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A CONFORMING DISCONTINUOUS GALERKIN FINITE ELEMENT METHOD: PART III

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Abstract. The conforming discontinuous Galerkin (CDG) finite element methods were introduced in [12] on simplicial meshes and in [13] on polytopal meshes. The CDG method gets its name by combining the features of both conforming finite element method and discontinuous Galerkin (DG) finite element method. The goal of this paper is to continue our efforts on simplifying formulations for the finite element method with discontinuous approximation by constructing new spaces for the gradient approximation. Error estimates of optimal order are established for the corresponding CDG finite element approximation in both a discrete H^1 norm and the L^2 norm. Numerical results are presented to confirm the theory.

Key words. Weak gradient, discontinuous Galerkin, stabilizer/penalty free, finite element methods, second order elliptic problem.

1. Introduction

Finite element methods with discontinuous approximation are flexible in finite element construction and mesh generation. However, when discontinuous approximation is used, finite element formulations tend to be more complex to ensure connection of discontinuous function across element boundary. For example, stabilizing/penalty terms are often needed in finite element methods with discontinuous approximations to enforce connection of discontinuous functions across element boundaries [2, 4, 5, 6, 7, 9, 10]. Removing stabilizing term from discontinuous finite element methods will reduce the complexity of formulation and computer programming.

Aiming on simplifying finite element formulation with discontinuous approximation, conforming discontinuous Galerkin finite element methods have been developed in [12] on simplicial mesh and in [13] on polytopal mesh for the following model problem: seek an unknown function u satisfying

$$-\Delta u = f \quad \text{in } \Omega, \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where Ω is a bounded polytopal domain in $\mathbb{R}^d$. The weak form of the problem (1)-(2) is given as follows: find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega). \tag{3}$$

Conforming discontinuous Galerkin finite element method by name maintains the flexibility of DG methods and the features of conforming finite element method such as simple formulation: find $u_h \in V_h$ such that

$$(\nabla_w u_h, \nabla_w v) = (f, v) \quad \forall v \in V_h, \tag{4}$$

where ∇_w is a approximation of gradient ∇. Construction of the space to approximate ∇ is the key of maintaining ultra simple formulation (4). In [13], gradient is approximated by a polynomial of order $j = k + n - 1$ with n the number of sides

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of polygonal element. This result has been improved in [1] by reducing the degree of polynomial j. In [8], Wachspress coordinates are used to approximate ∇, which are usually rational functions, instead of polynomials.

The goal of this paper is to develop a new CDG finite element method with a different philosophy to approximate gradient ∇. In this method, we use piecewise low order polynomial to approximate ∇_w instead of using one piece high order polynomial in [13]. Optimal order error estimates are established for the corresponding conforming DG approximations in both a discrete H^1 norm and the L^2 norm. Numerical results are presented verifying the theorem.

2. Preliminaries

For any given polygon $D \subseteq \Omega$, we use the standard definition of Sobolev spaces $H^s(D)$ with $s \geq 0$. The associated inner product, norm, and semi-norms in $H^s(D)$ are denoted by $(\cdot,\cdot)_{s,D}$, $\|\cdot\|_{s,D}$, and $|\cdot|_{s,D}$, respectively. When $s = 0$, $H^0(D)$ coincides with the space of square integrable functions $L^2(D)$. In this case, the subscript s is suppressed from the notation of norm, semi-norm, and inner products. Furthermore, the subscript D is also suppressed when $D = \Omega$.

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of polygons in two dimension or polyhedra in three dimension satisfying a set of conditions specified in [11]. Denote by $\mathcal{E}_h$ the set of all edges/faces in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ be the set of all interior edges/faces. For simplicity, we will use term edge for edge/face without confusion.

Let $P_k(K)$ consist all the polynomials degree less or equal to k defined on K. A finite element space V_h is defined for $k \geq 1$ as

$$V_h = \left\{ v \in L^2(\Omega) : \ v|_T \in P_k(T), \ \ T \in \mathcal{T}_h \right\}. \tag{5}$$

Let T_1 and T_2 be two polygons/polyhedrons in $\mathcal{T}_h$ sharing $e \in \mathcal{E}_h$. For $e \in \mathcal{E}_h$ and $v \in V_h + H^1(\Omega)$, the jump $[v]$ is defined as

$$[v] = v \quad \text{if } e \subset \partial\Omega, \quad [v] = v|_{T_1} - v|_{T_2} \ \text{ if } e \in \mathcal{E}_h^0. \tag{6}$$

The order of T_1 and T_2 is not essential. For $e \in \mathcal{E}_h$ and $v \in V_h + H^1(\Omega)$, the average $\{v\}$ is defined as

$$\{v\} = 0 \quad \text{if } e \subset \partial\Omega, \quad \{v\} = \frac{1}{2}(v|_{T_1} + v|_{T_2}) \ \text{ if } e \in \mathcal{E}_h^0. \tag{7}$$

