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THE WEAK GALERKIN FINITE ELEMENT METHOD FOR SOLVING THE TIME-DEPENDENT STOKES FLOW

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Abstract. In this paper, we solve the time-dependent Stokes problem by the weak Galerkin (WG) finite element method. Full-discrete WG finite element scheme is obtained by applying the implicit backward Euler method for time discretization. Optimal order error estimates are established for the corresponding numerical approximation in H^1 norm for the velocity, and L^2 norm for both the velocity and the pressure in semi-discrete forms and full-discrete forms, respectively. Some computational results are presented to demonstrate the accuracy, convergence and efficiency of the method.

Key words. Stokes problem, weak Galerkin finite element method, discrete weak gradient, discrete weak divergence.

1. Introduction

The Stokes problem [27] describes the dynamics of fluid flows in complex porous media. It has wide applications in industrial and scientific fields, such as, petroleum, biomedical engineering, and heat conduction model, etc. In this paper, we consider the time-dependent Stokes problem, which has been treated by various numerical methods, such as the finite element methods (FEMs) [9, 13], the finite volume methods [1, 21], the discontinuous Galerkin methods [2, 3, 10, 28], and the weak Galerkin finite element methods [4, 15]. We provide a new developed weak Galerkin finite element method in this paper. The concerning time-dependent Stokes equation seeks the velocity function $\mathbf{u}$ and pressure function p satisfying

$$\mathbf{u}_t - \mu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \text{ in } \Omega\times(0,T], \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0, \text{ in } \Omega\times(0,T], \tag{2}$$

$$\mathbf{u} = \mathbf{g}, \text{ on } \partial\Omega\times(0,T], \tag{3}$$

$$\mathbf{u}(\cdot,0) = \mathbf{u}^0, \text{ in } \Omega, \tag{4}$$

where Ω is a polygonal or polyhedral domain in $\mathbb{R}^d$ $(d=2,3)$. $\mathbf{f}$ is a momentum source term, $\mu>0$ is the kinematic viscosity, and $\mathbf{u}_t$ is the time partial derivation of $\mathbf{u}(x,t)$. We assume that $\mathbf{f},\mathbf{g}$ and $\mathbf{u}^0$ are given, sufficiently smooth functions. For simplicity, we consider (1) and (3) with $\mu=1$ and $\mathbf{g}=\mathbf{0}$.

The weak forms in the primary velocity-pressure formulations for the Stokes problems (1)-(4) find $(\mathbf{u};p)\in L^2(0,T;[H_0^1(\Omega)]^d)\times L^2(0,T;L_0^2(\Omega))$, for any $(\mathbf{v};q)\in [H_0^1(\Omega)]^d\times L_0^2(\Omega)$ with $t\in(0,T]$ satisfying

$$\begin{aligned}(\mathbf{u}_t,\mathbf{v})+(\nabla\mathbf{u},\nabla\mathbf{v})-(p,\nabla\cdot\mathbf{v}) &= (\mathbf{f},\mathbf{v}),\\ (q,\nabla\cdot\mathbf{u}) &= 0.\end{aligned}$$

For the discretization of the Stokes equation, we use the weak Galerkin (WG) finite element method. The WG method was first introduced in 2011 [14] for the second order elliptic problem and further applied to other partial differential equations, for example, the second order elliptic equation[12, 16, 22], the Stokes equation

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[17, 18, 24], linear elasticity equations [11], the parabolic equations [22, 25, 29] the Brinkman equation [19, 23], the Biharmon problem [6, 7, 26] and the Helmholtz equation [8], etc. The WG method refers to a general finite element technique for partial differential equations in which differential operators are approximated by weak forms as distributions for generalized functions. The main idea of the WG method is the use of weak functions and their corresponding weak derivatives defined as distributions. Weak functions and weak derivatives can be approximated by polynomials with arbitrary degrees. Thus, there are three prominent features: (1) The usual derivatives are replaced by distributions or discrete approximations of distributions; (2) The approximating functions are discontinuous; (3) The WG method allows the use of finite element partitions with arbitrary shape of polygons in $2D$ or polyhedra in $3D$ with certain shape regularity. These features make the WG method have many advantages, such as high order of accuracy, high flexibility, and easy handling of complicated geometries.

In this paper, we provide an effective WG finite element method for the time-dependent Stokes equation. The weak Galerkin finite element space consists of discontinuous piecewise polynomials of degree $k \geq 1$ for the velocity $\mathbf{u}$ and polynomials of degree $k-1$ for the pressure p, respectively. The paper is organized as follows. In Section 2, we introduce some standard notations in Sobolev space and then develop the semi-discrete and full-discrete WG finite element scheme for the Stokes equation (1)-(4). For time discretization, we use the backward Euler method, which is an implicit method. In Section 3, we derive the semi-discrete and full-discrete error equations for the WG approximations. Optimal order error estimates for both the semi-discrete and full-discrete backward Euler WG finite element approximations are established in Section 4 in H^1 norm for the velocity and L^2 norm for both the velocity and the pressure functions. Finally, in Section 5, we present some numerical experiments to confirm the theoretical analysis.

2. The Weak Galerkin Finite Element Method

In this section, we introduce some preliminaries and notations for Sobolev space, the semi-discrete and full-discrete WG finite element schemes for the Stokes problem (1)-(4).

Let D be any open bounded domain with Lipschitz continuous boundary in $\mathbb{R}^d$ $(d = 2, 3)$. We use the standard notations for the Sobolev space $H^s(D)$, and the associated inner product $(\cdot,\cdot)_{s,D}$, norm $\|\cdot\|_{s,D}$, and semi-norm $|\cdot|_{s,D}$ for any $s \geq 0$. The space $H^0(D)$ coincides with $L^2(D)$, for which the norm and the inner product are denoted by $\|\cdot\|_D$ and $(\cdot,\cdot)_D$, respectively. When $D = \Omega$, we shall drop the subscript D in the norm and inner product notation.

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of polygons in $\mathbb{R}^2$ or polyhedral in $\mathbb{R}^3$ satisfying a set of conditions [5], and T be each element with ∂T as its boundary. $\mathcal{E}_h$ is the set of all edges or flat faces in $\mathcal{T}_h$, and $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ is the set of all interior edges or flat faces. For each $T \in \mathcal{T}_h$, denote by h_T the diameter of T, and $h = \max_{T\in\mathcal{T}_h}$ is the mesh size of $\mathcal{T}_h$.

