}\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}) + [\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^{hk} - \mathcal{P}_{Q^h}\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0 - \boldsymbol{u}_0^{hk})] \\ &+\mathcal{P}_{Q^h}[\int_0^{t_n}\mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))ds - \sum_{j=1}^n k_j\mathcal{G}(\boldsymbol{\sigma}_{j-1}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}))] \\ &+\mathcal{P}_{Q^h}\sum_{j=1}^n k_j[\mathcal{G}(\boldsymbol{\sigma}_{j-1}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1})) - \mathcal{G}(\boldsymbol{\sigma}_{j-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk}))],\end{aligned}$$

where I_Q is the identity operator defined on Q. Denote

$$I_n = \|\int_0^{t_n}\mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))ds - \sum_{j=1}^n k_j\mathcal{G}(\boldsymbol{\sigma}_{j-1}, \boldsymbol{\varepsilon}(u_{j-1}))\|_Q, \quad n = 1, \cdots, N,$$

and

$$e_n = \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}\|_Q + \|\boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q \quad n = 0, \cdots, N.$$

Using assumptions H($\mathcal{E}$)(i), H($\mathcal{G}$)(i) and the property of projection, we can derive the following inequality:

$$\|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\|_Q \le \|(I_Q-\mathcal{P}_{Q^h})(\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_0)\|_Q + c\|\boldsymbol{\varepsilon}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q + ce_0 + I_n + c\sum_{j=1}^{n} k_j(\|\boldsymbol{\sigma}_{j-1}-\boldsymbol{\sigma}_{j-1}^{hk}\|_Q + \|\boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}-\boldsymbol{u}_{j-1}^{hk})\|_Q). \tag{26}$$

We combine (24) and (25) with $\boldsymbol{v}=\boldsymbol{u}_n^{hk}$ to obtain

$$(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n),\boldsymbol{\varepsilon}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}))_Q \le (\boldsymbol{f}_n,\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})_V + \int_{\Gamma_3} j_\nu^0(u_{n\nu};u_{n\nu}^{hk}-u_{n\nu})d\Gamma + \Big(\int_0^{t_n}\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))ds,\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}-\boldsymbol{u}_n)\Big)_Q + \Big(\boldsymbol{\sigma}_0-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0),\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}-\boldsymbol{u}_n)\Big)_Q. \tag{27}$$

We combine (22) and (23) with any $\boldsymbol{v}^h=\boldsymbol{v}_n^h\in U^h$ to obtain

$$\begin{aligned}
&-(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}))_Q \\
&= -(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{u}_n-\boldsymbol{v}_n^{h}))_Q - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{v}_n^h-\boldsymbol{u}_n^{hk}))_Q \\
&\le -(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{u}_n-\boldsymbol{v}_n^{h}))_Q - (\boldsymbol{f}_n,\boldsymbol{v}_n^h-\boldsymbol{u}_n^{hk})_V + \int_{\Gamma_3} j_\nu^0(u_{n\nu}^{hk};v_{n\nu}^h-u_{n\nu}^{hk})d\Gamma \\
&\quad + \Big(\sum_{j=1}^{n} k_j\mathcal{G}(\boldsymbol{\sigma}_{j-1}^{hk},\boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk})),\boldsymbol{\varepsilon}(\boldsymbol{v}_n^h-\boldsymbol{u}_n^{hk})\Big)_Q + \Big(\boldsymbol{\sigma}_0^{hk}-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{v}_n^h-\boldsymbol{u}_n^{hk})\Big)_Q.
\end{aligned} \tag{28}$$

Combining (24), (27) and (28), we have the following inequality by using assumption H($\mathcal{E}$)(ii) and (4).

