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WEAK GALERKIN METHOD FOR THE HELMHOLTZ EQUATION WITH DTN BOUNDARY CONDITION

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Abstract. In this article, we consider a weak Galerkin finite element method for the two dimensional exterior Helmholtz problem. After introducing a nonlocal boundary condition by means of the exact Dirichlet to Neumann (DtN) operator for the exterior problem, we prove that the existence and uniqueness of the weak Galerkin finite element solution for this problem. Then, applying some projection techniques, we establish a priori error estimate, which include the effect of truncation of the DtN boundary condition as well as the spatial discretization. Finally, some numerical examples are presented to confirm the theoretical predictions.

Key words. Helmholtz equation, weak Galerkin method, Dirichlet to Neumann operator, error estimates.

1. Introduction

We consider the exterior Helmholtz problem:

$$-\Delta u - k^2 u = f \quad \text{in}\, \Omega^c, \tag{1}$$

$$u = g \quad \text{on}\, \Gamma_0, \tag{2}$$

$$\lim_{r\to+\infty} r^{1/2}\left(\frac{\partial u}{\partial r} - \mathrm{i}ku\right) = 0 \quad r = |x|, \tag{3}$$

where i denotes the imaginary unit and $k \in \mathbb{R}$ is known as the wave number. $\Omega \subset \mathbb{R}^2$, be an open, bounded domain with smooth boundary Γ_0, and let $\Omega^c = \mathbb{R}^2 \backslash \overline{\Omega}$ be the unbounded exterior domain in $\mathbb{R}^2$. The equation (3) is known as the standard Sommerfeld radiation condition.

In order to solve the exterior problem in an infinite domain numerically, it is usual to limit the computation to a finite domain by introducing an artificial boundary. The original exterior domain problem reduced to a boundary value problem by enforcing a boundary condition on the artificial boundary. In [5, 7, 11, 12, 13, 14, 17, 19, 21, 29, 31, 33, 43, 46, 52], the authors developed different numerical methods for the Helmholtz equation with the lowest order absorbing boundary condition. However, the lowest order absorbing boundary conditions will lead to large errors caused by the reflections from the artificial boundary, unless the computational domain is large. To decrease the errors caused by boundary reflection, several numerical approaches have been considered. These contain high order absorbing boundary condition [16], perfectly matched layer [3], and the Dirichlet to Neumann (DtN) boundary condition [18, 22, 24, 25, 44] and the references therein. The DtN boundary condition is called exact nonreflecting boundary condition because it allows waves to propagate outward without producing any spurious reflection from the artificial boundary. The DtN condition was derived by Keller and Givoili [23], for the Helmholtz equation when the artificial boundary is a circle or sphere. In [24], Koyama established a priori error estimates for the DtN finite element method, including the effects of truncation of the DtN boundary condition and the finite

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element discretization. Hsiao et. al [18] derived for the DtN finite element method and present numerical results that show optimal convergence in the L^2 and H^1 norms using conforming piecewise linear finite elements. Kapita and Monk, in [22], considered an approximate boundary value problem with a truncated DtN series by the plane wave discontinuous Galerkin method and derived error estimates with respect to the truncation order of the DtN map and mesh width. In [44], Wang et. al coupled the DG method and spectral method to solve the DtN boundary value problem and gave DG-norm and L^2 norm errors analysis explicitly with wave number.

The numerical studies of the Helmholtz equation have been become an extraordinary popular research field in recent years. Many numerical schemes have been developed and analyzed for the Helmholtz equation, for instance, conforming finite element method [19, 21], interior penalty discontinuous Galerkin method [11], continuous interior penalty Galerkin method [46, 52], local discontinuous Galerkin method [12], hybridizable discontinuous Galerkin method [5, 14], plane wave discontinuous Galerkin method [13, 17, 22], weak Galerkin(WG) finite element method [7, 31, 33, 43], spectral method [29], finite difference method [15, 39, 40, 41, 42], immersed finite element method [26, 27] and the references therein. Weak Galerkin finite element methods were first introduced by Wang and Ye in [37] for second order elliptic equations. The main idea behind weak Galerkin method lies in the classical gradient operator replaced by a discrete weak gradient operator for weak functions on a partition of the domain. Weak Galerkin methods have been widely used to solve a variety of partial differential equations[34, 35, 36, 45, 47, 48, 49, 50, 51]. The methods have been applied for solving the Helmholtz equation with the lowest order absorbing boundary condition in [7, 31, 33, 43]. In [33], due to use of the RT and BDM elements, the WG finite element methods were limited to classical finite element partitions. Following the stabilization technique of [38], Mu et. al developed a new WG method that admits general finite element partitions with a mix of arbitrary shape of polygons and polyhedrons.