We define weak Galerkin finite element space for the velocity function $\mathbf{u}$ and the pressure function p, as follows

$$
\begin{aligned}
V_h &= \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\}, \mathbf{v}_0|_T \in [P_k(T)]^d, \mathbf{v}_b|_e \in [P_{k-1}(e)]^d, \forall T \in \mathcal{T}_h, \forall e \in \partial T\},\\
V_h^0 &= \{\mathbf{v} \in V_h, \mathbf{v}_b = 0 \text{ on } \partial\Omega\},\\
W_h &= \{q : q \in L_0^2(\Omega), q|_T \in P_{k-1}(T), \forall T \in \mathcal{T}_h\}.
\end{aligned}
$$

We would like to emphasize that any function $\mathbf{v} \in V_h$ has a single value $\mathbf{v}_b$ on each edge $e \in \mathcal{E}_h$.

A discrete weak gradient $\nabla_{w,r,T}$ is defined as a linear operator by

$$(\nabla_{w,r,T}\mathbf{v}, q)_T = -(\mathbf{v}_0, \nabla \cdot q) + \langle \mathbf{v}_b, q \cdot \mathbf{n} \rangle_{\partial T}, \ \forall q \in [P_r(T)]^{d\times d}.$$

A discrete weak divergence $\nabla_{w,r,T}\cdot$ is defined as a linear operator by

$$(\nabla_{w,r,T} \cdot \mathbf{v}, \varphi)_T = -(\mathbf{v}_0, \nabla\varphi) + \langle \mathbf{v}_b \cdot \mathbf{n}, \varphi \rangle_{\partial T}, \ \forall \varphi \in [P_r(T)]^{d\times d}.$$

Next, we introduce three bilinear forms

$$\begin{aligned} s(\mathbf{v}, \mathbf{w}) &= \sum_{T\in\mathcal{T}_h} h_T^{-1} \langle Q_b\mathbf{v}_0 - \mathbf{v}_b, Q_b\mathbf{w}_0 - \mathbf{w}_b \rangle_{\partial T}, \\ a(\mathbf{v}, \mathbf{w}) &= (\nabla_w \mathbf{v}, \nabla_w \mathbf{w}) + s(\mathbf{v}, \mathbf{w}), \\ b(\mathbf{v}, q) &= (\nabla_w \cdot \mathbf{v}, q). \end{aligned}$$

For any $\mathbf{v} \in V_h^0$, we have

$$|||\mathbf{v}|||^2 = a(\mathbf{v}, \mathbf{v}) = \|\nabla_w \mathbf{v}\|^2 + \sum_{T\in\mathcal{T}_h} h_T^{-1} \|Q_b\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T}^2.$$

Weak Galerkin Algorithm 1. *Find* $\boldsymbol{u}_h = \{\boldsymbol{u}_0(\cdot,t), \boldsymbol{u}_b(\cdot,t)\} \in L^2(0,T;V_h^0)$ *and* $p_h \in L^2(0,T;W_h^0)$*, such that*

$$((\boldsymbol{u}_h)_t, \boldsymbol{v}_0) + a(\boldsymbol{u}_h, \boldsymbol{v}) - b(\boldsymbol{v}, p_h) = (\boldsymbol{f}, \boldsymbol{v}_0), \ \forall \boldsymbol{v} = \{\boldsymbol{v}_0, \boldsymbol{v}_b\} \in V_h^0, \tag{5}$$

$$b(\boldsymbol{u}_h, q) = 0, \ \forall q \in W_h, \tag{6}$$

$$\boldsymbol{u}_h(x, 0) = Q_h \boldsymbol{u}^0(x), \ \forall x \in \Omega. \tag{7}$$

Then, we introduce some projections. Denote by Q_0 the L^2 projection operator from $[L^2(T)]^d$ onto $[P_k(T)]^d$. For each edge/face $e \in \mathcal{E}_h$, denote by Q_b the L^2 projection from $[L^2(e)]^d$ onto $[P_{k-1}(e)]^d$. We shall combine Q_0 with Q_b by writing $Q_h = \{Q_0, Q_b\}$. Let L_h and R_h be two local L^2 projections onto $P_{k-1}(T)$ and $[P_{k-1}(T)]^{d\times d}$, respectively.

Lemma 2.1. [15] *The projection operators satisfy the following commutative properties*

$$\nabla_w(Q_h\boldsymbol{u}) = R_h(\nabla \boldsymbol{u}), \quad \forall \boldsymbol{u} \in [H^1(\Omega)]^d, \tag{8}$$

$$\nabla_w \cdot (Q_h\boldsymbol{u}) = L_h(\nabla \cdot \boldsymbol{u}), \quad \forall \boldsymbol{u} \in H(div, \Omega). \tag{9}$$

Lemma 2.2. [15] *For any* $\boldsymbol{v}, \boldsymbol{w} \in V_h^0$*, we have the boundedness and coercivity*

$$\begin{aligned} |a(\boldsymbol{v}, \boldsymbol{w})| &\leq |||\boldsymbol{v}||| \cdot |||\boldsymbol{w}|||, \\ a(\boldsymbol{v}, \boldsymbol{v}) &= |||\boldsymbol{v}|||^2. \end{aligned}$$

Theorem 2.1. *For the numerical solution of the weak Galerkin algorithm (5)-(6) with the initial setting (7), there is a stability as follows*

$$\|\boldsymbol{u}_h(t)\|^2 \leq e^{Ct}(\|\boldsymbol{u}_h(0)\|^2 + \int_0^t \|\boldsymbol{f}(\tau)\|^2 \ d\tau).$$

Proof. Taking $\mathbf{v} = \mathbf{u}_h$ in (5) and $q = q_h$ in (6), we have

$$((\mathbf{u}_h)_t, \mathbf{u}_h) + a(\mathbf{u}_h, \mathbf{u}_h) = (\mathbf{f}, \mathbf{u}_h).$$

Considering the definition of $a(\cdot,\cdot)$, it follows that

$$a(\mathbf{u}_h, \mathbf{u}_h) \geq 0.$$

Thus, it is to know that

$$((\mathbf{u}_h)_t, \mathbf{u}_h) \leq (\mathbf{f}, \mathbf{u}_h),$$

i.e.

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \mathbf{u}_h^2(t)\ dx \leq \int_\Omega \mathbf{f}\mathbf{u}_h\ dx.$$

Using Canchy-Schwarz inequality and Young inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \mathbf{u}_h^2\ dx \leq C(\int_\Omega \mathbf{f}^2\ dx + \int_\Omega \mathbf{u}_h^2\ dx).$$

Integrating the above inequality with respect to t, we get

$$\|\mathbf{u}_h(t)\|^2 \leq \|\mathbf{u}_h(0)\|^2 + C\int_0^t \|\mathbf{f}(\tau)\|^2\ d\tau + C\int_0^t \|\mathbf{u}(\tau)\|^2\ d\tau.$$

Based on the Gronwall lemma, we complete the proof of the theorem. □

Next, for time discretization, we use the implicit backward Euler method to discrete the semi-discrete WG finite element scheme (5)-(7). Let τ denote the time step, and $t_n = n\tau(n = 0, 1, \cdots), \mathbf{u}_h^n = \mathbf{u}_h(t_n)$. The backward Euler method for time discretization is given by

$$\overline{\partial}_t \mathbf{u}_h^n = \frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\tau}.$$

Therefore, we obtain the full-discrete weak Galerkin finite element scheme for the Stokes equation (1)-(4).