The space $H(div;\Omega)$ is defined as the set of vector-valued functions on Ω which, together with their divergence, are square integrable; i.e.,

$$H(div;\Omega) = \left\{ \mathbf{v} \in [L^2(\Omega)]^d : \ \nabla \cdot \mathbf{v} \in L^2(\Omega) \right\}.$$

For any $T \in \mathcal{T}_h$, it can be divided in to a set of disjoint triangles T_i with $T = \cup T_i$. Then $\Lambda_h(T)$ can be defined as

$$\Lambda_k(T) = \{\mathbf{v} \in H(div, T), \ \mathbf{v}|_{T_i} \in RT_k(T)\}, \tag{8}$$

where $RT_k(T)$ is the usual Raviart-Thomas element of order k [3].

For a function $v \in V_h + H^1(\Omega)$, its weak gradient $\nabla_w v$ is defined as a piecewise vector valued polynomial such that $\nabla_w v|_T \in \Lambda_k(T)$ and satisfies the following equation,

$$(\nabla_w v, \mathbf{q})_T = -(v, \nabla \cdot \mathbf{q})_T + \langle \{v\}, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T} \qquad \forall \mathbf{q} \in \Lambda_k(T). \tag{9}$$

For simplicity, we adopt the following notations,

$$(v,w)_{\mathcal{T}_h} = \sum_{T\in\mathcal{T}_h}(v,w)_T = \sum_{T\in\mathcal{T}_h}\int_T vwd\mathbf{x},$$
$$\langle v,w\rangle_{\partial\mathcal{T}_h} = \sum_{T\in\mathcal{T}_h}\langle v,w\rangle_{\partial T} = \sum_{T\in\mathcal{T}_h}\int_{\partial T} vwds.$$

3. CDG Method on Polytopal Mesh

In this section, we introduce the conforming DG Method and investigate the well-posedness of the method.

Algorithm 1. *A conforming DG finite element method for the problem (1)-(2) seeks $u_h \in V_h$ satisfying*

$$(\nabla_w u_h, \nabla_w v)_{\mathcal{T}_h} = (f,\ v) \quad \forall v \in V_h. \tag{10}$$

Next we introduce a semi-norms $|||v|||$ and a norm $\|v\|_{1,h}$ for any $v \in V_h + H_0^1(\Omega)$ as follows:

$$|||v|||^2 = \sum_{T\in\mathcal{T}_h}(\nabla_w v, \nabla_w v)_T, \tag{11}$$

$$\|v\|_{1,h}^2 = \sum_{T\in\mathcal{T}_h}\|\nabla v\|_T^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[v]\|_e^2. \tag{12}$$

For any function $\varphi \in H^1(T)$, the following trace inequality holds true (see [11] for details):

$$\|\varphi\|_e^2 \le C\left(h_T^{-1}\|\varphi\|_T^2 + h_T\|\nabla\varphi\|_T^2\right). \tag{13}$$

Lemma 3.1. *For a given $v \in V_h$ and $T \in \mathcal{T}_h$, there exists a $\mathbf{q}_0 \in \Lambda_k(T)$ such that,*

$$(\nabla v, \mathbf{q}_0)_T = 0, \qquad \text{and } \ \langle\{v\} - v, \mathbf{q}_0\cdot\mathbf{n}\rangle_{\partial T} = \|\{v\} - v\|_{\partial T}^2, \tag{14}$$

and

$$\|\mathbf{q}_0\|_T \le Ch_T^{1/2}\|\{v\} - v\|_{\partial T}. \tag{15}$$

Proof. This lemma is from the definition of RT finite elements. Let $T = \cup_{i=1}^n T_i$ be decomposed to n triangles/tetrahedra and $\Lambda_k = \{\mathbf{q} \in H(\mathrm{div}, T) \mid \mathbf{q}|_{T_i} \in RT_k\}$. We define a $\mathbf{q}_0 \in \Lambda_k$ by

$$\int_e(\mathbf{q}_0\cdot\mathbf{n} - \{v\} + v)p_k ds = 0 \quad \forall p_k \in P_k(e),\ e \in \partial T \cap \cup\partial T_i,$$
$$\int_e(\mathbf{q}_0\cdot\mathbf{n} - 0)p_k ds = 0 \quad \forall p_k \in P_k(e),\ e \in \cup\partial T_i \setminus \partial T,$$
$$\int_{T_i}(\mathbf{q}_0 - \mathbf{0})\cdot\mathbf{p}_{k-1} d\mathbf{x} = 0 \quad \forall \mathbf{p}_{k-1} \in (P_{k-1}(T_i))^d, 1 \le i \le n,$$

where e is a face-edge/polygon of T_i. e can be part of a face of T. For example, a polygon face of T has to be subdivided in to several triangles, faces of some subtetrahedra T_i. Since $\mathbf{q}_0\cdot\mathbf{n}$ is a P_k polynomial, $\mathbf{q}_0\cdot\mathbf{n} = \{v\} - v$ and

$$\langle\{v\} - v, \mathbf{q}_0\cdot\mathbf{n}\rangle_{\partial T} = \sum_{e\in\partial T}\int_e(\{v\} - v)^2 ds = \|\{v\} - v\|_{\partial T}^2.$$

