INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 17, Number 5, Pages 662–678

MODIFYING THE SPLIT-STEP θ-METHOD WITH HARMONIC-MEAN TERM FOR STOCHASTIC DIFFERENTIAL EQUATIONS

KAZEM NOURI, HASSAN RANJBAR AND JUAN CARLOS CORTÉS LÓPEZ

Abstract. In this paper, we design a class of general split-step methods for solving Itô stochastic differential systems, in which the drift or deterministic increment function can be taken from special ordinary differential equations solver, based on the harmonic-mean. This method is justified to have a strong convergence order of $\frac{1}{2}$. Further, we investigate mean-square stability of the proposed method for linear scalar stochastic differential equation. Finally, some examples are included to demonstrate the validity and efficiency of the introduced scheme.

Key words. Itô stochastic differential system, split-step method, ODE solver, harmonic-mean, strong convergence, mean-square stability.

1. Introduction

Many phenomena in various branches of science like physics, chemistry and engineering can be modeled more efficiently by the stochastic differential equations (SDEs) [3,5,6,15]. Since analytical solutions of SDEs are generally not available, we are forced to use numerical methods that give approximated solutions [8,9,13,15,21, 25,29,40]. First attempt in this direction was made by Maruyama [17], who established the well-known Euler-Maruyama (EM) method, then Milstein [18] presented an important numerical scheme with faster convergence than EM method [9,10,35]. Based on EM and Milstein methods, many numerical schemes have been presented and developed later, see for example [2, 11, 12, 19, 22, 23, 30, 36, 37].
In [24], Platen and Wagner proposed a stochastic generalization of the Taylor formula for Itô diffusions. This generalization, called the Itô-Taylor expansions, was based upon the use of multiple stochastic integrals. The Itô-Taylor expansions are characterized by the choice of multiple integrals which appear in them. Many numerical methods based on Itô-Taylor expansions have been presented for simulating the approximate solutions to SDEs [15,20]. In this paper we will consider numerical methods for strong solution of Itô stochastic differential systems of the form

$$(1)\qquad \mathrm{d}X(t) = f(X(t))\mathrm{d}t + \sum_{j=1}^{m} g_j(X(t))\mathrm{d}B^j(t), \quad X(t_0) = X_0,\ t \in [t_0, T],$$

where $X \in \mathbb{R}^d$, $f : \mathbb{R}^d \to \mathbb{R}^d$, is a drift vector, $g = (g_1, \ldots, g_m) : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ is a diffusion matrix and $B = (B_1, \ldots, B_m)^T$ is an m–dimensional Brownian motion process. Similarly to contributions [8, 25, 34], we design and analyze the strong convergence of a class of general split-step methods for solving the Itô stochastic differential system (1).
Nowadays, stability is judged better to account the efficiency of numerical methods for solving SDEs. Several kinds of these stabilities have been proposed in [13,14,16].

Received by the editors March 28, 2018 and, in revised form, June 08, 2020.
2000 *Mathematics Subject Classification.* 60H10, 65C20, 65C30, 65L20.

Throughout this paper the so-called mean-square (MS) stability will be considered. This kind of stability, which is based on the second statistical moment of the (exact or numerical) solution, has been considered in the literature so far [4, 7, 26, 33, 39]. In order to discuss mean-square stability properties of our proposed method, we will focus on the special linear scalar Itô test equation

$$\mathrm{d}X(t) = aX(t)\mathrm{d}t + bX(t)\mathrm{d}B(t), \quad t \geq t_0, \quad X(t_0) = X_0, \tag{2}$$

where $a,\ b \in \mathbb{C}$ and $X_0 \neq 0$ are constants. For numerical step size h and $a, b \in \mathbb{R}$, Saito and Mitsui [26] plot MS-stability region in the $(\overline{h}, k)$-plane, with $\overline{h} = ah$, $k = -\frac{b^2}{a}$. Then, Higham [11] performed the analysis in the (x, y)-plane with $x = ah$, $y = b^2h$, which has been accepted by researchers for presenting MS-stability domain of numerical stochastic methods [4, 7, 11, 33]. Moreover, some MS-stability domains have been plotted with $x = ah$, $y = b\sqrt{h}$ and $a, b \in \mathbb{R}$ (see [8, 30, 34, 37, 39]).
The paper is organized as follows. Section 2 is devoted to introducing the proposed method. Convergence properties of the method are discussed in Section 3. Mean-square stability properties and numerical results of the method are reported in Sections 4 and 5, respectively.

2. General split-step method

For solving stochastic differential system (1), thereupon we present general split-step methods, based on EM numerical scheme, of the form

$$\begin{cases} \overline{Y}_k &= Y_k + h\varphi(Y_k, \overline{Y}_k), \\ Y_{k+1} &= \overline{Y}_k + \sum_{j=1}^{m} g_j(\overline{Y}_k)\Delta B_k^j, \end{cases} \tag{3}$$

where $\varphi(Y_k, \overline{Y}_k)$ is an increment function of the deterministic ordinary differential equation (ODE) solver. This idea was first presented by Higham in [12], as a modification of the classical EM method, which is usually referred to as split-step methods. This approach is a class of fully implicit methods which allows us the incorporation of implicitness in the stochastic part of the system with relatively little additional cost. Then, Wang and Li in [36] presented two types of split-step methods, drifting split-step Euler and diffused split-step Euler methods, for SDEs by a single noise term. Ding et al. [4] have analysed the split-step θ-methods for solving nonlinear non-autonomous SDEs. Guo et al. in [7] improved split-step θ-methods for solving SDEs systems by a single noise term. Recently, error corrected EM method, which is constructed by adding an error correction term to the EM method, was introduced in [39].
Instead of using the above methods on the increment function, we replace them by a method based on different means to solve ODEs. In this paper, based on the concept of averaging the harmonic-mean functional [27, 38], we consider the ODE solver in the form,

$$\varphi(Y_k, \overline{Y}_k) = (1-\theta)f(\overline{Y}_k) + 2\theta\left(f^{-1}(Y_k) + f^{-1}(\overline{Y}_k)\right)^{-1}, \quad \theta \in [0, 1]. \tag{4}$$

Here $f^{-1}(\cdot) = \frac{1}{f(\cdot)}$ and we assume that $f(Y_k) + f(\overline{Y}_k) \neq 0$. The choice $\theta = 0$ and $m = 1$ becomes the method introduced in [12]. Note that by inserting ODEs solver harmonic-mean θ (HMT) (4) into general split-step method (3), we have the following split-step harmonic-mean θ–method (SSHMT method)

$$\overline{Y}_k = Y_k + h\left((1-\theta)f(\overline{Y}_k) + 2\theta\left(f^{-1}(Y_k) + f^{-1}(\overline{Y}_k)\right)^{-1}\right), \tag{5a}$$

$$Y_{k+1} = \overline{Y}_k + \sum_{j=1}^{m} g_j(\overline{Y}_k)\Delta B_k^j, \tag{5b}$$

where $k = 0, 1, 2, \ldots, N$, $\theta \in [0, 1]$, $Y_0 = X_0$, $Y_k \approx X(t_k)$, and for $k = 1, 2, \ldots, N$ the constant step size h is defined as $h = t_k - t_{k-1}$, and each $\Delta B_k^j = B_{t_k}^j - B_{t_{k-1}}^j$ are independent $\mathcal{N}(0, h)$–distributed Gaussian random variables of zero mean and variance $h > 0$.

Analogously to [4,7,8,12,25,34,37], numerical method (5) is implicit. Then, $\overline{Y}_k$ needs to be computed in order to determine the intermediate approximation Y_k. The zero of nonlinear equation (5a) in each time step is computed by the Newton's iterative method, approximately.