$$\begin{aligned}
& m_{\mathcal{E}}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2 \\
\le\ & (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}))_Q \\
\le\ & (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{u}_n-\boldsymbol{v}_n^{h}))_Q + \int_{\Gamma_3} j_\nu^0(u_{n\nu}^{hk};v_{n\nu}^h-u_{n\nu})\ d\Gamma \\
& + \int_{\Gamma_3} j_\nu^0(u_{n\nu}^{hk};u_{n\nu}-u_{n\nu}^{hk})\ d\Gamma + \int_{\Gamma_3} j_\nu^0(u_{n\nu};u_{n\nu}^{hk}-u_{n\nu})\ d\Gamma \\
& + \big(\ \boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0^{hk}-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0-\boldsymbol{u}_0^{hk})\ ,\ \boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}-\boldsymbol{v}_n^h)\ \big)_Q \\
& + \Big(\ \int_0^{t_n}\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))ds - \sum_{j=1}^{n} k_j\mathcal{G}(\boldsymbol{\sigma}_{j-1}^{hk},\boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk}))\ ,\ \boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}-\boldsymbol{v}_n^h)\ \Big)_Q \\
& + \big(\boldsymbol{f}_n,\boldsymbol{u}_n-\boldsymbol{v}_n^h)_V - (\boldsymbol{\sigma}_n,\boldsymbol{\varepsilon}(\boldsymbol{u}_n-\boldsymbol{v}_n^h)\big)_Q.
\end{aligned}$$

Note that, from H(j)(iii), (3) and Theorem 6, we have

$$\int_{\Gamma_3} j_\nu^0(u_{n\nu}^{hk};v_{n\nu}^h-u_{n\nu})d\Gamma \le c\int_{\Gamma_3}\|\gamma(\boldsymbol{v}_n^h-\boldsymbol{u}_n)\|_{\mathbb{R}^d}d\Gamma \le c\sqrt{\mathrm{meas}(\Gamma_3)}\|\boldsymbol{v}_n^h-\boldsymbol{u}_n\|_{L^2(\Gamma_3;\mathbb{R}^d)}.$$

Further combined with assumptions H($\mathcal{E}$)(i) and H(j)(iv), we have

$$\begin{aligned} & m_{\mathcal{E}}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2 \\ \leq\ & c\ \|\boldsymbol{\varepsilon}(\boldsymbol{u}_n-\boldsymbol{v}_n^h)\|_Q\ \|\boldsymbol{\varepsilon}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q+\widetilde{m}_j\|\gamma\|^2\ \|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V^2 \\ & +c\|\boldsymbol{v}_n^h-\boldsymbol{u}_n\|_{L^2(\Gamma_3;\mathbb{R}^d)}+(\boldsymbol{f}_n,\boldsymbol{u}_n-\boldsymbol{v}_n^h)_V-(\boldsymbol{\sigma}_n,\boldsymbol{\varepsilon}(\boldsymbol{u}_n-\boldsymbol{v}_n^h))_Q \\ & +c\ \big(\|\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0^{hk}\|_Q+\|\boldsymbol{\varepsilon}(\boldsymbol{u}_0-\boldsymbol{u}_0^{hk})\|_Q\big)\ \|\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}-\boldsymbol{v}_n^h)\|_Q \\ & +\Big(\int_0^{t_n}\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))ds-\sum_{j=1}^n k_j\mathcal{G}(\boldsymbol{\sigma}_{j-1}^{hk},\boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk}))\ ,\ \boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}-\boldsymbol{v}_n^h)\Big)_Q. \end{aligned}$$

By definition of the norm in Hilbert space V and assumption of H($\mathcal{G}$)(i), the above inequality can be rewritten as:

$$\begin{aligned} & (m_{\mathcal{E}}-\widetilde{m}_j\|\gamma\|^2)\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V^2 \\ \leq\ & c\|\boldsymbol{u}_n-\boldsymbol{v}_n^h\|_V\ \|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V+c\|\boldsymbol{v}_n^h-\boldsymbol{u}_n\|_{L^2(\Gamma_3;\mathbb{R}^d)}+(\boldsymbol{f}_n,\boldsymbol{u}_n-\boldsymbol{v}_n^h)_V \\ & -(\boldsymbol{\sigma}_n,\boldsymbol{\varepsilon}(\boldsymbol{u}_n-\boldsymbol{v}_n^h))_Q\ +\ ce_0\cdot\|\boldsymbol{u}_n^{hk}-\boldsymbol{v}_n^h\|_V+\ I_n\cdot\|\boldsymbol{u}_n^{hk}-\boldsymbol{v}_n^h\|_V \\ & +c\sum_{j=1}^n k_j\big(\|\boldsymbol{\sigma}_{j-1}-\boldsymbol{\sigma}_{j-1}^{hk}\|_Q+\|\boldsymbol{u}_{j-1}-\boldsymbol{u}_{j-1}^{hk}\|_V\big)\ \|\boldsymbol{u}_n^{hk}-\boldsymbol{v}_n^h\|_V. \end{aligned}$$