Weak Galerkin Algorithm 2. *A full-discrete numerical approximation for (1)-(4) can be obtained by finding $(\boldsymbol{u}_h^n; p_h^n) \in V_h^0 \times W_h$ with any positive integer n and $0 \leq t \leq T$ such that*

$$(\overline{\partial}_t \boldsymbol{u}_h^n, \boldsymbol{v}_0) + a(\boldsymbol{u}_h^n, \boldsymbol{v}) - b(\boldsymbol{v}_0, p_h^n) = (\boldsymbol{f}^n, \boldsymbol{v}_0), \tag{10}$$

$$b(\boldsymbol{u}_h^n, q) = 0, \tag{11}$$

$$\boldsymbol{u}_h^0 = Q_h \boldsymbol{u}^0. \tag{12}$$

Lemma 2.3. *The full-discrete weak Galerkin finite element scheme (10)-(12) has a unique solution.*

Proof. It is sufficient to prove the following homogenous equation has a unique zero solution

$$(\overline{\partial}_t \mathbf{u}_h^n, \mathbf{v}_0) + a(\mathbf{u}_h^n, \mathbf{v}) - b(\mathbf{v}_0, p_h^n) = 0, \tag{13}$$

$$b(\mathbf{u}_h^n, q) = 0, \tag{14}$$

$$\mathbf{u}_h^0 = 0. \tag{15}$$

Taking $\mathbf{v} = \mathbf{u}_h^n$ and $q = p_h^n$ in (13) and (14), respectively, we have

$$(\overline{\partial}_t \mathbf{u}_h^n, \mathbf{u}_h^n) + a(\mathbf{u}_h^n, \mathbf{u}_h^n) = 0.$$

Since

$$\begin{aligned}
(\overline{\partial}_t \mathbf{u}_h^n, \mathbf{u}_h^n) &= \frac{1}{\tau}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{u}_h^n) \\
&= \frac{1}{2\tau}((\mathbf{u}_h^n, \mathbf{u}_h^n) - (\mathbf{u}_h^{n-1}, \mathbf{u}_h^{n-1}) + (\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{u}_h^n - \mathbf{u}_h^{n-1})) \\
&\geq \frac{1}{2\tau}(\|\mathbf{u}_h^n\|^2 - \|\mathbf{u}_h^{n-1}\|^2),
\end{aligned}$$

we obtain

$$\frac{1}{2\tau}(\|\mathbf{u}_h^n\|^2 - \|\mathbf{u}_h^{n-1}\|^2) + |\!|\!|\mathbf{u}_h^n|\!|\!|^2 \le 0.$$

Repeating application yields

$$\|\mathbf{u}_h^n\|^2 + 2\tau \sum_{i=1}^{n} |\!|\!|\mathbf{u}_h^i|\!|\!|^2 \le 0.$$

It follows from the positive definiteness of norm that

$$|\!|\!|\mathbf{u}_h^i|\!|\!| = 0, \quad 1 \le i \le n, \quad \text{on} \quad \mathbf{u}_h^n = 0.$$

From the rationality of the definition of $|\!|\!|\cdot|\!|\!|$ in paper [15], we get

$$\mathbf{u}_0^i = \mathbf{u}_b^i = \mathbf{0}, \quad 1 \le i \le n.$$

With the Lemma 4.3 in paper [15], $\mathbf{u}_h^n = 0$ and the form (13), we arrive at

$$p_h^i = 0, \quad 1 \le i \le n,$$

which completes the proof of this lemma. □

3. Error Equations

In this section, we will introduce error equations for the semi-discrete scheme (5)-(7) and full-discrete scheme (10)-(12).

3.1. Semi-discrete weak Galerkin error equation. About all, we first define the projections $E_h\mathbf{u}$ and $\mathcal{E}_h p$ onto V_h^0 and W_h respectively for the exact solution of the Stokes problem (1)-(4).

$$a(E_h\mathbf{u}, \mathbf{x}) - b(\mathbf{x}, \mathcal{E}_h p) = (-\Delta\mathbf{u}, \mathbf{x}) + (\nabla p, \mathbf{x}), \tag{16}$$
$$b(E_h\mathbf{u}, \mathbf{x}) = (\nabla\cdot\mathbf{u}, \mathbf{x}). \tag{17}$$

Those similar definitions of Wheeler's projections are introduced in [5, 20]. $E_h\mathbf{u}$ and $\mathcal{E}_h p$ are standard weak Galerkin finite element solution of the Stokes problem.

Lemma 3.1. *Assume* $(\mathbf{u}; p) \in L^2(0, t; [H_0^1(\Omega)]^d) \times L^2(0, t; L_0^2(\Omega))$ *is the exact solution of the Stokes problem (1)-(4), there exists a constant* C *satisfying*

$$|\!|\!|Q_h\mathbf{u} - E_h\mathbf{u}|\!|\!| \le Ch^k(\|\mathbf{u}\|_{k+1} + \|p\|_k), \tag{18}$$
$$\|Q_h\mathbf{u} - E_h\mathbf{u}\| \le Ch^{k+1}(\|\mathbf{u}\|_{k+1} + \|p\|_k), \tag{19}$$
$$\|\mathbf{u} - Q_h\mathbf{u}\| \le Ch^{k+1}\|\mathbf{u}\|_{k+1}. \tag{20}$$

Denote

$$\begin{aligned} \mathbf{e} &= E_h\mathbf{u} - \mathbf{u}_h,\ \varepsilon = \mathcal{E}_h p - p_h, \\ \eta &= Q_h\mathbf{u} - E_h\mathbf{u},\ \eta_t = Q_h\mathbf{u}_t - E_h\mathbf{u}_t, \\ \rho &= \mathbf{u} - Q_h\mathbf{u},\ \rho_t = \mathbf{u}_t - Q_h\mathbf{u}_t. \end{aligned}$$

Lemma 3.2. *Assume* $(\mathbf{u}; p) \in L^2(0, t; [H_0^1(\Omega)]^d) \times L^2(0, t; L_0^2(\Omega)]^d)$ *is the exact solution of (1)-(4), and* $(\boldsymbol{u}_h; p_h) \in V_h^0 \times W_h$ *is the numerical solution of (5)-(7) with any* $(\boldsymbol{v}; q) \in V_h^0 \times W_h$ *and* $0 \le t \le T$. *Then, we have*

$$(\mathbf{e}_t, \boldsymbol{v}) + a(\mathbf{e}, \boldsymbol{v}) - b(\boldsymbol{v}, \varepsilon) = -(\rho_t, \boldsymbol{v}) - (\eta_t, \boldsymbol{v}), \tag{21}$$
$$b(\mathbf{e}, q) = 0. \tag{22}$$