In practical computations, the use of the Fourier series representation of the DtN operator requires truncating the infinite series at a finite order to obtain an approximation DtN operator. So in this paper, we firstly investigate the truncating order of the series. Then, after studying the Gårding inequality and the unique of the weak solution, we establish a priori error estimates in the tri-norm and L^2 norm for the exterior Helmholtz problem by using weak Galerkin finite element method, including the effect of the truncation order and the finite element discretization. Following the stabilization approach of [38], we also add a parameter-free stabilier in the weak Galerkin formulation. The present weak Galerkin method is more flexible in terms of choosing approximation functions and finite element partitions. What's more, it is well suited to handling obstacles with complicated geometries and is easy to handle DtN boundary condition.

An outline of the remainder of the paper follows: In Section 2, we describe the nonlocal boundary value problem and modified nonlocal boundary value problem. Section 3 is devoted to a description of weak Galerkin method and algorithm. We shall design a weak Galerkin formulation for the modified nonlocal boundary value problem given in (10)–(12) and demonstrate the well-posedness of the formulation. We present an error equation and the error estimates in Section 4. In Section 5, we report some numerical results in order to demonstrate the accurate and efficient of our method. In this section, we consider the first approximation of the DtN operator. The reason is that, the first approximation will generate a sparse linear

(a) The sketch of Ω (b) The sketch of Ω_R and Ω

FIGURE 1. The sketch of the boundary value problem and the nonlocal boundary value problem.

system while the DtN operator results a density system. We end in Section 6 with some conclusions.

2. Nonlocal Boundary Value Problem

To seek approximate solution of problem (1)–(3), we introduce an artificial domain Ω_R with artificial boundary $\Gamma_R = \{x \in \mathbb{R}^2 : |x| = R\}$ enclosing the entire region Ω (see Figure 1). The artificial domain Ω_R is the annular region between Γ_0 and Γ_R, and $\Omega_R^c = \mathbb{R}^2 \setminus \overline{\Omega \cup \Omega_R}$ is the unbounded exterior domain. The problem (1)–(3) is reduced equivalently to the following nonlocal boundary value problem [20]:

$$-\Delta u - k^2 u = f \qquad \text{in}\, \Omega_R, \tag{4}$$

$$u = g \qquad \text{on}\, \Gamma_0, \tag{5}$$

$$\frac{\partial u}{\partial n} = Tu \qquad \text{on}\, \Gamma_R, \tag{6}$$

where the Dirichlet to Neumann (DtN) operator $T: \ H^{\frac{1}{2}}(\Gamma_R) \to H^{-\frac{1}{2}}(\Gamma_R)$, is defined as

$$T\varphi = {\sum_{n=0}^{\infty}}' \frac{kH_n^{(1)'}(kR)}{\pi H_n^{(1)}(kR)} \int_0^{2\pi} \varphi(R,\phi)\cos(n(\theta-\phi))d\phi \qquad \forall \varphi \in H^{\frac{1}{2}}(\Gamma_R). \tag{7}$$

The prime behind the summation means that the first term in the summation is multiplied by 1/2 and $H_n^{(1)}(\cdot)$ is the first-kind Hankel function of order n. The following lemma show the boundedness of the DtN operator [18].

Lemma 2.1. *The DtN operator T in (7) is a bounded linear operator from $H^s(\Gamma_R)$ to $H^{s-1}(\Gamma_R)$ for any constants $s \geq \frac{1}{2}$.*