Denote $R_n(\boldsymbol{u}_n,\boldsymbol{v}_n^h)=(\boldsymbol{f}_n,\boldsymbol{u}_n-\boldsymbol{v}_n^h)_V-(\boldsymbol{\sigma}_n,\boldsymbol{\varepsilon}(\boldsymbol{u}_n-\boldsymbol{v}_n^h))_Q$. Using assumption (14) and taking some manipulations, we can show that

$$\begin{aligned} \|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V\leq c\Big[&\|\boldsymbol{u}_n-\boldsymbol{v}_n^h\|_V+|R_n(\boldsymbol{u}_n,\boldsymbol{v}_n^h)|^{1/2}+\|\boldsymbol{v}_n^h-\boldsymbol{u}_n\|_{L^2(\Gamma_3;\mathbb{R}^d)}^{1/2} \\ & +e_0+I_n+\sum_{j=1}^n k_j(\|\boldsymbol{\sigma}_{j-1}-\boldsymbol{\sigma}_{j-1}^{hk}\|_Q+\|\boldsymbol{u}_{j-1}-\boldsymbol{u}_{j-1}^{hk}\|_V)\Big]. \end{aligned} \tag{29}$$

For any $\boldsymbol{v}_n^h\in U^h$, we can use (26) and (29) to find the expression

$$\begin{aligned} & \|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\|_Q+\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V \\ \leq\ & c\Big[\|\boldsymbol{u}_n-\boldsymbol{v}_n^h\|_V+|R_n(\boldsymbol{u}_n,\boldsymbol{v}_n^h)|^{1/2}+\|\boldsymbol{v}_n^h-\boldsymbol{u}_n\|_{L^2(\Gamma_3;\mathbb{R}^d)}^{1/2}+e_0+I_n \\ & +\|(I_Q-\mathcal{P}_{Q^h})(\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_0)\|_Q\Big]+c\sum_{j=1}^n k_j\big(\|\boldsymbol{\sigma}_{j-1}-\boldsymbol{\sigma}_{j-1}^{hk}\|_Q+\|\boldsymbol{u}_{j-1}-\boldsymbol{u}_{j-1}^{hk}\|_V\big). \end{aligned} \tag{30}$$

Denote $g_n=\|\boldsymbol{u}_n-\boldsymbol{v}_n^h\|_V+|R_n(\boldsymbol{u}_n,\boldsymbol{v}_n^h)|^{1/2}+\|\boldsymbol{v}_n^h-\boldsymbol{u}_n\|_{L^2(\Gamma_3;\mathbb{R}^d)}^{1/2}+e_0+I_n+\|(I_Q-\mathcal{P}_{Q^h})(\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_0)\|_Q$. Then (30) can be rewritten as

$$e_n\leq cg_n+c\sum_{j=1}^n k_je_{j-1},\quad n=1,\cdots,N. \tag{31}$$

From Lemma 2, it implies that

$$\max_{1\leq n\leq N}(\|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\|_Q+\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V)\leq c\max_{1\leq n\leq N}g_n. \tag{32}$$

What's more, we give an estimate for the I_n:

$$I_n \leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|\mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t))) - \mathcal{G}(\boldsymbol{\sigma}_{j-1}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}))\|_Q dt \tag{33}$$
$$\leq c\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (\|\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}_{j-1}\|_Q + \|\boldsymbol{u}(t) - \boldsymbol{u}_{j-1}\|_V) dt$$
$$\leq c t_n k(\|\dot{\boldsymbol{\sigma}}\|_{L^\infty(0,T;Q)} + \|\dot{\boldsymbol{u}}\|_{L^\infty(0,T;V)}).$$

Combining (32) and (33), we have the following result.

