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INTERIOR-EXTERIOR PENALTY APPROACH FOR SOLVING ELASTO-HYDRODYNAMIC LUBRICATION PROBLEM: PART I

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Abstract. A new interior-exterior penalty method for solving quasi-variational inequality and pseudo-monotone operator arising in two-dimensional point contact problem is analyzed and developed in discontinuous Galerkin finite volume (DG-FVEM) framework. We derive a discrete DG-FVEM formulation of the problem and prove existence and uniqueness results for it. Optimal error estimates in H^1 and L^2 norm are derived under a light load parameter assumptions. In addition, the article provides a complete algorithm to tackle all numerical complexities appear in the solution procedure. Numerical outcomes are presented for light, moderate and relative high load conditions. The variations of load parameter and its effect on the evolution of deformations and pressure profile are evaluated and described. This method is well suited for solving elasto-ehydrodynamic lubrication point contact problems and can probably be treated as commercial software. Furthermore, the results give a hope for the further development of the scheme for extreme load condition, observes in a more realistic operating situation which will be discussed in part II.

Key words. Elasto-hydrodynamic lubrication, discontinuous finite volume method, interior-exterior penalty method, pseudo-monotone operators, variational inequality.

1. Introduction

In the last century, a various attempts have been devoted by a large scientific community in shaping a more solid mathematical foundation, modeling and developing a robust scientific tool in the area of lubrication theory (study of thin-film flows). In particular, elasto-hydrodynamic lubrication (EHL) has picked up a notable innovation pace since its acceptance as the essential physical phenomenon behind the flourishing operation of many important industrial devices such as journal bearings, rolling contact bearings, gears etc. An extensive list of contributions of EHL model and theoretical development can be found mainly in [6, 29, 37, 12, 11, 24, 25].

EHL is indispensable mechanism of thin fluid-film lubrication characterized by high contact pressure. As a consequence an exceptional elastic deformation and piezo-viscous increase in lubricant viscosity. The mechanical action (squeezing, shearing etc) changes the lubricants film thickness, viscosity and density which account for the variation of bearing performance characteristics. Therefore, a qualitative and precise prediction of the elasto-hydrodynamic lubrication model requires consideration of the constitutive equation for the lubricant. In literature, numerous models [21, 29] have been introduced to describe the basic aspects of the EHL theory, where the three main attributes of this kind problems are quoted; the fluid hydrodynamic displacement (govern by Reynolds equation), the solid elastic deformation and the cavitation generation. Typical lubricated devices consist of a thin flow of lubricant between two contacting geometries in relative movement. Classically, this equation (known as Reynolds equation) is described by making heuristic

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rational argument (see [35], for example) and later by deducing from Stokes equation with the help of asymptotic techniques (see Bayada and Chambat [15]).

In the current study, two important features of the lubricating fluid are density and viscosity. We consider density of lubricant as function of applied pressure (that is non-homogeneous fluid) and it is governed by the empirical relation proposed by Dowson and Higginson [14] (see eqn (6)). We examine piezoviscous property of lubricant into account, ie. the viscosity is no longer constant and it obeys the Barus law [5] (see eqn (5)). The piezoviscous regimes has led to many outcomes based on mathematical analysis that describe the existence and uniqueness of the solution (see [12, 37, 24, 25], for example), as well as to design precise numerical methods for approximating the corresponding solutions, which have no analytical illustrations. An alternate pressure-viscosity relation also has been considered by many authors in literature (see [38, 21], for example).

It is well known that cavitation is one of the crucial phenomenon in EHL problems and it is interpreted as the rupture of the continuous fluid film due to formation of air bubbles inside the region. This phenomenon has been experimentally observed in many lubricated devices such as journal-bearings, ball-bearings, etc. Different cavitation models have been suggested for physical and mathematical analysis (e.g., see [6, 29]). One common ingredient of most of these models is the decomposition of the EHL region into two parts a lubricated part where Reynolds equation is governed and a cavitation region where the lubricant pressure is constant and equal to the saturation pressure. Consequently, the boundary splitting both regions is also priori not known in the problem, therefore modeling of the cavitation was used to impose a free boundary based on the following condition,

$$u_c = \nabla u.\mathbf{n} = 0,$$

where u_c stands for the cavitation pressure and $\mathbf{n}$ stands for the unit normal vector to the free boundary. This condition leads to a formulations in terms of a complementarity problem associated with the corresponding nonlinear Reynolds equation, or equivalently to a variational inequality (see eqns (1)–(3)).

Over last few decades, renowned interest has been paid to study Elastohydrodynamic lubrication (EHL) model problems in terms of theoretical as well as practical points of view. A significant amount of numerical techniques are available in literature for the EHL model problems [30, 21, 20, 19, 23, 36, 1, 32, 42, 43, 22, 11, 40, 31, 24, 45, 41]. Recently, theoretical study of finite volume element method (FVEM) [28] and discontinuous Galerkin finite volume element method (DG-FVEM) such as optimal error analysis is gaining momentum. In particular, these methods can be derived from a firm theoretical foundation and understanding similar to finite element, see for example [25, 7, 8, 9, 27, 17, 18]. Formulation of these methods is derived by integrating the partial differential equation (PDE) model over a control volume element. Due to its natural conservation characteristic feature, adaptivity and parallelizability, DG-FVEM gained popularity in development of scientific computing software for many real life complex phenomenon such as fluid mechanics, contact problems in mechanics, mathematical finance and hyperbolic conservation laws (traffic flow etc.) where analytical solution has minimum regularity of in nature. In many cases, implementing high order methods is not straight forward and it requires huge computational storage and time. On the other hand, using low order scheme we pay the price in the form of low accuracy when discretization grid are not small. To achieve the numerical accuracy one choice we have is, refine the mesh and use the parallelization. Hence, it is quite reasonable to demand the advantage of nonconforming or Discontinuous Galerkin finite element method (DG-FEM) (see for example [2, 34, 4, 44, 13, 3]) can be incorporated into DG-FVEM (see for example [46, 39, 26, 7]). However, there is hardly any numerical results available on DG-FVEM for solving nonlinear variational inequalities or for solving EHL problems. Even though numerical study of EHL model problems have been largely available in many papers, a very little attention have been drawn in analyzing the optimal error estimate study using DG-FVEM. Therefore, the present article has been devoted to investigated convergence and optimal error estimates for DG-FVEM for solving EHL model problem with the help of interior-exterior penalty procedure. This approach is quite natural and reduces the ambiguity in connecting the exterior penalty in DG-FVEM setting to capture free boundary and helps us to prove convergence and optimal error estimate of penalized problems not only EHL problems but also general variational inequality. However, in this paper, we do focus our attention in devising and in analyzing DG-FVEM (more about in theoretical perspective) for the EHL problems only. More practical results discussion will be given in the second part of this paper.

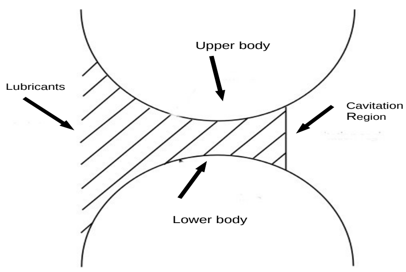


FIGURE 1. Undeformed surface body.

1.1. Model Problem. Let two contacting elastic bodies separated by a lubricant (like oil, liquid etc.), having film thickness size h_d, (assume that $h_d << R$, where R is equivalent radius of both ball and defined as $R^{-1} = R_1^{-1} + R_2^{-1}$.) rolling in positive x-direction give rise well known EHL model. Mathematical model of such model is described in the form of strongly non-linear variational inequality (VI) as

$$\frac{\partial}{\partial x}\Big(\epsilon^*\frac{\partial u}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(\epsilon^*\frac{\partial u}{\partial y}\Big) \le \frac{\partial(\rho h_d)}{\partial x} \tag{1}$$

$$u \ge 0 \tag{2}$$

$$u.\Big[\frac{\partial}{\partial x}\Big(\epsilon^*\frac{\partial u}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(\epsilon^*\frac{\partial u}{\partial y}\Big) - \frac{\partial(\rho h_d)}{\partial x}\Big] = 0, \tag{3}$$

here term ϵ^* is defined as

$$\epsilon^* = \frac{\rho h_d^3}{\eta\lambda},$$

where u denote the dimensionless pressure of lubricant, ρ is dimensionless density of lubrication, η is dimensionless viscosity of lubrication and speed parameter

$$\lambda = \frac{6\eta_0 v_s R^2}{a^3 p_H}, \tag{4}$$

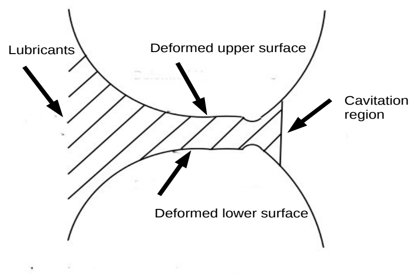


FIGURE 2. Deformed surface body.

where η_0 ambient pressure viscosity, v_s sum of contact velocity, p_H maximum Hertzian pressure and a is radius of Hertzian contact. The detail of its specific value used in computation are described in appendix (B). The non-dimensionless viscosity η is defined according to [5]

$$\eta(u) = \exp\Big(\alpha p_H u\Big), \tag{5}$$

where α $(1 \times 10^{-8} \backsimeq 2 \times 10^{-8})$ is pressure viscosity coefficient. Dimensionless density ρ is given by [14]

$$\rho(u) = \frac{0.59 \times 10^9 + 1.34 u p_H}{0.59 \times 10^9 + u p_H}. \tag{6}$$

We consider above non linear VI in a bounded, but large domain [1]

$$\Omega = \Omega_c \cup \Omega_0 \subset \mathbb{R}^2,$$

here Ω_c denotes the contact domain (lubricated domain like liquid, oil etc.) where $u > 0$, Ω_0 denotes the non contact domain where $u = 0$ and $\Omega_c \cap \Omega_0$ denotes the free boundary. Since lubricant pressure u is sufficiently small on the boundary $\partial\Omega$ (as it is almost equal to the atmospheric pressure). We take

$$u = 0 \quad \text{on} \quad \partial\Omega. \tag{7}$$

The film thickness equation is in dimensionless form is written as follows

$$h_d(x, y) = h_{00} + \frac{x^2}{2} + \frac{y^2}{2} + \frac{2}{\pi^2}\int_\Omega \frac{u(x^{'}, y^{'})dx^{'}dy^{'}}{\sqrt{(x - x')^2 + (y - y')^2}}, \tag{8}$$

where h_{00} is an integration constant.
The dimensionless force balance equation is defined as follows

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x', y')dx'dy' = \frac{3\pi}{2}. \tag{9}$$

Then system (1)–(9) forms an Elasto-hydrodynamic Lubrication. Schematic diagrams of EHL model is given in (1) and (2) in the form of undeformed and deformed contacting body structure respectively.
The rest of the article is organized in following way. In section (2) variational inequality and some notation is presented; Furthermore, existence results are explained for our model problem; In section. (3) DG-FVEM formulation, existence

[1] $\Omega = [-3, 3] \times [-3, 3]$ works fine in our numerical computation.

and uniqueness results are stablished; In section. (4) error estimates are derived in L^2 and H^1 norm; In section. (5) numerical experiments are performed; Section. (6) conclusion and future directions are mentioned.

2. Variational Inequality

We consider space $\mathscr{V} = H_0^1(\Omega)$ and its dual space as $\mathscr{V}^* = (H_0^1(\Omega))^* = H^{-1}(\Omega)$. Also define notion $\langle ., . \rangle$ as duality pairing on $\mathscr{V}^* \times \mathscr{V}$. Furthermore, we assume that $\mathscr{C}$ is closed convex subset of $\mathscr{V}$ defined by

$$\mathscr{C} = \Big\{ v \in \mathscr{V} : v \geq 0 \text{ a.e. } \in \Omega \Big\}. \tag{10}$$

Additionally, we define the operator $\mathscr{T}$ as

$$\mathscr{T} : u \to -\Big[\frac{\partial}{\partial x}\Big(\epsilon^* \frac{\partial u}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(\epsilon^* \frac{\partial u}{\partial y}\Big)\Big] + \frac{\partial(\rho h_d)}{\partial x}. \tag{11}$$

Then, for a given $f \in \mathscr{V}^*$, the problem of finding an element $u \in \mathscr{C}$ such that

$$\langle \mathscr{T}(u) - f, v - u \rangle \geq 0, \quad \forall v \in \mathscr{C}. \tag{12}$$

Throughout this article, we shall assume that there exists $\epsilon_1, M_* \in \mathbb{R}_+$ such that

$$0 < \epsilon_1 \leq \epsilon(u) \leq M_* \quad \forall \varsigma \in \Omega \quad \text{and} \quad u \in \mathbb{R}. \tag{13}$$

Remark 1. Under light load parameter assumption we also assume that minimum range of ϵ_1 should not go below beyond $\approx 1.0 \times 10^{-6}$. Below the range non local effect of film thickness term dominates and allow to blow the iterated solution after few iterations.

Definition 2.1. Operator $\mathscr{T} : \mathscr{C} \subset \mathscr{V} \to \mathscr{V}^*$ is said to be pseudo-monotone if $\mathscr{T}$ is a bounded operator and whenever $u_k \rightharpoonup u$ in $\mathscr{V}$ as $k \to \infty$ and

$$\lim_{k\to\infty} \sup \langle \mathscr{T}(u_k), u_k - u \rangle \leq 0, \tag{14}$$

then it follows that

$$\lim_{k\to\infty} \inf \langle \mathscr{T}(u_k), u_k - v \rangle \geq \langle \mathscr{T}(u), u - v \rangle \quad \forall v \in \mathscr{C}. \tag{15}$$

Definition 2.2. Operator $\mathscr{T} : \mathscr{V} \to \mathscr{V}^*$ is said to be hemi-continuous if and only if the function $\phi : t \longmapsto \langle \mathscr{T}(tx + (1-t)y), x - y \rangle$ is continuous on $[0, 1] \quad \forall x, y \in \mathscr{V}$.

In 1985, Oden and Wu [25] has proved an existence theorem for the above EHL model by assuming constant density and constant viscosity of the lubricant. Later in 1993, Rodrigues et. al. [37] has proved an independent existence result to the model using an a priori L^∞-estimate which holds true for a wider class of problems, including those arising from the linear Hertzian theory, and yields new existence results for a pressure viscosity dependent case or the inclusion of a load constraint case. In 2011, Ciuperca and Tello (see [12], for example) have given a first theoretical results on the range of admissible displacements for this EHL model with load constraint. Recently, Bayada and Vazquez (see [6], for example) have presented a novel EHL cavitation model with load constraint and have proved its existence and uniqueness. However, above ideas of existence result are easily extendable for more realistic operating condition in which density and viscosity of the lubricant depend on its applied pressure. A straight forward modification of the analysis of [25] yields the theorem below and so we will omit the proof.

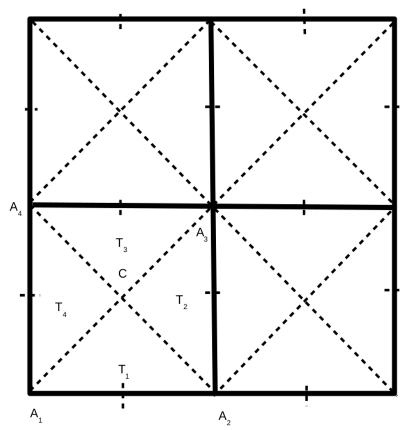


FIGURE 3. Rectangular partition $\mathscr{R}_h$ denoted by bold line and its dual partition denoted by dotted line.