Proof. First, testing (1) by $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_h^0$, we arrive at

$$(\mathbf{u}_t, \mathbf{v}) - (\Delta \mathbf{u}, \mathbf{v}) + (\nabla p, \mathbf{v}) = (\mathbf{f}, \mathbf{v}).$$

Using the definition of $a(E_h\mathbf{u}, \mathbf{v})$ and $b(\mathbf{v}, \mathcal{E}_h p)$, we obtain

$$(\mathbf{u}_t, \mathbf{v}) + a(E_h\mathbf{u}, \mathbf{v}) - b(\mathbf{v}, \mathcal{E}_h p) = (\mathbf{f}, \mathbf{v}). \tag{23}$$

The difference between (23) and (5) yields the following equation

$$\begin{aligned} & a(E_h\mathbf{u} - \mathbf{u}_h, \mathbf{v}) - b(\mathbf{v}, \mathcal{E}_h p - p_h) \\ = \; & ((\mathbf{u}_h)_t - \mathbf{u}_t, \mathbf{v}) \\ = \; & ((\mathbf{u}_h)_t - E_h\mathbf{u}_t, \mathbf{v}) + (E_h\mathbf{u}_t - Q_h\mathbf{u}_t, \mathbf{v}) + (Q_h\mathbf{u}_t - \mathbf{u}_t, \mathbf{v}). \end{aligned}$$

That is

$$(\mathbf{e}_t, \mathbf{v}) + a(\mathbf{e}, \mathbf{v}) - b(\mathbf{v}, \varepsilon) = -(\rho_t, \mathbf{v}) - (\eta_t, \mathbf{v}).$$

This completes the derivation of (21).

Next, we test the equation (2) by $q \in W_h$ and use the definition of $b(E_h\mathbf{u}, \mathbf{v})$ to obtain

$$b(E_h\mathbf{u}, q) = 0. \tag{24}$$

It follows from (24) and (6) that

$$b(\mathbf{e}, q) = 0,$$

which completes the derivation of (22). Therefore, we obtain the semi-discrete error equations. □

3.2. Full-discrete weak Galerkin error equation. In this part, we derive the full-discrete error equations for the Stokes problem (1)-(4). For simplicity, we only present the analysis for the full-discrete scheme with backward Euler time discretization.

Denote

$$\begin{aligned} \mathbf{e}^n &= E_h\mathbf{u}^n - \mathbf{u}_h^n, \ \varepsilon^n = \mathcal{E}_h p^n - p_h^n, \\ \eta^n &= Q_h\mathbf{u}^n - E_h\mathbf{u}^n, \ \rho^n = \mathbf{u}^n - Q_h\mathbf{u}^n. \end{aligned}$$

Lemma 3.3. *Let $(\boldsymbol{u}_h^n; p_h^n) \in V_h \times W_h$ be the numerical solution of (10) -(12), and $(\boldsymbol{u}; p) \in L^2(0, T; [H_0^1(\Omega)]^d) \times L^2(0, T; L_0^2(\Omega))$ be the exact solution of (1)-(4) for $0 \le t \le T$. Then, for any $\boldsymbol{v} \in V_h^0$ and $q \in W_h$, we have*

$$(\overline{\partial}_t\mathbf{e}^n, \mathbf{v}) + a(\mathbf{e}^n, \mathbf{v}) - b(\mathbf{v}, \varepsilon^n) = -(\overline{\partial}_t\eta^n, \mathbf{v}) - (\overline{\partial}_t\rho^n, \mathbf{v}) - (\mathbf{u}_t^n - \overline{\partial}_t\mathbf{u}_h^n, \mathbf{v}), \tag{25}$$

$$b(\mathbf{e}^n, q) = 0. \tag{26}$$

Proof. Considering (23) and (10), we arrive at

$$a(E_h\mathbf{u}^n - \mathbf{u}_h^n, \mathbf{v}) - b(\mathbf{v}, \mathcal{E}_h p^n - p_h^n) = -(\mathbf{u}_t^n - \overline{\partial}_t\mathbf{u}_h^n, \mathbf{v}).$$

Adding $(\overline{\partial}_t(E_h\mathbf{u}^n - \mathbf{u}_h^n), \mathbf{v})$ to both sides of the above equation gives

$$\begin{aligned} & (\overline{\partial}_t(E_h\mathbf{u}^n - \mathbf{u}_h^n), \mathbf{v}) + a(E_h\mathbf{u}^n - \mathbf{u}_h^n, \mathbf{v}) - b(\mathbf{v}, \mathcal{E}_h p^n - p_h^n) \\ = \; & (\overline{\partial}_t(E_h\mathbf{u}^n - Q_h\mathbf{u}_h^n), \mathbf{v}) + (\overline{\partial}_t(Q_h\mathbf{u}^n - \mathbf{u}_h^n), \mathbf{v}) - (\mathbf{u}_t - \overline{\partial}_t\mathbf{u}_h^n, \mathbf{v}). \end{aligned}$$

i.e.

$$(\overline{\partial}_t\mathbf{e}^n, \mathbf{v}) + a(\mathbf{e}^n, \mathbf{v}) - b(\mathbf{v}, \varepsilon^n) = -(\overline{\partial}_t\rho^n, \mathbf{v}) - (\overline{\partial}_t\eta^n, \mathbf{v}) - (\mathbf{u}_t - \overline{\partial}_t\mathbf{u}_h^n, \mathbf{v}).$$

This completes the derivation of (25). From the full-discrete form (11) and (24), we have the fact

$$b(\mathbf{e}^n, q) = 0.$$

This completes the derivation of (26). Thus, we get the full-discrete error equations. □

4. Error Estimates

In this section, we shall establish optimal order error estimates for the velocity approximation $\mathbf{u}_h$ in a norm which is equivalent to the usual H^1 norm and the standard L^2 norm, and for the pressure approximation p_h in the standard L^2 norm both for the semi-discrete scheme and full-discrete scheme, respectively.

4.1. Semi-discrete weak Galerkin error estimate. This section gives the error estimates for the semi-discrete weak Galerkin scheme.