In the following, we introduce an assumption that will be used throughout our discussion.

Assumption 2.1. *The functions f and g_j, $j = 1, \ldots, m$, in stochastic differential system* (1) *satisfy the Lipschitz condition*

$$|f(x) - f(y)|^2 \vee \sum_{j=1}^{m} |g_j(x) - g_j(y)|^2 \leq \mathcal{K}_1|x - y|^2, \tag{6}$$

and linear growth bounds

$$|f(x)|^2 \vee \sum_{j=1}^{m} |g_j(x)|^2 \leq \mathcal{K}_2(1 + |x|^2), \tag{7}$$

for all real $x, y \in \mathbb{R}^d$. Here, $\mathcal{K}_1$ and $\mathcal{K}_2$ are positive constants, and $\vee$ is the maximal operator.

Theorem 2.2. *([1]) Suppose that $G : \mathbb{R}^d \to \mathbb{R}^d$ is continuous and there exists a positive constant C such that*

$$\langle G(x_1) - G(x_2), x_1 - x_2\rangle \geq C|x_1 - x_2|^2,$$

for all $x_1,\ x_2 \in \mathbb{R}^d$. Then G is a homeomorphism with Lipschitz continuous inverse, in particular

$$\left|G^{-1}(y_1) - G^{-1}(y_2)\right| \leq \tfrac{1}{C}|y_1 - y_2|,$$

for all $y_1,\ y_2 \in \mathbb{R}^d$.

Similar to [4,12], for study on the existence and uniqueness of SSHMT method (5), we prove the following lemma.

Lemma 2.3. *Let us assume that Assumption 2.1 and Theorem 2.2 hold, and suppose that for $0 \leq \theta \leq 1$, $h < \left((1-\theta)\sqrt{\mathcal{K}_1} + \frac{2\theta}{C}\right)^{-1}$. Then for every $x, y \in \mathbb{R}^d$, implicit equation* (5a) *admits a unique solution.*

Proof. Writing (5a) as

$$\begin{aligned} x &= F(x) \\ &= y + h\left((1-\theta)f(x) + 2\theta\left(f^{-1}(x) + f^{-1}(y)\right)^{-1}\right). \end{aligned}$$

Observing that for $u, v \in \mathbb{R}^d$,

$$\begin{aligned}|F(u)-F(v)| &\le h\Bigg((1-\theta)|f(u)-f(v)| \\ &\quad +2\theta\left|\left(f^{-1}(u)+f^{-1}(y)\right)^{-1}-\left(f^{-1}(v)+f^{-1}(y)\right)^{-1}\right|\Bigg) \\ &\le h\left((1-\theta)\sqrt{\mathcal{K}_1}+\tfrac{2\theta}{C}\right)|u-v|,\end{aligned}$$

to obtain the above relation, we set $G(y_1) = f^{-1}(u) + f^{-1}(y)$ and $G(y_2) = f^{-1}(v) + f^{-1}(y)$ in Theorem 2.2. Then the result follows from the classical Banach contraction mapping theorem [31]. □

3. Convergence properties

In this section, we discuss the convergence analysis of the SSHMT method (5), under the Assumption 2.1 on f and g_j, $j = 1, \ldots, m$. We shall prove that this method has strong convergence of order $\frac{1}{2}$. To measure the strong convergence order of the SSHMT method derived in this paper, we introduce the following convergence lemma established in [19, 20] for the global error.

Lemma 3.1. *Assume that for a one-step discrete time approximation Y, the local mean error and mean-square error for all $N = 1, 2, \ldots,$ and $k = 0, 1, \ldots, N-1$ satisfy, respectively, the estimates*

$$|\mathbb{E}\left[Y_{k+1}-X(t_{k+1})|Y_k=X(t_k)\right]| \le \mathcal{K}(1+|Y_k|^2)^{1/2}h^{p_1}, \tag{8}$$

and

$$\left|\mathbb{E}\left[|Y_{k+1}-X(t_{k+1})|^2\,|Y_k=X(t_k)\right]\right|^{1/2} \le \mathcal{K}(1+|Y_k|^2)^{1/2}h^{p_2}, \tag{9}$$

where $p_2 \ge \frac{1}{2}$ and $p_1 \ge p_2 + \frac{1}{2}$. Then

$$\left|\mathbb{E}\left[|Y_n-X(t_n)|^2\,|Y_0=X(t_0)\right]\right|^{1/2} \le \mathcal{K}(1+|Y_0|^2)^{1/2}h^{p_2-1/2},$$

holds for each $n = 0, 1, \ldots, N$. Here, $\mathcal{K}$ is independent of h but dependent on the length of the time interval $T - t_0$.

The following theorem shows that, under Assumption 2.1, the strong convergence order of SSHMT method is equal to $\frac{1}{2}$, similar to SST (split-step θ–method) [4] and SSCT (split-step composite θ-method) [7].

Theorem 3.2. *Assume that Lemma 2.3 holds. Also let Y_n be the numerical approximation of $X(t_n)$ at time T after n steps with step size $h = (T - t_0)/N$. Under Assumption 2.1, for a sufficiently small step size,*

$$h < \min\left\{\left((1+\theta)K\sqrt{2\mathcal{K}_1}\right)^{-1}, \left((1-\theta)\sqrt{\mathcal{K}_1}+\frac{2\theta}{C}\right)^{-1}\right\}.$$

Then, applying the SSHMT method (5) to the stochastic differential system (1), for all $n = 0, 1, 2, \ldots, N$, one gets

$$\left|\mathbb{E}\left[\left|Y_n-X(t_n)\right|^2\Big|Y_0=X(t_0)\right]\right|^{1/2} = \mathcal{O}(h^{\frac{1}{2}}).$$

Proof. A similar proof can be found in [7]. We use the Lipschitz condition (6) and linear growth bounds (7) of the drift and diffusion functions. At first, we show that the estimate (8) holds for the SSHMT method with $p_1 = 2$. For the local EM approximation step

$$Y_{k+1}^{\mathrm{EM}} = Y_k^{\mathrm{EM}} + hf(Y_k^{\mathrm{EM}}) + \sum_{j=1}^{m} g_j(Y_k^{\mathrm{EM}})\Delta B_k^j, \tag{10}$$

one can show that [20],

$$\left|\mathbb{E}\left[\left(Y_{k+1}^{\mathrm{EM}} - X(t_{k+1})\right)\middle|Y_k^{\mathrm{EM}} = X(t_k)\right]\right| = \mathcal{O}(h^2), \tag{11a}$$

$$\left|\mathbb{E}\left[\left(Y_{k+1}^{\mathrm{EM}} - X(t_{k+1})\right)^2\middle|Y_k^{\mathrm{EM}} = X(t_k)\right]\right| = \mathcal{O}(h^2). \tag{11b}$$