Since ∇v is a vector P_{k-1} polynomial, we have

$$(\nabla v, \mathbf{q}_0)_T = \sum_{i=1}^n \int_{T_i}\mathbf{q}_0\cdot\mathbf{p}_{k-1}d\mathbf{x} = 0.$$

Assuming T is a shape regular, size one polygon/polyhedron, by the finite dimensional matrix norm, we have $\|\mathbf{q}_0\|_T \le C\|\{v\} - v\|_{\partial T}$. (15) follows by scaling T. $\square$

Lemma 3.2. *There exist two positive constants C_1 and C_2 independent of mesh size h such that for any $v \in V_h$, we have*

$$C_1\|v\|_{1,h} \le |\!|\!| v |\!|\!| \le C_2\|v\|_{1,h}. \tag{16}$$

Proof. For any $v \in V_h$, it follows from the definition of weak gradient (9) and integration by parts that for all $\mathbf{q} \in \Lambda_k(T)$

$$\begin{aligned}(\nabla_w v, \mathbf{q})_T &= -(v, \nabla\cdot\mathbf{q})_T + \langle\{v\}, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T}\\ &= (\nabla v, \mathbf{q})_T - \langle v - \{v\}, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T}.\end{aligned} \tag{17}$$

By letting $\mathbf{q} = \nabla_w v$ in (17) we arrive at

$$(\nabla_w v, \nabla_w v)_T = (\nabla v, \nabla_w v)_T - \langle v - \{v\}, \nabla_w v\cdot\mathbf{n}\rangle_{\partial T}.$$

It is easy to see that the following equations hold true for $\{v\}$ defined in (7) on T with $e \subset \partial T$,

$$\|v - \{v\}\|_e = \|[v]\|_e \quad \text{if } e \subset \partial\Omega, \quad \|v - \{v\}\|_e = \frac{1}{2}\|[v]\|_e \ \text{ if } e \in \mathcal{E}_h^0. \tag{18}$$

From (18), (13) and the inverse inequality we have

$$\begin{aligned}\|\nabla_w v\|_T^2 &\le \|\nabla v\|_T\|\nabla_w v\|_T + \|v - \{v\}\|_{\partial T}\|\nabla_w v\|_{\partial T}\\ &\le \|\nabla v\|_T\|\nabla_w v\|_T + Ch_T^{-1/2}\|v - \{v\}\|_{\partial T}\|\nabla_w v\|_T\\ &\le \|\nabla v\|_T\|\nabla_w v\|_T + C\alpha h_T^{-1/2}\|[v]\|_{\partial T}\|\nabla_w v\|_T\end{aligned}$$

which implies

$$\|\nabla_w v\|_T \le C\left(\|\nabla v\|_T + C\alpha h_T^{-1/2}\|[v]\|_{\partial T}\right),$$

and consequently

$$|\!|\!| v |\!|\!| \le C_2\|v\|_{1,h}.$$

Next we will prove $C_1\|v\|_{1,h} \le |\!|\!| v |\!|\!|$. For $v \in V_h$ and $\mathbf{q} \in \Lambda_h(T)$, by (9) and integration by parts, we have

$$(\nabla_w v, \mathbf{q})_T = (\nabla v, \mathbf{q})_T + \langle\{v\} - v, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T}. \tag{19}$$

Letting $\mathbf{q} = \mathbf{q}_0$ defined in Lemma 3.1 in (19) and using (14)-(15) give

$$\|\{v\} - v\|_e^2 \le C\|\nabla_w v\|_T\|\mathbf{q}_0\|_T \le Ch_T^{1/2}\|\nabla_w v\|_T\|\{v\} - v\|_e,$$

which gives

$$h_T^{-1/2}\|\{v\} - v\|_{\partial T} \le C\|\nabla_w v\|_T. \tag{20}$$

Using (18) and summing the both sides of (20) over T, we obtain

$$\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[v]\|_e^2 \le C|\!|\!| v |\!|\!|^2. \tag{21}$$

It follows from the trace inequality, the inverse inequality and (20),

$$\|\nabla v\|_T^2 \le \|\nabla_w v\|_T\|\nabla v\|_T + Ch_T^{-1/2}\|\{v\} - v\|_{\partial T}\|\nabla v\|_T \le C\|\nabla_w v\|_T\|\nabla v\|_T,$$

which implies

$$\sum_{T\in\mathcal{T}_h} \|\nabla v\|_T^2 \le C|\!|\!| v |\!|\!|^2. \tag{22}$$

Combining (21) and (22), we prove the lower bound of (16) and complete the proof of the lemma. □

4. Error Estimates

In this section, optimal order error estimates are established for the corresponding conforming DG approximations in both a discrete H^1 norm and the L^2 norm. First we will derive the equations errors satisfied.