Theorem 2. [25] *Let $\mathscr{C}(\neq \emptyset)$ be a closed, convex subset of a reflexive Banach space $\mathscr{V}$ and let $\mathscr{T} : \mathscr{C} \subset \mathscr{V} \to \mathscr{V}^*$ be a pseudo-monotone, bounded, and coercive operator from $\mathscr{C}$ into the dual $\mathscr{V}^*$ of $\mathscr{V}$, in the sense that there exists $y \in \mathscr{C}$ such that*

$$lim_{||x|| \to \infty} \frac{\langle \mathscr{T}(x), x - y \rangle}{||x||} = \infty. \tag{16}$$

Let f be given in $\mathscr{V}^*$ then there exists at least one $u \in \mathscr{C}$ such that

$$\langle \mathscr{T}(x) - f, y - x \rangle \geq 0 \quad \forall y \in \mathscr{C}. \tag{17}$$

In the next section, we derive a weak formulation and proof for existence of discrete DG-FVEM solution for the EHL model problem.

3. Discrete Formulation of DG-FVEM

Let $\mathscr{R}_h = \{K_i : 1 \leq i \leq N_h\}$ is a triangulations of domain Ω with $h = \max_{1 \leq i \leq N_h}\{h_i\}$, where h_i denotes the diameter of K_i. We assume that $\mathscr{R}_h$ is shape regular, quasi-uniform and it satisfies bounded local variation (i.e. $h_i/h_j \leq \kappa$, for all pairs of neighboring elements). We construct the dual partition $\mathscr{M}_h$ of $\mathscr{R}_h$ by dividing each $K_i \in \mathscr{R}_h$ into four triangles by connecting the barycenter and the four corners of the rectangle as shown in Figure 3. We define the finite dimensional space associated with $\mathscr{R}_h$ for trial functions as

$$\mathscr{V}_h = \{v \in L^2(\Omega) : v|_{K_i} \in \mathcal{P}_1(K_i), v|_{\partial\Omega} = 0 \quad K_i \in \mathscr{R}_h \quad \forall i\}. \tag{18}$$

We define the finite dimensional space $\mathscr{W}_h$ for test functions associated with the dual partition $\mathscr{M}_h$ as

$$\mathscr{W}_h = \{q \in L^2(\Omega) : q|_{T_j} \in \mathcal{P}_0(T_j), q|_{\partial\Omega} = 0 \quad T_j \in \mathscr{M}_h \quad \forall j\}, \tag{19}$$

where $\mathcal{P}_l(T_j)$ consist of all the polynomials with degree less than or equal to l defined on T_j. Let $\mathscr{V}(h) = \mathscr{V}_h + H^2(\Omega) \cap H_0^1(\Omega)$. Define a mapping

$$\gamma : \mathscr{V}(h) \longmapsto \mathscr{W}_h \quad \gamma v|_{T_j} = \frac{1}{h_{e_k}} \int_{e_k} v|_{T_j} ds, \quad T_j \in \mathscr{M}_h, \tag{20}$$

where e_k is an edge in K_i, T_j is the dual element in $\mathscr{M}_h$ containing e_k, and h_{e_k} is the length of the edge e_k (see Fig. 3). Let $e_{ii'}$ be an interior edge shared by two elements K_i and $K_{i'}$ in $\mathscr{R}_h$ and let $\mathbf{n_i}$ and $\mathbf{n_{i'}}$ be unit normal vectors on $e_{ii'}$

pointing exterior to K_i and $K_{i'}$ respectively. We define average $\{.\}$ and jump $[.]$ on $e_{ii'}$ for scalar q and vector w, respectively, as ([2])

$$\{q\} = \frac{1}{2}(q|_{\partial K_i} + q|_{\partial K_{i'}}), \quad [q] = (q|_{\partial K_i}\mathbf{n_i} + q|_{\partial K_{i'}}\mathbf{n_{i'}})$$

$$\{w\} = \frac{1}{2}(w|_{\partial K_i} + w|_{\partial K_{i'}}), \quad [w] = (w|_{\partial K_i}\mathbf{n_i} + w|_{\partial K_{i'}}\mathbf{n_{i'}})$$

If $e_{ii'}$ is a edge on the boundary of Ω, we define $q = q, \quad [w] = w.\mathbf{n}$. Let Γ denote the union of the boundaries of the triangle K_i of $\mathscr{R}_h$ and $\Gamma_0 := \Gamma \diagdown \partial\Omega$.

3.1. Weak Formulation. Reconsider the problem of the type

$$\frac{\partial}{\partial x}\Big(\epsilon^* \frac{\partial u}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(\epsilon^* \frac{\partial u}{\partial y}\Big) - \frac{\partial(\rho h_d)}{\partial x} = 0 \quad \text{in } \Omega \tag{21}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{22}$$

where all notation has their previously defined meaning.
Let $T_j \in \mathscr{M}_h (j = 1, 2, 3, 4)$ be four triangles in $K_i \in \mathscr{R}_h$. Multiply above equation (21) by $\gamma v \in \mathscr{W}_h$, where $v \in \mathscr{V}_h$. Integrating over the control volume $T_j \in \mathscr{M}_h$, applying Gauss's divergence theorem, summing up over all the control volume elements and using similar relation (2.3) and identity (2.5) as explained in [26, 46], we define bilinear form in following way. For given $u, v \in H^2(\Omega)$ and for fixed function $w_u \in H^2(\Omega)$, define bilinear form as

$$\begin{aligned}
\mathscr{B}_X(w_u; u, v) = &\sum_{K_i \in \mathscr{R}} \sum_{j=1}^{4} \int_{A_{j+1}CA_j} \epsilon(w_u)\nabla u.\mathbf{n}\gamma v ds \\
&+ \sum_{e_k \in \Gamma} \int_{e_k} [\gamma v]\{\epsilon(w_u)\nabla u\} ds + \alpha_1 \sum_{e_k \in \Gamma} [\gamma u]_{e_k} [\gamma v]_{e_k} \\
&- \sum_{K_i \in \mathscr{R}_h} \sum_{j=1}^{4} \int_{A_{j+1}CA_j} (\rho(w_u) h_d(u))(\vec{\beta}.\mathbf{n})\gamma v ds \\
&- \sum_{e_k \in \Gamma} \int_{e_k} [\gamma v]\{\vec{\beta}\rho(w_u) h_d(x)\} ds,
\end{aligned} \tag{23}$$

where $\vec{\beta} = (1, 0)^T$, $A_5 = A_1$ and $\mathbf{n}$ denotes unit outward normal vector to the boundary.
We define the following mesh dependent norm $|||.|||$ and $|||.|||_\nu$ as

$$|||v|||^2 := |v|_{1,h}^2 + \sum_{e_k} [\gamma v]_{e_k}^2 \tag{24}$$

$$|||v|||_\nu^2 := |v|_{1,h}^2 + \sum_{e_k} h_{e_k} \int_{e_k} \Big\{\frac{\partial v}{\partial \nu}\Big\}^2 ds + \sum_{e_k} [\gamma v]_{e_k}^2, \tag{25}$$

where $|v|_{1,h}^2 := \sum_{K_i} |v|_{1,K_i}^2$. Now we will state few lemmas and inequalities without proof which will be later helpful in our subsequent analysis.

Lemma 3. *For $u \in H^s(K_i)$, there exist a positive constant C_A and an interpolation value $u_I \in \mathscr{V}_h$, such that*

$$||u - u_I||_{s,K_i} \leq C_A h^{2-s} |u|_{2,K_i}, \quad s = 0, 1. \tag{26}$$

Proof. See [10]. □

Trace inequality.[2] We state without proof the following trace inequality. Let $\phi \in H^2(K_i)$ and for an edge e_k of K_i,

$$||\phi||^2_{e_k} \leq C(h^{-1}_{e_k}|\phi|^2_{K_i} + h_{e_k}|\phi|^2_{1,K_i}). \tag{27}$$

Lemma 4. *Let for any $u, v \in \mathscr{V}_h$, then we have following relation*

$$\langle h^3_d \rho e^{-au} \nabla_h u, \nabla_h v\rangle \leq \langle \mathscr{T}_1(u; u_h), v\rangle + C_1 h |||u||| |||v|||, \tag{28}$$

where

$$\langle \mathscr{T}_1(u; u_h), v\rangle = \sum_{K_i \in \mathscr{R}} \sum_{j=1}^{4} \int_{A_{j+1}CA_j} h^3_d \rho e^{-au} \nabla u.\mathbf{n}\gamma v ds$$

Proof. Proof of lemma follows using similar argument as mentioned in [46],lemma 2.1. □

Next lemma provides us a bound of film thickness term and later helpful in proving coercivity and error analysis.

Lemma 5. *For h_d defined in equation* (8), $0 < \beta_* < 1, s = 2 - \beta_*/(1 - \beta_*) > 2$ *there exist C_1 and $C_2 > 0$ such that*

$$\max_{x,y \in \Omega} |h_d(u)| \leq C_1 + C_2\|u\|_{L^s} \quad 0 < \beta_* < 1, \quad \forall (x, y) \in \bar{\Omega}. \tag{29}$$

Proof. Proof follows by straightforward modification of the proof as mentioned in [25, Lemma 1]. □

Lemma 6. *The bilinear operator $\mathscr{B}_X$ defined in equation* (23) *is bounded under the norm defined in equation* (24) *and* (25).

Lemma 7. *The bilinear operator $\mathscr{B}_X$, defined in equation* (23) *is hemi-continuous, that is $\forall u, v, w_1, w_2 \in \mathscr{V}_h$,*

$$\lim_{t \to 0^+} \mathscr{B}_X(u + tv, w_1, w_2) = \mathscr{B}_X(u, w_1, w_2).$$

Lemma 8. *The bilinear operator defined on equation* (23) *is coercive under the norm defined in equation* (24) *i.e. there is a constant C independent of h such that for α_1 large enough and h is small enough*

$$\langle \mathscr{B}_X(u; u_h, u_h)\rangle \geq C|||u_h|||^2 \quad \forall u_h \in \mathscr{V}_h. \tag{30}$$

Proof. Proof follows from lemma (3.2), lemma (3.3) and using similar argument as explained in [26]. □

3.2. Exterior penalty solution approximation. In this section, we introduce an exterior penalty term to regularize the inequality constraint (1)–(9). We define a exterior penalty operator $\xi_\sigma : H^1_0(\Omega) \to H^{-1}$ as

$$\xi_\sigma(u) = u^-/\sigma \quad \text{with } \sigma > 0, \tag{31}$$

where $u^- = u - \max(u, 0) = \dfrac{u - |u|}{2}$. Let us define exterior penalty problem, $(\mathscr{U}_\sigma)$: for $\sigma > 0$, find $u_\sigma \in \mathscr{V}_h$ such that

$$\langle \mathscr{T}(u_\sigma), v\rangle + \langle \xi_\sigma(u_\sigma), v\rangle = \langle f, v\rangle \quad \forall v \in \mathscr{V}_h, \tag{32}$$

This approach can be used in our DG-FVEM case and modified discrete weak formulation is rewritten as

$$\langle \mathscr{T}_1(u), \gamma v\rangle + \frac{1}{\sigma} \sum_{i=1}^{N_h} \sum_{j=1}^{4} \int_{T_j \subset K_i} u_- \gamma v dx - \langle \mathscr{T}_2(u), \gamma v\rangle = 0, \forall v \in \mathscr{V}_h, \tag{33}$$

where $\mathscr{T}_1$ and $\mathscr{T}_2$ correspond the diffusion and convection term of Reynolds equation (21). Here σ is an arbitrary small positive number ($\sigma = 1.0 \times 10^{-6}$).

Lemma 9. *Under suitable choice of quadrature rule, penalty operator $\xi_\sigma : \mathscr{V}_h \longmapsto \mathscr{V}_h$ is monotone, coercive and bounded under the norm defined in* (24) *and* (24).

Proof. Now define domains $\Omega_1 = \{x \in \Omega_1 : u_1 > 0\}$ and $\Omega_2 = \{x \in \Omega_2 : u_2 > 0\}$ and their compliments as Ω_1^c and Ω_2^c respectively. Also consider

$$u_i^- = \begin{cases} u_i \in \Omega_i^c & \forall i = 1,2 \\ \\ 0 \in \Omega_i & \forall i = 1,2. \end{cases} \tag{34}$$

For proving monotonicity we consider

$$\begin{aligned}
\langle \xi_\sigma(u_1) - \xi_\sigma(u_2), u_1 - u_2 \rangle &= \sum_{K_i \in \mathscr{R}_h} \int_{K_i} u_1^-(\gamma(u_1 - u_2)) - u_2^-(\gamma(u_1 - u_2))dx \\
&= \sum_{K_i \in \mathscr{R}_h} \int_{K_i} u_1^-(\gamma(u_1 - u_2) - (u_1 - u_2) + (u_1 - u_2))dx \\
&\quad - \sum_{K_i \in \mathscr{R}_h} \int_{K_i} u_2^-(\gamma(u_1 - u_2) - (u_1 - u_2) + (u_1 - u_2))dx \\
&= \sum_{K_i \in \mathscr{R}_h} \int_{K_i \cap \Omega_1^c} u_1^-(u_1 - u_2)dx - \int_{K \cap \Omega_2^c} u_2^-(u_1 - u_2)dx \\
&\quad + \sum_{K_i \in \mathscr{R}_h} \int_{K_i \cap \Omega_1^c \cap \Omega_2^c} u_1^-(u_1 - u_2) - u_2^-(u_1 - u_2)dx \\
&\quad + \sum_{K_i \in \mathscr{R}_h} \int_{K_i \cap \Omega_1 \cap \Omega_2} u_1^-(u_1 - u_2) - u_2^-(u_1 - u_2)dx \\
&= \sum_{K_i \in \mathscr{R}_h} \int_{K_i \cap \Omega_1^c} u_1^-(u_1 - u_2) - u_2^-(u_1 - u_2)dx \\
&\quad - \sum_{K_i \in \mathscr{R}_h} \int_{K_i \cap \Omega_2^c} u_1^-(u_1 - u_2) - u_2^-(u_1 - u_2)dx \\
&\quad + \sum_{K_i \in \mathscr{R}_h} \int_{K_i \cap \Omega_1^c \cap \Omega_2^c} (u_1 - u_2)^2 dx \geq 0
\end{aligned}$$

Hence, operator is monotone. Also, coercivity follows from the fact that

$$\begin{aligned}
\langle \xi_\sigma(u), u \rangle =& \langle u^-, u \rangle = \sum_{K_i \in \mathscr{R}_h} \int_{K_i(u \leq 0)} u^- \gamma u dx \\
=& \sum_{K_i \in \mathscr{R}_h} \int_{K_i(u \leq 0)} u^-(\gamma u - u + u)dx \\
=& \sum_{K_i \in \mathscr{R}_h} \int_{K_i(u \leq 0)} (u^-)^2 dx = |||(u^-)|||^2 \geq 0.
\end{aligned} \tag{35}$$

Furthermore, since

$$|\langle \xi_\sigma(u), v \rangle| = |u^- \gamma v| \leq |||u||| |||v|||. \tag{36}$$

This implies that ξ_σ is bounded. $\square$

Remark 10. The equality in above proof of lemma 3.7 is proved using the fact as explained below.

