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UNIQUE SOLVABILITY AND DECOMPOSITION METHOD FOR ONE NONLINEAR MULTI-DIMENSIONAL INTEGRO-DIFFERENTIAL PARABOLIC EQUATION

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Abstract. The paper is devoted to the construction and study of the decomposition type semi-discrete scheme for one nonlinear multi-dimensional integro-differential equation of parabolic type. Unique solvability of the first type initial-boundary value problem is given as well. The studied equation is some generalization of integro-differential model, which is based on the well-known Maxwell system arising in mathematical simulation of electromagnetic field penetration into a medium.

Key words. Additive averaged semi-discrete scheme, nonlinear integro-differential multi-dimensional equation, unique solvability.

1. Introduction

The process of electromagnetic field penetration into a medium is described by Maxwell system of nonlinear partial differential equations [16]. In [7], reduction to the integro-differential form of mentioned Maxwell system is proposed

$$\frac{\partial H}{\partial t} = -rot\left[a\left(\int_0^t |rot\, H|^2\, d\tau\right) rot\, H\right], \tag{1}$$

where $H = (H_1, H_2, H_3)$ is a vector of magnetic field and $a = a(S)$ is defined for $S \in [0, \infty)$.

One must note that in one-component case $H = (0, 0, U)$ from (1) we get the following integro-differential equation

$$\frac{\partial U}{\partial t} = \nabla\left[a\left(\int_0^t |\nabla U|^2\, d\tau\right) \nabla U\right]. \tag{2}$$

The main characteristic feature of models of type (1) is associated with the appearance of the nonlinear coefficient at the higher derivatives depending on the time integral. This circumstance requires different approach than it is necessary to solve local differential problems. Along with its origin from the applied problems, the studied integro-differential equation (2) may be considered as a natural generalization of the well-known p-Laplacian models (see, for example, [20], [27]). The model (1) for scalar one-dimensional space case was first investigated in [4], [7] and scalar multi-dimensional space case (2) was studied in [5]. Later, these types of integro-differential models were considered in the numerous papers as well (see, for example, [1], [6], [9] - [11], [15], [17] - [19], [26], [29]). The asymptotic behavior and existence of solution by means of Galerkin method for multi-dimensional case and for non-homogeneous right side was studied in [26].

In [4], [5], [7] the solvability of the first boundary value problem is studied using a modified version of the Galerkin method and compactness arguments that are

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used in [20], [27] for investigation of nonlinear elliptic and parabolic models. The uniqueness of the solutions is investigated also in [4], [5], [7]. The asymptotic behavior of solutions is discussed in [6], [10] - [12], [15] and in a number of other works as well. Note also that to numerical resolution of (1) type one-dimensional equations were devoted many works as well (see, for example, [8], [9], [12], [15] and references therein). Many authors study the Rothe scheme, semi-discrete scheme with space variable, finite element and finite difference approximation for a differential and integro-differential models (see, for example, [12], [14] and references therein).

It is very important to study decomposition analogs for the above mentioned multi-dimensional differential and integro-differential models as well. There are some effective algorithms for solving the multi-dimensional problems (see, for example, [2], [3], [21] - [25], [28] and references therein).

Our work is dedicated to the unique solvability of the initial-boundary value problem. We shall focus our attention on (2) multi-dimensional type model Investigations are given in usual Sobolev spaces. Main attention is paid to investigation of semi-discrete additive average scheme. Reduction of multi-dimensional model to one-dimensional ones is proposed and corresponding convergence theorem with rate of convergence is given.

This article is organized as follows. In Section 2 the formulation of the problem and some of its properties are given. Particularly, unique solvability of the stated problem is considered there. Main attention is paid to the construction and investigation of the semi-discrete additive average scheme. This question is discussed in Section 3. Finite difference scheme for one-dimensional case and its implementation are discussed in Sections 4 and 5 respectively. Numerical experiments confirming the theoretical findings are carried out and some of them are given in Section 6. Some conclusions are given in Section 7.

2. Formulation of the problem and unique solvability

Let Ω be a bounded domain in the n-dimensional Euclidean space R^n, with a sufficiently smooth boundary $\partial\Omega$. In the domain $Q = \Omega \times (0, T)$ let us consider the following first type initial-boundary value problem:

$$\frac{\partial U}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[\left(1 + \int_0^t \left| \frac{\partial U}{\partial x_i} \right|^2 d\tau \right) \frac{\partial U}{\partial x_i} \right] = f(x,t), \quad (x,t) \in Q, \tag{3}$$

$$U(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \tag{4}$$

$$U(x,0) = 0, \quad x \in \overline{\Omega}, \tag{5}$$

where $x = (x_1, x_2, ..., x_n)$, T is a fixed positive constant and f is the given function of its arguments.

Theorem 2.1. *If*

$$f, \quad \frac{\partial f}{\partial t}, \quad \sqrt{\psi} \frac{\partial f}{\partial x_i} \in L_2(Q), \quad f(x,0) = 0,$$

where $\psi \in C^\infty(\overline{\Omega})$, $\psi(x) > 0$, for $x \in \Omega$; $\frac{\partial \psi}{\partial \nu} = 0$, for $x \in \partial\Omega$ and ν is the outer normal of $\partial\Omega$, then there exists the unique solution U of problem (3) - (5)

satisfying:

$$\int_0^T\int_\Omega\left[\frac{\partial U}{\partial t}V+\sum_{i=1}^n\left(1+\int_0^t\left|\frac{\partial U}{\partial x_i}\right|^2d\tau\right)\frac{\partial U}{\partial x_i}\frac{\partial V}{\partial x_i}\right]dxdt=\int_0^T\int_\Omega fVdxdt, \tag{6}$$

for all $V\in L_4\left(0,T;\overset{\circ}{W}{}_4^1(\Omega)\right)$.