The variational formulation of the problem (4)–(6) is as follows: find $u \in H^1(\Omega_R)$ and $u = g$ on Γ_0 such that

$$(\nabla u, \nabla v) - k^2(u,v) - \langle Tu, v\rangle_{\Gamma_R} = (f,v) \qquad \forall v \in H_0^1(\Omega_R), \tag{8}$$

where $H_0^1(\Omega_R) = \{v \in H^1(\Omega_R),\ v|_{\Gamma_0} = 0\}$. Throughout the paper, we shall use the standard Sobolev space $H^s(D)$ for any $s \geq 0$ [4, 6]. In particular, $(\cdot,\cdot)_D$ and $\langle\cdot,\cdot\rangle_{\Gamma_R}$ are used for L^2–inner product on complex valued spaces $L^2(D)$ and $L^2(\Gamma_R)$. We use $\|\cdot\|_D$ and $|\cdot|_D$ to denote the norm and the seminorm in $L^2(D)$. When $D = \Omega_R$, we shall drop the subscript D in the norm and inner product notation. The well–posedness of problem (8) addressed in the following lemma [20, 28].

Lemma 2.2. *For every $f \in L^2(\Omega_R)$, problem* (8) *has a unique solution which is the restriction to Ω_R of the solution of problem* (4)–(6).

For the practical computations, we need to truncate the infinite series of the exact DtN operator to obtain an approximate DtN operator written as

$$T^N\varphi = {\sum_{n=0}^{N}}{}' \frac{kH_n^{(1)'}(kR)}{\pi H_n^{(1)}(kR)} \int_0^{2\pi} \varphi(R,\phi)\cos(n(\theta-\phi))d\phi, \tag{9}$$

for $\forall\varphi \in H^{\frac{1}{2}}(\Gamma_R)$. Here, the non-negative integer N is called the truncation order of the DtN operator. Accordingly, the boundary value problem (4)–(6) is replaced by a modified nonlocal boundary value problem reads:

$$-\Delta u - k^2 u = f \qquad \text{in}\,\Omega_R, \tag{10}$$

$$u = g \qquad \text{on}\,\Gamma_0, \tag{11}$$

$$\frac{\partial u}{\partial n} = T^N u \quad \text{on}\,\Gamma_R. \tag{12}$$

Clearly, the variational formulation of the modified nonlocal boundary value problem is: find $u_N \in H^1(\Omega_R)$ and $u_N = g$ on Γ_0 such that

$$(\nabla u_N, \nabla v) - k^2(u_N, v) - \langle T^N u_N, v\rangle_{\Gamma_R} = (f, v) \qquad \forall v \in H_0^1(\Omega_R). \tag{13}$$

The well-posedness of problem (13) can be found in [18].

Next, we recall the estimate for the difference of T and T^N in the next lemma [18]. This estimate will be needed in the error estimates.

Lemma 2.3. *Suppose DtN operator T and T^N are defined as in* (7) *and* (9)*, respectively. Then, for given $\varphi \in H^s(\Gamma_R), s \in \mathbb{R}$, there holds, for $t = 1$,*

$$\|(T - T^N)\varphi\|_{H^{s-1}(\Gamma_R)} \leq c\frac{\epsilon(N,\varphi)}{N}\|\varphi\|_{H^{s+1}(\Gamma_R)}, \tag{14}$$

where $c > 0$ is a constant dependent on kR but independent of ϕ and N, and

$$\epsilon(N,\varphi) = \frac{\left\{\sum\limits_{n=N+1}^{\infty}(1+n^2)^{s+1}(|a_n|^2+|b_n|^2)\right\}^{\frac{1}{2}}}{\left\{\frac{|a_0|^2}{2} + \sum\limits_{n=1}^{\infty}(1+n^2)^{s+1}(|a_n|^2+|b_n|^2)\right\}^{\frac{1}{2}}} \leq 1. \tag{15}$$

is a function of the truncation order N and the function φ satisfying $\epsilon(N,\varphi) \to 0$ as $N \to \infty$, and a_n, b_n are Fourier coefficients of φ.

3. Weak Galerkin Finite Element Method

Let $\mathcal{T}_h$ be a quasi uniform and shape regular [38] triangular of the domain Ω_R. Each $E \in \mathcal{T}_h$ has at most one curved edge on $\partial\Omega_R$. Let $\widehat{\mathcal{T}}_h = \cup\{\widehat{E}\}$ is a body-fitted mesh of Ω_R, where $\widehat{E}$ is the triangle sharing vertices with E. Define $\widehat{\Omega}_R =$ interior$(\cup_{\widehat{E}\in\widehat{\mathcal{T}}_h}\widehat{E})$. The mapping $\mathcal{F} : \Omega_R \to \widehat{\Omega}_R$ is C^2 smooth on each $E \in \mathcal{T}_h$ and satisfies $\mathcal{F}(E) = \widehat{E}$. Therefore, the partition $\mathcal{T}_h = \{E = \mathcal{F}^{-1}(\widehat{E}) : \forall\widehat{E} \in \widehat{\mathcal{T}}_h\}$.