4.1. Error Equations. Let $\mathbb{Q}_h$ be the element-wise defined L^2 projection onto $\Lambda_k(T)$ on each element $T \in \mathcal{T}_h$.

Lemma 4.1. *Let $\phi \in H_0^1(\Omega)$, then one has that for any $\mathbf{q} \in \cup_{T\in\mathcal{T}_h}\Lambda_k(T)$*

$$(\nabla_w \phi, \mathbf{q})_{\mathcal{T}_h} = (\mathbb{Q}_h \nabla\phi, \mathbf{q})_{\mathcal{T}_h}. \tag{23}$$

Proof. Using (9) and integration by parts, we have that for any $\mathbf{q} \in \cup_{T\in\mathcal{T}_h}\Lambda_k(T)$

$$\begin{aligned}(\nabla_w \phi, \mathbf{q})_{\mathcal{T}_h} &= -(\phi, \nabla\cdot\mathbf{q})_{\mathcal{T}_h} + \langle\{\phi\}, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}\\ &= -(\phi, \nabla\cdot\mathbf{q})_{\mathcal{T}_h} + \langle\phi, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}\\ &= (\nabla\phi, \mathbf{q})_{\mathcal{T}_h}\\ &= (\mathbb{Q}_h\nabla\phi, \mathbf{q})_{\mathcal{T}_h},\end{aligned}$$

which proves the lemma. □

Let Q_h be the element-wise defined L^2 projection onto $P_k(T)$ on each element $T \in \mathcal{T}_h$. Let $e_h = u - u_h$ and $\epsilon_h = Q_h u - u_h$. Next we derive the error equations that e_h and ϵ_h satisfy.

Lemma 4.2. *For any $v \in V_h$, one has,*

$$(\nabla_w e_h, \nabla_w v)_{\mathcal{T}_h} = \ell_1(u, v), \tag{24}$$

$$(\nabla_w \epsilon_h, \nabla_w v)_{\mathcal{T}_h} = \ell_1(u, v) + \ell_2(u, v), \tag{25}$$

where

$$\begin{aligned}\ell_1(u, v) &= \langle(\nabla u - \mathbb{Q}_h\nabla u)\cdot\mathbf{n}, v - \{v\}\rangle_{\partial T_h},\\ \ell_2(u, v) &= (\nabla_w(Q_h u - u), \nabla_w v)_{\mathcal{T}_h}.\end{aligned}$$

Proof. It follows (7) that

$$\sum_{T\in\mathcal{T}_h}\langle\nabla u\cdot\mathbf{n}, \{v\}\rangle_{\partial T} = 0. \tag{26}$$

Testing (1) by any $v \in V_h$, using integration by parts and (26), we arrive at

$$(\nabla u, \nabla v)_{\mathcal{T}_h} - \langle\nabla u\cdot\mathbf{n}, v - \{v\}\rangle_{\partial\mathcal{T}_h} = (f, v). \tag{27}$$

It follows from integration by parts, (9) and (23) that

$$\begin{aligned}(\nabla u, \nabla v)_{\mathcal{T}_h} &= (\mathbb{Q}_h\nabla u, \nabla v)_{\mathcal{T}_h}\\ &= -(v, \nabla\cdot(\mathbb{Q}_h\nabla u))_{\mathcal{T}_h} + \langle v, \mathbb{Q}_h\nabla u\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}\\ &= (\mathbb{Q}_h\nabla u, \nabla_w v)_{\mathcal{T}_h} + \langle v - \{v\}, \mathbb{Q}_h\nabla u\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}\\ &= (\nabla_w u, \nabla_w v)_{\mathcal{T}_h} + \langle v - \{v\}, \mathbb{Q}_h\nabla u\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}.\end{aligned} \tag{28}$$

Using (27) and (28), we have

$$(\nabla_w u, \nabla_w v)_{\mathcal{T}_h} = (f, v) + \ell_1(u, v). \tag{29}$$

The error equation (24) follows from subtracting (10) from (29),

$$(\nabla_w e_h, \nabla_w v)_{\mathcal{T}_h} = \ell_1(u, v) \quad \forall v \in V_h.$$

By adding $\nabla_w Q_h u$ to the both sides of the equation above, we obtain (25). This completes the proof of the lemma. □