Here and thereafter, the usual C^∞, $C^{m,k}$, L_p and Sobolev $\overset{\circ}{W}{}_p^k$ spaces are used.

Proof. To prove the existence of solution the similar procedure can be used as given in [5], [10] that is based on an approach from [20], [27].

As for uniqueness of solution of problem (3) - (5), let us assume that there are two solutions U_1 and U_2 and introduce the following notation $Z(x,t)=U_2(x,t)-U_1(x,t)$. From (3) - (5) we get:

$$\begin{aligned}\frac{\partial Z}{\partial t}-\sum_{i=1}^n\frac{\partial}{\partial x_i}&\left[\left(1+\int_0^t\left|\frac{\partial U_2}{\partial x_i}\right|^2d\tau\right)\frac{\partial U_2}{\partial x_i}\right.\\&\left.-\left(1+\int_0^t\left|\frac{\partial U_1}{\partial x_i}\right|^2d\tau\right)\frac{\partial U_1}{\partial x_i}\right]=0,\quad (x,t)\in Q,\end{aligned} \tag{7}$$

$$Z(x,t)=0,\quad (x,t)\in\partial\Omega\times[0,T], \tag{8}$$

$$Z(x,0)=0,\quad x\in\overline{\Omega}. \tag{9}$$

Let us multiply (7) by Z and integrate the obtained identity on $\Omega\times(0,\,t)$. Taking into account (8) and (9) we have

$$\begin{aligned}\frac{1}{2}\int_\Omega Z^2(x,t)\,dx+\sum_{i=1}^n\int_0^t\int_\Omega&\left[\left(1+\int_0^t\left|\frac{\partial U_2}{\partial x_i}\right|^2d\tau\right)\frac{\partial U_2}{\partial x_i}\right.\\&\left.-\left(1+\int_0^t\left|\frac{\partial U_1}{\partial x_i}\right|^2d\tau\right)\frac{\partial U_1}{\partial x_i}\right]\frac{\partial Z}{\partial x_i}dxd\tau=0.\end{aligned} \tag{10}$$

Note that,

$$\begin{aligned}&\left[\left(1+\int_0^t\left|\frac{\partial U_2}{\partial x_i}\right|^2d\tau\right)\frac{\partial U_2}{\partial x_i}\right.\\&\left.-\left(1+\int_0^t\left|\frac{\partial U_1}{\partial x_i}\right|^2d\tau\right)\frac{\partial U_1}{\partial x_i}\right]\left(\frac{\partial U_2}{\partial x_i}-\frac{\partial U_1}{\partial x_i}\right)\\&=\frac{1}{2}\left[2+\int_0^t\left|\frac{\partial U_2}{\partial x_i}\right|^2d\tau+\int_0^t\left|\frac{\partial U_1}{\partial x_i}\right|^2d\tau\right]\left[\frac{\partial U_2}{\partial x_i}-\frac{\partial U_1}{\partial x_i}\right]^2\\&+\frac{1}{2}\left[\int_0^t\left|\frac{\partial U_2}{\partial x_i}\right|^2d\tau-\int_0^t\left|\frac{\partial U_1}{\partial x_i}\right|^2d\tau\right]\left[\left|\frac{\partial U_2}{\partial x_i}\right|^2-\left|\frac{\partial U_1}{\partial x_i}\right|^2\right]\end{aligned} \tag{11}$$

$$\geq \frac{1}{2}\left[\int_0^t \left|\frac{\partial U_2}{\partial x_i}\right|^2 d\tau - \int_0^t \left|\frac{\partial U_1}{\partial x_i}\right|^2 d\tau\right]\left[\left|\frac{\partial U_2}{\partial x_i}\right|^2 - \left|\frac{\partial U_1}{\partial x_i}\right|^2\right].$$

Introducing the following notation

$$W_i(x,t) = \int_0^t \left(\left|\frac{\partial U_2}{\partial x_i}\right|^2 - \left|\frac{\partial U_1}{\partial x_i}\right|^2\right) d\tau,$$

from (10) and (11) we get

$$\int_\Omega Z^2(x,t)\,dx + \sum_{i=1}^n \int_0^t \int_\Omega W_i(x,t)\frac{\partial W_i(x,t)}{\partial t}dxd\tau \leq 0$$

or

$$\int_\Omega Z^2(x,t)\,dx + \frac{1}{2}\sum_{i=1}^n \int_\Omega W_i^2(x,t)\,dx \leq 0.$$

The last estimation implies that $Z \equiv 0$. Thus, Theorem 2.1 has been proved.

3. Semi-discrete additive averaged scheme

The main goal of this section is the construction and investigation of the Rothe type semi-discrete averaged scheme of sum approximation for the problem (3) - (5). On $[0,\, T]$ let us introduce a net with mesh points denoted by $t_j = j\tau$, $j = 0,\, 1\, \ldots\, J$, with $\tau = T/J$.