Denote by $\mathcal{E}_h$ the set of all edges in $\mathcal{T}_h$, $\mathcal{E}_h^0 = \mathcal{E}_h / \partial\Omega_R$ the set of all interior edges and $\mathcal{E}_h^R = \mathcal{E}_h \cap \Gamma_R$. For any element $E \in \mathcal{T}_h$, denote by h_E the diameter of E, and $h = \max_{E\in\mathcal{T}_h} h_E$. For any $e \in \mathcal{E}_h$, denote by h_e the length of e. For the simplicity of notation, we shall use c for a generic positive constant independent of h which may stand for different values at its different occurrences.

For each triangle E, E^0 and ∂E denote the interior and boundary of E respectively. For any non-negative integer $l \geq 0$, denote $P_l(E^0)$ the set of polynomials on E^0 with degree no more than l, and $P_l(e)$ the set of polynomials on each edge $e \in \partial E$ with degree no more than l. In this paper, we only consider the case of linear elements, i.e., $l = 1$. A discrete weak function $v_h = \{v_0, v_b\}$ refers to a polynomial with two components. The first component v_0 can be understood as the value of v_h in E^0, and v_b represents the value of v_h on the edge $e, e \in \partial E$. Next, we define the weak Galerkin finite element space V_h and $V_{0,h}$,

$$V_h = \{v_h = \{v_0, v_b\} : \ v_0|_E \in P_1(E^0), \ v_b|_e \in P_1(e), \ e \in \partial E, \ E \in \mathcal{T}_h\}, \tag{16}$$

$$V_{0,h} = \{v_h \in V_h, v_b|_{\Gamma_0} = 0\}. \tag{17}$$

We would like to emphasize that any function $v_h \in V_h$ has a single value v_b on each edge $e \in \mathcal{E}_h$ and v_b may not necessarily be related to the trace of v_0 on ∂E. For any $v_h \in V_h$, we define the discrete weak gradient of v_h, denote by $\nabla_d v_h \in [P_0(E)]^2$ on each element E, satisfying the following equation

$$(\nabla_d v_h, q)_E = -(v_0, \nabla \cdot q)_E + \langle v_b, q \cdot n\rangle_{\partial E} \qquad \forall q \in [P_0(E)]^2. \tag{18}$$

For any $u_h, v_h \in V_h$, we introduce the following bilinear form:

$$a_h^N(u_h, v_h) = \sum_{E\in\mathcal{T}_h} (\nabla_d u_h, \nabla_d v_h)_E + s(u_h, v_h) + d^N(u_h, v_h) - k^2(u_0, v_0), \tag{19}$$

where

$$s(u_h, v_h) = \sum_{E\in\mathcal{T}_h} h_E^{-1} \langle u_0 - u_b, v_0 - v_b\rangle_{\partial E}, \tag{20}$$

$$d^N(u_h, v_h) = -\langle T^N u_b, v_b\rangle_{\Gamma_R}. \tag{21}$$

The bilinear form $s(\cdot, \cdot)$ is called stabilizer or smoother.

Then, we arrive at the weak Galerkin finite element formulation: find $u_h = \{u_0, u_b\} \in V_h$ and $u_b = Q_b g$ on Γ_0 such that

$$a_h^N(u_h, v_h) = (f, v_0) \quad \forall v_h = \{v_0, v_b\} \in V_{0,h}, \tag{22}$$

where Q_b is the L^2 projection operator from $L^2(e)$ onto $P_1(e)$. In particular, we define a seminorm for $v_h \in V_h$ by

$$|\!|\!| v_h |\!|\!|^2 = \sum_{E\in\mathcal{T}_h} \|\nabla_d v_h\|_E^2 + \sum_{E\in\mathcal{T}_h} h_E^{-1} \|v_0 - v_b\|_{\partial E}^2 + k^2 \|v_b\|_{\frac{1}{2},\Gamma_R}^2, \tag{23}$$

where

$$\|v_h\|_{\frac{1}{2},\Gamma_R}^2 = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (1 + n^2)^{\frac{1}{2}} (|a_n|^2 + |b_n|^2),$$

where a_n and b_n are Fourier coefficients of v_h.