4.2. Error Estimates in Energy Norm.

Lemma 4.3. *For any $w \in H^{k+1}(\Omega)$ and $v \in V_h$, we have*

$$|\ell_1(w,v)| \leq Ch^k |w|_{k+1} |||v|||, \tag{30}$$

$$|||w - Q_h w||| \leq Ch^k |w|_{k+1}, \tag{31}$$

$$|\ell_2(w,v)| \leq Ch^k |w|_{k+1} |||v|||. \tag{32}$$

Proof. Using the Cauchy-Schwarz inequality, the trace inequality (13), (18) and (16), we have

$$\begin{aligned} |\ell_1(w,v)| &= \left| \sum_{T\in\mathcal{T}_h} \langle (\nabla w - \mathbb{Q}_h \nabla w)\cdot \mathbf{n}, v - \{v\}\rangle_{\partial T} \right| \\ &\leq C \sum_{T\in\mathcal{T}_h} \|\nabla w - \mathbb{Q}_h \nabla w\|_{\partial T} \|v - \{v\}\|_{\partial T} \\ &\leq C \left(\sum_{T\in\mathcal{T}_h} h_T \|(\nabla w - \mathbb{Q}_h \nabla w)\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{e\in\mathcal{E}_h} h_e^{-1} \|[v]\|_e^2 \right)^{\frac{1}{2}} \\ &\leq Ch^k |w|_{k+1} |||v|||, \end{aligned}$$

which proves (30). It follows from (9), integration by parts, (13) and (18),

$$\begin{aligned} & |(\nabla_w (w - Q_h w), \mathbf{q})_T| \\ =\ & |-(w - Q_h w, \nabla\cdot \mathbf{q})_T + \langle \{w - Q_h w\}, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T}| \\ =\ & |(\nabla (w - Q_h w), \mathbf{q})_T - \langle w - Q_h w - \{w - Q_h w\}, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T}| \\ \leq\ & \|\nabla (w - Q_h w)\|_T \|\mathbf{q}\|_T + Ch^{-1/2} \|[w - Q_h w]\|_{\partial T} \|\mathbf{q}\|_T \\ \leq\ & Ch^k |w|_{k+1,T} \|\mathbf{q}\|_T. \end{aligned}$$

Letting $\mathbf{q} = \nabla_w (w - Q_h w)$ in the above equation and taking summation over T, we have

$$|||w - Q_h w||| \leq Ch^k |w|_{k+1}.$$

The estimate above implies

$$\begin{aligned} \ell_2(w,v) &= (\nabla_w (Q_h w - w), \nabla_w v)_{\mathcal{T}_h} \\ &\leq |||Q_h w - w||| \, |||v||| \\ &\leq Ch^k |w|_{k+1} |||v|||, \end{aligned}$$

which implies (32). We have proved the lemma. □

Theorem 4.1. *Let $u_h \in V_h$ be the finite element solution of (10). Assume the exact solution $u \in H^{k+1}(\Omega)$. Then, there exists a constant C such that*

$$|||u - u_h||| \leq Ch^k |u|_{k+1}, \tag{33}$$

$$\|u - u_h\|_{1,h} \leq Ch^k |u|_{k+1}. \tag{34}$$

Proof. Letting $v = \epsilon_h$ in (25) gives

$$|||\epsilon_h|||^2 = \ell_1(u, \epsilon_h) + \ell_2(u, \epsilon_h). \tag{35}$$

Applying (30) and (32) to the equation (35) yields

$$|||\epsilon_h||| \leq Ch^k |u|_{k+1}. \tag{36}$$

It follows from the triangle inequality, (36) and (31)

$$\|\!|u - u_h\|\!| \le \|\!|Q_h u - u_h\|\!| + \|\!|u - Q_h u\|\!| \le Ch^k |u|_{k+1}, \tag{37}$$

which implies (35). Using (16) and (36), we have

$$\begin{aligned} \|u - u_h\|_{1,h} &\le \|u - Q_h u\|_{1,h} + \|Q_h u - u_h\|_{1,h} \\ &\le C(\|u - Q_h u\|_{1,h} + \|\!|Q_h u - u_h\|\!|) \\ &\le Ch^k |u|_{k+1}, \end{aligned}$$

which completes the proof of the lemma. □

4.3. Error Estimates in L^2 Norm. Consider a dual problem that seeks $\Phi \in H_0^1(\Omega)$ satisfying

$$-\Delta\Phi = e_h \quad \text{in } \Omega. \tag{38}$$

Assume that the following H^2-regularity holds

$$\|\Phi\|_2 \le C\|e_h\|. \tag{39}$$

Recall $e_h = u - u_h$ and $\epsilon_h = Q_h u - u_h$.