Let us construct the additive average Rothe type semi-discrete scheme:

$$\eta_i \frac{u_i^{j+1} - u^j}{\tau} = \frac{\partial}{\partial x_i}\left[\left(1 + \tau\sum_{k=1}^{j+1}\left|\frac{\partial u_i^k}{\partial x_i}\right|^2\right)\frac{\partial u_i^{j+1}}{\partial x_i}\right] + f_i^{j+1}, \tag{12}$$

$$u_i^0 = u^0 = 0, \quad i = 1,...,n, \quad j = 0,\, 1\, \ldots\, J-1,$$

with homogeneous boundary conditions. Here $u_i^j(x)$ is the solution of problem (12) and the following notations are also introduced:

$$u^j(x) = \sum_{i=1}^n \eta_i u_i^j(x), \quad \sum_{i=1}^n f_i^{j+1}(x) = f^{j+1}(x) = f(x, t_{j+1}),$$

where η_i, $i = 1,...,n$ are arbitrary positive constants satisfying $\sum\limits_{i=1}^n \eta_i = 1$ and $u^j(x)$ denotes approximation of exact solution of problem (3) - (5) at t_j.

The object of this section is to prove one main statement of this paper. We use usual scalar product $(\,,\,)$ and norm $\|\cdot\|$ of the space $L_2(\Omega)$ as follows:

$$(U,\, V) = \int_\Omega U(x)V(x)dx, \quad \|U\| = (U,\, U)^{1/2}.$$

Theorem 3.1. *Let us $U \in C_{x,\,t}^{2,\,2}(\overline{Q})$, then functions u^j defined by the solutions of problems (12) converge to the solution of problem (3) - (5) and the following estimate is true*

$$\left\|U^j - u^j\right\| = O(\tau^{1/2}), \quad j = 1\, \ldots\, J,$$

where $U^j(x) = U(x, t_j)$.

Proof. Let us introduce the following notations:

$$z^j = U^j - u^j, \quad z_i^j = U^j - u_i^j.$$

For the exact solution of problem (3) - (5) we have

$$\eta_i \frac{U^{j+1} - U^j}{\tau} = \eta_i \sum_{\ell=1}^{n} \frac{\partial}{\partial x_\ell} \left[\left(1 + \tau \sum_{k=1}^{j+1} \left| \frac{\partial U^k}{\partial x_\ell} \right|^2 \right) \frac{\partial U^{j+1}}{\partial x_\ell} \right] + \eta_i f^{j+1} + O(\tau).$$

After subtracting (12) from the above relation we get

$$\eta_i \left(\frac{U^{j+1} - U^j}{\tau} - \frac{u_i^{j+1} - u^j}{\tau} \right) = \eta_i \sum_{\ell=1}^{n} \frac{\partial}{\partial x_\ell} \left[\left(1 + \tau \sum_{k=1}^{j+1} \left| \frac{\partial U^k}{\partial x_\ell} \right|^2 \right) \frac{\partial U^{j+1}}{\partial x_\ell} \right]$$

$$- \frac{\partial}{\partial x_i} \left[\left(1 + \tau \sum_{k=1}^{j+1} \left| \frac{\partial u_i^k}{\partial x_i} \right|^2 \right) \frac{\partial u_i^{j+1}}{\partial x_i} \right] + \eta_i f^{j+1} - f_i^{j+1} + O(\tau).$$

Thus, introducing the well-known notation [25]

$$\frac{z_i^{j+1} - z^j}{\tau} = z_{i\,\bar{t}}^{j+1}$$

and using equalities (3) and (12) we have the following problem:

$$(13) \quad \begin{gathered} \eta_i z_{i\,\bar{t}}^{j+1} = \frac{\partial}{\partial x_i} \left[\left(1 + \tau \sum_{k=1}^{j+1} \left| \frac{\partial U^k}{\partial x_i} \right|^2 \right) \frac{\partial U^{j+1}}{\partial x_i} \right. \\ \left. - \left(1 + \tau \sum_{k=1}^{j+1} \left| \frac{\partial u_i^k}{\partial x_i} \right|^2 \right) \frac{\partial u_i^{j+1}}{\partial x_i} \right] + \psi_i^{j+1}(x), \\ z_i^0 = 0, \end{gathered}$$

with homogeneous boundary conditions where

$$\psi_i^{j+1}(x) = -\frac{\partial}{\partial x_i} \left[\left(1 + \tau \sum_{k=1}^{j+1} \left| \frac{\partial U^k}{\partial x_i} \right|^2 \right) \frac{\partial U^{j+1}}{\partial x_i} \right]$$

$$+ \eta_i \sum_{\ell=1}^{n} \frac{\partial}{\partial x_\ell} \left[\left(1 + \tau \sum_{k=1}^{j+1} \left| \frac{\partial U^k}{\partial x_\ell} \right|^2 \right) \frac{\partial U^{j+1}}{\partial x_\ell} \right]$$

$$+ \eta_i f^{j+1}(x) - f_i^{j+1}(x) + O(\tau) = \overline{\psi}_i^{j+1}(x) + O(\tau).$$