The following lemma derive an analogy of Gårding's inequality.

Lemma 3.1. *There exist positive constants α and β satisfying*

$$\mathrm{Re}\{a_h^N(v_h, v_h)\} + \alpha\|v_0\|^2 \geq \beta(|\!|\!| v_h |\!|\!|^2 + \|v_0\|^2), \tag{24}$$

for all $v_h \in V_h$.

Proof. Since

$$
\begin{aligned}
a_h^N(v_h, v_h) &= \sum_{E\in\mathcal{T}_h} (\nabla_d v_h, \nabla_d v_h)_E + s(v_h, v_h) + d^N(v_h, v_h) - k^2(v_0, v_0)\\
&= \sum_{E\in\mathcal{T}_h} \|\nabla_d v_h\|_E^2 + \sum_{E\in\mathcal{T}_h} h_E^{-1}\|v_0 - v_b\|_{\partial E}^2 - k^2\|v_0\|^2 - \langle T^N v_b, v_b\rangle_{\Gamma_R}. \qquad (25)
\end{aligned}
$$

Using the estimate in [18], one has

$$\mathrm{Re}\{-\langle T^N v_b, v_b\rangle_{\Gamma_R}\} \geq -c\|v_b\|_{0,\Gamma_R}^2.$$

By the Sobolev embedding theorem, we have

$$\|v_b\|_{0,\Gamma_R}^2 \leq c\|v_b\|_{\frac{1}{2},\Gamma_R}^2.$$

Therefore, we can obtain

$$\mathrm{Re}\{-\langle T^N v_b, v_b\rangle_{\Gamma_R}\} \geq -c\|v_b\|_{\frac{1}{2},\Gamma_R}^2.$$

Substituting the above inequality into (25) yields

$$
\begin{aligned}
\mathrm{Re}\{a_h^N(v_h, v_h)\} &\geq \sum_{E\in\mathcal{T}_h} \|\nabla_d v_h\|_E^2 + \sum_{E\in\mathcal{T}_h} h_E^{-1}\|v_0 - v_b\|_{\partial E}^2 - k^2\|v_0\|^2 - c\|v_b\|_{\frac{1}{2},\Gamma_R}^2\\
&\geq c|\!|\!|v_h|\!|\!|^2 - k^2\|v_0\|^2 - c\frac{1}{k^2}|\!|\!|v_h|\!|\!|^2 = c(1 - \frac{1}{k^2})|\!|\!|v_h|\!|\!|^2 - k^2\|v_0\|^2.
\end{aligned}
$$

Then there exist two positive constants α and β such that

$$\mathrm{Re}\{a_h^N(v_h, v_h)\} + \alpha\|v_0\|^2 \geq \beta(|\!|\!|v_h|\!|\!|^2 + \|v_0\|^2).$$

This completes the proof. □

We end this section with the well-possedness of the weak Galerkin discretization scheme (22).

Theorem 3.2. *The weak Galerkin finite element discretization scheme* (22) *has a unique solution in the finite element space* V_h.

Proof. Since the weak Galerkin finite element scheme (22) is a linear system, the well-posedness is equivalent to the uniqueness of the solution. To prove the uniqueness, let $u_{h,1}$ and $u_{h,2}$ be two solutions of scheme (22), then $\xi = u_{h,1} - u_{h,2}$ is the solution of (22) with the condition that $f = 0$.

Letting $v_h = \xi$ in (22), we have

(26)

$$a_h^N(\xi, \xi) = \sum_{E\in\mathcal{T}_h} \|\nabla_d \xi\|_E^2 + \sum_{E\in\mathcal{T}_h} h_E^{-1}\|\xi_0 - \xi_b\|_{\partial E}^2 - k^2\|\xi_0\|^2 - \langle T^N \xi_b, \xi_b\rangle_{\Gamma_R} = 0.$$