From (11a), we arrive at

$$\begin{aligned} H_1 &= \left|\mathbb{E}\left[(Y_{k+1} - X(t_{k+1}))\left|Y_k = X(t_k)\right.\right]\right| \\ &\leq \left|\mathbb{E}\left[\left(Y_{k+1}^{\mathrm{EM}} - X(t_{k+1})\right)\left|Y_k^{\mathrm{EM}} = X(t_k)\right.\right]\right| \\ &\quad + \left|\mathbb{E}\left[\left(Y_{k+1} - Y_{k+1}^{\mathrm{EM}}\right)\left|Y_k = Y_k^{\mathrm{EM}}\right.\right]\right| \\ &= \mathcal{O}(h^2) + H_2, \end{aligned} \tag{12}$$

being

$$\begin{aligned} H_2 &= \left|\mathbb{E}\left[(Y_{k+1} - Y_{k+1}^{\mathrm{EM}})\middle|Y_k = Y_k^{\mathrm{EM}}\right]\right| \\ &= \left|\mathbb{E}\Big[Y_k - Y_k^{\mathrm{EM}} + h(1-\theta)f(\overline{Y}_k)\right. \\ &\quad + h\left(2\theta\left(f^{-1}(Y_k) + f^{-1}(\overline{Y}_k)\right)^{-1} - f(Y_k^{\mathrm{EM}})\right) \\ &\quad \left. + \textstyle\sum_{j=1}^m \Delta B_k^j\left(g_j(\overline{Y}_k) - g_j(Y_k^{\mathrm{EM}})\right)\left|Y_k = Y_k^{\mathrm{EM}}\right.\Big]\right| \\ &= h\left|(1-\theta)f(\overline{Y}_k) + 2\theta\left(f^{-1}(Y_k) + f^{-1}(\overline{Y}_k)\right)^{-1} - f(Y_k)\right| \\ &= h\frac{|(1-\theta)f(\overline{Y}_k) + f(Y_k)|}{|f(\overline{Y}_k) + f(Y_k)|}|f(\overline{Y}_k) - f(Y_k)| \end{aligned}$$

where we have used that $\mathbb{E}[\Delta B_k^j] = 0$, $1 \leq j \leq m$. For sufficiently large $K > 0$, we get

$$H_2 \leq (1+\theta)Kh\left|f(\overline{Y}_k) - f(Y_k)\right|,$$

from Lipschitz condition (6), we have

$$H_2 \leq (1+\theta)hK\sqrt{\mathcal{K}_1}\left|\overline{Y}_k - Y_k\right|. \tag{13}$$

Furthermore, from (5a) and the previous process to obtain the bound (13) of H_2, one gets

$$\begin{aligned} |\overline{Y}_k - Y_k| &\leq h\frac{|(1-\theta)f(\overline{Y}_k) + f(Y_k)|}{|f(\overline{Y}_k) + f(Y_k)|}|f(\overline{Y}_k) - f(Y_k)| + h|f(Y_k)| \\ &\leq (1+\theta)Kh|f(\overline{Y}_k) - f(Y_k)| + h|f(Y_k)|. \end{aligned} \tag{14}$$

Now, considering (6) and (7) one follows

$$|\overline{Y}_k - Y_k| \leq (1+\theta)K\sqrt{\mathcal{K}_1}h|\overline{Y}_k - Y_k| + h\sqrt{\mathcal{K}_2}(1 + |Y_k|^2)^{1/2},$$

and thus

$$|\overline{Y}_k - Y_k|\left(1 - (1+\theta)K\sqrt{\mathcal{K}_1}h\right) \leq h\sqrt{\mathcal{K}_2}(1 + |Y_k|^2)^{1/2}.$$

Since by hypothesis $1-2(1+\theta)^2K^2\mathcal{K}_1h^2>0$, hence $1-(1+\theta)K\sqrt{\mathcal{K}_1}h>0$, one gets

$$|\overline{Y}_k-Y_k|\le\frac{h\sqrt{\mathcal{K}_2}(1+|Y_k|^2)^{1/2}}{1-(1+\theta)K\sqrt{\mathcal{K}_1}h}. \tag{15}$$

Accordingly, from (13) and (15), one obtains

$$H_2\le h^2\frac{(1+\theta)K\sqrt{\mathcal{K}_1\mathcal{K}_2}(1+|Y_k|^2)^{1/2}}{1-(1+\theta)K\sqrt{\mathcal{K}_1}h}=\mathcal{O}(h^2). \tag{16}$$

Therefore, by (12) and (16) one concludes that

$$\begin{aligned}H_1&=\left|\mathbb{E}\left[(Y_{k+1}-X(t_{k+1}))|Y_k=X(t_k)\right]\right|\\&=\mathcal{O}(h^2).\end{aligned}$$

Consequently, the estimate with $p_1=2$ in Lemma 3.1 is satisfied for the SSHMT method.

Next, we prove formula (9) for the local mean-square error to the SSHMT method for $k=0,1,2,\dots,N-1$. First, let us apply inequality $(a+b)^2\le 2a^2+2b^2$ with (11b), this leads

$$\begin{aligned}H_3&=\mathbb{E}\left[|Y_{k+1}-X(t_{k+1})|^2\,\middle|Y_k=X(t_k)\right]\\&\le 2\mathbb{E}\left[\left|Y_{k+1}^{\mathrm{EM}}-X(t_{k+1})\right|^2\,\middle|Y_k^{\mathrm{EM}}=X(t_k)\right]\\&\quad+2\mathbb{E}\left[\left|Y_{k+1}-Y_{k+1}^{\mathrm{EM}}\right|^2\,\middle|Y_k=Y_k^{\mathrm{EM}}\right]\\&\le\mathcal{O}(h^2)+2H_4,\end{aligned} \tag{17}$$

being

$$\begin{aligned}H_4&=\mathbb{E}\left[\left|Y_{k+1}-Y_{k+1}^{\mathrm{EM}}\right|^2\,\middle|Y_k=Y_k^{\mathrm{EM}}\right]\\&=\mathbb{E}\Bigg[\Bigg|h(1-\theta)f(\overline{Y}_k)+h\left(2\theta\left(f^{-1}(Y_k)+f^{-1}(\overline{Y}_k)\right)^{-1}-f(Y_k)\right)\\&\quad+\sum_{j=1}^m\Delta B_k^j\left(g_j(\overline{Y}_k)-g_j(Y_k^{\mathrm{EM}})\right)\Bigg|^2\Bigg|Y_k=Y_k^{\mathrm{EM}}\Bigg]\\&=h^2\left|(1-\theta)f(\overline{Y}_k)+2\theta\left(f^{-1}(Y_k)+f^{-1}(\overline{Y}_k)\right)^{-1}-f(Y_k)\right|^2\\&\quad+h\sum_{j=1}^m\left|g_j(\overline{Y}_k)-g_j(Y_k)\right|^2\\&=h^2\frac{|(1-\theta)f(\overline{Y}_k)+f(Y_k)|^2}{|f(\overline{Y}_k)+f(Y_k)|^2}|f(\overline{Y}_k)-f(Y_k)|^2\\&\quad+h\sum_{j=1}^m\left|g_j(\overline{Y}_k)-g_j(Y_k)\right|^2\end{aligned} \tag{18}$$

where we have used that $\mathbb{E}[\Delta B_k^j]=0$ and

$$\mathbb{E}\left[\Delta B_k^l\Delta B_k^r\right]=\begin{cases}h, & \text{if } l=r,\\ 0, & \text{if } l\ne r.\end{cases}$$

Now, using sufficiently large $K > 0$ and (6) for the (18)

$$\begin{aligned} H_4 &= \mathbb{E}\left[\left|Y_{k+1} - Y_{k+1}^{\mathrm{EM}}\right|^2 \middle| Y_k = Y_k^{\mathrm{EM}}\right] \\ &\le (1+\theta)^2 K^2 h^2 |f(\overline{Y}_k) - f(Y_k)|^2 + h\sum_{j=1}^{m} \left|g_j(\overline{Y}_k) - g_j(Y_k)\right|^2 \qquad (19) \\ &\le h(1 + h(1+\theta)^2 K^2)\mathcal{K}_1 \left|\overline{Y}_k - Y_k\right|^2. \end{aligned}$$