Theorem 4.2. *Let $u_h \in V_h$ be the finite element solution of (10). Assume that the exact solution $u \in H^{k+1}(\Omega)$ and (39) holds true. Then, there exists a constant C such that*

$$\|u - u_h\| \le Ch^{k+1}|u|_{k+1}. \tag{40}$$

Proof. Testing (38) by e_h and using the fact that $\sum_{T\in\mathcal{T}_h} \langle \nabla\Phi\cdot\mathbf{n}, \{e_h\}\rangle_{\partial T} = 0$ and (9) give

$$\begin{aligned} \|e_h\|^2 &= -(\Delta\Phi, e_h) = (\nabla\Phi,\ \nabla e_h)_{\mathcal{T}_h} - \langle\nabla\Phi\cdot\mathbf{n},\ e_h - \{e_h\}\rangle_{\partial T_h} \\ &= (\mathbb{Q}_h\nabla\Phi,\ \nabla e_h)_{\mathcal{T}_h} + (\nabla\Phi - \mathbb{Q}_h\nabla\Phi,\ \nabla e_h)_{\mathcal{T}_h} - \langle\nabla\Phi\cdot\mathbf{n},\ e_h - \{e_h\}\rangle_{\partial T_h} \\ &= -(\nabla\cdot\mathbb{Q}_h\nabla\Phi,\ e_h)_{\mathcal{T}_h} + \langle\mathbb{Q}_h\nabla\Phi\cdot\mathbf{n},\ e_h\rangle_{\partial T_h} \\ &+ (\nabla\Phi - \mathbb{Q}_h\nabla\Phi,\ \nabla e_h)_{\mathcal{T}_h} - \langle\nabla\Phi\cdot\mathbf{n},\ e_h - \{e_h\}\rangle_{\partial T_h} \\ &= (\mathbb{Q}_h\nabla\Phi,\ \nabla_w e_h)_{\mathcal{T}_h} + \langle\mathbb{Q}_h\nabla\Phi\cdot\mathbf{n},\ e_h - \{e_h\}\rangle_{\partial T_h} \\ &+ (\nabla\Phi - \mathbb{Q}_h\nabla\Phi,\ \nabla e_h)_{\mathcal{T}_h} - \langle\nabla\Phi\cdot\mathbf{n},\ e_h - \{e_h\}\rangle_{\partial T_h} \\ &= (\mathbb{Q}_h\nabla\Phi,\ \nabla_w e_h)_{\mathcal{T}_h} + (\nabla\Phi - \mathbb{Q}_h\nabla\Phi,\ \nabla e_h)_{\mathcal{T}_h} - \ell_1(\Phi, e_h). \end{aligned}$$

It follows from (23), (24) and the fact $\Phi = 0$ on $\partial\Omega$ that

$$\begin{aligned} (\mathbb{Q}_h\nabla\Phi,\ \nabla_w e_h)_{\mathcal{T}_h} &= (\nabla_w\Phi,\ \nabla_w e_h)_{\mathcal{T}_h} \\ &= (\nabla_w Q_h\Phi,\ \nabla_w e_h)_{\mathcal{T}_h} + (\nabla_w(\Phi - Q_h\Phi),\ \nabla_w e_h)_{\mathcal{T}_h} \\ &= \ell_1(u, Q_h\Phi) + (\nabla_w(\Phi - Q_h\Phi),\ \nabla_w e_h)_{\mathcal{T}_h}. \end{aligned}$$

Combining the two equations above gives

$$\begin{aligned} \|e_h\|^2 &= \ell_1(u, Q_h\Phi) + (\nabla_w(\Phi - Q_h\Phi),\ \nabla_w e_h)_{\mathcal{T}_h} \\ &+ (\nabla\Phi - \mathbb{Q}_h\nabla\Phi,\ \nabla e_h)_{\mathcal{T}_h} - \ell_1(\Phi, e_h). \end{aligned} \tag{41}$$

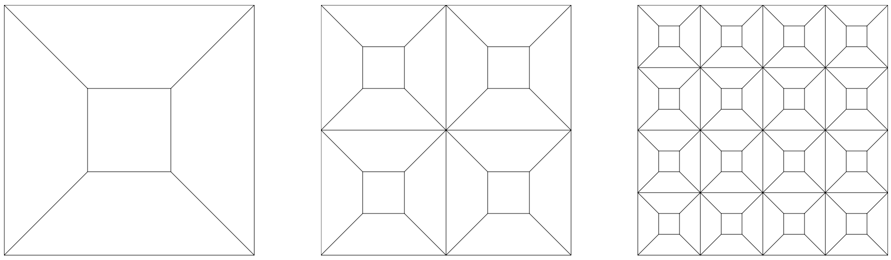

FIGURE 1. The first three levels of quadrilateral grids for Table 1.