Lemma 11. *Under the suitable choice of quadrature rule, the following results hold true for $u_1, u_2 \in \mathscr{V}_h$ and $\forall K_i \in \mathscr{R}_h$ then*

$$\int_{K_i} u_1^-(\gamma(u_1 - u_2) - (u_1 - u_2))dx = 0$$

and

$$\int_{K_i} u_2^-(\gamma(u_1 - u_2) - (u_1 - u_2))dx = 0.$$

In particular, it is can be shown that if $u \in \mathscr{V}_h$ and $\forall K_i \in \mathscr{R}_h$ then

$$\int_{K_i} u^-(\gamma u - u)dx = 0.$$

Proof. Take quadrature rule as mentioned in [26, Eqn 2.12] and use [26, lemma 2.1] for the proof. It is interesting to note that u^- is treated as weight function here. □

3.3. Linearizion. Let us consider a fix function of $w_u \in H^2(\Omega)$ and also take $w, v \in H^2(\Omega)$. Furthermore, consider bilinear form $\mathscr{B}(w_u; w, v)$ solving EHL problem defined in (1)–(9) as

$$\begin{aligned}\mathscr{B}(w_u; w, v) := & \sum_{K_i \in \mathscr{R}_h} \sum_{j=1}^{4} \int_{A_{j+1}CA_j} \epsilon(w_u)\nabla w.\mathbf{n}\gamma v ds \\ & + \sum_{e_k \in \Gamma} \int_{e_k} [\gamma v]\{\epsilon(w_u)\nabla w\} ds + \alpha_1 \sum_{e_k \in \Gamma} [\gamma v]_{e_k}[\gamma w]_{e_k} + \sigma^{-1}\langle w^-, v\rangle \\ & - \sum_{K_i \in \mathscr{R}_h} \sum_{j=1}^{4} \int_{A_{j+1}CA_j} (\rho(w_u)h_d(x))(\vec{\beta}.\mathbf{n})\gamma v ds \\ & - \sum_{e_k \in \Gamma} \int_{e_k} [\gamma v]\{(\rho(w_u)h_d(x))\vec{\beta}\} ds \end{aligned} \tag{37}$$

Remark 12. Note that bilinear form $\mathscr{B}$ and $\mathscr{B}_X$ are related here as

$$\mathscr{B}(w_u; w, v) = \mathscr{B}_X(w_u; w, v) + \sigma^{-1}\langle w^-, v\rangle,$$

here σ stands for exterior penalty parameter.

Now define weak formulation for solving DG-FVEM for solving problem (1)–(9) as find $u \in H^2(\Omega, \mathscr{R}_h)$ such that

$$\mathscr{B}(u; u, v) = 0. \tag{38}$$

and corresponding DG-FVEM approximation of u is to find $u_h \in \mathscr{V}_h$ such that

$$\mathscr{B}(u_h; u_h, v_h) = 0,$$

here $\mathscr{B}(u_h; u_h, v_h)$ is the discrete representation of bilinear form ((37)).
Also $u_h \in \mathscr{V}_h \subset H^2(\Omega, \mathscr{R}_h)$ so we have

$$\mathscr{B}(u; u, v_h) = \mathscr{B}(u_h; u_h, v_h) \quad \forall v_h \in \mathscr{V}_h, \tag{39}$$

Since we are solving non-linear type of operator and so an appropriate linearizion is required for further analysis. We use following Taylor series expansion to linearize the problem as

$$\epsilon(w) = \epsilon(u) + \tilde{\epsilon}_u(w)(w - u), \tag{40}$$

where $\tilde{\epsilon}_u(w) = \int_0^1 \epsilon_u(w + \tau[w-u])d\tau$ and

$$\epsilon(w) = \epsilon(u) + \epsilon_u(w)(w-u) + \tilde{\epsilon}_{uu}(w)(w-u)^2, \tag{41}$$

where $\tilde{\epsilon}_{uu}(w) = \int_0^1 (1-\tau)\epsilon_{uu}(w+\tau[w-u])d\tau$. It is easy to check that $\tilde{\epsilon}_u \in C_b^1(\bar{\Omega}, \mathcal{R})$ and $\tilde{\epsilon}_{uu} \in C_b^0(\bar{\Omega}, \mathcal{R})$.
Now consider the following bilinear form $\bar{\mathscr{B}}(:,.)$ as

$$\begin{aligned}\bar{\mathscr{B}}(w_u, w, v) =& \mathscr{B}(w_u, w, v) + \sum_{K_i\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}(\epsilon_u(w_u)\nabla w_u)w.\mathbf{n}\gamma v ds\\ &+ \sum_{e_k\in\Gamma}\int_{e_k}[\gamma v]\Big\{\epsilon_u(w_u)\nabla w_u w\Big\}ds\\ &+ \sum_{K_i\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\rho_u h_d w\vec{\beta}.\mathbf{n}\gamma v ds + \sum_{e_k\in\Gamma}\int_{e_k}[\gamma v]\Big\{\rho_u h_d\vec{\beta}w\Big\}ds.\end{aligned} \tag{42}$$

It is easy to check that $\bar{\mathscr{B}}$ is linear in w and v and for fixed value of $w_u \in H^2(\Omega)$. Also as $\epsilon(w_u) \in C_b^2(\bar{\Omega}, \mathcal{R})$ and $u \in C^2(\bar{\Omega})$, there is a unique solution $w_u \in H^2(\Omega)$ to the following elliptic problem

$$\begin{aligned}\nabla.(\epsilon(u)\nabla\varphi + \epsilon_u\varphi\nabla u) - \nabla(\vec{\beta}(\rho h_d + \rho_u h_d\varphi)) = \psi_h \text{ in } \Omega\\ \varphi = 0 \text{ on } \partial\Omega,\end{aligned} \tag{43}$$

where $\psi_h = -\xi_\sigma(u) = -u^-/\sigma$.
Also from well known elliptic regularity property [16] we have

$$\|\varphi\|_{H^2(\Omega)} \le C\|\psi_h\|. \tag{44}$$

Now for showing existence, uniqueness and for analyzing intermediate stage error analysis of discrete DG-FVEM solution we linearize weak formulation of equation (39) around $\Pi_h u$, where interpolation map $\Pi_h : \mathscr{V} \longmapsto \mathscr{V}_h$. Let $e = u - u_h$ be an error term for exact and approximated DG-FVEM solution. Now by subtracting $\mathscr{B}(u; u_h, v_h)$ from both side of equation (39), we get

$$\begin{aligned}\mathscr{B}(u; e, v_h) =& \sum_{K_i\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}(\epsilon(u_h) - \epsilon(u))\nabla u_h.\mathbf{n}\gamma v_h ds\\ &+ \sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h](\epsilon(u_h) - \epsilon(u))\nabla u_h ds\\ &- \sum_{K_i\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}(\rho(u_h)h_d(x) - \rho(u)h_d(x))\vec{\beta}.\mathbf{n}\gamma v_h ds\\ &- \sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h]\Big\{\rho(u_h)h_d(x) - \rho(u)h_d(x)\vec{\beta}\Big\}ds\end{aligned} \tag{45}$$

Now adding both side in above equation following term

$$\sum_{K_i\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\epsilon_u(u_h)(u_h-u)\nabla u.\mathbf{n}\gamma v_h ds$$
$$+\sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h]\Big\{\epsilon_u(u_h)(u_h-u)\nabla u\Big\}ds$$
$$-\sum_{K_i\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}(\rho h_d)_u(u_h)(u_h-u)\vec{\beta}.\mathbf{n}\gamma v_h ds$$
$$-\sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h]\{(\rho h_d)_u(u_h)(u_h-u)\vec{\beta}\}ds. \tag{46}$$

Now we split error term as

$$e=u-u_h=u-\Pi_h u+\Pi_h u-u_h$$

and using Taylor's formula for linearizion given in (40)–(41), we rewrite equation (45) as

$$\bar{\mathscr{B}}(u;\Pi_h u-u_h,v_h)=\bar{\mathscr{B}}(u;\Pi_h u-u,v_h)+\mathscr{F}(u_h;u_h-u,v_h), \tag{47}$$

where

$$\mathscr{F}(u_h;u_h-u,v_h)=\sum_{K_i\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\widetilde{\epsilon}_u(u_h)e\nabla e.\mathbf{n}\gamma v_h ds$$
$$+\sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h]\Big\{\epsilon_u(u_h)e\nabla e\Big\}ds$$
$$+\sum_{K_i\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\widetilde{\epsilon}_{uu}(u_h)e^2\nabla u.\mathbf{n}\gamma v_h ds$$
$$-\sum_{K_i\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\widetilde{(\rho h_d)}_{uu}(u_h)e^2\vec{\beta}.\mathbf{n}\gamma v_h ds$$
$$-\sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h]\Big\{\widetilde{(\rho h_d)}_{uu}(u_h)e^2\vec{\beta}\Big\}ds. \tag{48}$$

Note that solving (39) is equivalent to solve equation (47). Now for showing there exist at least one $u_h\in\mathscr{V}_h$ solution to the above equation (47) we consider a map

$$\mathcal{S}:\mathscr{V}_h\to\mathscr{V}_h$$

defined as $\mathcal{S}(u_\varphi)=\varphi\in\mathscr{V}_h,\quad\forall u_\varphi\in\mathscr{V}_h$ such that

$$\bar{\mathscr{B}}(u;\Pi_h u-\varphi,v_h)=\bar{\mathscr{B}}(u;\Pi_h u-u,v_h)+\mathscr{F}(u_\varphi;u_\varphi-u,v_h) \tag{49}$$

holds. Consider the closed neighborhood $\mathcal{Q}_\delta(\Pi_h u)$ of the diameter $\delta>0$.

$$\mathcal{Q}_\delta(\Pi_h u)=\Big\{u_\varphi\in\mathscr{V}_h:|||u_\varphi-\Pi_h u|||\le\delta\Big\}.$$

Now we first show that $\mathcal{S}$ map closed neighborhood $\mathcal{Q}_\delta(\Pi_h u)$ into itself and then prove existence of DG-FVEM solution by exploiting Browder's fixed point theorem. The proof can be break using following lemmas.

Lemma 13. *Let $u_\varphi, v_h \in \mathscr{V}_h$ also set $\chi = u_\varphi - \Pi_h u$ and $\eta = u - \Pi_h u$. Then there exists a constant $C \geq 0$ (independent of h) such that*

$$
\begin{aligned}
|\mathscr{F}(u_\varphi; u_\varphi - u, v_h)| \leq & C_\epsilon \Big[|||\chi|||^2 + C_u(h^{5/3} + h^{1/2} + h + h^{2/3} + h^{3/2}) |||\chi||| \\
& + C_u(h^2 + h + h^{3/2}) |||\eta||| \Big] |||v_h||| + C_{\rho h_d} \Big[|||\chi|||^2 \\
& + C_u(h^{5/3} + h^{3/2}) |||\chi||| + C_u(h^{3/2} + h) |||\eta||| \Big] |||v_h|||.
\end{aligned} \tag{50}
$$

Proof. Let $u_\varphi \in \mathscr{V}_h$ and take $\zeta = u_\varphi - u$ in equation (48) we write u_φ in place of u_h and $\zeta = u_\varphi - u$ to get

$$
\begin{aligned}
\mathscr{F}(u_\varphi; \zeta, v_h) = & \sum_{K \in \mathscr{R}_h} \sum_{j=1}^{4} \int_{A_{j+1} C A_j} \widetilde{\epsilon}_u(u_\varphi) \zeta \nabla \zeta . \mathbf{n} \gamma v_h ds \\
& + \sum_{e_k \in \Gamma} \int_{e_k} [\gamma v_h] \Big\{ \epsilon_u(u_\varphi) \zeta \nabla \zeta \Big\} ds \\
& + \sum_{K \in \mathscr{R}_h} \sum_{j=1}^{4} \int_{A_{j+1} C A_j} \widetilde{\epsilon}_{uu}(u_\varphi) \zeta^2 \nabla u . \mathbf{n} \gamma v_h ds \\
& - \sum_{K \in \mathscr{R}_h} \sum_{j=1}^{4} \int_{A_{j+1} C A_j} \widetilde{(\rho h_d)}_{uu}(u_\varphi) \zeta^2 \vec{\beta} . \mathbf{n} \gamma v_h ds \\
& - \sum_{e_k \in \Gamma} \int_{e_k} [\gamma v_h] \Big\{ \widetilde{(\rho h_d)}_{uu}(u_\varphi) \zeta^2 \Big\} ds.
\end{aligned} \tag{51}
$$

Now split $\zeta = \chi - \eta$ where $\chi = u_\varphi - \Pi_h u$ and $\eta = u - \Pi_h u$. Then right hand side is estimated in following way. The right hand side of First term of equation (51) is estimated as

$$
\begin{aligned}
& \Big| \sum_{K \in \mathscr{R}_h} \sum_{j=1}^{4} \int_{A_{j+1} C A_j} \widetilde{\epsilon}_u(u_\varphi) \zeta \nabla \zeta . \mathbf{n} \gamma v_h ds \Big| \\
\leq & \Big| \sum_{K \in \mathscr{R}_h} \sum_{j=1}^{4} \int_{A_{j+1} C A_j} \widetilde{\epsilon}_u(u_\varphi) \chi \nabla \chi . \mathbf{n} \gamma v_h ds \Big| \\
& + \Big| \sum_{K \in \mathscr{R}_h} \sum_{j=1}^{4} \int_{A_{j+1} C A_j} \widetilde{\epsilon}_u(u_\varphi) \chi \nabla \eta . \mathbf{n} \gamma v_h ds \Big| \\
& + \Big| \sum_{K \in \mathscr{R}_h} \sum_{j=1}^{4} \int_{A_{j+1} C A_j} \widetilde{\epsilon}_u(u_\varphi) \eta \nabla \chi . \mathbf{n} \gamma v_h ds \Big| \\
& + \Big| \sum_{K \in \mathscr{R}_h} \sum_{j=1}^{4} \int_{A_{j+1} C A_j} \widetilde{\epsilon}_u(u_\varphi) \eta \nabla \eta . \mathbf{n} \gamma v_h ds \Big|.
\end{aligned} \tag{52}
$$

Right hand side of second term of equation (51) is estimated as

$$\begin{aligned}\Big|\sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h]\Big\{\epsilon_u(u_\varphi)\zeta\nabla\zeta\Big\}ds\Big| \leq&\Big|\sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h]\Big\{\epsilon_u(u_\varphi)\chi\nabla\chi\Big\}ds\Big|\\ &+\Big|\sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h]\Big\{\epsilon_u(u_\varphi)\eta\nabla\chi\Big\}ds\Big|\\ &+\Big|\sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h]\Big\{\epsilon_u(u_\varphi)\chi\nabla\eta\Big\}ds\Big|\\ &+\Big|\sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h]\Big\{\epsilon_u(u_\varphi)\eta\nabla\eta\Big\}ds\Big|.\end{aligned}\tag{53}$$

Right hand side of third term of equation (51) is estimated as

$$\begin{aligned}&\Big|\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\widetilde{\epsilon}_{uu}(u_\varphi)\zeta^2\nabla u.\mathbf{n}\gamma v_h ds\Big|\\ \leq&\Big|\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\widetilde{\epsilon}_{uu}(u_\varphi)\eta^2\nabla u.\mathbf{n}\gamma v_h ds\Big|\\ &+2\Big|\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\widetilde{\epsilon}_{uu}(u_\varphi)\eta.\chi\nabla u.\mathbf{n}\gamma v_h ds\Big|\\ &+\Big|\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\widetilde{\epsilon}_{uu}(u_\varphi)\chi^2\nabla u.\mathbf{n}\gamma v_h ds\Big|.\end{aligned}\tag{54}$$