Theorem 4.1. *Let $(\boldsymbol{u}_h; p_h) \in V_h^0 \times W_h$ be the numerical solution of the Stokes problem (1)-(4). Assume the exact solution is smooth enough so that $(\boldsymbol{u}; p) \in L^2(0,T;[H_0^1(\Omega)]^d) \times L^2(0,T;L_0^2(\Omega))$. Then the following error estimate holds true*

$$\|\mathbf{e}\|^2 \leq \|\mathbf{e}(\cdot,0)\|^2 + Ch^{2(k+1)} \int_0^t (\|\mathbf{u}_\tau\|_{k+1}^2 + \|p\|_k^2)\, d\tau. \tag{27}$$

Proof. Letting $\mathbf{v} = \mathbf{e}$ in (21) and $q = \varepsilon$ in (22), we obtain

$$(\mathbf{e}_t, \mathbf{e}) + a(\mathbf{e}, \mathbf{e}) = -(\eta_t, \mathbf{e}) - (\rho_t, \mathbf{e}). \tag{28}$$

From the definition of $|\!|\!| \cdot |\!|\!|$, Young's inequality and the relation between $\|\cdot\|$ and $|\!|\!| \cdot |\!|\!|$, we get

$$\begin{aligned} (\mathbf{e}_t, \mathbf{e}) &= \frac{1}{2}\frac{d}{dt}(\mathbf{e},\mathbf{e}) = \frac{1}{2}\frac{d}{dt}\|\mathbf{e}\|^2, \\ a(\mathbf{e},\mathbf{e}) &= |\!|\!|\mathbf{e}|\!|\!|^2, \\ -(\rho_t, \mathbf{e}) &\leq \frac{1}{2}\|\rho_t\|^2 + \frac{1}{2}\|\mathbf{e}\|^2 \leq \frac{1}{2}\|\rho_t\|^2 + \frac{1}{2}|\!|\!|\mathbf{e}|\!|\!|^2, \\ -(\eta_t, \mathbf{e}) &\leq \frac{1}{2}\|\eta_t\|^2 + \frac{1}{2}\|\mathbf{e}\|^2 \leq \frac{1}{2}\|\eta_t\|^2 + \frac{1}{2}|\!|\!|\mathbf{e}|\!|\!|^2. \end{aligned}$$

Combining the forms above into (28) yields

$$\frac{d}{dt}\|\mathbf{e}\|^2 \leq \|\eta_t\|^2 + \|\rho_t\|^2.$$

Integrating above inequality with respect to t, we have

$$\|\mathbf{e}\|^2 \leq \|\mathbf{e}(\cdot,0)\|^2 + \int_0^t (\|\eta_\tau\|^2 + \|\rho_t\|^2)\, d\tau.$$

We complete the proof of (27) by (19) and (20). □

Theorem 4.2. *Let $(\boldsymbol{u}; p) \in L^2(0,T;[H_0^1(\Omega)]^d) \times L^2(0,T;L_0^2(\Omega))$ be the exact solution of the Stokes problem (1)-(4), and $(\boldsymbol{u}_h; p_h) \in V_h^0 \times W_h$ be the numerical solution of the semi-discrete WG (5)-(7). Then there is a constant C satisfying*

$$\int_0^t \|\mathbf{e}_\tau\|^2\, d\tau \leq |\!|\!|\mathbf{e}(\cdot,0)|\!|\!|^2 + Ch^{2(k+1)} \int_0^t (\|\mathbf{u}_\tau\|_{k+1}^2 + \|p\|_k^2)\, d\tau, \tag{29}$$

$$|\!|\!|\mathbf{e}|\!|\!|^2 \leq |\!|\!|\mathbf{e}(\cdot,0)|\!|\!|^2 + Ch^{2(k+1)} \int_0^t (\|\mathbf{u}_\tau\|_{k+1}^2 + \|p\|_k^2)\, d\tau, \tag{30}$$

$$\int_0^t \|\varepsilon\|^2\, d\tau \leq |\!|\!|\mathbf{e}(\cdot,0)|\!|\!|^2 + Ch^{2(k+1)} \int_0^t (\|\mathbf{u}_\tau\|_{k+1}^2 + \|p\|_k^2)\, d\tau. \tag{31}$$

Proof. When $\mathbf{v} = \mathbf{e}_t$ in (21) and $q = \varepsilon$ in (22), we have

$$\begin{aligned} (\mathbf{e}_t, \mathbf{e}_t) + a(\mathbf{e}, \mathbf{e}_t) - b(\mathbf{e}_t, \varepsilon) &= -(\eta_t, \mathbf{e}_t) - (\rho_t, \mathbf{e}_t), \\ b(\mathbf{e}, \varepsilon) &= 0. \end{aligned}$$

We have the fact

$$\begin{aligned} 0 &= \frac{d}{dt} b(\mathbf{e}, q) = \frac{d}{dt} \int_\Omega (\nabla \cdot \mathbf{e}) q \ dx \\ &= \int_\Omega (\nabla \cdot \mathbf{e}_t) q \ dx + \int_\Omega (\nabla \cdot \mathbf{e}) q_t \ dx \\ &= b(\mathbf{e}_t, q) + b(\mathbf{e}, q_t). \end{aligned}$$

From (22), we get $b(\mathbf{e}_t, q) = 0$, for any $q \in W_h$ which yields

$$(\mathbf{e}_t, \mathbf{e}_t) + a(\mathbf{e}, \mathbf{e}_t) = -(\rho_t, \mathbf{e}_t) - (\eta_t, \mathbf{e}_t).$$

Using the definition of $\|\!|\cdot|\!\|$ and Young's inequality, we arrive at

$$\begin{aligned} \|\mathbf{e}_t\|^2 + \frac{1}{2}\frac{d}{dt} |\!\|\mathbf{e}|\!\|^2 &= (\mathbf{e}_t, \mathbf{e}_t) + a(\mathbf{e}, \mathbf{e}_t) = -(\eta_t, \mathbf{e}_t) - (\rho_t, \mathbf{e}_t) \\ &\leq \|\eta_t\|^2 + \frac{1}{4}\|\mathbf{e}_t\|^2 + \|\rho_t\|^2 + \frac{1}{4}\|\mathbf{e}_t\|^2, \end{aligned}$$

which leads to

$$\|\mathbf{e}_t\|^2 + \frac{d}{dt} |\!\|\mathbf{e}|\!\|^2 \leq 2\|\eta_t\|^2 + 2\|\rho_t\|^2.$$

Integrating above formula with respect to t, we obtain

$$\int_0^t \|\mathbf{e}_\tau\|^2 \ d\tau + |\!\|\mathbf{e}|\!\|^2 \leq |\!\|\mathbf{e}(\cdot, 0)|\!\|^2 + 2\int_0^t (\|\eta_\tau\|^2 + \|\rho_\tau\|^2) \ d\tau.$$

From the estimates of η_t and ρ_t, we get (29) and (30).