Next we will estimate all the terms on the right hand side of (41). Using the Cauchy-Schwarz inequality, the trace inequality (13) and (18) we obtain

$$
\begin{aligned}
& |\ell_1(u, Q_h\Phi)| \\
\le\ & |\langle(\nabla u - \mathbb{Q}_h\nabla u)\cdot\mathbf{n},\ Q_h\Phi - \{Q_h\Phi\}\rangle_{\partial T_h}| \\
\le\ & C\left(\sum_{T\in\mathcal{T}_h} h\|(\nabla u - \mathbb{Q}_h\nabla u)\|_{\partial T}^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h} h^{-1}\|[Q_h\Phi - \Phi]\|_{\partial T}^2\right)^{1/2} \\
\le\ & Ch^{k+1}|u|_{k+1}|\Phi|_2.
\end{aligned}
$$

It follows from (33) and (31) that

$$
|(\nabla_w e_h,\ \nabla_w(\Phi - Q_h\Phi))_{\mathcal{T}_h}| \le C|\!|\!|e_h|\!|\!||\!|\!|\Phi - Q_h\Phi|\!|\!| \le Ch^{k+1}|u|_{k+1}|\Phi|_2.
$$

The estimate (34) implies

$$
\begin{aligned}
|(\nabla\Phi - \mathbb{Q}_h\nabla\Phi,\ \nabla e_h)_{\mathcal{T}_h}| &\le C(\sum_{T\in\mathcal{T}_h}\|\nabla\Phi - \mathbb{Q}_h\nabla\Phi\|_T^2)^{1/2}\|e_h\|_{1,h} \\
&\le Ch^{k+1}|u|_{k+1}|\Phi|_2.
\end{aligned}
$$

Using (16), (18), (36), and (31), we obtain

$$
\begin{aligned}
|\ell_1(\Phi, e_h)| &= \left|\sum_{T\in\mathcal{T}_h}\langle(\mathbb{Q}_h\nabla\Phi - \nabla\Phi)\cdot\mathbf{n},\ e_h - \{e_h\}\rangle_{\partial T}\right| \\
&\le \sum_{T\in\mathcal{T}_h} h_T^{1/2}\|\mathbb{Q}_h\nabla\Phi - \nabla\Phi\|_{\partial T} h_T^{-1/2}\|[e_h]\|_{\partial T} \\
&\le Ch\|\Phi\|_2(\sum_{T\in\mathcal{T}_h} h_T^{-1}(\|[\varepsilon_h]\|_{\partial T}^2 + \|[u - Q_h u]\|_{\partial T}^2)^{1/2} \\
&\le Ch\|\Phi\|_2(|\!|\!|\varepsilon_h|\!|\!| + (\sum_{T\in\mathcal{T}_h} h_T^{-1}\|[u - Q_h u]\|_{\partial T}^2)^{1/2} \\
&\le Ch^{k+1}|u|_{k+1}\|\Phi\|_2.
\end{aligned}
$$

Combining all the estimates above with (41) yields

$$
\|e_h\|^2 \le Ch^{k+1}|u|_{k+1}\|\Phi\|_2.
$$

The estimate (40) follows from the above inequality and the regularity assumption (39). We have completed the proof. □

TABLE 1. Example 5.1. Error profile for (42) on quadrilateral grids (Figure 1).

Grid	$\|Q_h u - u_h\|$	Rate	$\|\|\|Q_h u - u_h\|\|\|$	Rate
	P_1 CDG finite element with RT_1 gradient			
6	0.2575E-03	2.00	0.1593E-01	1.03
7	0.6440E-04	2.00	0.7921E-02	1.01
8	0.1610E-04	2.00	0.3955E-02	1.00
	P_2 CDG finite element with RT_2 gradient			
6	0.1118E-05	3.01	0.6222E-03	2.02
7	0.1395E-06	3.00	0.1549E-03	2.01
8	0.1744E-07	3.00	0.3869E-04	2.00
	P_3 CDG finite element with RT_3 gradient			
4	0.1961E-05	4.32	0.3487E-03	3.72
5	0.1133E-06	4.11	0.3142E-04	3.47
6	0.6941E-08	4.03	0.3429E-05	3.20
	P_4 CDG finite element with RT_4 gradient			
2	0.1300E-03	4.81	0.1633E-01	3.75
3	0.2187E-05	5.89	0.4589E-03	5.15
4	0.4354E-07	5.65	0.1610E-04	4.83

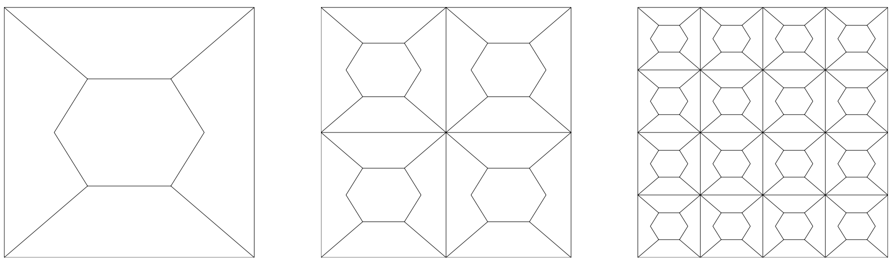

FIGURE 2. The first three levels of quadrilateral grids for Table 2.