Also, from (14) one gets

$$|\overline{Y}_k - Y_k|^2 \le \left((1+\theta)Kh|f(\overline{Y}_k) - f(Y_k)| + h|f(Y_k)|\right)^2,$$

hence applying (6) and (7) with inequality $(a+b)^2 \le 2a^2 + 2b^2$, one follows that

$$\begin{aligned} |\overline{Y}_k - Y_k|^2 &\le 2(1+\theta)^2 K^2 \mathcal{K}_1 h^2 \left|\overline{Y}_k - Y_k\right|^2 + 2h^2 |f(Y_k)|^2 \\ &\le 2(1+\theta)^2 K^2 \mathcal{K}_1 h^2 |\overline{Y}_k - Y_k|^2 + 2h^2 \mathcal{K}_2 (1 + |Y_k|^2), \end{aligned}$$

or equivalently

$$\left(1 - 2(1+\theta)^2 K^2 \mathcal{K}_1 h^2\right) |\overline{Y}_k - Y_k|^2 \le 2h^2 \mathcal{K}_2 (1 + |Y_k|^2),$$

$$|\overline{Y}_k - Y_k|^2 \le \frac{2h^2 \mathcal{K}_2 (1 + |Y_k|^2)}{1 - 2(1+\theta)^2 K^2 \mathcal{K}_1 h^2}, \qquad (20)$$

where in the last step we have used the hypothesis $1 - 2(1+\theta)^2 K^2 \mathcal{K}_1 h^2 > 0$. Substituting this inequality into (19), one obtains

$$\begin{aligned} H_4 &\le \frac{2h^3(1 + h(1+\theta)^2 K^2)\mathcal{K}_1 \mathcal{K}_2 (1 + |Y_k|^2)}{1 - 2(1+\theta)^2 K^2 \mathcal{K}_1 h^2} \\ &= \mathcal{O}(h^3). \qquad (21) \end{aligned}$$

Finally, applying (21) in (17) means that

$$\begin{aligned} H_3 &= \mathbb{E}\left[|Y_{k+1} - X(t_{k+1})|^2 \middle| Y_k = X(t_k)\right] \\ &= \mathcal{O}(h^2). \end{aligned}$$

We choose in Lemma 3.1 the exponent $p_2 = 1$ together with $p_1 = 2$ and apply it to finally prove the strong order $p = p_2 - \frac{1}{2} = \frac{1}{2}$ of the SSHMT method. □

4. Mean-square stability analysis

The exact solution of (2) is given by

$$X(t) = X_0 \exp\left(\left(a - \tfrac{1}{2}b^2\right)t + bB(t)\right). \qquad (22)$$

This solution is said to be mean-square stable if $\lim_{t\to\infty} \mathbb{E}\left[|X(t)|^2\right] = 0$. It is known that the mean-square stability for (22) is equivalent to [11, 26, 28],

$$\Re(a) + \tfrac{1}{2}|b|^2 < 0, \qquad (23)$$

where $\Re(a)$ denotes the real part of the complex number a. In the following we assume $a \in \mathbb{R}$, hence $\Re(a) = a$ and from (23), $a < 0$ [26]. When the numerical method is applied to the test equation (2) with $f(Y,t) = aY$, $g(Y,t) = bY$, the one step difference equation

$$Y_{k+1} = R_{SSHMT}(a, b, h, \xi_k) Y_k, \qquad (24)$$

is obtained, where for $k = 0, 1, \ldots, N$, $\xi_k = \frac{\Delta B_k}{\sqrt{h}} \sim \mathcal{N}(0,1)$. For this purpose, from (5a), we have

$$\overline{Y}_k = Y_k + h\left((1-\theta)a\overline{Y}_k + 2\theta\frac{(aY_k)(a\overline{Y}_k)}{aY_k + a\overline{Y}_k}\right).$$

By multiplying $a(Y_k + \overline{Y}_k) \neq 0$ in both sides of the above equation, we obtain

$$(1-(1-\theta)ah)\,\overline{Y}_k^2 - (1+\theta)ahY_k\overline{Y}_k - Y_k^2 = 0,$$

and thus

$$\overline{Y}_k = \frac{(1+\theta)ah \pm \sqrt{(1+\theta)^2a^2h^2 + 4(1-(1-\theta)ah)}}{2(1-(1-\theta)ah)}Y_k.$$

Also, from (5b) we can write

$$\begin{aligned} Y_{k+1} &= \overline{Y}_k + b\overline{Y}_k\sqrt{h}\xi_k \\ &= (1+b\sqrt{h}\xi_k)\frac{(1+\theta)ah \pm \sqrt{(1+\theta)^2a^2h^2 + 4(1-(1-\theta)ah)}}{2(1-(1-\theta)ah)}Y_k, \end{aligned}$$

accordingly, we achieve

$$R_{\text{SSHMT}\pm}(a,b,h,\xi_k) = \frac{(1+\theta)ah \pm \sqrt{((1+\theta)h)^2a^2 + 4(1-(1-\theta)ah)}}{2(1-(1-\theta)ah)}(1+b\sqrt{h}\xi_k). \tag{25}$$

Definition 4.1. ([26]) The numerical method is said to be MS-stable if

$$\overline{R}_{\text{method}}(a,b,h) = \mathbb{E}[R^2_{\text{method}}(a,b,h,\xi_k)] < 1.$$

In the above definition, $\overline{R}_{\text{method}}(a,b,h)$ is called MS-stability function and the set $\mathfrak{D}_{MS} = \{(a,b) \in \mathbb{C}\times\mathbb{C} : \overline{R}_{\text{method}}(a,b,h) < 1\}$ is called the MS-stability domain of the numerical method [33].

Now, by using relation (25) and Definition 4.1, we obtain stability domain of the SSHMT method by restricting our attention on $b \in \mathbb{R}$ and $a < 0$, i.e. condition

$$\begin{aligned} \overline{R}_{\text{SSHMT}\pm}(a,b,h) &= \mathbb{E}[R^2_{\text{SSHMT}\pm}(a,b,h,\xi_k)] \\ &= \frac{\left((1+\theta)ah \pm \sqrt{(1+\theta)^2h^2a^2 + 4(1-(1-\theta)ah)}\right)^2(b^2h+1)}{4(1-(1-\theta)ah)^2} < 1, \end{aligned} \tag{26}$$

where we have used that $\mathbb{E}[\xi_k] = 0$ and $\mathbb{E}[\xi_k^2] = 1$.

Theorem 4.2. *Let us assume that condition* (23) *holds,* $0 \le \theta \le 1$ *and step size* $h > 0$*, then the SSHMT method* (5) *for the SDE* (22) *is MS-stable, if positive sense of condition* (26) *holds (i.e.* $\overline{R}_{SSHMT+}(a,b,h) < 1$*).*

Proof. Let us introduce the notation

$$\alpha = (1+\theta)ah, \quad \beta = 1-(1-\theta)ah, \tag{27}$$

then relation (26) can be written, after some algebraic manipulations, as

$$\begin{aligned} \frac{(\alpha \pm \sqrt{\alpha^2+4\beta})^2(b^2h+1)}{4\beta^2} &= \frac{(\alpha \pm \sqrt{\alpha^2+4\beta})^2(\alpha \mp \sqrt{\alpha^2+4\beta})^2(b^2h+1)}{4\beta^2(\alpha \mp \sqrt{\alpha^2+4\beta})^2} \\ &= \frac{4(b^2h+1)}{(\alpha \mp \sqrt{\alpha^2+4\beta})^2}. \end{aligned}$$

As a consequence, by (26) we have to prove

$$\overline{R}_{\text{SSHMT}\pm}(a,b,h) = \frac{4\left(b^2h+1\right)}{\left(\alpha \mp \sqrt{\alpha^2+4\beta}\right)^2} < 1,$$

which is equivalent to

$$\psi_\pm(a,b,h) = 4hb^2 - 2\alpha^2 + 4(1-\theta)ah \pm 2\alpha\sqrt{\alpha^2+4\beta} < 0. \tag{28}$$