From the error equations (21), we have

$$b(\mathbf{v}, \varepsilon) = (\eta_t, \mathbf{v}) + (\rho_t, \mathbf{v}) + a(\mathbf{e}, \mathbf{v}) + (\mathbf{e}_t, \mathbf{v}).$$

Using inf-sup Condition, Cauchy-Schwarz inequality and the property of $a(\cdot, \cdot)$, we obtain

$$\begin{aligned} C\|\varepsilon\| \cdot |\!\|\mathbf{v}|\!\| &\leq b(\mathbf{v}, \varepsilon) = (\rho_t, \mathbf{v}) + (\eta_t, \mathbf{v}) + (\mathbf{e}_t, \mathbf{v}) + a(\mathbf{e}, \mathbf{v}) \\ &\leq C\|\rho_t\| \cdot |\!\|\mathbf{v}|\!\| + C\|\eta_t\| \cdot |\!\|\mathbf{v}|\!\| + C\|\mathbf{e}_t\| \cdot |\!\|\mathbf{v}|\!\| + |\!\|\mathbf{e}|\!\| \cdot |\!\|\mathbf{v}|\!\|. \end{aligned}$$

Furthermore

$$\|\varepsilon\|^2 \leq C(\|\eta_t\| + \|\rho_t\| + |\!\|\mathbf{e}|\!\| + \|\mathbf{e}_t\|)^2.$$

We get the following inequality by integrating the above inequality from 0 to t

$$\int_0^t \|\varepsilon\|^2 \ d\tau \leq |\!\|\mathbf{e}(\cdot, 0)|\!\|^2 + Ch^{2(k+1)} \int_0^t (\|\mathbf{u}_\tau\|_{k+1}^2 + \|p\|_k^2) \ d\tau.$$

This completes the proof of (29)-(31). □

4.2. Full-discrete weak Galerkin error estimates. This part presents the error estimates for the backward Euler weak Galerkin scheme (10)-(12).

Theorem 4.3. *Assume* $(\boldsymbol{u};p) \in L^2(0,T;[H_0^1(\Omega)]^d) \times L^2(0,T;L_0^2(\Omega))$ *is the exact solution of the Stokes problem (1)-(4), and* $(\boldsymbol{u}_h, p_h) \in V_h^0 \times W_h$ *is the numerical solution of the full-discrete WG scheme (10)-(12), then the following error estimate holds true*

$$\|\mathbf{e}^n\| \le \|\mathbf{e}^0\| + C(\tau \int_0^{t_n} \|\mathbf{u}_{tt}\|\ dt + h^{k+1} \int_0^{t_n} (\|\mathbf{u}_t\|_{k+1} + \|p\|_k)\ dt). \tag{32}$$

Proof. Letting $\mathbf{v} = \mathbf{e}^n$ in (25) and $q = \varepsilon^n$ in (26), we obtain

$$(\overline{\partial}_t \mathbf{e}^n, \mathbf{e}^n) + a(\mathbf{e}^n, \mathbf{e}^n) = -(\overline{\partial}_t \eta^n, \mathbf{e}^n) - (\overline{\partial}_t \rho^n, \mathbf{e}^n) - (\mathbf{u}_t^n - \overline{\partial}_t \mathbf{u}_h^n, \mathbf{e}^n).$$

By the backward Euler form, the definition of $\|\!|\cdot|\!\|$ and Young's inequality, we have

$$\begin{aligned}
(\overline{\partial}_t \mathbf{e}^n, \mathbf{e}^n) &= (\frac{\mathbf{e}^n - \mathbf{e}^{n-1}}{\tau}, \mathbf{e}^n) \\
&= \frac{1}{\tau}((\mathbf{e}^n, \mathbf{e}^n) - (\mathbf{e}^{n-1}, \mathbf{e}^n)) \\
&= \frac{1}{\tau}\|\mathbf{e}^n\|^2 - \frac{1}{\tau}(\mathbf{e}^{n-1}, \mathbf{e}^n), \\
a(\mathbf{e}^n, \mathbf{e}^n) &= |\!|\!|\mathbf{e}^n|\!|\!|^2, \\
-(\overline{\partial}_t \eta^n, \mathbf{e}^n) &\le \|\overline{\partial}_t \eta^n\|^2 + \frac{1}{4}\|\mathbf{e}^n\|^2 \\
&\le \|\overline{\partial}_t \eta^n\|^2 + \frac{1}{4}|\!|\!|\mathbf{e}^n|\!|\!|^2, \\
-(\overline{\partial}_t \rho^n, \mathbf{e}^n) &\le \|\overline{\partial}_t \rho^n\|^2 + \frac{1}{4}\|\mathbf{e}^n\|^2 \\
&\le \|\overline{\partial}_t \rho^n\|^2 + \frac{1}{4}|\!|\!|\mathbf{e}^n|\!|\!|^2,
\end{aligned}$$

and

$$-(\mathbf{u}_t^n - \overline{\partial}_t \mathbf{u}^n, \mathbf{e}^n) \le \frac{1}{2}\|\mathbf{u}_t^n - \overline{\partial}_t \mathbf{u}_h^n\|^2 + \frac{1}{2}|\!|\!|\mathbf{e}^n|\!|\!|^2.$$

Combining with all these formulas and repeating application, it follows

$$\begin{aligned}
\|\mathbf{e}^n\|^2 &\le \|\mathbf{e}^{n-1}\|^2 + C\tau(\|\overline{\partial}_t \eta^n\|^2 + \|\overline{\partial}_t \rho^n\|^2 + \|\mathbf{u}_t^n - \overline{\partial}_t \mathbf{u}^n\|^2) \\
&\ \vdots \\
&\le \|\mathbf{e}^0\|^2 + C\tau \sum_{i=1}^{n} (R_1^{2i} + R_2^{2i} + R_3^{2i}),
\end{aligned}$$

where

$$\begin{aligned}
R_1^i &= \|\overline{\partial}_t \eta^i\| = \frac{1}{\tau}\| \int_{t_{i-1}}^{t_i} (Q_h - E_h)\mathbf{u}_t\ dt\| \le \frac{1}{\tau} \int_{t_{i-1}}^{t_i} Ch^{k+1}(\|\mathbf{u}_t\|_{k+1} + \|p\|_k)\ dt, \\
R_2^i &= \|\overline{\partial}_t \rho^i\| = \frac{1}{\tau}\| \int_{t_{i-1}}^{t_i} (Q_h - I)\mathbf{u}_t\ dt\| \le \frac{1}{\tau} \int_{t_{i-1}}^{t_i} Ch^{k+1}(\|\mathbf{u}_t\|_{k+1} + \|p\|_k)\ dt, \\
R_3^i &= \|\mathbf{u}_t(t_i) - \overline{\partial}_t \mathbf{u}(t_i)\| = \frac{1}{\tau}\| \int_{t_{i-1}}^{t_i} (s - t_{i-1})\mathbf{u}_{tt}(s)\ ds\| \le \int_{t_{i-1}}^{t_i} \|\mathbf{u}_{tt}\|\ dt.
\end{aligned}$$