THEOREM 7. *Let $(\boldsymbol{u}, \boldsymbol{\sigma})$ be the solution of the Problem P_V and $\{\boldsymbol{u}_n^{hk}, \boldsymbol{\sigma}_n^{hk}\}_{n=1}^N$ be the solution of Problem P_V^{hk}. Then we have the error estimate*

$$\max_{1\leq n\leq N} (\|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}\|_Q + \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V) \tag{34}$$
$$\leq c(\|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h\|_Q + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V) + ck\big(\|\dot{\boldsymbol{\sigma}}\|_{L^\infty(0,T;Q)} + \|\dot{\boldsymbol{u}}\|_{L^\infty(0,T;V)}\big)$$
$$+c \max_{1\leq n\leq N} \Bigg(\|(I_Q - \mathcal{P}_{Q^h})(\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_0)\|_Q + \inf_{\boldsymbol{v}_n^h \in V_h} \Big\{ \|\boldsymbol{u}_n - \boldsymbol{v}_n^h\|_V + |R_n(\boldsymbol{u}_n, \boldsymbol{v}_n^h)|^{1/2} + \|\boldsymbol{v}_n^h - \boldsymbol{u}_n\|_{L^2(\Gamma_3;\mathbb{R}^d)}^{1/2} \Big\} \Bigg).$$

Theorem 7 is the basis for error estimation. For brevity, we assume that Ω is a polygonal or polyhedral domain. let $\mathcal{T}^h$ be a regular family of finite element triangulations of $\overline{\Omega}$ into triangles or tetrahedrons. We now specify the finite element spaces V^h and Q^h. According to assumption (21), we naturally use continuous linear elements for the finite element space V^h and piecewise constants for Q^h.

THEOREM 8. *Assume g is concave on each line segment of Γ_3. Let $(\boldsymbol{u}, \boldsymbol{\sigma})$ be the solution of Problem P_V and $\{\boldsymbol{u}_n^{hk}, \boldsymbol{\sigma}_n^{hk}\}_{n=0}^N$ be the solution of Problem P_V^{hk}. Assume*

$$\boldsymbol{u} \in C(0,T; H^2(\Omega;\mathbb{R}^d)), \quad \boldsymbol{\sigma} \in C(0,T; H^1(\Omega;\mathbb{S}^d)),$$
$$\boldsymbol{u}|_{\Gamma_{3,i}} \in C(0,T; H^2(\Gamma_{3,i};\mathbb{R}^d)), \quad 1 \leq i \leq I.$$

The initial values $\boldsymbol{u}_0^h \in U^h$ and $\boldsymbol{\sigma}_0^h \in Q^h$ are chosen in such a way that

$$\|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h\|_Q \leq ch, \quad \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V \leq ch.$$

Then we have the optimal order error estimate

$$\max_{1\leq n\leq N} (\|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}\|_Q + \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V) \leq c(k+h). \tag{35}$$

Proof. For $t \in [0, T]$, let $\Pi^h \boldsymbol{u}(t) \in V^h$ be the piecewise linear interpolant of $\boldsymbol{u}(t)$. Then we have the error estimates ([6]):

$$\|\boldsymbol{u}(t) - \Pi^h \boldsymbol{u}(t)\|_V \leq ch\|\boldsymbol{u}(t)\|_{H^2(\Omega;\mathbb{R}^d)},$$
$$\|\boldsymbol{u}(t) - \Pi^h \boldsymbol{u}(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)} \leq ch^2 \sum_{i=1}^{I} \|\boldsymbol{u}(t)\|_{H^2(\Gamma_{3,i};\mathbb{R}^d)},$$
$$\|(I_Q - \mathcal{P}_{Q^h})\boldsymbol{\sigma}(t)\|_Q \leq ch\|\boldsymbol{\sigma}(t)\|_{H^1(\Omega;\mathbb{S}^d)}.$$

Thus,

$$\|(I_Q - \mathcal{P}_{Q^h})(\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_0)\|_Q \leq ch\|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_0\|_{H^1(\Omega;\mathbb{S}^d)}.$$

Since g is concave on each line segment of Γ_3, we have $\Pi^h \boldsymbol{u}(t) \in U$. Using integration by parts and the trace theorem, we have

$$\begin{aligned} & |R(t; \boldsymbol{u}(t), \Pi^h \boldsymbol{u}(t))| \\ = \ & |(\boldsymbol{f}(t), \boldsymbol{u}(t) - \Pi^h \boldsymbol{u}(t))_V - (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(u(t) - \Pi^h \boldsymbol{u}(t)))_Q| \\ = \ & |\int_{\Gamma_3} (\boldsymbol{\sigma}(t)\boldsymbol{\nu}) \cdot (\boldsymbol{u}(t) - \Pi^h \boldsymbol{u}(t)) d\Gamma| \\ \leq \ & c\|\boldsymbol{\sigma}(t)\|_{H^1(\Omega;\mathbb{S}^d)} \|\boldsymbol{u}(t) - \Pi^h \boldsymbol{u}(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)}. \end{aligned}$$

Combining above inequalities and (34), we have the error estimate (35). □

5. Numerical results

In this section, we provide numerical simulation results on Problem P_V^{hk}. We use iterative scheme based on primal-dual active set approach to solve the discrete problem. More details can be found in [3].