Right hand side of fourth term of equation (51) is estimated as

$$\begin{aligned}&\Big|\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\widetilde{(\rho h_d)}_{uu}(u_\varphi)\zeta^2\vec{\beta}.\mathbf{n}\gamma v_h ds\Big|\\ \leq&\Big|\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\widetilde{(\rho h_d)}_{uu}(u_\varphi)\chi^2\vec{\beta}.\mathbf{n}\gamma v_h ds\Big|\\ &+2\Big|\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\widetilde{(\rho h_d)}_{uu}(u_\varphi)\eta.\chi\vec{\beta}.\mathbf{n}\gamma v_h ds\Big|\\ &+\Big|\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\widetilde{(\rho h_d)}_{uu}(u_\varphi)\eta^2\vec{\beta}.\mathbf{n}\gamma v_h ds\Big|.\end{aligned}\tag{55}$$

Right hand side of fifth term of equation (51) is estimated as

$$\begin{aligned}&\Big|\sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h]\Big\{(\rho h_d)_{uu}(u_\varphi)\zeta^2\Big\}ds\Big|\\ \leq&\Big|\sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h]\Big\{(\rho h_d)_{uu}(u_\varphi)\chi^2\Big\}ds\Big|\\ &+2\Big|\sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h]\Big\{(\rho h_d)_{uu}(u_\varphi)\eta.\chi\Big\}ds\Big|\\ &+\Big|\sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h]\Big\{(\rho h_d)_{uu}(u_\varphi)\eta^2\Big\}ds\Big|.\end{aligned}\tag{56}$$

In equation (52), right hand side of first term is estimated as

$$\begin{aligned}
&\Big|\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\widetilde{\epsilon}_u(u_\varphi)\chi\nabla\chi.\mathbf{n}\gamma v_h ds\Big|\\
&\le\Big|\sum_K\langle\epsilon_u(u_\varphi)\chi\nabla\chi,\nabla v_h\rangle\Big|+\Big|\sum_K\int_{\partial K}[\gamma v_h-v_h]\Big\{\epsilon_u(u_\varphi)\chi\nabla\chi.\mathbf{n}\Big\}ds\Big|\\
&\quad+\Big|\sum_K\langle\nabla\Big(\epsilon_u(u_\varphi)\chi\nabla\chi\Big),v_h-\gamma v_h\rangle\Big|. \qquad (57)
\end{aligned}$$

Right hand side of first part of equation (57) is estimated as

$$\Big|\sum_K\langle\epsilon_u(u_\varphi)\chi\nabla\chi,\nabla v_h\rangle\Big|\le C_\epsilon\sum_K\int_K|\chi.\nabla\chi.\nabla v_h|dx.$$

Now using Holder's inequality we get

$$\begin{aligned}
C_\epsilon\sum_K\int_K|\chi.\nabla\chi.\nabla v_h|dx \le& C_\epsilon\sum_K\|\chi\|_{L^6(K)}\|\chi\|_{L^3(K)}\|\nabla v_h\|_{L^2(K)}\\
\le& C_\epsilon|||\chi|||\,|||\chi|||\,|||v_h|||. \qquad (58)
\end{aligned}$$

Now right hand side of second part of equation (57) is estimated using Holder's inequality and trace inequality

$$\begin{aligned}
&\Big|\sum_K\int_{\partial K}[\gamma v_h-v_h]\Big\{\epsilon_u(u_\varphi)\chi\nabla\chi.\mathbf{n}\Big\}ds\Big|\\
\le& C_\epsilon\sum_K\Big(\int_{\partial K}[\gamma v_h-v_h]^2\Big)^{1/2}\|\chi\|_{L^4(\partial K)}\|\nabla\chi\|_{L^4(\partial K)}.
\end{aligned}$$

Now using trace inequality defined as

$$\|\nabla\chi\|_{L^4(\partial K)}\le C_h\Big(h^{-1}\|\nabla\chi\|^4_{L^4(K)}+\|\nabla\chi\|^3_{L^6(K)}\|\nabla.\nabla\chi\|_{L^2(K)}\Big) \qquad (59)$$

and

$$\|\chi\|_{L^4(\partial K)}\le C_h\Big(h^{-1}\|\chi\|^4_{L^4(K)}+\|\chi\|^3_{L^6(K)}\|\chi\|_{L^2(K)}\Big). \qquad (60)$$

We get that

$$\begin{aligned}
&\le C_\epsilon\Big(h^{-1}|\gamma v_h-v_h|^2_{L^2(K)}+h|\gamma v_h-v_h|^2_{H^1(K)}\Big)^{1/2}\\
&\quad\times\Big(h^{-1}\|\chi\|^4_{L^4(K)}+\|\chi\|^3_{L^6(K)}\|\chi\|_{L^2(K)}\Big)^{1/4}\\
&\quad\times\Big(h^{-1}\|\nabla\chi\|^4_{L^4(K)}+\|\nabla\chi\|^3_{L^6(K)}\|\nabla.\nabla\chi\|_{L^2(K)}\Big)^{1/4}\\
&\le C_\epsilon|||\chi|||\,|||\chi|||\,|||v_h||| \qquad (61)
\end{aligned}$$

Right hand side of third term of equation (57) is estimated in similar way and it is written as

$$\Big|\sum_K\langle\nabla\Big(\epsilon_u(u_\varphi)\chi\nabla\chi\Big),v_h-\gamma v_h\rangle\Big|\le C_\epsilon|||\chi|||\,|||\chi|||\,|||v_h|||. \qquad (62)$$

Now right hand side of second term equation (52) is estimated as

$$\begin{aligned}
&\Big|\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\widetilde{\epsilon}_u(u_\varphi)\chi\nabla\eta.\mathbf{n}\gamma v_h ds\Big| \\
&\leq\Big|\sum_K\langle\epsilon_u(u_\varphi)\chi\nabla\eta,\nabla v_h\rangle\Big|+\Big|\sum_K\int_{\partial K}[\gamma v_h-v_h]\Big\{\epsilon_u(u_\varphi)\chi\nabla\eta.\mathbf{n}\Big\}ds\Big| \\
&\quad+\Big|\sum_K\langle\nabla\Big(\epsilon_u(u_\varphi)\chi\nabla\eta\Big),v_h-\gamma v_h\rangle\Big|. \qquad (63)
\end{aligned}$$

Now right hand side of first term of equation (63) is estimated using Holder's inequality as

$$\begin{aligned}
\Big|\sum_K\langle\epsilon_u(u_\varphi)\chi\nabla\eta,\nabla v_h\rangle\Big| &\leq C_\epsilon\sum_K\int_K|\chi.\nabla\eta.\nabla v_h|dx \\
&\leq C_\epsilon\sum_K\|\chi\|_{L^6(K)}\|\nabla\eta\|_{L^3(K)}\|\nabla v_h\|_{L^2(K)} \qquad (64)
\end{aligned}$$

Now using **inverse inequality** defined as

$$\|v_h\|_{L^r(K)}\leq Ch^{2/r-1}\|v_h\|_{L^2(K)}\quad\forall r\geq 2. \qquad (65)$$

and also using approximation property we get

$$\begin{aligned}
&\leq C_\epsilon C_u h^{-1/3}\|\nabla\eta\|_{L^2(K)}|||\chi|||\,|||v_h||| \\
&\leq C_\epsilon C_u h^{2/3}\|u\|_{H^2(\Omega)}|||\chi|||\,|||v_h|||. \qquad (66)
\end{aligned}$$

Right hand side of second term of equation (63) is estimated as using Holder's inequality and trace inequality

$$\begin{aligned}
&\Big|\sum_K\int_{\partial K}[\gamma v_h-v_h]\Big\{\epsilon_u(u_\varphi)\chi\nabla\eta.\mathbf{n}\Big\}ds\Big| \\
&\leq C_\epsilon\sum_K\Big(\int_{\partial K}[\gamma v_h-v_h]^2\Big)^{1/2}\|\chi\|_{L^4(\partial K)}\|\nabla\eta\|_{L^4(\partial K)} \\
&\leq C_\epsilon h^{-1}\sum_K\Big(|\gamma v_h-v_h|^2_{L^2(K)}+h^2|\gamma v_h-v_h|^2_{H^1(K)}\Big)^{1/2} \\
&\quad\times\Big(\|\chi\|^4_{L^4(K)}+h\|\chi\|^3_{L^6(K)}\|\nabla\chi\|_{L^2(K)}\Big)^{1/4} \\
&\quad\times\Big(\|\nabla\eta\|^4_{L^4(K)}+h\|\nabla\eta\|^3_{L^6(K)}\|\nabla.\nabla\eta\|_{L^2(K)}\Big)^{1/4} \\
&\leq C_\epsilon h^{1/2}\|u\|_{H^2(\Omega)}|||v_h|||\,|||\chi|||. \qquad (67)
\end{aligned}$$

Right hand side of third term of equation (63) is estimated as

$$\begin{aligned}
&\Big|\sum_K\langle\nabla\Big(\epsilon_u(u_\varphi)\chi\nabla\eta\Big),v_h-\gamma v_h\rangle\Big| \\
&\leq C_\epsilon C_u(h^{2/3}\|u\|_{H^2(K)}|||\chi|||\,|||v_h|||+h^{1/2}\|u\|_{H^2(\Omega)}|||v_h|||\,|||\chi|||). \qquad (68)
\end{aligned}$$

Now right hand side of third term of equation (52) is estimated as

$$
\begin{aligned}
&\Big|\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\widetilde{\epsilon}_u(u_\varphi)\eta\nabla\chi.\mathbf{n}\gamma v_h ds\Big| \\
&\leq\Big|\sum_K\langle\epsilon_u(u_\varphi)\eta\nabla\chi,\nabla v_h\rangle\Big|+\Big|\sum_K\int_{\partial K}[\gamma v_h-v_h]\Big\{\epsilon_u(u_\varphi)\eta\nabla\chi.\mathbf{n}\Big\}ds\Big| \\
&\quad+\Big|\sum_K\langle\nabla\Big(\epsilon_u(u_\varphi)\eta\nabla\chi\Big),v_h-\gamma v_h\rangle\Big|. \qquad (69)
\end{aligned}
$$

Right hand side of first part of equation (69) is estimated by using Holder's inequality as

$$
\begin{aligned}
\Big|\sum_K\langle\epsilon_u(u_\varphi)\eta\nabla\chi,\nabla v_h\rangle\Big| &\leq C_\epsilon\sum_K\|\eta\|_{L^6(K)}\|\nabla\chi\|_{L^3(K)}\|\nabla v_h\|_{L^2(K)} \\
&\leq C_\epsilon\sum_K h^{2/6-1}\|\eta\|_{L^2(K)}h^{2/3-1}\|\nabla\chi\|_{L^2(K)}\|\nabla v_h\|_{L^2(K)} \\
&\leq C_\epsilon C_u h\|u\|_{H^2(\Omega)}|||\chi||| |||v_h|||. \qquad (70)
\end{aligned}
$$

Right hand side of second part of equation (69) is estimated using trace inequality we have

$$
\begin{aligned}
&\Big|\sum_K\int_{\partial K}[\gamma v_h-v_h]\Big\{\epsilon_u(u_\varphi)\eta\nabla\chi.\mathbf{n}\Big\}ds\Big| \\
&\leq C_\epsilon\sum_K\Big(\int_{\partial K}[\gamma v_h-v_h]^2ds\Big)^{1/2}\|\eta\|_{L^4(\partial K)}\|\nabla\chi\|_{L^4(\partial K)} \\
&\leq C_\epsilon h^{3/2}\|u\|_{H^2(\Omega)}|||\chi||| |||v_h|||. \qquad (71)
\end{aligned}
$$

Right hand side of third part of equation (69) is estimated as

$$
\begin{aligned}
&\Big|\sum_K\langle\nabla\Big(\epsilon_u(u_\varphi)\eta\nabla\chi\Big),v_h-\gamma v_h\rangle\Big| \\
&\leq C_\epsilon C_u h\|u\|_{H^2(\Omega)}|||\chi||| |||v_h|||+C_\epsilon h^{3/2}\|u\|_{H^2(\Omega)}|||\chi||| |||v_h|||. \qquad (72)
\end{aligned}
$$

Right hand side of fourth term of equation (52) is estimated as

$$
\begin{aligned}
&\Big|\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\tilde{\epsilon}_u(u_\varphi)\eta\nabla\eta.\mathbf{n}\gamma v_h ds\Big| \\
&\leq\Big|\sum_K\langle\epsilon_u(u_\varphi)\eta\nabla\eta,\nabla v_h\rangle\Big|+\Big|\sum_K\int_{\partial K}[\gamma v_h-v_h]\Big\{\epsilon_u(u_\varphi)\eta\nabla\eta.\mathbf{n}\Big\}ds\Big| \\
&\quad+\Big|\sum_K\langle\nabla\Big(\epsilon_u(u_\varphi)\eta\nabla\eta\Big),v_h-\gamma v_h\rangle\Big|. \qquad (73)
\end{aligned}
$$

Right hand side of first part of equation (73) is estimated using Holder's inequality as

$$
\begin{aligned}
\Big|\sum_K\langle\epsilon_u(u_\varphi)\eta\nabla\eta,\nabla v_h\rangle\Big| &\leq C_\epsilon\sum_K\|\eta\|_{L^6(K)}\|\nabla\eta\|_{L^3(K)}\|\nabla v_h\|_{L^2(K)} \\
&\leq C_\epsilon C_u h\|u\|_{H^2(\Omega)}|||\eta||| |||v_h||| \qquad (74)
\end{aligned}
$$

Right hand side of second part of equation (73) is estimated as

$$
\begin{aligned}
&\Big|\sum_K \int_{\partial K} [\gamma v_h - v_h]\Big\{\epsilon_u(u_\varphi)\eta\nabla\eta.\mathbf{n}\Big\}ds\Big| \\
\leq & C_\epsilon \sum_K \Big(\int_{\partial K}[\gamma v_h - v_h]^2 ds\Big)^{1/2}\|\eta\|_{L^4(\partial K)}\|\nabla\eta\|_{L^4(\partial K)} \\
\leq & C_\epsilon h^{3/2}\|u\|_{H^2(\Omega)}|||\eta||| |||v_h||| \qquad (75)
\end{aligned}
$$

Right hand side of third part of equation (73) is estimated as

$$
\begin{aligned}
&\Big|\sum_K \langle\nabla\Big(\epsilon_u(u_\varphi)\eta\nabla\eta\Big), v_h - \gamma v_h\rangle\Big| \\
\leq & C_\epsilon C_u h\|u\|_{H^2(\Omega)}|||\eta||| |||v_h||| + C_\epsilon h^{3/2}\|u\|_{H^2(\Omega)}|||\eta||| |||v_h|||. \qquad (76)
\end{aligned}
$$