Since $a, \alpha < 0$, $\beta \geq 1$ and $0 \leq \theta \leq 1$,

$$\alpha + 2 \leq \sqrt{\alpha^2 + 4\beta},$$

and so

$$2\alpha\sqrt{\alpha^2 + 4\beta} \leq 2\alpha^2 + 4\alpha. \tag{29}$$

On the other hand, taking into account that $0 \leq \theta \leq 1$, (27) and inequality $\sqrt{x^2 + y^2} \leq |x| + |y|$, we have

$$\begin{aligned}\sqrt{\alpha^2 + 4\beta} &= \sqrt{\alpha^2 + 4(1 - (1-\theta)ah)} \\ &= \sqrt{\left(\alpha - 2\tfrac{1-\theta}{1+\theta}\right)^2 + 4 - 4\tfrac{(1-\theta)^2}{(1+\theta)^2}} \\ &= \sqrt{\left(\alpha - 2\tfrac{1-\theta}{1+\theta}\right)^2 + \tfrac{16\theta}{(1+\theta)^2}} \\ &\leq \left|\alpha - 2\tfrac{1-\theta}{1+\theta}\right| + \tfrac{4\sqrt{\theta}}{1+\theta} \\ &= -\alpha + 2\tfrac{1-\theta+2\sqrt{\theta}}{1+\theta} \\ &\leq -\alpha + 2\tfrac{3-\theta}{1+\theta}.\end{aligned}$$

Hence, by multiplying by $-2\alpha > 0$ one gets

$$-2\alpha\sqrt{\alpha^2 + 4\beta} \leq 2\alpha^2 - 4(3-\theta)ah. \tag{30}$$

Eventually, by (27) and substituting (29) and (30) as the positive and negative senses respectively into (28), we obtain

$$\begin{aligned}\psi_+(a,b,h) &\leq 4hb^2 + 4(1-\theta)ah + 4(1+\theta)ah \\ &= 4h(b^2 + 2a)\end{aligned}$$

and

$$\begin{aligned}\psi_-(a,b,h) &\leq 4hb^2 + 4(1-\theta)ah - 4(3-\theta)ah \\ &= 4h(b^2 - 2a).\end{aligned}$$

Finally, by (23), since $b^2+2a < 0$ concludes $\psi_+(a,b,h) < 0$, but $2a-b^2 \leq b^2+2a < 0$ yields $\psi_-(a,b,h) \geq 0$, and the proof is complete. □

Corollary 4.3. *Given $b \in \mathbb{R}$ and $a < 0$ and let $x = ah$, $y = b^2h$, then using 26 the SSHMT method is mean-square stable if*

$$y < \frac{4(1-(1-\theta)x)^2}{\left((1+\theta)x + \sqrt{((1+\theta)x)^2 + 4(1-(1-\theta)x)}\right)^2} - 1.$$

For the test equation (2), Figure 1 displays the MS-stable regions $y < -2x$ (areas below dashed borders), and the MS-stable regions of the SSHMT method (areas below solid borders), with $\theta = 0, 0.25, 0.5, 0.75, 1.0$ and $h > 0$. It is noted that the MS-stable regions of the SSHMT method have covered the regions obtained by condition (23), and the region is wider for the greater value of θ. While, SST method is MS-stable for $\theta = 1.0$ and $h > 0$, see Theorem 4.1 in [4] for more details. Also, from Theorem 4 in [7], we can found that SSCT method is MS-stable for $\frac{1}{2} \leq \theta = \lambda \leq 1$ and $h > 0$.

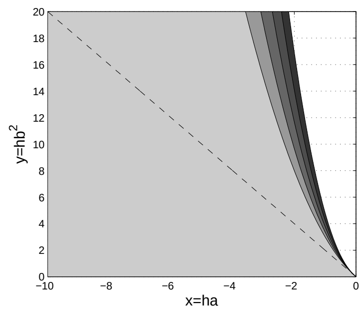


FIGURE 1. The MS-stability regions (areas below dashed borders), and the MS-stability regions of the SSHMT method for $\theta = 0.0, 0.25, 0.5, 0.75, 1.0$ (from lighter to darker areas) of the test equation (2).

5. Numerical results

In this section numerical results are reported to confirm the efficiency and superiority of the SSHMT method against other available methods that will be specified in the subsequent discussion. If $Y_N^{(i)}$ and $X_{t_N}^{(i)}$ denote the numerical solutions and the exact solution at step point t_N in the i-th simulation, respectively, then accuracy and convergence properties of the EM, SSHMT, SST and SSCT methods will be measured by mean absolute and mean square errors defined by

$$\varepsilon_{\mathrm{MA}} = \frac{1}{K}\sum_{i=1}^{K} |Y_N^{(i)} - X_{t_N}^{(i)}|, \tag{31}$$

$$\varepsilon_{\mathrm{MS}} = \left(\frac{1}{K}\sum_{i=1}^{K}\left(Y_N^{(i)} - X_{t_N}^{(i)}\right)^2\right)^{\frac{1}{2}}, \tag{32}$$

respectively. In simulations of examples 5.2–5.4, we assume that $K = 5000$.

Example 5.1. Consider the following scalar test equation

$$\mathrm{d}X(t) \ = aX(t)\mathrm{d}t + \sum_{j=1}^{m} b_j X(t)\mathrm{d}B^j(t), \quad X_0 = 1. \tag{33}$$

with exact solution

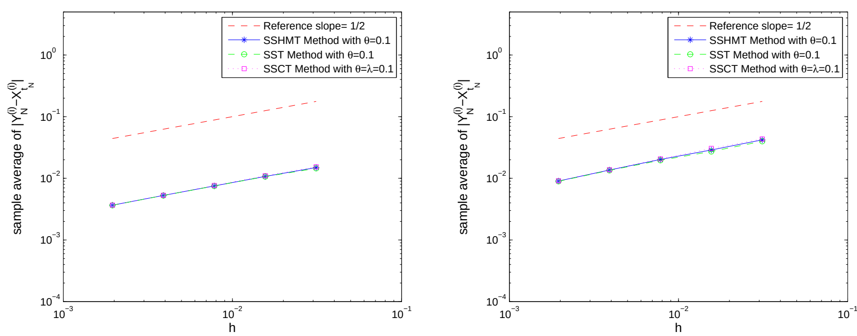
$$X(t) \ = X_0 \exp\left(\left(a - \tfrac{1}{2}\sum_{j=1}^{m} b_j^2\right)t + \sum_{j=1}^{m} b_j B^j(t)\right).$$

We use two groups of parameters, as follows

- Case I: $a = -b_1 = -\frac{1}{2}$ (hence $m = 1$ in (33)), [4, 34]
- Case II: $a = -1.5$, $b_1 = 1$, $b_2 = b_3 = 0.1$, $b_4 = b_5 = -0.5$ (hence $m = 5$ in (33)),

to demonstrate the strong convergence rates of the SSHMT and SST [4] methods with $\theta = 0.1$, and SSCT method [7] with $\theta = \lambda = 0.1$ at the terminal time $T = Nh = 1$. Figure 2 shows a log-log plot of the sample average approximation $|Y_N^{(i)} -$

$X_{t_N}^{(i)}|$ against h, based on the 1000 different discretized Brownian paths over $[0,1]$ with the step size $\delta t = 2^{-9}$. For each realization, we have applied the SSHMT, SST and SSCT methods with five different step sizes $h = 2^{j-1}\delta t$ for $1 \leq j \leq 5$. A reference line of slope $\frac{1}{2}$ is added in a dashed line type. We can see that this is consistent with the result that strong error is arbitrarily close to order $\frac{1}{2}$.