The physical setting of the contact is depicted in Figure 1. Let $\Omega = (0, L_1) \times (0, L_2)$ be the rectangle with a boundary Γ which is divided into three parts

$$\Gamma_1 = \{0\} \times [0, L_1], \Gamma_2 = (\{L_1\} \times (0, L_2]) \cup ((0, L_1) \times \{L_2\}), \Gamma_3 = (0, L_1] \times \{0\}.$$

The domain Ω represents the cross section of a three-dimensional viscoplastic body subjected to the action of tractions in such a way that a plane stress hypothesis is assumed.

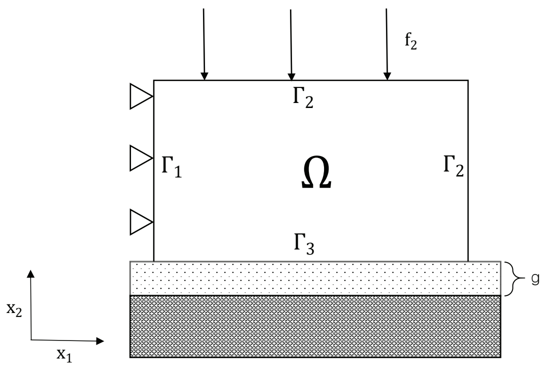


FIGURE 1. Initial configuration of the two-dimensional example.

The contact boundary conditions on Γ_3 are characterized as follows:

$$-\sigma_\nu^1 = \begin{cases} 0 & \text{if} \quad u_\nu < 0, \\ c_\nu^1 u_\nu & \text{if} \quad u_\nu \in (0, u_\nu^1], \\ c_\nu^1 u_\nu^1 + c_\nu^2 (u_\nu - u_\nu^1) & \text{if} \quad u_\nu \geq u_\nu^1, \end{cases} \tag{36}$$

$$u_\nu \leq g, \quad \sigma_\nu^2 \leq 0, \quad (u_\nu - g)\sigma_\nu^2 = 0, \tag{37}$$

$$\boldsymbol{\sigma}_\tau = \mathbf{0}. \tag{38}$$

To better appreciate the nonmonotone character of the normal response, we show the dependence of $-\sigma_\nu$ as a function of the normal displacement u_ν related to the relations (36) and (37), see Figure 2.

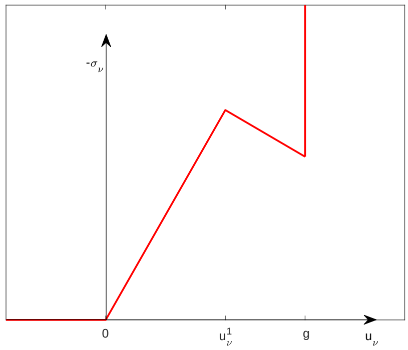
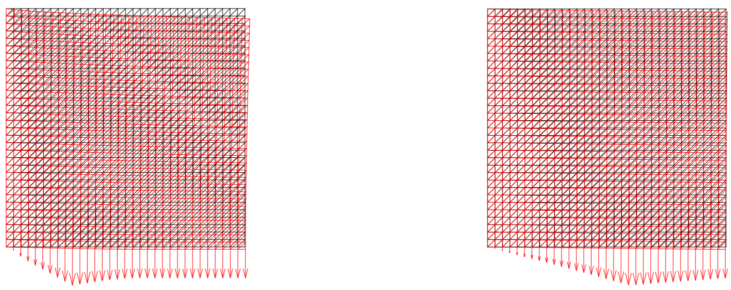


FIGURE 2. Dependence of $-\sigma_\nu$ on u_ν.



(a) with plastic term (b) without plastic term

FIGURE 3. Deformed meshes and contact interface forces with and without plastic term.