Now right hand side of first part of equation (53) is estimated as

$$
\begin{aligned}
&\Big|\sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h]\Big\{\epsilon_u(u_\varphi)\chi\nabla\chi\Big\}ds\Big| \\
\leq & C_\epsilon \sum_K \Big([\gamma v_h]^2\Big)^{1/2}\|\chi\|_{L^4(\partial K)}\|\nabla\chi\|_{L^4(\partial K)} \\
\leq & C_\epsilon \sum_K \Big([\gamma v_h]^2\Big)^{1/2}\Big(\|\chi\|^4_{L^4(K)} + h\|\chi\|^3_{L^6(K)}\|\nabla\chi\|_{L^2(K)}\Big)^{1/4} \\
& \times\Big(\|\nabla\chi\|^4_{L^4(K)} + h\|\nabla\chi\|^3_{L^6(K)}\|\nabla.\nabla\chi\|_{L^2(K)}\Big)^{1/4} \\
\leq & C_\epsilon|||v_h||| |||\chi||| |||\chi||| \qquad (77)
\end{aligned}
$$

In similar way we can show that right hand side of second, third and fourth part of equation (53) is estimated as

$$
\Big|\sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h]\Big\{\epsilon_u(u_\varphi)\chi\nabla\eta\Big\}ds\Big| \leq C_\epsilon C_u h^{1/2}\|u\|_{H^2(\Omega)}|||v_h||| |||\chi||| \qquad (78)
$$

$$
\Big|\sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h]\Big\{\epsilon_u(u_\varphi)\eta\nabla\chi\Big\}ds\Big| \leq C_\epsilon C_u h^{3/2}\|u\|_{H^2(\Omega)}|||v_h||| |||\chi||| \qquad (79)
$$

$$
\Big|\sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h]\Big\{\epsilon_u(u_\varphi)\eta\nabla\eta\Big\}ds\Big| \leq C_\epsilon C_u h^{3/2}\|u\|_{H^2(\Omega)}|||v_h||| |||\eta|||. \qquad (80)
$$

Right hand side of first part of equation (54) is estimated using similar argument as

$$
\Big|\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\widetilde{\epsilon}_{uu}(u_\varphi)\chi^2\nabla u.\mathbf{n}\gamma v_h ds\Big| \leq C_\epsilon C_u|||\chi|||^2|||v_h||| \qquad (81)
$$

Right hand side of second part of equation (54) is estimated using similar argument as

$$
\begin{aligned}
&\Big|\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\widetilde{\epsilon}_{uu}(u_\varphi)\chi.\eta\nabla u.\mathbf{n}\gamma v_h ds\Big| \\
\leq & C_\epsilon C_u\Big(h^{5/3}\|u\|_{H^2(\Omega)}|||\chi||| |||v_h||| + h^{3/2}\|u\|_{H^2(\Omega)}|||\chi||| |||v_h|||\Big) \qquad (82)
\end{aligned}
$$

Right hand side of third part of equation (54) is estimated using similar argument as

$$\Big|\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\widetilde{\epsilon}_{uu}(u_\varphi)\eta^2\nabla u.\mathbf{n}\gamma v_h ds\Big|$$
$$\leq C_\epsilon C_u\Big(h^2\|u\|_{H^2(\Omega)}|||\eta|||\,|||v_h||| + h^{3/2}\|u\|_{H^2(\Omega)}|||\eta|||\,|||v_h|||\Big). \tag{83}$$

Right hand side of first part of equation (55) is estimated as

$$\Big|\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\widetilde{(\rho h_d)}_{uu}(u_\varphi)\chi^2\vec{\beta}.\mathbf{n}\gamma v_h ds\Big| \leq C_{\rho h_d}|||v_h|||\,|||\chi|||^2. \tag{84}$$

Right hand side of second part of equation (55) is estimated as

$$\Big|\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\widetilde{(\rho h_d)}_{uu}(u_\varphi)\eta.\chi\vec{\beta}.\mathbf{n}\gamma v_h ds\Big|$$
$$\leq C_{\rho h_d}\Big(h^{5/3}\|u\|_{H^2(\Omega)}|||\chi|||\,|||v_h||| + h^{3/2}\|u\|_{H^2(\Omega)}|||\chi|||\,|||v_h|||\Big). \tag{85}$$

Right hand side of third part of equation (55) is estimated as

$$\Big|\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\widetilde{(\rho h_d)}_{uu}(u_\varphi)\eta^2\vec{\beta}.\mathbf{n}\gamma v_h ds\Big|$$
$$\leq C_{\rho h_d}h\|u\|_{H^2(\Omega)}|||\eta|||\,|||v_h||| + C_{\rho h_d}h^{3/2}\|u\|_{H^2(\Omega)}|||\eta|||\,|||v_h||| \tag{86}$$

Now equation (56) is estimated using similar argument as

$$\Big|\sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h]\Big\{(\rho h_d)_{uu}(u_\varphi)\zeta^2\Big\}ds\Big|$$
$$\leq\Big|\sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h]\Big\{(\rho h_d)_{uu}(u_\varphi)\chi^2\Big\}ds\Big|$$
$$+2\Big|\sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h]\Big\{(\rho h_d)_{uu}(u_\varphi)\eta.\chi\Big\}ds\Big| + \Big|\sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h]\Big\{(\rho h_d)_{uu}(u_\varphi)\eta^2\Big\}ds\Big|$$
$$\leq C_{\rho h_d}\Big(|||v_h|||\,|||\chi|||^2 + h^{3/2}\|u\|_{H^2(\Omega)}|||\chi|||\,|||v_h||| + h^{3/2}\|u\|_{H^2(\Omega)}|||\eta|||\,|||v_h|||\Big). \tag{87}$$

Now combining the right hand side of equation (51) we get the desire results. □

Now we are interested in deriving upper bound of $|||\Pi_h u-\varphi|||$ and it is explained in next lemma.

Lemma 14. *Let $u_\varphi\in\mathscr{V}_h$ and take $\varphi=\mathcal{S}u_\varphi$.Then there exist a positive constant C (independent of h) such that*

$$\begin{aligned}|||\Pi_h u-\varphi||| \leq& C_\epsilon\Big[|||\Pi_h u-u_\varphi|||^2 + C_u\Big(h^{5/3}+h^{1/2}+h^{2/3}+h(1+h^{1/2})\Big)|||\Pi_h u-u_\varphi|||\\ &+ C_u(h^2+h+h^{3/2})|||\eta|||\Big] + C_{\rho h_d}\Big[|||\Pi_h u-u_\varphi|||^2\\ &+ C_u(h^{5/3}+h^{3/2})|||\Pi_h u-u_\varphi||| + C_u(h^{3/2}+h)|||\eta|||\Big] + C|||\eta|||.\end{aligned} \tag{88}$$

holds.

Proof. In equation (47) we redefine the term $\chi = \Pi_h u - u_\varphi$, $\eta = \Pi_h u - u$, and $\vartheta = \Pi_h u - \varphi$. Now consider the first term in the right hand side of equation (47) and replace $v_h = \vartheta$ and use the boundedness property of the operator to get

$$\left|\bar{\mathscr{B}}(u;\eta,\vartheta)\right| \leq C|||\eta||| \, |||\vartheta||| \tag{89}$$

Also by replacing $v_h = \vartheta$ in previous lemma 3.10 we obtain

$$\begin{aligned}\left|\mathscr{F}(u_\varphi;u_\varphi-u,\vartheta)\right| \leq & C_\epsilon\Big[|||\chi|||^2 + C_u\Big(h^{5/3}+h^{1/2}+h^{2/3}+h(1+h^{1/2})\Big)|||\chi||| \\ & + C_u(h^2+h+h^{3/2})|||\eta|||\Big]|||\vartheta||| \\ & + C_{\rho h_d}\Big[|||\chi|||^2 + C_u(h^{5/3}+h^{3/2})|||\chi||| + C_u(h^{3/2}+h)|||\eta|||\Big]|||\vartheta|||. \end{aligned} \tag{90}$$

Now putting the value of equation (89) and (90) in equation (47) we get

$$\begin{aligned}\bar{\mathscr{B}}(u;\vartheta,\vartheta) \leq & C_\epsilon\Big[|||\chi|||^2 + C_u\Big(h^{5/3}+h^{1/2}+h^{2/3}+h(1+h^{1/2})\Big)|||\chi||| \\ & + C_u(h^2+h+h^{3/2})|||\eta|||\Big]|||\vartheta||| + C_{\rho h_d}\Big[|||\chi|||^2 + C_u(h^{5/3}+h^{3/2})|||\chi||| \\ & + C_u(h^{3/2}+h)|||\eta|||\Big]|||\vartheta||| + C|||\eta||| \, |||\vartheta|||. \end{aligned} \tag{91}$$

Now using coercive property we obtain

$$\begin{aligned}|||\vartheta|||^2 \leq & C_\epsilon\Big[|||\chi|||^2 + C_u\Big(h^{5/3}+h^{1/2}+h^{2/3}+h(1+h^{1/2})\Big)|||\chi||| \\ & + C_u(h^2+h+h^{3/2})|||\eta|||\Big]|||\vartheta||| + C_{\rho h_d}\Big[|||\chi|||^2 + C_u(h^{5/3}+h^{3/2})|||\chi||| \\ & + C_u(h^{3/2}+h)|||\eta|||\Big]|||\vartheta||| + C|||\eta||| \, |||\vartheta|||. \end{aligned} \tag{92}$$

Now eliminating ϑ from both sides we get the desire result. □

Theorem 15. *For sufficiently small h there is a $\delta > 0$ such that the map $\mathcal{S}$ maps $\mathcal{Q}_\delta(\Pi_h u)$ into itself.*

Proof. Let $u_\varphi \in \mathcal{Q}(\Pi_h u)$ and consider an element y such that $y = \mathcal{S}u_\varphi$. Furthermore, choose $\delta = h^{-\delta_0}|||\Pi_h u - u|||$, where $0 < \delta_0 \leq 1/4$. Then we get

$$\begin{aligned}|||\Pi_h u - u_\varphi|||^2 \leq & \delta^2 \\ |||\Pi_h u - u_\varphi|||^2 \leq & h^{-\delta_0}|||\Pi_h u - u|||\delta \\ |||\Pi_h u - u_\varphi|||^2 \leq & h^{1-\delta_0}C\|u\|_{H^2(\Omega)}\delta \\ |||\Pi_h u - u_\varphi|||^2 \leq & h^{1-\delta_0}C'_u C_1\delta. \end{aligned} \tag{93}$$

From lemma 3.11 and equation (93) we get

$$\begin{aligned}|||\Pi_h u - \varphi||| \leq & \Big[(C_\epsilon + C_{\rho h_d})h^{1-\delta_0}C'_u C_1\delta + \Big(C_\epsilon C_u(h^{1/2}+h^{2/3}+h) \\ & + C_u(C_{\rho h_d}+C_\epsilon)(h^{5/3}+h^{3/2})\Big)h^{\delta_0}\delta \\ & + \Big(C_u(C_{\rho h_d}+C_\epsilon)(h+h^{3/2}) + C_\epsilon C_u h^2 + C\Big)h^{\delta_0}\delta\Big]. \end{aligned} \tag{94}$$

Now choosing h small enough so that

$$\begin{aligned}&\Big[(C_\epsilon + C_{\rho h_d})h^{1-\delta_0}C'_u C_1 + \Big(C_\epsilon C_u(h^{1/2}+h^{2/3}+h) + C_u(C_{\rho h_d}+C_\epsilon)(h^{5/3} \\ & + h^{3/2})\Big)h^{\delta_0} + \Big(C_u(C_{\rho h_d}+C_\epsilon)(h+h^{3/2}) + C_\epsilon C_u h^2 + C\Big)h^{\delta_0}\Big] < 1 \end{aligned} \tag{95}$$

and so $\mathcal{S}$ maps $\mathcal{Q}_\delta(\Pi_h u)$ into itself. □

Theorem 16. *Let $\delta > 0$ and assume that $u_{\varphi_1}, u_{\varphi_2} \in \mathcal{Q}_\delta(\Pi_h u)$, then there exists a positive constant C such that the following condition holds for given $0 \le \delta_0 \le 1/4$*

$$|||\mathcal{S}u_{\varphi_1} - \mathcal{S}u_{\varphi_2}||| \le C_u C h^{\delta_0} |||u_{\varphi_1} - u_{\varphi_2}|||. \tag{96}$$

Proof. Consider $\delta = h^{-\delta_0}|||\eta|||$ for some $0 \le \delta_0 \le 1/4$, where $\eta = \Pi_h u - u$. Take $\varphi_1 = \mathcal{S}u_{\varphi_1}$ and $\varphi_2 = \mathcal{S}u_{\varphi_2}$. Then, we have

$$\bar{\mathscr{B}}(u; \varphi_1 - \varphi_2, v_h) = \mathscr{F}(u_{\varphi_1}; u_{\varphi_1} - u, v_h) - \mathscr{F}(u_{\varphi_2}; u_{\varphi_2} - u, v_h). \tag{97}$$

For proving condition (96), we first evaluate an upper bound of equation (97) as

$$\begin{aligned}
&\Big|\mathscr{F}(u_{\varphi_1}; u_{\varphi_1} - u, v_h) - \mathscr{F}(u_{\varphi_2}; u_{\varphi_2} - u, v_h)\Big| \\
\le&\Big|\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}(\tilde{\epsilon}_u(u_{\varphi_1})\zeta_1\nabla\zeta_1 - \tilde{\epsilon}_u(u_{\varphi_2})\zeta_2\nabla\zeta_2).\mathbf{n}\gamma v_h ds\Big| \\
&+\Big|\sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h]\Big\{\epsilon_u(u_{\varphi_1})\zeta_1\nabla\zeta_1 - \epsilon_u(u_{\varphi_2})\zeta_2\nabla\zeta_2\Big\}ds\Big| \\
&+\Big|\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\tilde{\epsilon}_{uu}(u_{\varphi_1})\zeta_1^2 - \tilde{\epsilon}_{uu}(u_{\varphi_2})\zeta_2^2\nabla u.\mathbf{n}\gamma v_h ds\Big| \\
&+\Big|\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}(\rho\tilde{h}_d)_{uu}(u_{\varphi_1})\zeta_1^2 - (\rho\tilde{h}_d)_{uu}(u_{\varphi_2})\zeta_2^2\vec{\beta}.\mathbf{n}\gamma v_h ds\Big| \\
&+\Big|\sum_{e_k\in\Gamma}\int_{e_k}[\gamma v_h]\Big\{(\rho h_d)_{uu}(u_{\varphi_1})\zeta_1^2 - (\rho h_d)_{uu}(u_{\varphi_2})\zeta_2^2\Big\}ds\Big|.
\end{aligned} \tag{98}$$

Now by using Taylor's formula we obtain

$$\begin{aligned}
\epsilon_u(u_{\varphi_1})(u_{\varphi_1} - u) - \epsilon_u(u_{\varphi_2})(u_{\varphi_2} - u) =&\epsilon(u_{\varphi_1}) - \epsilon(u_{\varphi_2}) \\
=&\tilde{R_{\epsilon_1}}(u_{\varphi_1}, u_{\varphi_2})(u_{\varphi_1} - u_{\varphi_2})
\end{aligned} \tag{99}$$

and

$$\begin{aligned}
&\tilde{\epsilon}_{uu}(u_{\varphi_1})(u_{\varphi_1} - u)^2 - \tilde{\epsilon}_{uu}(u_{\varphi_2})(u_{\varphi_2} - u)^2 \\
=&\tilde{R_{\epsilon_2}}(u_{\varphi_1}, u_{\varphi_2})(u_{\varphi_1} - u_{\varphi_2})^2 + \tilde{\epsilon}_{uu}(u_{\varphi_2})(u_{\varphi_2} - u)(u_{\varphi_1} - u_{\varphi_2}).
\end{aligned} \tag{100}$$

Now using (99) and (100) property and using similar argument of lemma 3.10 we can bound equation (98) as

$$\begin{aligned}
&\bar{\mathscr{B}}(u; \varphi_1 - \varphi_2, v_h) \\
\le&C_\epsilon\Big[|||\chi|||^2 + C_u\Big(h^{5/3} + h^{1/2} + h^{2/3} + h(1 + h^{1/2})\Big)|||\chi||| |||u_{\varphi_1} - \Pi_h u||| \\
&+ C_u(h^2 + h + h^{3/2})|||\chi||| |||u_{\varphi_2} - \Pi_h u|||\Big] |||v_h||| \\
&+ C_{\rho h_d}\Big[|||\chi|||^2 + C_u(h^{5/3} + h^{3/2})|||\chi||| |||u_{\varphi_1} - \Pi_h u||| \\
&+ C_u(h^{3/2} + h)|||\chi||| |||u_{\varphi_2} - \Pi_h u|||\Big] |||v_h||| \\
\le&CC_u h^{\delta_0}|||\chi||| |||v_h|||.
\end{aligned}$$

Now taking $v_h = \varphi_1 - \varphi_2$ and using coercive property we have the desire result. $\square$

Now we apply Browder's fixed point theorem to get unique DG-FVEM solution to the problem (39).