Multiplying (13) scalarly on $2\tau z_i^{j+1}$ we obtain

$$(14) \quad \begin{gathered} 2\tau\eta_i \left(z_{i\,\bar{t}}^{j+1}, z_i^{j+1} \right) + 2\tau \left(\left(1 + \tau \sum_{k=1}^{j+1} \left| \frac{\partial U^k}{\partial x_i} \right|^2 \right) \frac{\partial U^{j+1}}{\partial x_i} \right. \\ \left. - \left(1 + \tau \sum_{k=1}^{j+1} \left| \frac{\partial u_i^k}{\partial x_i} \right|^2 \right) \frac{\partial u_i^{j+1}}{\partial x_i}, \; \frac{\partial z_i^{j+1}}{\partial x_i} \right) - 2\tau \left(\psi_i^{j+1}, \, z_i^{j+1} \right) = 0. \end{gathered}$$

It can be easily checked that

$$\left(\left(1+\tau\sum_{k=1}^{j+1}\left|\frac{\partial U^k}{\partial x_i}\right|^2\right)\frac{\partial U^{j+1}}{\partial x_i}-\left(1+\tau\sum_{k=1}^{j+1}\left|\frac{\partial u_i^k}{\partial x_i}\right|^2\right)\frac{\partial u_i^{j+1}}{\partial x_i},\ \frac{\partial z_i^{j+1}}{\partial x_i}\right)$$

$$=\frac{1}{2}\left[\left(2+\tau\sum_{k=1}^{j+1}\left|\frac{\partial U^k}{\partial x_i}\right|^2+\tau\sum_{k=1}^{j+1}\left|\frac{\partial u_i^k}{\partial x_i}\right|^2,\ \left|\frac{\partial z_i^{j+1}}{\partial x_i}\right|^2\right)\right.$$

$$\left.+\left(\tau\sum_{k=1}^{j+1}\left[\left|\frac{\partial U^k}{\partial x_i}\right|^2-\left|\frac{\partial u_i^k}{\partial x_i}\right|^2\right],\ \left|\frac{\partial U^{j+1}}{\partial x_i}\right|^2-\left|\frac{\partial u_i^{j+1}}{\partial x_i}\right|^2\right)\right]$$

$$\geq\frac{1}{2}\left(\tau\sum_{k=1}^{j+1}\left[\left|\frac{\partial U^k}{\partial x_i}\right|^2-\left|\frac{\partial u_i^k}{\partial x_i}\right|^2\right],\ \left|\frac{\partial U^{j+1}}{\partial x_i}\right|^2-\left|\frac{\partial u_i^{j+1}}{\partial x_i}\right|^2\right).$$

From (14) for the error we get

$$2\tau\eta_i\left(z_{i\,\bar t}^{j+1},\ z_i^{j+1}\right)$$

$$+\tau\left(\tau\sum_{k=1}^{j+1}\left[\left|\frac{\partial U^k}{\partial x_i}\right|^2-\left|\frac{\partial u_i^k}{\partial x_i}\right|^2\right],\ \left|\frac{\partial U^{j+1}}{\partial x_i}\right|^2-\left|\frac{\partial u_i^{j+1}}{\partial x_i}\right|^2\right)$$

$$\leq 2\tau(\psi_i^{j+1}, z_i^{j+1}).$$

Using the well-known identities [25]:

$$z_i^{j+1}=z^j+\tau z_{i\,\bar t}^{j+1},\quad 2\tau\left(z_{i\bar t}^{j+1},\ z_i^{j+1}\right)=\left\|z_i^{j+1}\right\|^2+\tau^2\left\|z_{i\bar t}^{j+1}\right\|^2-\left\|z^j\right\|^2,$$

after simple transformations from the last inequality we have

$$\eta_i\left(\left\|z_i^{j+1}\right\|^2+\tau^2\left\|z_{i\bar t}^{j+1}\right\|^2\right)+\frac{1}{2}\left\|\tau\sum_{k=1}^{j+1}\left[\left|\frac{\partial U^k}{\partial x_i}\right|^2-\left|\frac{\partial u_i^k}{\partial x_i}\right|^2\right]\right\|^2$$

$$+\frac{\tau^2}{2}\left\|\left|\frac{\partial U^{j+1}}{\partial x_i}\right|^2-\left|\frac{\partial u_i^{j+1}}{\partial x_i}\right|^2\right\|^2$$

$$\leq\eta_i\left\|z^j\right\|^2+\frac{1}{2}\left\|\tau\sum_{k=1}^{j}\left[\left|\frac{\partial U^k}{\partial x_i}\right|^2-\left|\frac{\partial u_i^k}{\partial x_i}\right|^2\right]\right\|^2+2\tau\left(\psi_i^{j+1},\ z^j+\tau z_{i\bar t}^{j+1}\right).$$

Summing this equality from 1 to n we arrive at

$$\sum_{i=1}^{n}\eta_i\left(\left\|z_i^{j+1}\right\|^2+\tau^2\left\|z_{i\bar t}^{j+1}\right\|^2\right)+\frac{1}{2}\sum_{i=1}^{n}\left\|\tau\sum_{k=0}^{j+1}\left[\left|\frac{\partial U^k}{\partial x_i}\right|^2-\left|\frac{\partial u_i^k}{\partial x_i}\right|^2\right]\right\|^2$$

$$+\frac{\tau^2}{2}\sum_{i=1}^{n}\left\|\left|\frac{\partial U^{j+1}}{\partial x_i}\right|^2-\left|\frac{\partial u_i^{j+1}}{\partial x_i}\right|^2\right\|^2$$