(a) Case I (b) Case II

FIGURE 2. The convergence rates of the SSHMT and SST methods with $\theta = 0.1$, and the SSCT method with $\theta = \lambda = 0.1$, for linear test system (33).

Example 5.2. Consider the SDE [32],

$$\mathrm{d}X(t) = \left(\frac{1}{3}X^{\frac{1}{3}}(t) + 6X^{\frac{2}{3}}(t)\right)\mathrm{d}t + X^{\frac{2}{3}}(t)\mathrm{d}W(t), \tag{34}$$

for initial value $X_0 = 1$, whose exact solution is given by

$$X(t) = \left(1 + 2t + \frac{W(t)}{3}\right)^3.$$

In Figure 3 shows the results of the simulations for EM, SST, SSCT and SSHMT methods at various different parameters θ and λ in the interval $t \in [0,1]$. From them, we observe that SSHMT method improves approximations provided by EM, SST and SSCT methods for different values of θ and λ parameters when parameter θ tends to 1. Note that SSCT method is SST method if $\lambda = 1$ and SST method for $\theta = 0$ equal EM method. In addition, for h fixed both errors $\varepsilon_{\mathrm{MA}}$ (left of Figure 3) and $\varepsilon_{\mathrm{MS}}$ (right of Figure 3) decrease as θ tends to 1.

Example 5.3. We apply the SSHMT, SST and SSCT methods to the nonlinear SDE

$$\mathrm{d}X(t) = \left(\alpha\sin(x) - \tfrac{1}{4}\sin(2X(t))\right)\mathrm{d}t + \cos(X(t))\mathrm{d}W(t), \qquad X_0 = 1, \tag{35}$$

with exact solution

$$X(t) = \arcsin\left(1 - 2\left(1 + \exp\left(2\left(\alpha t + W(t) + \tfrac{1}{2}\ln\left(\tfrac{1+\sin(X_0)}{1-\sin(X_0)}\right)\right)\right)\right)^{-1}\right).$$

Figure 4, shows mean square errors ($\varepsilon_{\mathrm{MS}}$) of SSHMT, SST and SSCT methods for $\alpha = 0.0, 0.25$. As SST method for $\theta = 0.25, 0.75$ and SSCT method for $\theta = \lambda =$

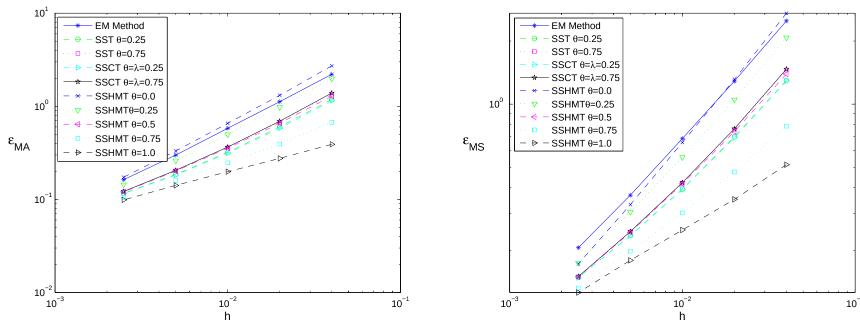


FIGURE 3. Value of the Mean of absolute errors $\varepsilon_{\mathrm{MA}}$ (left) and mean square errors, $\varepsilon_{\mathrm{MS}}$ (right) of the EM, SST, SSCT and SSHMT methods applied to the nonlinear SDE (34).

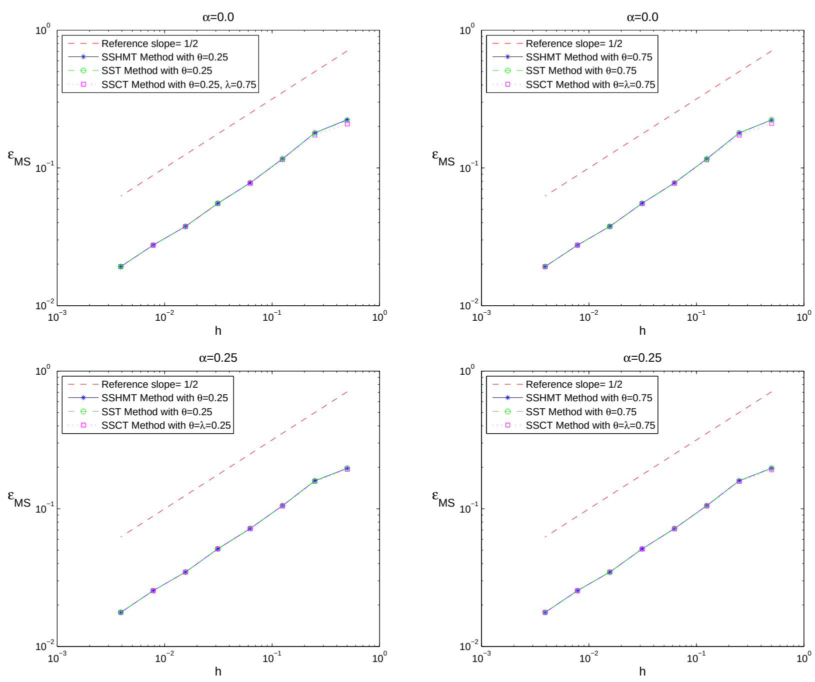


FIGURE 4. Value of the mean square error, $\varepsilon_{\mathrm{MS}}$ of the SSHMT, SST and SSCT methods applied to the nonlinear SDE (35).

$0.25, 0.75$, the results indicate that the SSHMT method for $\theta = 0.25, 0.75$ has a rate of convergence of order approximately equal to $\frac{1}{2}$.

Example 5.4. Now we consider the following 2-dimensional linear system [15]

$$\left\{ \begin{array}{ll} \mathrm{d}\mathbf{X}(t) & = \mathbf{M}_1\mathbf{X}(t)\mathrm{d}t + \mathbf{M}_2\mathbf{X}(t)\mathrm{d}W(t), \\ \mathbf{X}(0) & = (1,0)^T, \quad t \in [0,1], \end{array} \right. \tag{36}$$

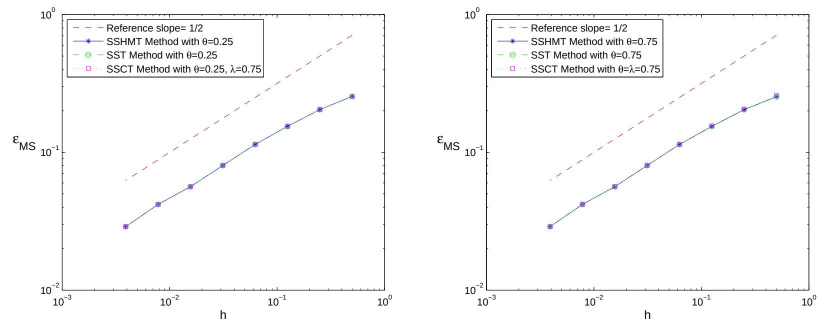


FIGURE 5. Value of the mean square error, $\varepsilon_{\mathrm{MS}}$ of the SSHMT, SST and SSCT methods applied to the 2-dimensional linear system (36).