5. Numerical Example

We compute two examples in 2D and one example in 3D.

5.1. Example 5.1. We solve the Poisson equation (1) on the 2D unit square domain $\Omega = (0, 1)^2$. The exact solution is

$$u(x, y) = \sin(\pi x) \sin(\pi y). \tag{42}$$

The first three quadrilateral grids are plotted in Figure 1. When computing weak gradient, we subdivide each quadrilateral in to four triangles. The computational results are listed in Table 1. Optimal order of convergence is achieved in all cases.

5.2. Example 5.2. We solve the Poisson equation (1) again in 2D with the exact solution (42). We use polygonal grids, consisting of quadrilaterals, pentagons and hexagons, shown in Figure 2. We subdivide each quadrilateral, pentagon and hexagon in to 4, 5 and 6 triangles, respectively, to define the piecewise RT_k space

TABLE 2. Example 5.2. Error profile for (42) on polygonal grids (Figure 2).

Grid	$\|Q_h u - u_h\|$	Rate	$\|\|\|Q_h u - u_h\|\|\|$	Rate
	P_1 CDG finite element with RT_1 gradient			
6	0.3574E-03	2.00	0.2070E-01	1.03
7	0.8939E-04	2.00	0.1030E-01	1.01
8	0.2235E-04	2.00	0.5143E-02	1.00
	P_2 CDG finite element with RT_2 gradient			
6	0.1083E-05	3.00	0.5480E-03	2.00
7	0.1354E-06	3.00	0.1370E-03	2.00
8	0.1692E-07	3.00	0.3424E-04	2.00
	P_3 CDG finite element with RT_3 gradient			
5	0.1110E-06	4.01	0.2999E-04	3.05
6	0.6927E-08	4.00	0.3713E-05	3.01
7	0.6613E-09	3.39	0.4629E-06	3.00
	P_4 CDG finite element with RT_4 gradient			
2	0.4829E-04	5.36	0.4935E-02	4.13
3	0.1192E-05	5.34	0.2112E-03	4.55
4	0.3421E-07	5.12	0.1086E-04	4.28

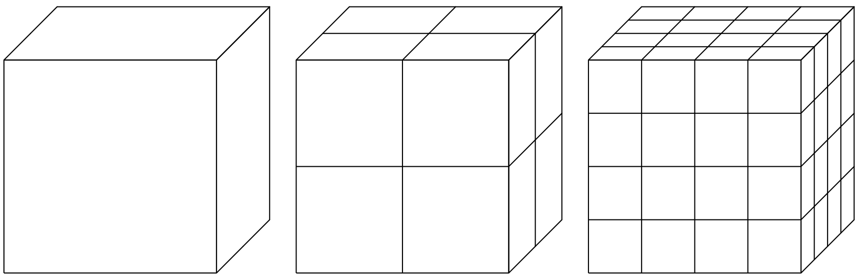

FIGURE 3. The first three levels of grids used for solving (43).

for computing the weak gradient. We list the errors and the order of convergence in Table 2. As we proved, we obtain the optimal order of convergence in both norms.

5.3. Example 5.3. We solve the Poisson equation (1) on the 3D unit cube domain $\Omega = (0,1)^3$, with a non-homogeneous Dirichlet boundary condition. The exact solution is

$$u(x,y,z) = \sin\frac{\pi x}{2}\sin\frac{\pi y}{2}\sin\frac{\pi z}{2}. \tag{43}$$

We use uniform cubic grids shown in Figure 3. To compute weak gradients, we subdivide each cube in to six tetrahedra. The computational results are listed in Table 3. Optimal order of convergence is achieved in all cases. It seems we have a half order superconvergence in H^1-like norm, based on numerical data on this test problem.

TABLE 3. Example 5.3. Error profile for (43) on cubic grids (Figure 3.)

Grid	$\|Q_h u - u_h\|$	Rate	$\|\|\|Q_h u - u_h\|\|\|$	Rate
	P_1 CDG finite element with RT_1 gradient			
5	0.5668E-03	1.75	0.8767E-02	1.68
6	0.1517E-03	1.90	0.2781E-02	1.66
7	0.3901E-04	1.96	0.9106E-03	1.61
	P_2 CDG finite element with RT_2 gradient			
4	0.5311E-04	3.76	0.4049E-02	2.86
5	0.3887E-05	3.77	0.5739E-03	2.82
6	0.3062E-06	3.67	0.9165E-04	2.65
	P_3 CDG finite element with RT_3 gradient			
3	0.1013E-03	4.24	0.3856E-02	3.46
4	0.3849E-05	4.72	0.3037E-03	3.67
5	0.1331E-06	4.85	0.2188E-04	3.79

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