(15)

$$\leq\sum_{i=1}^{n}\eta_i\left\|z^j\right\|^2+\frac{1}{2}\sum_{i=1}^{n}\left\|\tau\sum_{k=1}^{j}\left[\left|\frac{\partial U^k}{\partial x_i}\right|^2-\left|\frac{\partial u_i^k}{\partial x_i}\right|^2\right]\right\|^2$$

$$+2\tau\sum_{i=1}^{n}(\psi_i^{j+1},\, z^j)+2\tau\sum_{i=1}^{n}(\psi_i^{j+1},\, \tau z_{i\bar{t}}^{j+1}).$$

Note that,

$$\sum_{i=1}^{n}\eta_i z_i^{j+1}=\sum_{i=1}^{n}\eta_i(U^{j+1}-u_i^{j+1})=z^{j+1},\quad \sum_{i=1}^{n}\eta_i\left\|z^j\right\|^2=\left\|z^j\right\|^2. \tag{16}$$

Using the Jensen's inequality [13] we have

$$\left(\sum_{i=1}^{n}\eta_i z_i^{j+1}\right)^2\le\sum_{i=1}^{n}\eta_i\sum_{i=1}^{n}\eta_i\left(z_i^{j+1}\right)^2=\sum_{i=1}^{n}\eta_i\left(z_i^{j+1}\right)^2,$$

and thus,

$$\left\|z^{j+1}\right\|^2=\left\|\sum_{i=1}^{n}\eta_i z_i^{j+1}\right\|^2\le\sum_{i=1}^{n}\eta_i\left\|z_i^{j+1}\right\|^2. \tag{17}$$

Applying assumptions on f_i^{j+1} and η_i we have

$$\begin{aligned}\sum_{i=1}^{n}\overline{\psi}_i^{j+1}(x)=&-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[\left(1+\tau\sum_{k=1}^{j+1}\left|\frac{\partial U^k}{\partial x_i}\right|^2\right)\frac{\partial U^{j+1}}{\partial x_i}\right]\\ &+\sum_{i=1}^{n}\eta_i\sum_{\ell=1}^{n}\frac{\partial}{\partial x_\ell}\left[\left(1+\tau\sum_{k=1}^{j+1}\left|\frac{\partial U^k}{\partial x_\ell}\right|^2\right)\frac{\partial U^{j+1}}{\partial x_\ell}\right]\\ &+\sum_{i=1}^{n}\eta_i f^{j+1}(x)-\sum_{i=1}^{n}f_i^{j+1}(x)=0.\end{aligned} \tag{18}$$

So, we have property of sum approximation

$$\sum_{i=1}^{n}\psi_i^{j+1}(x)=O(\tau).$$

Taking into account relations (16) - (18) and Schwarz inequality we get from (15)

$$\begin{aligned}\left\|z^{j+1}\right\|^2+\sum_{i=1}^{n}\eta_i\tau^2\left\|z_{i\bar{t}}^{j+1}\right\|^2+\frac{1}{2}\sum_{i=1}^{n}\left\|\tau\sum_{k=1}^{j+1}\left[\left|\frac{\partial U^k}{\partial x_i}\right|^2-\left|\frac{\partial u_i^k}{\partial x_i}\right|^2\right]\right\|^2&\\ \le\left\|z^j\right\|^2+\frac{1}{2}\sum_{i=1}^{n}\left\|\tau\sum_{k=1}^{j}\left[\left|\frac{\partial U^k}{\partial x_i}\right|^2-\left|\frac{\partial u_i^k}{\partial x_i}\right|^2\right]\right\|^2&\\ +2\tau(O(\tau),\, z^j)+\tau^2\left\|\psi^{j+1}\right\|^2+\sum_{i=1}^{n}\eta_i\tau^2\left\|z_{i\bar{t}}^{j+1}\right\|^2.&\end{aligned}$$

Here we used the following inequality

$$\begin{aligned}2\tau\sum_{i=1}^{n}(\psi_i^{j+1},\, \tau z_{i\bar{t}}^{j+1})&=2\sum_{i=1}^{n}(\eta_i^{-1/2}\tau\psi_i^{j+1},\, \eta_i^{1/2}\tau z_{i\bar{t}}^{j+1})\\ &\le\sum_{i=1}^{n}\eta_i^{-1}\tau^2\left\|\psi_i^{j+1}\right\|^2+\sum_{i=1}^{n}\eta_i\tau^2\left\|z_{i\bar{t}}^{j+1}\right\|^2\end{aligned}$$

and the notation

$$\left\|\psi^{j+1}\right\|^2=\sum_{i=1}^{n}\eta_i^{-1}\left\|\psi_i^{j+1}\right\|^2.$$

Using boundedness of $\left\|\psi^{j+1}\right\|$ we find

$$\begin{aligned}&\left\|z^{j+1}\right\|^2+\frac{1}{2}\sum_{i=1}^{n}\left\|\tau\sum_{k=1}^{j+1}\left[\left|\frac{\partial U^k}{\partial x_i}\right|^2-\left|\frac{\partial u_i^k}{\partial x_i}\right|^2\right]\right\|^2\\ &\le\left\|z^j\right\|^2+\frac{1}{2}\sum_{i=1}^{n}\left\|\tau\sum_{k=1}^{j}\left[\left|\frac{\partial U^k}{\partial x_i}\right|^2-\left|\frac{\partial u_i^k}{\partial x_i}\right|^2\right]\right\|^2+2\tau(O(\tau),\,z^j)+O(\tau^2).\end{aligned}\tag{19}$$