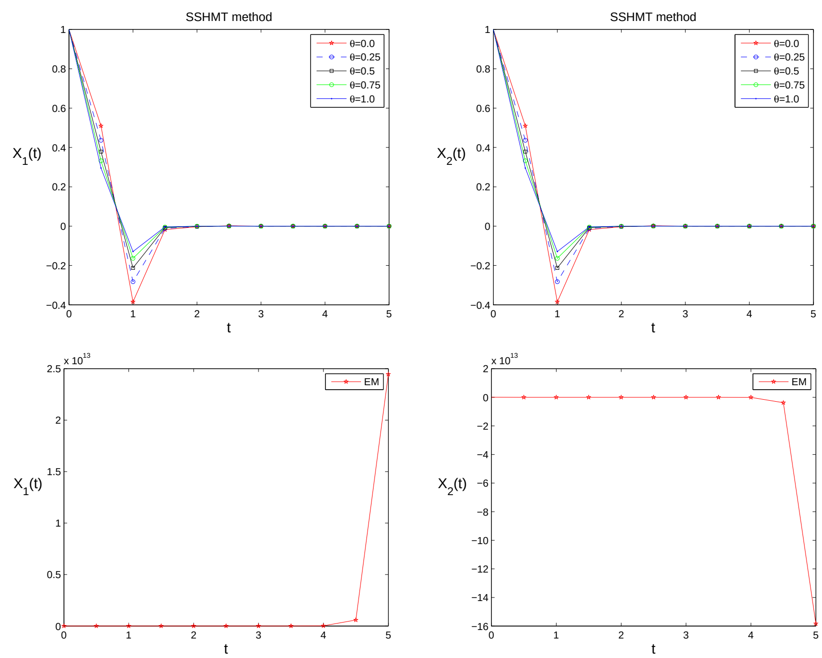


FIGURE 6. Numerical simulations of the system (37) by the SSHMT method (first row) and the EM method (second row).

where $\mathbf{M}_1 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ and $\mathbf{M}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The exact solution of (36) is given by $X(t) = \Big(\cos(W(t)), \sin(W(t))\Big)^T$. In Figure 4 we plot mean square error, $\varepsilon_{\mathrm{MS}}$ (32) for SSHMT, SST methods with $\theta = 0.25, 0.75$ and SSCT methods with $\theta = \lambda = 0.25, 0.75$. As SST and SSCT methods, it is seen that the strong convergence order of SSHMT method is $\frac{1}{2}$, which is agreement with the results in Theorem 3.2.

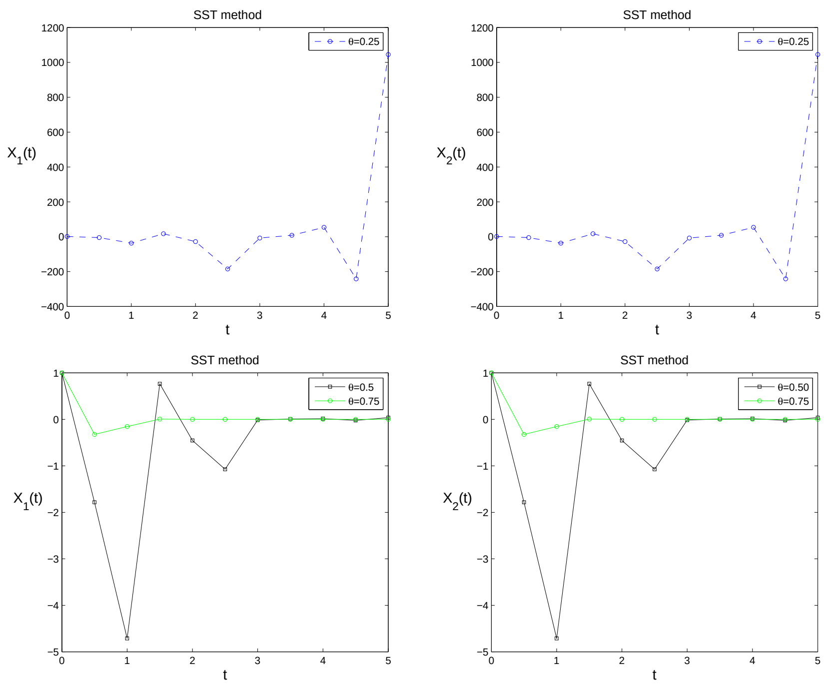


FIGURE 7. Numerical simulations of the system (37) by the SST method.

Example 5.5. Finally, we consider the following two-dimensional linear SDE [29],

$$\left.\begin{aligned} \mathrm{d}X_1(t) &= -12X_1(t)\mathrm{d}t + 4X_2(t)\mathrm{d}B(t), \\ \mathrm{d}X_2(t) &= -12X_2(t)\mathrm{d}t + 4X_1(t)\mathrm{d}B(t), \end{aligned}\right\} t \in [0,5], \quad \begin{aligned} X_1(0) &= 1, \\ X_2(0) &= 1. \end{aligned} \tag{37}$$

For linear system (37), Figures 6-8 illustrate the numerical simulations of the EM, SSHMT SST and SSCT methods when $h = \frac{1}{2}$. As it can be seen from Figure 6-8, the SSHMT method gives the stable solution for the system (37), while the EM method and SST, SSCT method for some parameter θ and λ become the unstable solutions.

6. Conclusions

In this paper, we introduce a class of general split-step method for solving Itô stochastic differential systems. The methods of this class are obtained by changing the drift increment function $\varphi(Y_k, \overline{Y}_k)$ which can be taken from special ODE solver, namely the harmonic-mean θ-method. Also, we have established strong order convergence and mean-square stability properties for presented method. Meantime, we compared MS-stability region of our method against MS-stability regions of SST [4] and SSCT [7] methods and showed that SSHMT method is MS-stable for any $\theta \in [0,1]$. Finally we have shown, through several illustrative examples that consider one and two-dimensional SDEs, that the proposed methods are valid for linear and nonlinear SDEs.

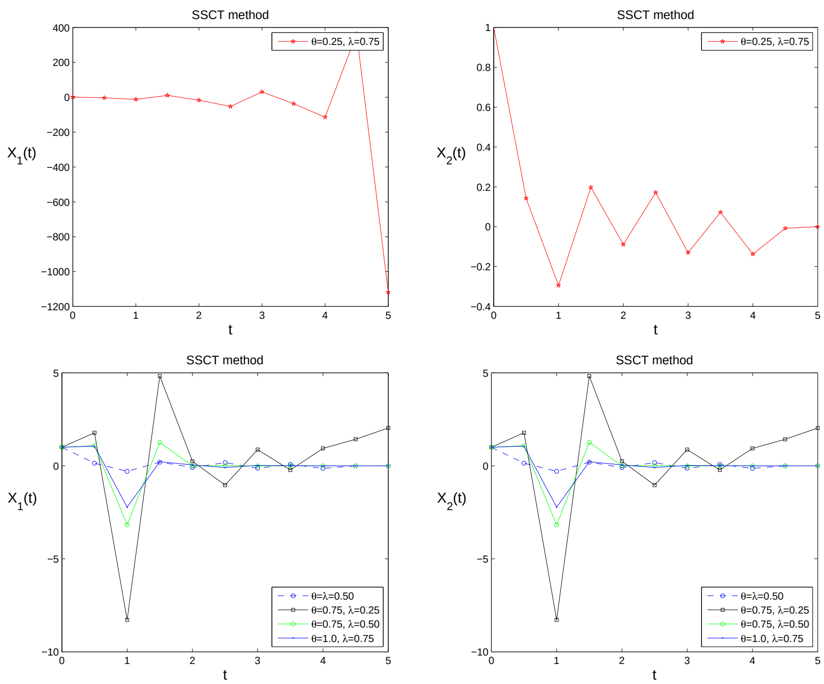


Figure 8. Numerical simulations of the system (37) by the SSCT method.

Acknolwedgments

This research was in part supported by the Research Council of Semnan University, and in part by a grant 97007950 from Iran National Science Foundation (INSF). Also, the third author acknowledges the support by the Spanish Ministerio de Economía, Industria y Competitividad (MINECO), the Agencia Estatal de Investigación (AEI) and Fondo Europeo de Desarrollo Regional (FEDER UE) grant MTM2017-89664-P. The authors are grateful to the anonymous reviewers for their careful reading, insightful comments and helpful suggestions which have led to improvement of the paper.