4. Error Estimates

In this section, we prove that under light load operating condition optimal order estimate in H^1 can be achieved in the defined norm $|||.|||$. Let $\Pi_h u \in \mathscr{V}_h$ be an interpolant of u, for which the following well known approximation property holds

$$|u - \Pi_h u|_{l,K} \leq Ch^{2-l}|u|_{2,K} \quad \forall K \in \mathscr{R}_h, \quad l = 0, 1. \tag{101}$$

where C depends only on the angle K. The following theorem provide H^1-norm estimate.

Theorem 17. *Suppose $u \in H^2(\Omega) \cap H^1_0(\Omega)$ and $u_h \in \mathscr{V}_h$ be the solution of (43) and (38) respectively. Then there exists a constant C (independent of h but may be dependent on α_1) such that*

$$|||u - u_h||| \leq Ch|u|_2 \tag{102}$$

Proof. By definition of norm define in equation (24) we get

$$|||u - \Pi_h u|||^2 = |u - \Pi_h u|^2_{1,h} + \sum_{e_k} [\gamma u - \gamma \Pi_h u]^2_{e_k}$$

$$\leq C\Big(|u - \Pi_h u|^2_{1,h} + \sum_{e_k} \int_{e_k} h^{-1}_{e_k} [u - \Pi_h u]^2_{e_k} ds\Big).$$

Now using trace inequality we have

$$|||u - \Pi_h u|||^2 \leq C\Big(|u - \Pi_h u|^2_{1,h} + \sum_{K} h^{-2}\|u - \Pi_h u\|^2_{K}\Big)$$

Now by using equation (101) we obtain

$$|||u - \Pi_h u|||^2 \leq Ch^2|u|^2_{2,K} \tag{103}$$

Now as $u - u_h = u - \Pi_h u + \Pi_h u - u_h$ and also using triangle inequality we have the following

$$|||u - u_h||| \leq |||u - \Pi_h u||| + |||\Pi_h u - u_h|||$$

It is important to note that operator as defined in (42) is coercive, bounded and monotone under the norm define in equation (24) So for h sufficiently small, there exist a positive constant $C = C(\alpha_1)$ (independent of h but may be dependent on α_1) such that following condition holds

$$|||\Pi_h u - u_h|||^2 \leq C|\langle \bar{\mathscr{B}}(u_h; u_h - \Pi_h u, u_h - \Pi_h u)|$$

$$\leq C\Big\{|||u_h - \Pi_h u||| + \Big(\sum_{K \in \mathscr{R}_h} h^2|u - \Pi_h u|^2_{2,K}\Big)^{1/2}\Big\}|||u_h - \Pi_h u|||$$

$$\leq Ch\|u\|_2.$$

So we have the desire result. $\square$

4.1. L^2-Error Estimates.

In this section, L^2-error estimate is evaluated for the light load parameter case by exploiting the Aubin-Nitsche "trick".

Theorem 18. *Let $u \in H^2(\Omega) \cap H^1_0(\Omega)$ and $u_h \in \mathscr{V}_h$ be the solution of problem (43) and (39) respectively. Then there exists a positive constant C independent of h but may be dependent on α_1 such that*

$$\|u - u_h\| \leq Ch^2\|u\|_2 \tag{104}$$

Proof. Consider $\phi \in H^2(\Omega)$ and for fix value of u and $h_d \in H^2(\Omega)$ we write the adjoint problem of (1.1) as

$$\nabla\Big(\epsilon(u)\nabla\phi - \phi\epsilon_u\nabla u\Big) + \vec{\beta}\Big(\rho h_d + (\rho h_d)_u\Big)\nabla\phi = e \quad \text{in } \Omega \tag{105}$$

$$\phi = 0 \quad \text{on } \partial\Omega. \tag{106}$$

also we have

$$\begin{aligned}\|e\|^2 =& \mathscr{B}(u;e,\phi) - \sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\epsilon_u e\nabla.\mathbf{n}\gamma\phi ds - \sum_{e_k\in\Gamma}\int_{e_k}[\gamma\phi]\Big\{\epsilon_u e\nabla u\Big\}ds\\ &+ \sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}(\rho h_d)_u e\vec{\beta}.\mathbf{n}\gamma\phi ds + \sum_{e_k\in\Gamma}\int_{e_k}[\gamma\phi]\Big\{(\rho h_d)_u e\vec{\beta}\Big\}ds\end{aligned} \tag{107}$$

First term of equation (107) is rewritten as

$$\begin{aligned}\mathscr{B}(u;e,\phi) =& \mathscr{B}(u;u,\phi) - \mathscr{B}(u_h;u_h,\phi) + \mathscr{B}(u_h;u_h,\phi) - \mathscr{B}(u;u_h,\phi)\\ =& \underbrace{\mathscr{B}(u;u,\phi-\vartheta) - \mathscr{B}(u_h;u_h,\phi-\vartheta)}_{I} + \underbrace{\mathscr{B}(u_h;u_h,\phi) - \mathscr{B}(u;u_h,\phi)}_{II},\end{aligned}$$

where $\vartheta = \mathcal{I}_h^k\phi$ such that $\vartheta|_{\partial\Omega} = 0$ (Here $\mathcal{I}_h^k u \in \mathscr{V}_h \cap H^2(\Omega) \cap C^0(\Omega)$). We notice that

$$\begin{aligned}I =& \mathscr{B}(u;u,\phi-\vartheta) - \mathscr{B}(u_h;u,\phi-\vartheta) + \mathscr{B}(u_h;u,\phi-\vartheta) - \mathscr{B}(u_h;u_h,\phi-\vartheta)\\ =& \sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}(\epsilon(u)-\epsilon(u_h))\nabla u.\mathbf{n}\gamma(\phi-\vartheta)ds\\ &+ \sum_{e_k\in\Gamma}\int_{e_k}[\gamma(\phi-\vartheta)]\Big\{(\epsilon(u)-\epsilon(u_h))\nabla u\Big\}ds\\ &- \sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}(\rho(u)h_d(x)-\rho(u_h)h_d(x))\vec{\beta}.\mathbf{n}\gamma(\phi-\vartheta)ds\\ &- \sum_{e_k\in\Gamma}\int_{e_k}[\gamma(\phi-\vartheta)]\Big\{(\rho(u)h_d(x)-\rho(u_h)h_d(x))\vec{\beta}\Big\}ds\\ &+ \sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}\epsilon(u_h)\nabla(u-u_h).\mathbf{n}\gamma(\phi-\vartheta)ds\\ &+ \sum_{e_k\in\Gamma}\int_{e_k}[\gamma(\phi-\vartheta)]\Big\{\epsilon(u_h)\nabla(u-u_h)\Big\}ds\\ =& J_{I_1} + J_{I_2} + J_{I_3} + J_{I_4} + J_{I_5} + J_{I_6}.\end{aligned} \tag{108}$$

First term, J_{I_1} of equation (108) is approximated as

$$\begin{aligned}|J_{I_1}| \leq& \Big|\sum_K\langle\epsilon(u)-\epsilon(u_h)\nabla u, \nabla(\phi-\vartheta)\rangle\Big|\\ &+ \Big|\sum_K\int_{\partial K}[\gamma(\phi-\vartheta)-(\phi-\vartheta)](\epsilon(u)-\epsilon(u_h))\nabla u.\mathbf{n}ds\Big|\\ &+ \Big|\sum_K\langle\nabla(\epsilon(u)-\epsilon(u_h))\nabla u, (\phi-\vartheta)-\gamma(\phi-\vartheta)\rangle\Big|\\ =& J_{01} + J_{02} + J_{03}.\end{aligned} \tag{109}$$

We bound first term, J_{01} of equation (109) as

$$\sum_K \Big| \int_K \epsilon(u) - \epsilon(u_h)\nabla u.\nabla(\phi - \vartheta)dx \Big| \leq C_u C_\epsilon |||e||| \|\phi - \vartheta\|. \tag{110}$$

Second term, J_{02} of equation (109) is approximated bounded above as

$$\begin{aligned} J_{02} \leq & C_u C_\epsilon \sum_K \Big(h^{-1}\|\gamma(\phi-\vartheta) - (\phi-\vartheta)\|_K^2 + h\|\gamma(\phi-\vartheta) - (\phi-\vartheta)\|_{1,K}^2 \Big)^{1/2} \times \|e\| \\ \leq & C_u C_\epsilon \|\phi - \vartheta\|_{H^1(\Omega)} |||e|||. \end{aligned} \tag{111}$$

Similarly, third term J_{03} of equation (109) is estimated as

$$J_{03} \leq C_\epsilon C_u |||e||| \|\phi - \vartheta\| + C_\epsilon C_u \|\phi - \vartheta\|_{H^1(\Omega)} |||e|||. \tag{112}$$

Using Holder's inequality and trace inequality we estimate second term, J_{I_2} of equation (108) as

$$\begin{aligned} J_{I_2} \leq & C_\epsilon \sum_{e_k \in \Gamma} \Big(\int_{e_k} [\gamma(\phi - \vartheta)]^2 ds \Big)^{1/2} \Big(\int_{e_k} |e|^4 ds \Big)^{1/4} \Big(\int_{e_k} |\nabla u|^4 ds \Big)^{1/4} \\ \leq & C_\epsilon \sum_{e_k \in \Gamma} \Big(\int_{e_k} h_{e_k}^{-1} [\gamma(\phi - \vartheta)]^2 ds \Big)^{1/2} \Big(\|e\|_{L^4(K)}^4 + h\|e\|_{L^6(K)}^3 \|\nabla e\|_{L^2(K)} \Big)^{1/4} \\ & \times \Big(\|\nabla u\|_{L^4(K)}^4 + h\|\nabla u\|_{L^6(K)}^3 \|\nabla.\nabla u\|_{L^2(K)} \Big)^{1/4} \\ \leq & C_u C_\epsilon |||e|||^2 |||\phi - \vartheta|||. \end{aligned} \tag{113}$$

By using Similar argument we bound the following terms as

$$|J_{I_3}| \leq C_u |||e||| |||\phi - \vartheta|||, \tag{114}$$
$$|J_{I_4}| \leq C_u |||e||| |||\phi - \vartheta|||, \tag{115}$$
$$|J_{I_5}| \leq C_u |||e||| |||\phi - \vartheta|||, \tag{116}$$
$$|J_{I_6}| \leq C_u |||e||| |||\phi - \vartheta|||. \tag{117}$$

We note that

$$
\begin{aligned}
II =& \sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}(\epsilon(u_h)-\epsilon(u))\nabla u_h.\mathbf{n}\gamma\phi ds + \sum_{e_k\in\Gamma}\int_{e_k}[\gamma\phi]\Big\{(\epsilon(u_h)\\
&-\epsilon(u))\nabla u_h\Big\}ds - \sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}(\rho(u_h)h_d(x)-\rho(u)h_d(x))\vec{\beta}.\mathbf{n}\gamma\phi ds\\
&-\sum_{e_k\in\Gamma}\int_{e_k}[\gamma\phi]\Big\{\Big(\rho(u_h)h_d(x)-\rho(u)h_d(x)\Big)\vec{\beta}\Big\}ds\\
=& \sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}(\epsilon(u_h)-\epsilon(u))\nabla(u_h-u).\mathbf{n}\gamma\phi ds\\
&+\sum_{e_k\in\Gamma}\int_{e_k}[\gamma\phi]\Big\{(\epsilon(u_h)-\epsilon(u))\nabla(u_h-u)\Big\}ds\\
&+\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}(\epsilon(u_h)-\epsilon(u))\nabla u.\mathbf{n}\gamma\phi ds\\
&+\sum_{e_k\in\Gamma}\int_{e_k}[\gamma\phi]\Big\{(\epsilon(u_h)-\epsilon(u))\nabla u\Big\}ds\\
&-\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j}(\rho(u_h)h_d(x)-\rho(u)h_d(x))\vec{\beta}.\mathbf{n}\gamma\phi ds\\
&-\sum_{e_k\in\Gamma}\int_{e_k}[\gamma\phi]\Big\{\Big(\rho(u_h)h_d(x)-\rho(u)h_d(x)\Big)\vec{\beta}\Big\}ds
\end{aligned}
$$

$$=J_{II_1}+J_{II_2}+J_{II_3}+J_{II_4}+J_{II_5}+J_{II_6} \tag{118}$$

First term J_{II_1} of equation (118) is approximated as

$$
\begin{aligned}
J_{II_1}\le&\Big|\sum_{K}\langle\epsilon(u_h)-\epsilon(u)\nabla(u_h-u),\nabla\phi\rangle\Big|+\Big|\sum_{e\in\Gamma}\int_{\partial K}[\gamma\phi-\phi]\Big\{(\epsilon(u_h)\\
&-\epsilon(u))\nabla(u_h-u)\Big\}ds\Big|+\Big|\sum_{K}\langle\nabla(\epsilon(u_h)-\epsilon(u))\nabla(u_h-u),\phi-\gamma\phi\rangle\Big|
\end{aligned}
$$

$$=J^1_{II_1}+J^2_{II_1}+J^3_{II_1}. \tag{119}$$

First term $J^1_{II_1}$ of equation (119) is estimated by using Holder's inequality

$$
\begin{aligned}
J^1_{II_1}\le&C_u\Big(\sum_K\int_K|e|^3dx\Big)^{1/3}\Big(\sum_K\int_K|\nabla e|^2dx\Big)^{1/2}\Big(\sum_K\int_K|\nabla\phi|^6dx\Big)^{1/6}\\
\le&C_uC\|\!|e|\!\|^2\|\phi\|_{H^2(\Omega)}
\end{aligned} \tag{120}
$$

Using trace inequality second term $J^2_{II_1}$ of equation (119) is estimated as

$$
\begin{aligned}
J^2_{II_1}\le&C_u\Big(\int_{\partial K}[\gamma\phi-\phi]^2ds\Big)^{1/2}\Big(\int_{\partial K}|e|^4ds\Big)^{1/4}\Big(\int_{\partial K}|\nabla e|^4ds\Big)^{1/4}\\
\le&C_uC\|\!|e|\!\|^2\|\phi\|_{H^2(\Omega)}
\end{aligned} \tag{121}
$$