Summing (19) with respect to j from 0 to $m-1$ we get

$$\begin{aligned}&\left\|z^m\right\|^2+\frac{\tau}{2}\sum_{i=1}^{n}\left\|\left|\frac{\partial U^m}{\partial x_i}\right|^2-\left|\frac{\partial u_i^m}{\partial x_i}\right|^2\right\|^2\le 2\tau\sum_{j=0}^{m-1}\left(O(\tau),\,z^j\right)+C\tau\\ &\le\tau\sum_{j=0}^{m-1}\left[O(\tau^2)+\left\|z^j\right\|^2\right]+O(\tau)\le\tau\sum_{j=0}^{m-1}\left\|z^j\right\|^2+C\tau.\end{aligned}\tag{20}$$

The desired result of Theorem 3.1 now follows from (20) by the standard discrete Gronwall lemma (see, for example [25]).

4. Finite difference scheme for one-dimensional case

Let us consider one-dimensional case of problem (3) - (5). So, let $\Omega=(0,\,1)$ and in the domain $(0,\,1)\times(0,T)$ consider the following one-dimensional initial-boundary value problem:

$$\frac{\partial U}{\partial t}-\frac{\partial}{\partial x}\left[\left(1+\int_0^t\left|\frac{\partial U}{\partial x}\right|^2d\tau\right)\frac{\partial U}{\partial x}\right]=f(x,t),\quad (x,t)\in(0,\,1)\times(0,T),\tag{21}$$

$$U(0,t)=U(1,t)=0,\quad t\in[0,T],\tag{22}$$

$$U(x,0)=0.\quad x\in[0,\,1],\tag{23}$$

The numerical resolution of multi-dimensional integro-differential model (3) can be now reduced to the one-dimensional (21) type models, numerical realization of which can be done using early investigated finite difference and finite element schemes (see, for example, [9], [12], [15], [18]). Here we give numerical realization algorithm for carrying out the numerical experiments using finite difference scheme. Results of those numerical experiments is given in Section 6.

In order to describe the finite difference method we introduce a net whose mesh points are denoted by $(x_m,t_j)=(mh,j\tau)$, where $m=0,1,...,M$; $j=0,1,...,J$ with $h=1/M$ and again $\tau=T/J$. The initial line is denoted by $j=0$. The discrete approximation at (x_m,t_j) is denoted by u_m^j and the exact solution to problem (21) - (23) at those points by U_m^j. We will use the following known notations [25]:

$$u_{x,m}^j=u_{\bar{x},m+1}^j=\frac{u_{m+1}^j-u_m^j}{h},\quad (u,v)_h=\sum_{m=1}^{M-1}u_mv_mh,\quad \|u\|_h=(u,u)_h^{1/2}.$$

Let us correspond to problem (21) - (23) the difference scheme:

$$\frac{u_m^{j+1}-u_m^j}{\tau}-\left\{\left[1+\tau\sum_{k=1}^{j+1}(u_{\bar{x},m}^k)^2\right]u_{\bar{x},m}^{j+1}\right\}_x=f_m^j, \tag{24}$$

$$m=1,2,...,M-1; \quad j=0,1,...,J-1,$$

$$u_0^j=u_M^j=0, \quad j=0,1,...,J, \tag{25}$$

$$u_m^0=0, \quad m=0,1,...,M. \tag{26}$$

In [9], [12] the following theorem of convergence was proved.

Theorem 4.1. *Let us* $U\in C_{x,\,t}^{4,\,2}(\overline{Q})$*, then the solution* $u^j=(u_1^j,u_2^j,\dots,u_{M-1}^j)$*,* $j=1,2,\dots,J$ *of the difference scheme (24) - (26) tends to solution of problem (21) - (23)* $U^j=(U_1^j,U_2^j,\dots,U_{M-1}^j)$*,* $j=1,2,\dots,J$ *as* $\tau\to 0$*,* $h\to 0$*, and the following estimate holds*

$$\|u^j-U^j\|_h\le C(\tau+h),\ j=1,2,\dots,J.$$

5. Numerical implementation remarks

We now comment on the numerical implementation of the discrete problem (24) - (26). Note that (24) can be rewritten as:

$$\frac{u_m^{j+1}-u_m^j}{\tau}-\frac{1}{h}\left\{\left[1+\tau\sum_{k=1}^{j+1}\left(\frac{u_{m+1}^k-u_m^k}{h}\right)^2\right]\frac{u_{m+1}^{j+1}-u_m^{j+1}}{h}\right.$$

$$\left.-\left[1+\tau\sum_{k=1}^{j+1}\left(\frac{u_m^k-u_{m-1}^k}{h}\right)^2\right]\frac{u_m^{j+1}-u_{m-1}^{j+1}}{h}\right\}=f_m^j,$$

$$m=1,\dots,M-1.$$

Let

$$A_m^\ell=1+\tau\sum_{k=1}^{\ell}\left[\left(\frac{u_{m+1}^k-u_m^k}{h}\right)^2\right], \qquad m=0,1,\dots,M-1,$$

then (24) becomes

$$\frac{u_m^{j+1}-u_m^j}{\tau}-\frac{1}{h}\left\{A_m^{j+1}\frac{u_{m+1}^{j+1}-u_m^{j+1}}{h}-A_{m-1}^{j+1}\frac{u_m^{j+1}-u_{m-1}^{j+1}}{h}\right\}=f_m^j, \tag{27}$$

$$m=1,2,\dots,M-1.$$

System (27) can be written in a matrix form

$$\mathbf{H}\left(\mathbf{u}^{j+1}\right)\equiv\mathbf{G}\left(\mathbf{u}^{j+1}\right)-\frac{1}{\tau}\mathbf{u}^j-\mathbf{f}^j=0.$$