The above equation implies that

$$k^2\|\xi_0\|^2 = \sum_{E\in\mathcal{T}_h} \|\nabla_d \xi\|_E^2 + \sum_{E\in\mathcal{T}_h} h_E^{-1}\|\xi_0 - \xi_b\|_{\partial E}^2 - \mathrm{Re}\langle T^N \xi_b, \xi_b\rangle_{\Gamma_R},$$

$$\mathrm{Im}\langle T^N \xi_b, \xi_b\rangle_{\Gamma_R} = 0. \qquad (27)$$

According to (see Lemma 3.3 in [30])

$$\mathrm{Re}\langle T^N \xi_b, \xi_b\rangle_{\Gamma_R} \leq 0, \qquad (28)$$

and the boundness of the DtN operator

$$
\begin{aligned}
\langle T^N \xi_b, \xi_b\rangle_{\Gamma_R} &\leq \|T^N \xi_b\|_{-\frac{1}{2},\Gamma_R}\|\xi_b\|_{\frac{1}{2},\Gamma_R}\\
&\leq c\|\xi_b\|_{\frac{1}{2},\Gamma_R}^2 \leq c\frac{1}{k^2}|\!|\!|\xi|\!|\!|^2,
\end{aligned}
$$

there exists a constant $c > 0$ such that

$$\|\xi_0\|^2 \le \frac{c(1+\frac{1}{k^2})}{k^2} \|\!|\xi|\!\|^2. \tag{29}$$

Now we have from Gårding's inequality (24) of Lemma 3.1 that

$$\mathrm{Re}\{a_h^N(\xi,\xi)\} + \alpha\|\xi_0\|^2 \ge \beta(\|\!|\xi|\!\|^2 + \|\xi_0\|^2). \tag{30}$$

Thus, it follows from the estimate (29) that

$$\beta(\|\!|\xi|\!\|^2 + \|\xi_0\|^2) \le \alpha\|\xi_0\|^2 \le \frac{c\alpha(1+\frac{1}{k^2})}{k^2}\|\!|\xi|\!\|^2 \le \frac{\beta}{2}\|\!|\xi|\!\|^2, \tag{31}$$

In view of $k > 1$, it yields

$$c\alpha(1+\frac{1}{k^2}) \le \frac{k^2\beta}{2}.$$

Therefore,

$$\|\!|\xi|\!\|^2 + \|\xi_0\|^2 = 0, \tag{32}$$

which implies that $\xi_0 = 0$ and

$$\|\!|\xi|\!\|^2 = \sum_{E\in\mathcal{T}_h} \|\nabla_d \xi\|_E^2 + \sum_{E\in\mathcal{T}_h} h_E^{-1}\|\xi_0 - \xi_b\|_{\partial E}^2 + \sum_{e\in\mathcal{E}_h^R} h_e^{-1}\|\xi_b\|_{\frac{1}{2},e}^2 = 0.$$

This shows that ξ is a constant and $\xi_b = \xi_0|_{\partial E} = 0$. Thus, $\xi = 0$ and consequently, $u_{h,1} = u_{h,2}$. This completes the proof. □

4. Error Estimates

In this section, we will establish some error estimates for the weak Galerkin finite element solution u_h arising from (22). First, we define some local projection operators and then derive some approximation properties which are useful in error analysis. For each element $E \in \mathcal{T}_h$, denote by Q_0 the L^2 projection from $L^2(E)$ onto $P_1(E)$. Similarly, for each edge $e \in \mathcal{E}_h$, let Q_b be the L^2 projection operator from $L^2(e)$ onto $P_1(e)$. Denote by R_h the L^2 projection onto the local discrete gradient space $[P_0(E)]^2$. What's more, we define a projection operator $Q_h v = \{Q_0 v, Q_b v\}$: $H^1(\Omega_R) \to V_h$ for any $v \in H^1(\Omega_R)$. Then, on each element $E \in \mathcal{T}_h$, we have

$$\nabla_d(Q_h\omega) = R_h\nabla\omega \qquad \forall\omega \in H^1(\Omega_R). \tag{33}$$

The following lemmas provide some estimates for the projection operators Q_h and R_h [32, 38].