Third term, $J^3_{II_1}$ of equation (119) is bounded using Holder's and trace inequality as

$$\begin{aligned} J^3_{II_1} \leq & C_u\Big(\sum_K \int_K |e|^3 dx\Big)^{1/3}\Big(\sum_K \int_K |\nabla e|^2 dx\Big)^{1/2}\Big(\sum_K \int_K |\nabla(\phi - \gamma\phi)|^6 dx\Big)^{1/6} \\ & + C_u\Big(\int_{\partial K} |\gamma\phi - \phi|^2 ds\Big)^{1/2}\Big(\int_{\partial K} |e|^4 ds\Big)^{1/4}\Big(\int_{\partial K} |\nabla e|^4 ds\Big)^{1/4} \\ \leq & C_u C |||e|||^2 \|\phi\|_{H^2(\Omega)} \end{aligned} \tag{122}$$

We bound the second term J_{II_2} of equation (118) by using trace as well as Holder's inequality to obtain

$$J_{II_2} \leq C_u C |||e|||^2 \|\phi\|_{H^2(\Omega)}. \tag{123}$$

Now consider the third term J_{II_3} of equation (118) and take second term of equation (107) and using Taylor's formula get

$$\Big|\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j} \tilde{\epsilon}_{uu}(u_h) e^2 \nabla u.\mathbf{n}\gamma\phi ds\Big| \leq C_u C_\epsilon |||e|||^2 \|\phi\|_{H^2(\Omega)}. \tag{124}$$

Take fourth term J_{II_4} of equation (118) and third term of equation (107) and use Taylor's formula to obtain

$$\begin{aligned} \Big|\sum_{e_k\in\Gamma}\int_{e_k}[\gamma\phi]\Big\{\tilde{\epsilon}_{uu}(u_h)e^2\nabla u\Big\}ds\Big| =& \Big|\sum_{e_k\in\Gamma}\int_{e_k}[\gamma\phi - \phi]\Big\{\tilde{\epsilon}_{uu}(u_h)e^2\nabla u\Big\}ds\Big| \\ \leq & C_u C_\epsilon |||e|||^2 \|\phi\|_{H^2(\Omega)}. \end{aligned} \tag{125}$$

We take fifth term J_{II_5} of equation (118) and fourth term of equation (107) and use Taylor's formula to get

$$\Big|\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{4}\int_{A_{j+1}CA_j} \rho\tilde{h}_{d_{uu}} e^2 \vec{\beta}.\mathbf{n}\gamma\phi ds\Big| \leq C_u C_{\rho h_d} |||e|||^2 \|\phi\|_{H^2(\Omega)} \tag{126}$$

Finally, taking sixth term J_{II_6} of equation (118), using the fact $[\phi] = 0$ and fifth term of equation (107) and by using Taylor's formula we get bound as

$$\begin{aligned} \Big|\sum_{e_k\in\Gamma}\int_{e_k}[\gamma\phi]\Big\{\rho\tilde{h}_{d_{uu}}(u_h)e^2\Big\}ds\Big| =& \Big|\sum_{e_k\in\Gamma}\int_{e_k}[\gamma\phi - \phi]\Big\{\rho\tilde{h}_{d_{uu}}(u_h)e^2\Big\}ds\Big| \\ \leq & C_{\rho h_d} |||e|||^2 \|\phi\|_{H^2(\Omega)}. \end{aligned} \tag{127}$$

Now using elliptic regularity of ϕ and combining right hand side of estimated results, we get the desire result. □

5. Numerical test of Discontinuous Galerkin finite volume method

In this section, numerical experiments are performed for EHL point contact cases. Numerical solution of EHL problems is obtained here using relaxation procedure explained in appendix. (A). Optimal error estimates for pressure $(u - u_h)$ are achieved in broken H^1 norm $|||.|||$ and L^2 norm which are plotted in Fig. (4) with the red line and the blue line respectively.

Remark 19. Note that we have taken a very fine mesh($\approx 1025\times1025$) and then solve the problem by linearizing DG-FVEM formulation (discrete version of eqn.(37)). We start relaxation upto appropriate tolernece (tolerence critera $(|||U_{i+1} - U_i|||)/|||U_i||| \approx 10^{-4}$ to terminate the relaxation process in our case) using initial Hertzian pressure guess as mentioned below. Then we have treated this solution as our exact solution

u for penalized problem for plotting error estimates in Fig. (4) by interpolating this exact solution on coarse grid.

For all test cases, we fix the parameter $\alpha = 1.7 \times 10^{-8}$ over computational domain $\Omega = [-2.5, 2.5] \times [-2.5, 2.5]$. The following Hertzian initial guess is taken here for computation

$$u(x,y) = \begin{cases} \sqrt{1-x^2-y^2}, & x^2+y^2 \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

We have taken interior penalty parameter $\alpha_1 = 10$ in our numerical study. Numerical results confirm the theoretical order of convergence derived in Theorem (16) and Theorem (17) which are almost equal to 1 and 2 respectively.
In the present analysis, we have taken Moes ([33, 32, 42, 43]) dimensionless parameters ($M = W(2U)^{-\frac{3}{4}}$ and $L = G(2U)^{\frac{1}{4}}$, where W is load parameter and $U = 1.0 \times 10^{-11}$ speed parameter (fixed value) see appendix. ((B)) for parameters detail) to measure the variation effect on film thickness and pressure profile. The load variation W is measured by keeping fixed L and varying M. In our case study, we take $L = 10$ and M varies from 7 to 100. The contour plot plot for pressure are shown graphically for light (see Fig. ((12)), for example) , moderate (see Fig. ((13)), for example) and high load case (see Fig. ((14)), for example). The deformation of film thickness contour plots are represented for light (Fig. ((15)), see for example), moderate (Fig. ((16)), see for example) and high load case (see in Fig. ((17)), for example). It is clearly observed from figures that as M increases from 7 to 100 pressure contour starts converging toward Hertzian pressure. It is also observed that height of the pressure peak starts decreasing as we increase load W. It is also noted that film thickness contour plot almost getting flat when load increases from low to high. Graphical figures of pressure u for the case $M = 7, 20, 100$ are represented in Fig. ((5)), Fig. ((7)) and Fig. ((9)). Inverted form of film thickness profile H plots for case $M = 7, 20, 100$ are described in Fig. ((6)) – Fig. ((8)) and Fig. ((10)).

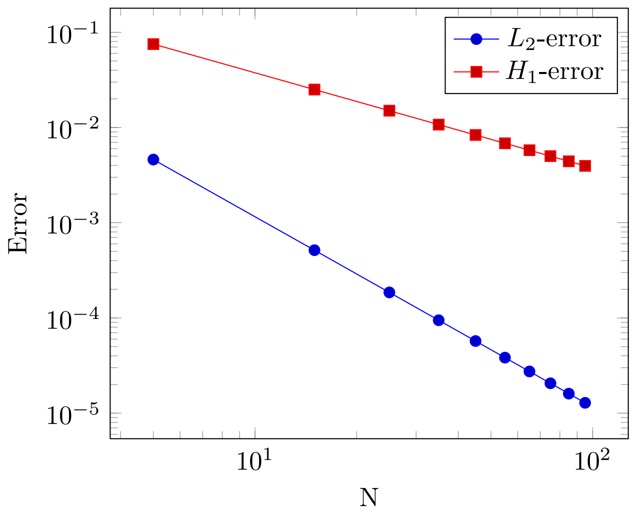


FIGURE 4. L^2 (in blue line) and H^1 (in red line) error $\|u - u_h\|$ plot.

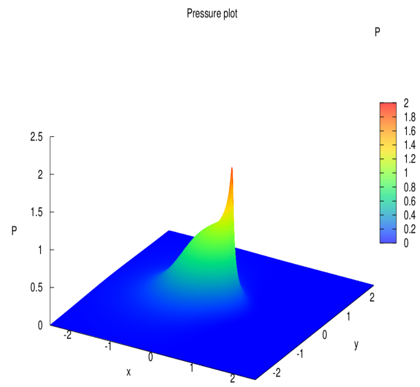


FIGURE 5. Pressure profile for light load case $M = 7$ and $L = 10$ (Moe's parameter).

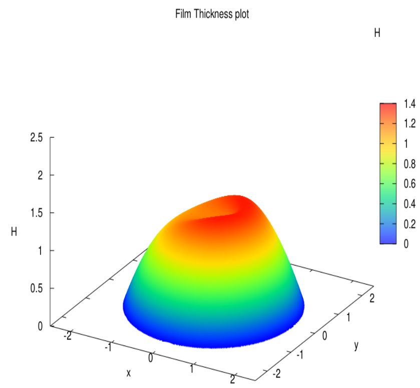


FIGURE 6. Film thickness profile in inverted form for $M = 7$ and $L = 10$.

5.1. Film thickness calculation. Accurate film thickness h_d computation is very important for stable relaxation procedure and require extra care in its computation. Film thickness is calculated as follows

$$\begin{aligned} h_d(x,y) =& h_0 + \frac{x^2+y^2}{2} + \frac{2}{\pi^2}\int_{x_-}^{x_+}\int_{y_-}^{y_+}\frac{u(x',y')dx'dy'}{\sqrt{(x-x')^2+(y-y')^2}} \\ =& h_0 + \frac{x^2+y^2}{2} + \frac{2}{\pi^2}\sum_{e=1}^{N}\int_e \frac{\sum_{i=0}^{p_e+1} U_i^e \mathscr{N}_i^e(x',y')}{\sqrt{(x-y')^2+(y-y')^2}}dx'dy' \\ =& h_0 + \frac{x^2+y^2}{2} + \frac{2}{\pi^2}\sum_{e=1}^{N}\sum_{i=0}^{p_e+1}\int_e \frac{U_i^e \mathscr{N}_i^e(x',y')dx'dy'}{\sqrt{(x-y')^2+(y-y')^2}} \\ =& h_0 + \frac{x^2+y^2}{2} + \frac{2}{\pi^2}\sum_{e=1}^{N}\sum_{i=0}^{p_e+1}\mathscr{G}_i^e(x,y)U_i^e, \end{aligned} \tag{128}$$

where U_i^e is updated solution.

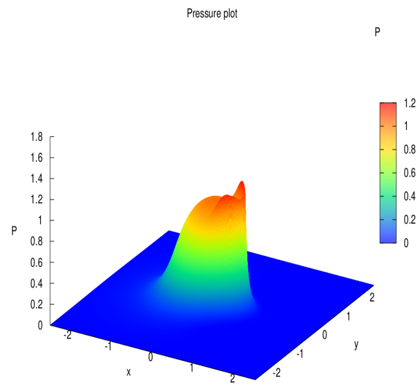


FIGURE 7. Pressure profile for moderately high load case $M = 20$ and $L = 10$.

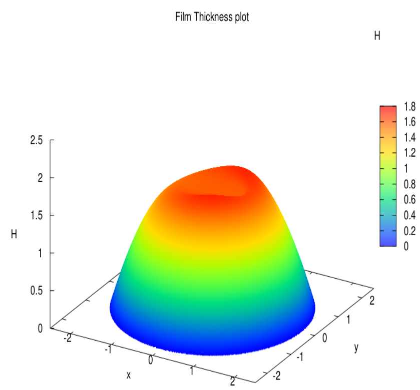


FIGURE 8. Film thickness profile in inverted form for $M = 20$ and $L = 10$.

5.2. Mild singular integral computation. The mild singularity appear in kernal at $(x', y') = (x, y)$ is resolved as below. We rewrite kernel $\mathscr{G}_i^e(x, y)$ in the following form

$$\begin{aligned}
\mathscr{G}_i^e(x,y) &= \int_{\Omega_e} \frac{\mathscr{N}_i^e(x',y')dx'dy'}{\sqrt{(x-y')^2+(y-y')^2}} \\
&= \frac{h_x^e}{2}\frac{h_y^e}{2}\int_{-1}^{1}\int_{-1}^{1} \frac{\mathscr{N}_i^e(x'(\xi,\chi),y'(\xi,\chi))d\xi d\chi}{\sqrt{(x-x'(\xi,\chi))^2+(y-y'(\xi,\chi))^2}} \\
&\approx \frac{h_x^e}{2}\frac{h_y^e}{2}\sum_{j=1}^{m}\sum_{k=1}^{m} \frac{\mathscr{N}_i^e(x'(\xi_j,\chi_k),y'(\xi_j,\chi_k))w_j w_k}{\sqrt{(x-x'(\xi_j,\chi_k))^2+(y-y'(\xi_j,\chi_k))^2}}, \qquad (129)
\end{aligned}$$

where $h_x^e = x_2 - x_1$ and $h_y^e = y_2 - y_1$ are the step sizes of element e in the x direction and y direction respectively and $\xi \in [-1, 1]$ and $\chi \in [-1, 1]$ are the coordinate directions for the reference element. Here we apply m point quadrature in x and y direction of discretization. Singular quadrature procedure is implemented here to resolve the singularity appeared in term $\mathscr{G}_i^e(x, y) = \frac{1}{\sqrt{(x-x')^2+(y-y')^2}}$ at the point (x, y). The key idea is divide the element e into four subpart elements $\mathscr{F}_k, k = 1, 2, 3, 4$ for calculating integrals of $\mathscr{G}_i^{\mathscr{F}_k}(x, y) =$

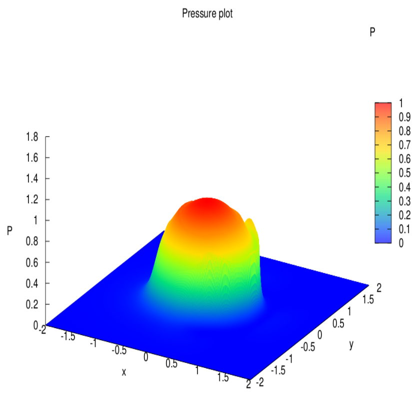


FIGURE 9. Pressure profile for limit case $M = 100$ and $L = 10$.

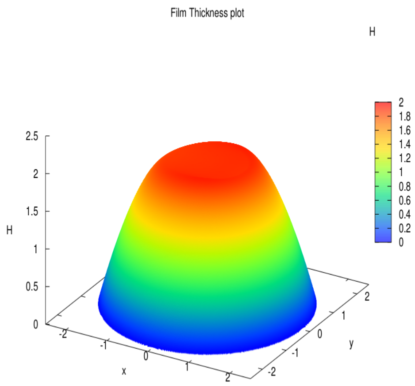
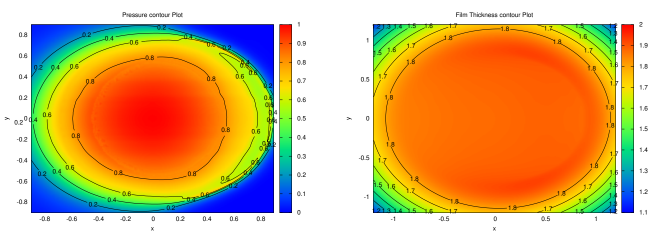


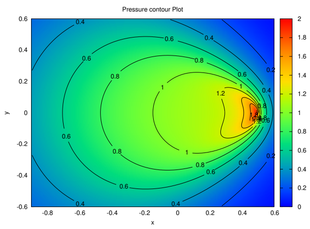
FIGURE 10. Limit case film thickness profile for $M = 100$ and $L = 10$.