We get the estimate (32) by summing over all elements $T \in \mathcal{T}_h$ in above inequalities. □

Theorem 4.4. *Let $(\boldsymbol{u};p) \in L^2(0,T;[H_0^1(\Omega)]^d) \times L^2(0,T;L_0^2(\Omega))$ be the exact solution of the problem (1)-(4), and $(\boldsymbol{u}_h;p_h) \in V_h^0 \times W_h$ be the numerical solution arising from full-discrete WG scheme (10)-(12) satisfying*

$$\|\!|\mathbf{e}^n|\!\| \leq \|\!|\mathbf{e}^0|\!\| + C(\tau \int_0^{t_n} \|\mathbf{u}_{tt}\|\ dt + h^{k+1} \int_0^{t_n} (\|\mathbf{u}_t\|_{k+1} + \|p\|_k)\ dt), \tag{33}$$

$$\|\overline{\partial}_t \mathbf{e}^n\| \leq \|\!|\mathbf{e}^0|\!\| + C(\tau \int_0^{t_n} \|\mathbf{u}_{tt}\|\ dt + h^{k+1} \int_0^{t_n} (\|\mathbf{u}_t\|_{k+1} + \|p\|_k)\ dt), \tag{34}$$

$$\|\varepsilon^n\| \leq \|\!|\mathbf{e}^0|\!\| + C(\tau \int_0^{t_n} \|\mathbf{u}_{tt}\|\ dt + h^{k+1} \int_0^{t_n} (\|\mathbf{u}_t\|_{k+1} + \|p\|_k)\ dt). \tag{35}$$

Proof. Taking $\mathbf{v} = \overline{\partial}_t \mathbf{e}^n$ in (25) and $q = \varepsilon^n$ in (26), we have

$$\begin{aligned} & \|\overline{\partial}_t \mathbf{e}^n\|^2 + a(\mathbf{e}^n, \overline{\partial}_t \mathbf{e}^n) \\ = & -(\overline{\partial}_t \eta^n, \overline{\partial}_t \mathbf{e}^n) - (\overline{\partial}_t \rho^n, \overline{\partial}_t \mathbf{e}^n) - (\mathbf{u}_t^n - \overline{\partial}_t \mathbf{u}^n, \overline{\partial}_t \mathbf{e}^n), \end{aligned} \tag{36}$$

where using the fact $b(\mathbf{e}^{n-1}, \varepsilon^n) = 0$.

From the backward Euler form, the definition of $\|\!|\cdot|\!\|$ and Young's inequality, we obtain

$$\begin{aligned} a(\mathbf{e}^n, \overline{\partial}_t \mathbf{e}^n) &= a(\mathbf{e}^n, \frac{\mathbf{e}^n - \mathbf{e}^{n-1}}{\tau}) \\ &= \frac{1}{\tau} a(\mathbf{e}^n, \mathbf{e}^n) - \frac{1}{\tau} a(\mathbf{e}^n, \mathbf{e}^{n-1}), \end{aligned} \tag{37}$$

$$-(\overline{\partial}_t \eta^n, \overline{\partial}_t \mathbf{e}^n) \leq \|\overline{\partial}_t \eta^n\|^2 + \frac{1}{4}\|\overline{\partial}_t \mathbf{e}^n\|^2, \tag{38}$$

$$-(\overline{\partial}_t \rho^n, \overline{\partial}_t \mathbf{e}^n) \leq \|\overline{\partial}_t \rho^n\|^2 + \frac{1}{4}\|\overline{\partial}_t \mathbf{e}^n\|^2, \tag{39}$$

and

$$(\mathbf{u}_t^n - \overline{\partial}_t \mathbf{u}^n, \overline{\partial}_t \mathbf{e}^n) \leq \frac{1}{2}\|\mathbf{u}_t^n - \overline{\partial}_t \mathbf{u}^n\|^2 + \frac{1}{2}\|\overline{\partial}_t \mathbf{e}^n\|^2.$$

Combining all these estimate forms and repeating application yields

$$\begin{aligned} \|\!|\mathbf{e}^n|\!\| &\leq \|\!|\mathbf{e}^{n-1}|\!\| + C\tau(R_1^n + R_2^n + R_3^n) \\ &\vdots \\ &\leq \|\!|\mathbf{e}^0|\!\| + C\tau \sum_{i=1}^n (R_1^i + R_2^i + R_3^i). \end{aligned}$$

From the estimates of R_1^i, R_2^i and R_3^i, we get (33).

Then, we have the fact

$$(\mathbf{u}_t^n - \overline{\partial}_t \mathbf{u}^n, \overline{\partial}_t \mathbf{e}^n) \leq \|\mathbf{u}_t^n - \overline{\partial}_t \mathbf{u}^n\|^2 + \frac{1}{4}\|\overline{\partial}_t \mathbf{e}^n\|^2. \tag{40}$$

Combining (36)-(40), we arrive at

$$\|\overline{\partial}_t \mathbf{e}^n\| \leq \|\!|\mathbf{e}^0|\!\| + C\tau \sum_{i=1}^n (R_1^i + R_2^i + R_3^i).$$

We get (34) from (33) and the estimates of R_1^i, R_2^i and R_3^i.

Finally, from (25), inf-sup condition, and full-discrete estimates (32)-(34), we have (35). ◻

5. Numerical Experiments

In this section, we provide some numerical experiments to illustrate the weak Galerkin finite element method for the time-dependent Stokes flow in this paper.

The numerical experiments are based on the weak Galerkin finite element space for the velocity function $\mathbf{u}$

$$V_h = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\}, \mathbf{v}_0 \in [P_k(T)]^d, \mathbf{v}_b \in P[_{k-1}(e)]^d, \forall T \in \mathcal{T}_h, \forall e \in \partial T\}.$$

The weak Galerkin finite element space for the pressure function p

$$W_h = \{q \in L_0^2(\Omega), q|_T \in P_{k-1}(T), \forall T \in \mathcal{T}_h\}.$$