Remark. From (37) and Figure 2, we use a multivalued normal compliance response on Γ_3. In fact, $-\sigma_\nu$ take the value from $[\, c_\nu^1 u_\nu^1 + c_\nu^2(g - u_\nu^1),\ +\infty\,]$ when $u_\nu = g$. In order to better observe the nonmonotone behavior on the boundary, we made some modifications when plotting the normal compliance response $-\sigma_\nu$. If u_ν reaches g, no matter what the value of $-\sigma_\nu$ is, we plot it as $c_\nu^1 u_\nu^1 + c_\nu^2(g - u_\nu^1)$.

The elasticity tensor $\mathcal{E}$ has the following form:

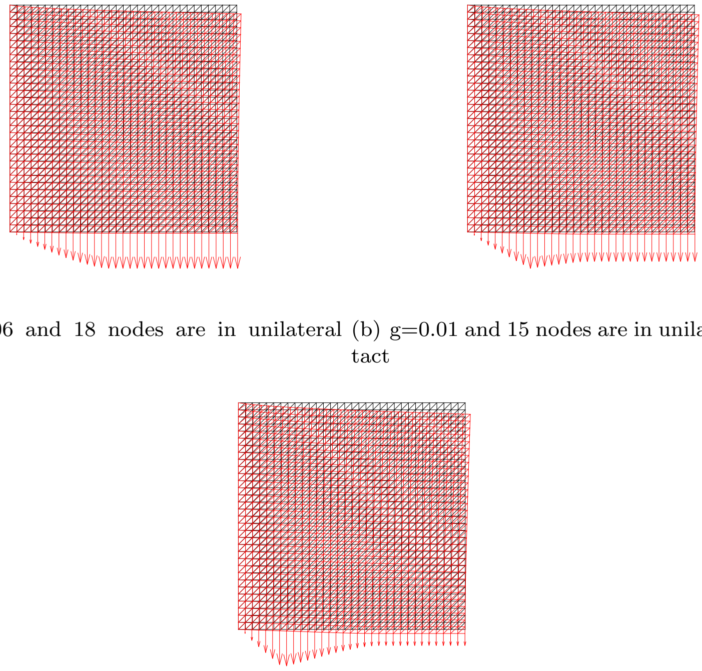
$$(\mathcal{E}\tau)_{ij} = \frac{E\kappa}{1-\kappa^2}(\tau_{11} + \tau_{22})\delta_{ij} + \frac{E}{1+\kappa}\tau_{ij}, \quad 1 \le i,j \le 2.$$

The coefficients E and κ denote the Young's modulus and the Poisson's ratio of the material and δ_{ij} denotes the Kronecker symbol. The viscoplastic function $\mathcal{G}$ is assumed to be of the classical Perzyna type ([9]):

$$\mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})) = -\frac{1}{2\lambda}\mathcal{E}(\boldsymbol{\sigma} - \mathcal{P}_K \boldsymbol{\sigma}),$$

where $\lambda > 0$ is the viscosity coefficient and $\mathcal{P}_K$ is the orthogonal projection operator (with respect to the norm $\|\boldsymbol{\tau}\| = (\mathcal{E}\boldsymbol{\tau}, \boldsymbol{\tau})^{1/2}$) over the convex subset $K \subset \mathbb{S}^2$. The subset K is defined by

$$K = \{\, \boldsymbol{\tau} \in \mathbb{S}^2 \mid \tau_{11}^2 + \tau_{22}^2 - \tau_{11}\tau_{22} + 3\tau_{12}^2 \le \sigma_Y^2 \,\},$$



(a) g=0.006 and 18 nodes are in unilateral contact

(b) g=0.01 and 15 nodes are in unilateral contact

(c) g=0.018 and 13 nodes are in unilateral contact

FIGURE 4. Deformed meshes and contact interface forces for different values of g.

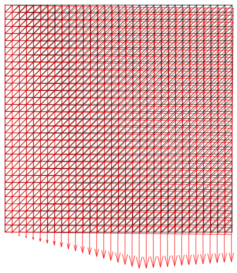
σ_Y being the uniaxial yield stress.