Lemma 4.1. *Let $\mathcal{T}_h$ be a shape regular polygonal finite element partition of the domain Ω_R. Then for any $\phi \in H^2(\Omega_R)$, we have*

$$\sum_{E\in\mathcal{T}_h} \|\phi - Q_0\phi\|_E^2 + \sum_{E\in\mathcal{T}_h} h_E^2 \|\nabla(\phi - Q_0\phi)\|_E^2 \le ch^4\|\phi\|_2^2, \tag{34}$$

$$\sum_{E\in\mathcal{T}_h} \|\nabla\phi - R_h\nabla\phi\|_E^2 = \sum_{E\in\mathcal{T}_h} \|\nabla\phi - \nabla_d(Q_h\phi)\|_E^2 \le ch^2\|\phi\|_2^2. \tag{35}$$

Lemma 4.2. *Let $\mathcal{T}_h$ be a shape regular polygonal finite element partition of the domain Ω_R. Then for any $\phi \in H^2(\Omega_R)$ and $v_h = \{v_0, v_b\} \in V_h$, we have*

$$\left|\sum_{E\in\mathcal{T}_h} h_E^{-1}\langle Q_0\phi - Q_b\phi, v_0 - v_b\rangle_{\partial E}\right| \le ch\|\phi\|_2 |||v_h|||, \tag{36}$$

$$\begin{aligned}&\left|\sum_{E\in\mathcal{T}_h}\langle(\nabla\phi - R_h\nabla\phi)\cdot n, v_0 - v_b\rangle_{\partial E}\right|\\ =&\left|\sum_{E\in\mathcal{T}_h}\langle(\nabla\phi - \nabla_d(Q_h\phi))\cdot n, v_0 - v_b\rangle_{\partial E}\right|\\ \le& ch\|\phi\|_2 |||v_h|||.\end{aligned}\tag{37}$$

4.1. Error Equation. Define an error function $e_h = \{e_0, e_b\} = \{Q_0u - u_0, Q_bu - u_b\} = Q_hu - u_h$. We derive an error equation for e_h in the following lemma.

Lemma 4.3. *The error function e_h satisfies the following equation for all $v_h = \{v_0, v_b\} \in V_{0,h}$,*

$$\begin{aligned}&\sum_{E\in\mathcal{T}_h}(\nabla_d e_h, \nabla_d v_h)_E + s(e_h, v_h) - k^2(e_0, v_0) + d^N(e_h, v_h)\\ &= \sum_{E\in\mathcal{T}_h}\langle(\nabla u - R_h\nabla u)\cdot n, v_0 - v_b\rangle_{\partial E} + s(Q_hu, v_h) + d^N(u, v_h) - d(u, v_h).\end{aligned}\tag{38}$$

Proof. Testing (4) by v_0 with $v_h = \{v_0, v_b\} \in V_{0,h}$, we have

$$\sum_{E\in\mathcal{T}_h}(\nabla u, \nabla v_0)_E - \sum_{E\in\mathcal{T}_h}\langle\nabla u\cdot n, v_0\rangle_{\partial E} - k^2(u, v_0) = (f, v_0). \tag{39}$$

It follows from (6) that

$$\sum_{E\in\mathcal{T}_h}\langle\nabla u\cdot n, v_b\rangle_{\partial E} = \langle\nabla u\cdot n, v_b\rangle_{\Gamma_R} = \langle Tu, v_b\rangle_{\Gamma_R}. \tag{40}$$

Substituting (40) into (39) gives

$$\begin{aligned}&\sum_{E\in\mathcal{T}_h}(\nabla u, \nabla v_0)_E - \sum_{E\in\mathcal{T}_h}\langle\nabla u\cdot n, v_0 - v_b\rangle_{\partial E} - k^2(u, v_0) - \langle T^Nu, v_b\rangle_{\Gamma_R}\\ &\qquad = (f, v_0) + \langle(T - T^N)u, v_b\rangle_{\Gamma_R}.\end{aligned}\tag{41}$$

By (33) and the definitions of Q_h, R_h and the discrete weak gradient, we have

$$\begin{aligned}(\nabla_d Q_hu, \nabla_d v_h)_E &= (R_h\nabla u, \nabla_d v_h)_E\\ &= -(v_0, \nabla\cdot(R_h\nabla u))_E + \langle v_b, (R_h\nabla u)\cdot n\rangle_{\partial E}\\ &= (\nabla v_0, R_h\nabla u)_E - \langle v_0 - v_b, (R_h\nabla u)\cdot n\rangle_{\partial E}\\ &= (\nabla v_0, \nabla u)_E - \langle v_0 - v_b, (R_h\nabla u)\cdot n\rangle_{\partial E}.\end{aligned}\tag{42}$$