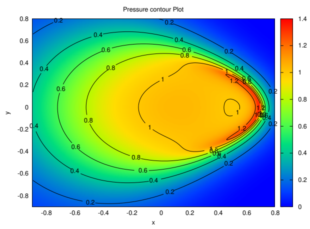
(A) (B)

FIGURE 11. Contour plot for limit case $M = 100$ and $L = 10$.

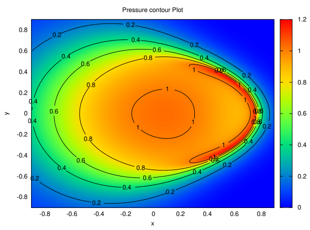
$\frac{1}{\sqrt{(x-x')^2+(y-y')^2}}$. Each four integrals have chosen in a such way that they have only one singular point in the domain of integration. Four integrals defined above can be evaluated



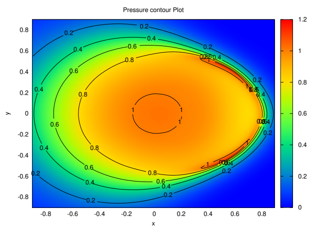
(A) M=7,L=10



(B) M=14,L=10

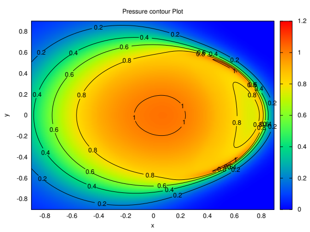


(C) M=20,L=10

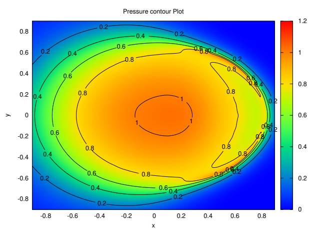


(D) M=28,L=10

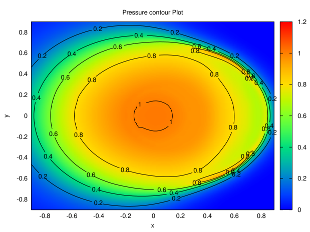
FIGURE 12. Light load case pressure contour.



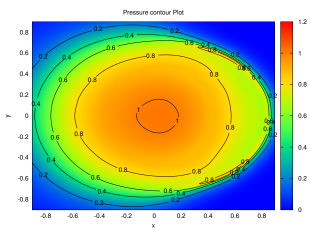
(A) M=35,L=10



(B) M=42,L=10



(C) M=49,L=10



(D) M=56,L=10

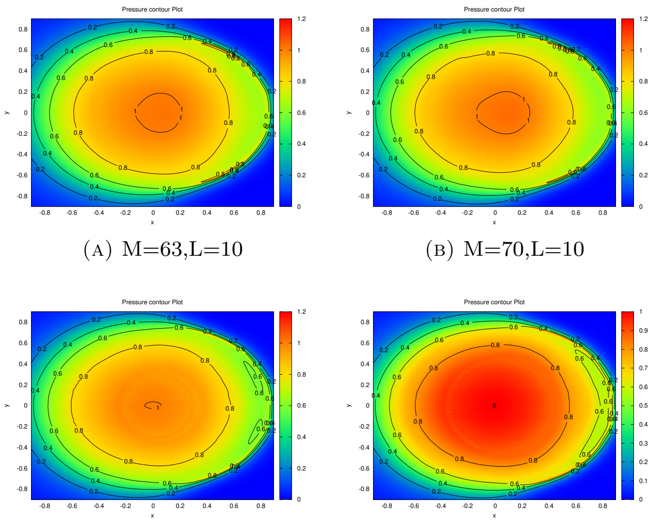
FIGURE 13. Moderate load case pressure contour.

as in general integral form:

$$\mathscr{S}^* = \int_0^1 \int_0^1 \mathscr{F}^*(x,y)\mathscr{G}^*(x,y)dxdy, \tag{130}$$

where $\mathscr{F}^*$ is analytic function and $\mathscr{G}^*$ is a function having a mild singularity at only one point.

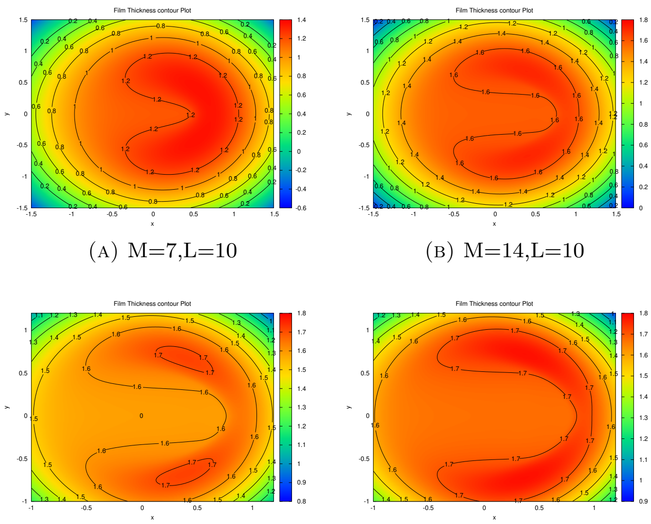
$$\mathscr{S} \approx \mathscr{S}_n^* = \sum_{i=1}^{n} \mathscr{I}_i, \tag{131}$$



(A) M=63,L=10 (B) M=70,L=10

(C) M=80,L=10 (D) M=90,L=10

FIGURE 14. High load case pressure contour.



(A) M=7,L=10 (B) M=14,L=10

(C) M=20,L=10 (D) M=28,L=10

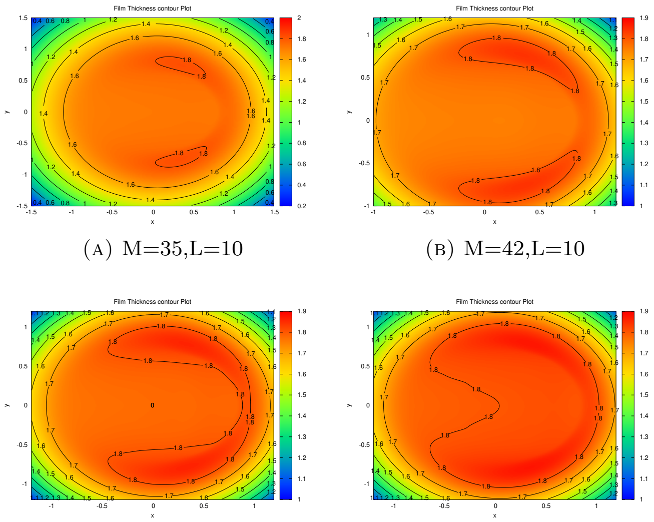
FIGURE 15. Light load case film thickness deformation.

where

$$\mathscr{I}_i = \int_{x_i}^{x_{i-1}} \int_{y_i}^{y_{i-1}} \mathscr{F}^*(x,y)\mathscr{G}^*(x,y)dxdy, (i \geq 1). \tag{132}$$

Where $(x_0, y_0) = (1,1)$ and $(x_n, y_n) \to (0,0)$ as $n \to \infty$ for the value $(x_n, y_n) = (\theta^n, \theta^n), (0 < \theta < 1)$.

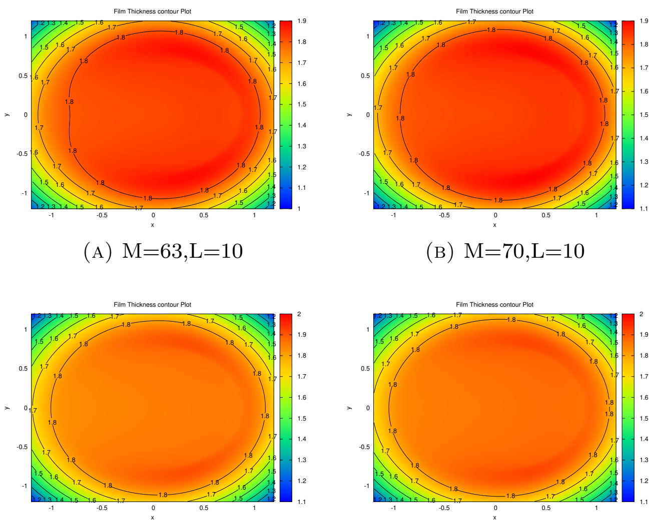
Remark 20. Adoptive integration is applied here to calculate accurate singular integration to achieve accuracy.



(A) M=35,L=10 (B) M=42,L=10

(C) M=49,L=10 (D) M=56,L=10

FIGURE 16. Moderate load case film thickness deformation.



(A) M=63,L=10 (B) M=70,L=10

(C) M=80,L=10 (D) M=90,L=10

FIGURE 17. High load case film thickness deformation.

5.3. Load balance equation calculation. The force balance equation is discretized according to

$$\sum_{e=1}^{N} \int_{\Omega_e} \sum_{i=0}^{p_e+1} \mathscr{G}_i^e(x,y) U_i^e \, dxdy - \frac{2\pi}{3} = 0 \tag{133}$$

By introducing another kernel $\mathscr{N}_i^e$ as

$$\mathscr{N}_i^e = \int_{\Omega_e} \mathscr{G}_i^e(x,y) \, dxdy \tag{134}$$

the discrete force balance equation can be rewritten as

$$\sum_{e=1}^{N}\sum_{i=0}^{p_e+1}(\mathscr{N}_i^{\,e})U_i^e-\frac{2\pi}{3}=0 \tag{135}$$

Remark 21. We implement discrete force balance equation implicitly during the relaxation procedure by updating h_{00} in following way

$$h_{00}\leftarrow h_{00}-c1\Big(\sum_{e=1}^{N}\sum_{i=0}^{p_e+1}(\mathscr{N}_i^{\,e})U_i^e-\frac{2\pi}{3}\Big), \tag{136}$$

where U_i^e is updated solution and $c1\approx 0.01-0.1$ is positive relaxation factor.

6. Conclusion

New discontinuous Galerkin finite volume element method is investigated for solving EHL model problem with the help of interior-exterior penalty approach. Interior penalty comes naturally using formulation of DG-FVEM, however exterior penalty appears naturally due to transformation of variational inequality into equality. This exterior penalty formulation is obtained by regularizing the constraint $u\geq 0$. The existence and uniqueness for discrete DG-FVEM formulation is proved using Browder's fixed point theorem. This method is fully systematic and easily parallelized in MPI (Massage passing interface) environment. The stability estimates are achieved by showing operator as pseudo-monotone for moderate load condition. Optimal error estimates are derive under light load condition theoretically as well as by numerical computation in H^1 and L^2 norm respectively. More implementation issues and applications will be discussed in the second part of the paper.

Appendix A. Relaxation of EHL

For finding unique solution for discrete DG-FVEM formulation explained in equation (37), we update our nonlinear operator iterative manner by taking old and new pressure value in the following form

$$U_{\text{new}}=U_{\text{old}}+\Big(\frac{\partial\mathscr{T}_d(U)}{\partial U}\Big)^{-1}\mathcal{R}_s, \tag{A.1}$$

where $\mathcal{R}_s$ is the numerical residual value of the discretized Reynolds equation and, $\mathscr{T}_d$ is discretized nonlinear operator. The approximation of $\frac{\partial\mathscr{T}_d(U)}{\partial U}$ is evaluated in the following way,

$$\begin{aligned}\frac{\partial\mathscr{T}_d(U)}{\partial U}&\approx\frac{\partial\mathscr{T}_d^{*}(U)}{\partial U}-\frac{\partial\mathscr{T}_d^{**}(U)}{\partial U}\\&\approx\mathscr{A}_d^{*}(U)-\frac{\partial\mathscr{T}_d^{**}(U)}{\partial U},\end{aligned} \tag{A.2}$$

where $\mathscr{A}_d^{*}(U)$ and $\frac{\partial\mathscr{T}_d^{**}(U)}{\partial U}$ denote the discrete diffusion contribution and the discrete convection contribution respectively. In the above equation (A.2), we notice that term $\frac{\partial\mathscr{T}_d^{**}(U)}{\partial U}$ is a full dense matrix and it is evaluated in the following way,

$$\begin{aligned}\frac{\partial\mathscr{T}_d^{**}(U)}{\partial U}\Big|_{I,J}=&\sum_{K\in\mathscr{R}_h}\sum_{j=1}^{3}\int_{A_{j+1}CA_j}(\rho\frac{\partial h_d}{\partial U_j^f}+h_d\frac{\partial\rho}{\partial U_j^f}).(\vec{\beta}.\mathbf{n})\gamma v\,ds\\&+\sum_{e_k\in\Gamma}\int_{e_k}[\gamma v]\Big\{(\rho\frac{\partial h_d}{\partial U_j^f}+h_d\frac{\partial\rho}{\partial U_j^f}).(\vec{\beta}.\mathbf{n})\Big\}ds,\end{aligned}$$

where the I^{th} subscript denote the row generated with the test function $v=\mathscr{N}_i^{\,e}(\mathbf{X})$ and the J^{th} subscript correspond to the unknown $U_j^{\mathscr{F}}$. According to the discrete equation (128) we can evaluate the following expression

$$\frac{\partial h_d}{\partial U_j^f}=\mathscr{G}_j^f$$

which is pre-evaluated. It is worth mentioning that, from the discrete equation (128) the film thickness depends heavily on the local pressure and very less on the pressure for away. The value of $\mathscr{G}_j^{\mathscr{F}}$ is rapidly decreases as the position of element $\mathscr{F}$ is far away from the position of $\mathbf{X} = (x, y)$. From the above information we can reduce our computation cost by considering the following approximations of $\frac{\partial \mathscr{T}_d^{**}(U)}{\partial U}$:

- $\frac{\partial h_d(\mathbf{X})}{\partial U_j^f} = 0$ where $\mathbf{X} \in e_k$ if $f \neq e_k$ and f is not a adjacent element of e_k.
- $\frac{\partial h_d(\mathbf{X})}{\partial U_j^f} = 0$ where $\mathbf{X} \in \Gamma_{\text{int}}$ and if f is not a adjacent element of Γ_{int}.
- $\frac{\partial h_d(\mathbf{X})}{\partial U_j^f} = 0$ where $\mathbf{X} \in \Gamma_D$ and if f is not a adjacent element of Γ_D.
- $\frac{\partial h_d(\mathbf{X})}{\partial U_j^f} = \mathscr{G}_j^f(\mathbf{X})$, otherwise.

Appendix B. Parameters used in computation

Some frequently used notation in EHL model are denoted as below:
$p_H = \frac{Ea}{4R_x}$ maximum Hertzian pressure .
$\eta_0 = 0.04$ Ambient pressure viscosity.
$h_0 =$ central offset film thickness integral constant.
$a =$ Radius of point contact circle.
$\alpha =$ Pressure viscosity coefficient.
$u_s = u_1 + u_2$, where u_1 upper surface velocity and u_2 lower surface velocity respectively.
$p_0 =$ Constant ($p_0 = 1.98 \times 10^8$), z is pressure viscosity index ($z = 0.68$).
L and M are Moe's parameters and they are related as below.
$L = G(2U)^{\frac{1}{4}}, M = W(2U)^{-\frac{3}{4}}$, where
$2U = \dfrac{(\eta_0 u_s)}{(E'R)}, W = \dfrac{F}{E'R}, p_H = \dfrac{(3F)}{(2\pi a^2)}$.
For more detail explanation of parameter we refer to see [32, page 183].

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