Denote by $(\mathbf{u}; p)$ be the exact solution of the Stokes problem (1)-(4) and $(\mathbf{u}_h, p_h)$ be the numerical solution of the full-discrete weak Galerkin scheme (10)-(12), respectively. The errors between the exact solution and numerical solution are defined by $\mathbf{e}^n = E_h\mathbf{u}^n - \mathbf{u}_h^n$ and $\varepsilon^n = \mathcal{E}_h p^n - p_h^n$, where $E_h\mathbf{u}$ and $\mathcal{E}_h p$ are the elliptical projection of the exact solution $\mathbf{u}$ and p onto V_h^0 and W_h, respectively. Define three norms of the error for the weak Galerkin finite element solution as follows

$$\begin{aligned}
\|\mathbf{e}^n\| &\leq \|\mathbf{e}^0\| + C(\tau \int_0^{t_n} \|\mathbf{u}_{tt}\|\ dt + h^{k+1} \int_0^{t_n} (\|\mathbf{u}_t\|_{k+1} + \|p\|_k)\ dt), \\
|\!|\!|\mathbf{e}^n|\!|\!| &\leq |\!|\!|\mathbf{e}^0|\!|\!| + C(\tau \int_0^{t_n} \|\mathbf{u}_{tt}\|\ dt + h^{k+1} \int_0^{t_n} (\|\mathbf{u}_t\|_{k+1} + \|p\|_k)\ dt), \\
\|\varepsilon^n\| &\leq |\!|\!|\mathbf{e}^0|\!|\!| + C(\tau \int_0^{t_n} \|\mathbf{u}_{tt}\|\ dt + h^{k+1} \int_0^{t_n} (\|\mathbf{u}_t\|_{k+1} + \|p\|_k)\ dt).
\end{aligned}$$

5.1. Example 1. Consider the time dependent Stokes problem (1)-(4) in the square domain $\Omega = (0,1)^2$ with the Dirichlet boundary condition. We use the uniform triangle mesh with the mesh size h and time step τ. The weak Galerkin finite element space with $k = 2$ is employed in the numerical discretization. The analytic solution is

$$\mathbf{u} = \begin{pmatrix} \sin(2\pi x)\cos(2\pi y)e^{-t} \\ -\cos(2\pi x)\sin(2\pi y)e^{-t} \end{pmatrix} \text{ and } p = e^{-t}(2\pi\cos(2\pi x)\cos(2\pi y)).$$

The right-hand side function $\mathbf{f}$ in the equation (1) is computed to match the exact solution.

Table 5.1 shows the errors and convergence rates with respect to different h, when the time step is fixed $\tau = 1/512$ and $k = 2$. It is obvious that the convergence rates for the velocity function in H^1 norm and the pressure function in L^2 norm are of order $O(h^2)$. The convergence rates for the velocity function in L^2 norm is of order $O(h^2)$, which coincides with the theoretical analysis.

Table 5.2 shows the errors and convergence rates with respect to different τ, when the the mesh size is fixed $h = 1/64$ and $k = 2$. The convergence rates for the velocity function $\mathbf{u}$ and the pressure function p are of order $O(h^1)$ both in H^1 norm and L^2 norm.

5.2. Example 2. Consider the time dependent Stokes problem (1)-(4) in the square domain $\Omega = (0,1)^2$ with space interval h and time step τ. For convenience, we also study the Dirichlet boundary condition in the uniform triangle mesh with the weak Galerkin finite element space $k = 2$. It has the exact solution

$$\mathbf{u} = \begin{pmatrix} 2\pi\sin(\pi x)\sin(\pi x)\cos(\pi y)\sin(\pi y)\cos(t) \\ -2\pi\sin(\pi x)\sin(\pi y)\cos(\pi x)\sin(\pi y)\cos(t) \end{pmatrix} \text{ and } p = \cos(\pi x)\cos(\pi y)\cos(t).$$

The right-hand side function $\mathbf{f}$ is computed to match the exact solution.

TABLE 1. Example 1. Error and convergence rate for $k = 2$ with $\tau = 1/512$.

h	Error	Rate	Error	Rate	Error	Rate
1/2	2.2206E-02		2.0231E-02		2.0484E-02	
1/4	5.5369E-03	2.0038	2.5043E-03	3.0141	4.4565E-03	2.2005
1/8	1.3892E-03	1.9948	3.1338E-04	2.9984	1.0572E-03	2.0756
1/16	3.4832E-04	1.9958	3.9255E-05	2.9970	2.5815E-04	2.0340
1/32	8.7326E-05	1.9959	4.9204E-06	2.9960	6.3942E-05	2.0134
1/64	2.2248E-05	1.9727	6.5744E-07	2.9039	1.8048E-05	1.8249

TABLE 2. Example 1. Error and convergence rate for $k = 2$ with $h = 1/64$.

τ	Error	Rate	Error	Rate	Error	Rate
1/2	1.3176E-03		7.1020E-05		2.9119E-03	
1/4	6.0335E-04	1.1268	3.2514E-05	1.1272	1.3318E-03	1.1286
1/8	2.8928E-04	1.0605	1.5563E-05	1.0629	6.3728E-04	1.0633
1/16	1.4280E-04	1.0185	7.6331E-06	1.0278	3.1156E-04	1.0324
1/32	7.3129E-05	0.9655	3.8134E-06	1.0012	1.5407E-04	1.0159
1/64	4.0995E-05	0.8350	1.9703E-06	0.9527	7.7191E-05	0.9971

Table 5.3 presents the errors and convergence rates with respect to different h, when the time step is fixed $\tau = 1/512$ and $k = 2$. Table 5.4 presents the errors and the convergence rates with respect to different τ, when the mesh size is fixed $h = 1/300$ and $k = 2$. From the two tables, the optimal convergence orders are obtained both for the velocity function $\mathbf{u}$ and the pressure function p in different norms, which coincide with the theoretical analysis.

TABLE 3. Example 2. Error and convergence rate for $k = 2$ with $\tau = 1/512$.

h	Error	Rate	Error	Rate	Error	Rate
1/2	6.3415E+00		9.8662E+00		1.0214E+00	
1/4	2.0165E+00	1.6530	1.0205E+00	3.2732	4.8445E-01	1.0761
1/8	5.2254E-01	1.9482	1.0045E-01	3.3449	1.2297E-01	1.9780
1/16	1.3190E-01	1.9861	1.1441E-02	3.1341	3.0382E-02	2.0171
1/32	3.3066E-02	1.9961	1.3941E-03	3.0368	7.5424E-03	2.0101
1/64	8.2744E-03	1.9986	1.7282E-04	3.0120	1.8800E-03	2.0043

TABLE 4. Example 2. Error and convergence rate for $k = 2$ with $h = 1/300$.

τ	Error	Rate	Error	Rate	Error	Rate
1/2	4.4933E-02		2.5384E-03		4.4145E-03	
1/4	2.0618E-02	1.1238	1.1646E-03	1.1240	2.0272E-03	1.1227
1/8	9.7942E-03	1.0739	5.5289E-04	1.0748	9.6574E-04	1.0698
1/16	4.7726E-03	1.0371	2.6876E-04	1.0407	4.7591E-04	1.0210
1/32	2.3744E-03	1.0073	1.3242E-04	1.0212	2.4680E-04	0.9473
1/64	1.2229E-03	0.9573	6.5711E-05	1.0109	1.4333E-04	0.7840

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