The vector $\mathbf{u}$ containing all the unknowns $u_1,\dots,u_{M-1}$ at the level indicated. The vector $\mathbf{G}$ is given by

$$\mathbf{G}\left(\mathbf{u}^{j+1}\right)=\mathbf{T}^{j+1}\mathbf{u}^{j+1},$$

where the $(M-1)\times(M-1)$ matrix $\mathbf{T}$ is symmetric and tridiagonal with elements:

$$\mathbf{T}_{rs}^{\ell} = \begin{cases} -\frac{1}{h^2}A_{r-1}^{\ell}, & s=r-1, \\ \frac{1}{\tau}+\frac{1}{h^2}\left(A_r^{\ell}+A_{r-1}^{\ell}\right), & s=r, \\ -\frac{1}{h^2}A_r^{\ell}, & s=r+1, \\ 0, & \text{otherwise}, \end{cases} \tag{28}$$

where $r=0,1,\ldots,M-1$, $s=0,1,\ldots,M-1$, and $\ell=1,2,\ldots,J-1$.

Newton method for the system is given by

$$\nabla\mathbf{H}\left(\mathbf{u}^{j+1}\right)\Big|^{(k)}\left(\mathbf{u}^{j+1}\Big|^{(k+1)}-\mathbf{u}^{j+1}\Big|^{(k)}\right)=-\mathbf{H}\left(\mathbf{u}^{j+1}\right)\Big|^{(k)},$$

where k is the number of iterations. The elements of the matrix $\nabla\mathbf{H}\left(\mathbf{u}^{j+1}\right)$ require the derivative of A. The elements are

$$\frac{\partial A_{r-1}^{j+1}}{\partial u_s^{j+1}}=-\frac{\tau}{h^2}\frac{\partial}{\partial u_s^{j+1}}\left[\left(u_r^{j+1}-u_{r-1}^{j+1}\right)^2\right]=\begin{cases}\frac{2\tau}{h}\,u_{\bar{x},r}^{j+1}, & s=r-1,\\ -\frac{2\tau}{h}\,u_{\bar{x},r}^{j+1}, & s=r,\\ 0, & \text{otherwise},\end{cases}$$

and

$$\frac{\partial A_{r}^{j+1}}{\partial u_s^{j+1}}=-\frac{\tau}{h^2}\frac{\partial}{\partial u_s^{j+1}}\left[\left(u_{r+1}^{j+1}-u_{r}^{j+1}\right)^2\right]=\begin{cases}\frac{2\tau}{h}\,u_{x,r}^{j+1}, & s=r,\\ -\frac{2\tau}{h}\,u_{x,r}^{j+1}, & s=r+1,\\ 0, & \text{otherwise}.\end{cases}$$

Combining (28) and two relations above for partial derivatives we have

$$\nabla\mathbf{H}\left(\mathbf{u}^{j+1}\right)\Big|_{r\,s}=\begin{cases}-\frac{1}{h^2}A_{r-1}^{j+1}-\frac{2\tau}{h^2}\left(u_{\bar{x},r}^{j+1}\right)^2, & s=r-1,\\ \frac{1}{\tau}+\frac{1}{h^2}\left(A_r^{j+1}+A_{r-1}^{j+1}\right) & \\ +\frac{2\tau}{h^2}\left(u_{x,r}^{j+1}\right)^2+\frac{2\tau}{h^2}\left(u_{\bar{x},r}^{j+1}\right)^2, & s=r,\\ -\frac{1}{h^2}A_r^{j+1}-\frac{2\tau}{h^2}\left(u_{x,r}^{j+1}\right)^2, & s=r+1,\\ 0, & \text{otherwise}.\end{cases}$$

Let us state well-known statement (see, for example, [22]).

Theorem 4.2 *Given the nonlinear system of equations*

$$H_m\left(y_1,\ldots,y_{M-1}\right)=0,\ \ m=1,2,\ldots,M-1.$$

If H_m are three times continuously differentiable in a region containing the solution $\xi_1, \dots, \xi_{M-1}$ and the Jacobian does not vanish in that region, then Newton method converges at least quadratically.

The Jacobian in this theorem is the matrix ∇H computed above. The term $1/\tau$ on diagonal ensures that the Jacobian does not vanish. The differentiability is guaranteed, since ∇H is quadratic. Newton method is costly, because the matrix changes at every step of the iteration. One can use modified Newton method (keep the same matrix for several iterations) but the rate of convergence will be slower.

6. Results of numerical experiments

The study of operator splitting techniques has a long history and has been pursued with various methods. Since alternating-direction methods and fractional step methods these procedures, which reduce the time-stepping of multi-dimensional problems to locally one-dimensional computations, have been applied in the numerical simulation of many practically important problems.