References

[1] W.J. Beyn, E. Isaak and R. Kruse, Stochastic C-stability and B-consistency of explicit and implicit Euler-type schemes, J. Sci. Comput., 67(3) (2016) 955–987.

[2] K. Burrage and T. Tian, The composite Euler method for stiff stochastic differential equations, J. Comput. Appl. Math., 131(1-2) (2001) 407–426.

[3] P. Catuogno and C. Olivera, Strong solution of the stochastic Burgers equation, Appl. Anal., 93(3) (2014) 646–652.

[4] X. Ding, Q. Ma and L. Zhang, Convergence and stability of the split-step θ-method for stochastic differential equations, Comput. & Math. Appl., 60(5) (2010) 1310–1321.

[5] F.A. Dorini, M.S. Cecconello and L.B. Dorini, On the logistic equation subject to uncertainties in the environmental carrying capacity and initial population density, Comm. Nonlinear Sci. Num. Simul., 33 (2016) 160–173.

[6] F.A. Dorini and M.C.C. Cunha, On the linear advection equation subject to random velocity fields, Math. Comp. Simul., 82(4) (2011) 679–690.

[7] Q. Guo, H. Li and Y. Zhu, The improved split-step θ methods for stochastic differential equation, Math. Meth. Appl. Sci., 37(15) (2014) 2245–2256.

[8] A. Haghighi and S.M. Hosseini, A class of split step balanced methods for stiff stochastic differential equations, Numer. Algor., 61(1) (2012) 141–162.
[9] D.J. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, SIAM. Rev., 43 (2001) 525–546.
[10] D.J. Higham, A-stability and stochastic mean-square stability, BIT, 40(2) (2000) 404–409.
[11] D.J. Higham, Mean-square and asymptotic stability of the stochastic theta method, SIAM J. Numer. Anal., 38(3) (2000) 753–769.
[12] D.J. Higham, X. Mao and A.M. Stuart, Strong convergence of Euler-type methods for nonlinear stochastic differential equations, SIAM J. Numer. Anal., 40(3) (2003) 1041–1063.
[13] D.J. Higham, X. Mao and C. Yuan, Almost sure and moment exponential stability in the numerical simulation of stochastic differential equations, SIAM J. Numer. Anal., 45(2) (2007) 592–609.
[14] R. Khasminskii, Stochastic Stability of Differential Equations, Series: Stochastic Modelling and Applied Probability Vol. 66, Springer-Verlag, Berlin Heidelberg, 2012.
[15] P.E. Kloeden and E. Platen, Numerical Solution of Stochastic Differential Equations, Applications of Mathematics, Springer-Verlag, Berlin, 1999.
[16] X. Mao, Stochastic Differential Equations and their Applications, Horwood Publishing Ltd, New York, 1997.
[17] G. Maruyama, Continuous Markov processes and stochastic equations, Rend. Circ. Math. Palermo, 4 (1955) 48–90.
[18] G.N. Milstein, Approximate integration of stochastic differential equations, Theory Probab. Appl., 19(3) (1974) 557–562.
[19] G.N. Milstein, E. Platen and H. Schurz, Balanced implicit methods for stiff stochastic systems, SIAM J. Numer. Anal., 35 (1998) 1010–1019.
[20] G.N. Milstein and M.V. Tretyakov, Stochastic Numerics for Mathematical Physics, Springer-Verlag, Berlin, 2004.
[21] K. Nouri, Study on stochastic differential equations via modified Adomian decomposition method, U.P.B. Sci. Bull. Series A, 78 (2016) 81–90.
[22] K. Nouri, H. Ranjbar and L. Torkzadeh, Modified stochastic theta methods by ODEs solvers for stochastic differential equations, Comm. Nonlinear Sci. Num. Simul., 68 (2019) 336–346.
[23] K. Nouri, H. Ranjbar and L. Torkzadeh, Study on split-step Rosenbrock type method for stiff stochastic differential systems, Int. J. Comput. Math., 97(4) (2020) 816–836.
[24] E. Platen and W. Wagner, On a Taylor formula for a class of Itô processes, Prob. Math. Stat., 3(1) (1982) 37–51.
[25] V. Reshniak, A.Q.M. Khaliq, D.A. Voss and G. Zhang, Split-step Milstein methods for multichannel stiff stochastic differential systems, Appl. Numer. Math., 89 (2015) 1–23.
[26] Y. Saito and T. Mitsui, Stability analysis of numerical schemes for stochastic differential equations, SIAM J. Numer. Anal., 33(6) (1996) 2254–2267.
[27] B.B. Sanugi and D.J. Evans, A new fourth order Runge-Kutta formula based on the harmonic mean, Intern. J. Comp. Math., 50(1-2) (1994) 113–118.
[28] H. Schurz, Asymptotical mean square stability of an equilibrium point of some linear numerical solutions with multiplicative noise, Stoch. Anal. Appl., 14(3) (1996) 313–354.
[29] S. Singh, Split-step forward Milstein method for SDEs, Int. J. Numer. Anal. Model., 9(4) (2012) 970–981.
[30] S. Singh and S. Raha, Five-stage Milstein methods for SDEs, Intern. J. Comp. Math., 89(6) (2012) 760–779.
[31] D. Smart, Fixed Point Theorems, Cambridge University Press, Cambridge, 1974.
[32] A. Tocino and R. Ardanuy, Runge-Kutta methods for numerical solution of stochastic differential equations, J. Comput. Appl. Math., 138(2) (2002) 219–241.
[33] A. Tocino, R. Zeghdane and L. Abbaoui, Linear mean-square stability analysis of weak order 2.0 semi-implicit Taylor schemes for scalar stochastic differential equations, Appl. Num. Math., 68 (2013) 19–30.
[34] D.A. Voss and A.Q.M. Khaliq, Split-step Adams-Moulton Milstein methods for systems of stiff stochastic differential equations, Intern. J. Comp. Math., 92(5) (2015) 995–1011.
[35] X. Wang, S. Gan and D. Wang, A family of fully implicit Milstein methods for stiff stochastic differential equations with multiplicative noise, BIT, 52(3) (2012) 741–772.
[36] P. Wang and Y. Li, Split-step forward methods for stochastic differential equations, J. Comput. Appl. Math., 233(10) (2010) 2641–2651.

[37] P. Wang and Z. Liu, Split-step backward balanced Milstein methods for stiff stochastic systems, J. Appl. Numer. Math., 59(6) (2009) 1198–1213.
[38] A.M. Wazwaz, A modified third order Runge-Kutta method, Appl. Math. Letters, 3(3) (1990) 123–125.
[39] Z. Yin and S. Gan, An error corrected Euler-Maruyama method for stiff stochastic differential equations, Appl. Math. Comput., 256(1) (2015) 630–641.
[40] W. Zhao, L. Tian and L. Ju, Convergence analysis of a splitting method for stochastic differential equations, Int. J. Numer. Anal. Model., 5(4) (2008) 673–692.

(Corresponding Author) Department of Mathematics, Faculty of Mathematics, Statistics and Computer Sciences, Semnan University, P. O. Box 35195-363, Semnan, Iran ,
E-mail: knouri@semnan.ac.ir; knouri.h@gmail.com

Department of Mathematics, Faculty of Mathematics, Statistics and Computer Sciences, Semnan University, P. O. Box 35195-363, Semnan, Iran
E-mail: hranjbar@semnan.ac.ir

Instituto Universitario de Matemática Multidisciplinar, Universitat Politècnica de València, Camino de Vera s/n, 46022, Valencia, Spain
E-mail: jccortes@imm.upv.es