The following parameters are used in numerical experiments:

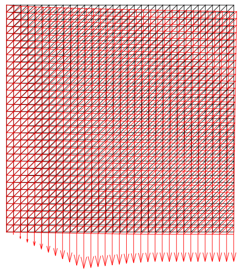
$$
\begin{aligned}
&L_1 = 1m, \quad L_2 = 1m, \quad T = 1s,\\
&E = 1000N/m^2, \quad \kappa = 0.4, \quad \lambda = 50Ns/m^2, \quad \sigma_Y = 2N/m^2,\\
&u_\nu^1 = 0.006m, \quad g = 0.01m, \quad c_\nu^1 = 300N/m^2, \quad c_\nu^2 = -100N/m^2,\\
&\boldsymbol{f}_0(t) = (0,0)N/m^2,\\
&\boldsymbol{f}_2(t) = \begin{cases} (0,-5)\ N/m & \text{on} \quad (0,L_1)\times\{L_2\},\\ (0,0)\ N/m & \text{on} \quad \{L_1\}\times(0,L_2],\end{cases}\\
&\boldsymbol{\sigma}_0 = \mathbf{0}N/m^2, \quad \boldsymbol{u}_0 = \mathbf{0}m.
\end{aligned}
$$

Our numerical results are presented in Figure 3-6 and Table 1. They are described in the following.

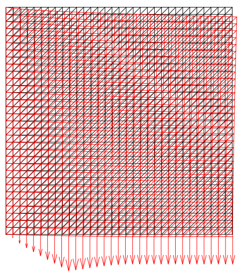
In Figure 3, the deformed configuration as well as the contact interface forces are plotted with and without plasticity at $T = 1s$. The time step and spatial step are chosen to be $h = \frac{1}{32}, k = \frac{1}{64}$. It is observed that the deformation without plasticity is smaller than that with plasticity, which is consistent with the fact under our physical setting. Moreover, in Figure 3(a), we have $0 \le u_\nu < u_\nu^1$ for part of the nodes , $u_\nu^1 \le u_\nu < g$ for other part and $u_\nu = g$ for the remainder part. It can be observed that the normal forces are increasing with respect to the penetration for



(a) t=1/64s and 1 node is in unilateral contact



(b) t=1/2s and 11 nodes are in unilateral contact



(c) t=1s and 15 nodes are in unilateral contact

FIGURE 5. Deformed meshes and contact interface forces for different time values.

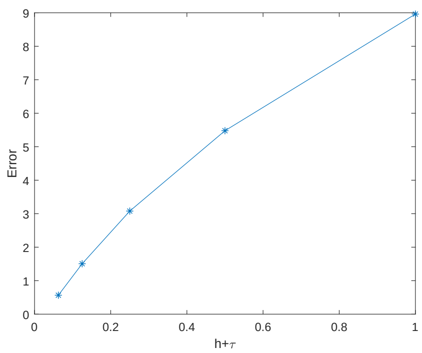


FIGURE 6. Dependence of Error on $h + \tau$.

$0 \leq u_\nu < u_\nu^1$ and are decreasing for $u_\nu^1 \leq u_\nu < g$. This also agrees with the theory since we have nonmonotone normal boundary condition.

In Figure 4, we plot the deformed configuration and the contact interface forces for different values of the maximal penetration g. The time step and spatial step are chosen to be $h = \frac{1}{32}, k = \frac{1}{64}$. Note that the number of nodes in unilateral contact status increases with the reduction of the value g. Especially, since $g = u_\nu^1$ in Figure 4(a), we have monotone boundary condition in this case.

TABLE 1. Error and convergence order.

h	τ	Error	Convergence order
1/2	1/2	8.9609	-
1/4	1/4	5.4782	0.71
1/8	1/8	3.0784	0.83
1/16	1/16	1.5060	1.03
1/32	1/32	0.5657	1.41

In Figure 5, we present the evolution of the contact during the process. More precisely, we plot the deformed meshes and the associated contact forces at three different time moments. The time step and spatial step are chosen to be $h = \frac{1}{32}, k = \frac{1}{64}$. We can find that at $t = \frac{1}{64}s$, almost all the contact nodes are in normal compliance contact. And at $t = 1s$, 15 of 32 nodes, since the penetration limit $g = 0.01m$ are reached, are into a unilateral contact.

The numerical errors are also computed for several values of the discretization parameter h and k, see Figure 6 and Table 1. Here the numerical error is in the form of

$$\text{Error} := \max_{0\le n\le N}\{\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V + \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}\|_Q\},$$

where $(\boldsymbol{u}, \boldsymbol{\sigma})$ is the 'reference' solution. Since the exact solution is unknown, we take the numerical solution corresponding to $h = \frac{1}{64}, k = \frac{1}{64}$ as the 'reference' solution. Note that the theoretically predicted first-order convergence of the numerical solution can be observed.

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