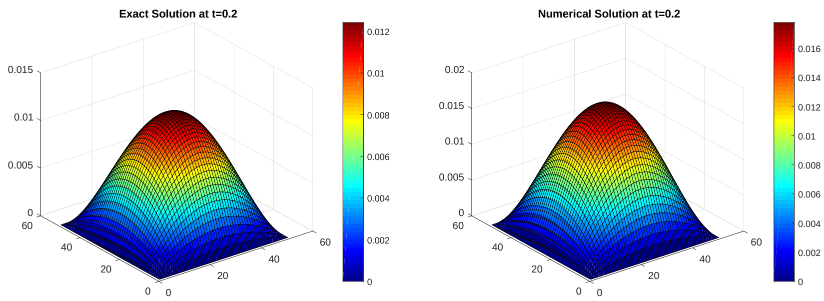

FIGURE 1. The exact (top) and numerical (bottom) solutions at $t = 0.2$.

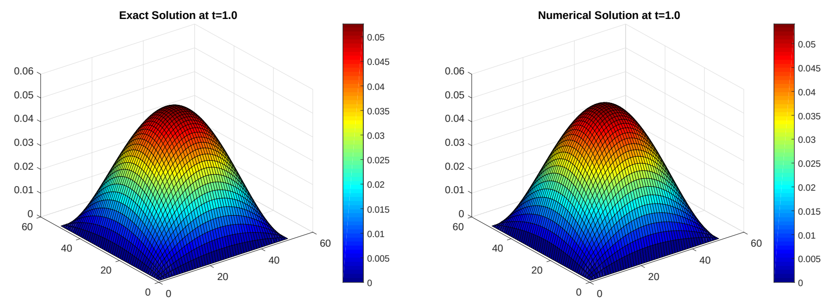

FIGURE 2. The exact (top) and numerical (bottom) solutions at $t = 1$.

Beginning from the basic works [2], [3], [23] the methods of constructing of effective algorithms for the numerical solution of the multi-dimensional problems of the mathematical physics and the sphere of problems solvable with the help of those algorithms were essentially extended. At present there are some effective algorithms for solving the multi-dimensional problems of the mathematical physics (see, for example, [21], [25], [28] and references therein). Those algorithms mainly belong to the methods of splitting-up or sum approximation according to their approximating properties.

Here we consider the following test example: let us consider case $n = 2$, $\Omega = (0, 1) \times (0, 1)$ and let us choose the right side of the equation (3) so that the exact solution is given by

$$U(x_1, x_2, t) = x_1 x_2 (1 - x_1)(1 - x_2)\sin(t).$$

Parameters used here are $T = 1$, $\tau = 0.004$ and for spatial discretization we used $h_1 = h_2 = 0.02$. Results of test experiment for exact and numerical solution are given on Figures 1 and 2.

For the errors analysis the maximum of the absolute values of errors between exact and numerical solutions for different time levels are shown in Table 1 and Figures 3.

TABLE 1. Errors.

t	Absolute Values of Errors Between Exact and Numerical Solutions
0.1	0.0021162280
0.2	0.0053472308
0.3	0.0072442167
0.4	0.0085204597
0.5	0.0079440224
0.6	0.0074552945
0.7	0.0064486412
0.8	0.0050587253
0.9	0.0038834107
1	0.0025198852

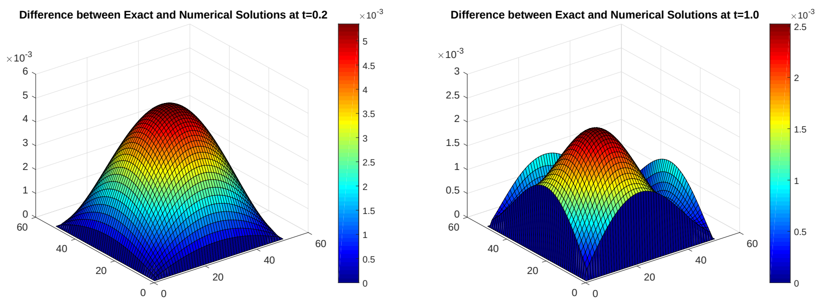

FIGURE 3. The differences between exact and numerical solutions at $t = 0.2$ (top) and at $t = 1$ (bottom).

We have carried out with several other numerical experiments and observed the same situation.

We also run test experiments for different values of time step τ and get the rate of convergence as in Theorem 3.1 (Table 2).

TABLE 2. The empirical rate of convergence in the error energy norm for τ.

τ	Error	Rate for τ
0.002	0.0084650404	0.4889140184
0.0005	0.0042980698	0.5420998137
0.0004	0.0038083651	0.5055851868
0.0002	0.0026825156	

7. Conclusions

We will try to make some comments on the results we obtained. The additive averaged semi-discrete scheme for nonlinear multi-dimensional integro-differential equation of parabolic type is studied. The investigated equation is some generalization of integro-differential model which is based on the well-known Maxwell system arising in mathematical simulation of electromagnetic field penetration into a medium. Unique solvability of the first type initial-boundary value problem is given. Reduction of multi-dimensional model to one-dimensional ones is discussed and convergence theorem with order of convergence is proved. For two-dimensional case the numerical experiments supporting the theoretical findings are presented.

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