Combining (41) and (42) yields

$$\begin{aligned}&\sum_{E\in\mathcal{T}_h}(\nabla_d Q_hu, \nabla_d v_h)_E - k^2(u, v_0) - \langle T^Nu, v_b\rangle_{\Gamma_R}\\ &= (f, v_0) + \sum_{E\in\mathcal{T}_h}\langle(\nabla u - R_h\nabla u)\cdot n, v_0 - v_b\rangle_{\partial E} + \langle(T - T^N)u, v_b\rangle_{\Gamma_R}.\end{aligned}\tag{43}$$

Adding $\sum_{E\in\mathcal{T}_h} h_E^{-1}\langle Q_0u - Q_bu, v_0 - v_b\rangle_{\partial E}$ on the both sides of the above equation and using the definitions of Q_0 and Q_b, we arrive at

$$\begin{aligned}
&\sum_{E\in\mathcal{T}_h}(\nabla_d Q_h u, \nabla_d v_h)_E + \sum_{E\in\mathcal{T}_h} h_E^{-1}\langle Q_0u - Q_bu, v_0 - v_b\rangle_{\partial E} - k^2(Q_0u, v_0)\\
&\qquad - \langle Q_b(T^N u), v_b\rangle_{\Gamma_R}\\
&= (f, v_0) + \sum_{E\in\mathcal{T}_h}\langle(\nabla u - R_h\nabla u)\cdot n, v_0 - v_b\rangle_{\partial E} + \langle(T - T^N)u, v_b\rangle_{\Gamma_R}\\
&\qquad + \sum_{E\in\mathcal{T}_h} h_E^{-1}\langle Q_0u - Q_bu, v_0 - v_b\rangle_{\partial E}.
\end{aligned}\tag{44}$$

Therefore, subtracting (22) from (44) leads to

$$\begin{aligned}
&\sum_{E\in\mathcal{T}_h}(\nabla_d e_h, \nabla_d v_h)_E + \sum_{E\in\mathcal{T}_h} h_E^{-1}\langle e_0 - e_b, v_0 - v_b\rangle_{\partial E} - k^2(e_0, v_0)\\
&\qquad - \langle T^N e_b, v_b\rangle_{\Gamma_R}\\
&= \sum_{E\in\mathcal{T}_h}\langle(\nabla u - R_h\nabla u)\cdot n, v_0 - v_b\rangle_{\partial E} + \langle(T - T^N)u, v_b\rangle_{\Gamma_R}\\
&\qquad + \sum_{E\in\mathcal{T}_h} h_E^{-1}\langle Q_0u - Q_bu, v_0 - v_b\rangle_{\partial E},
\end{aligned}\tag{45}$$

where $T^N e_b \triangleq Q_b(T^N u) - T^N u_b$. This completes the proof. □

4.2. Error Analysis. Using the error equation (38), we now turn to the error estimate for the weak Galerkin finite element solution.

Theorem 4.4. *Let $u \in H^2(\Omega_R)$ be the solution of (4)–(6) and $u_h \in V_h$ be the solution of scheme (22). Then the error function $e_h = Q_h u - u_h$ satisfies the following estimate*

$$|\!|\!| Q_h u - u_h |\!|\!| \le ck\|e_0\| + ch\|u\|_2 + c\frac{\epsilon(N,u)}{kN}\|u\|_2,\tag{46}$$

where $c > 0$ is a constant independent of h and N, and $\epsilon(N,u)$ is a function of the truncation order N and the function u satisfying $\epsilon(N,u) \to 0$ as $N \to \infty$.

Proof. We first take $v_h = e_h$ in the error equation (38) yields

$$\begin{aligned}
&\sum_{E\in\mathcal{T}_h}\|e_h\|_E^2 + \sum_{E\in\mathcal{T}_h} h_E^{-1}\|e_0 - e_b\|_{\partial E}^2 - \langle T^N e_b, e_b\rangle_{\Gamma_R}\\
&= k^2\|e_0\|^2 + \sum_{E\in\mathcal{T}_h}\langle(\nabla u - R_h\nabla u)\cdot n, e_0 - e_b\rangle_{\partial E} + d^N(u, e_h) - d(u, e_h)\\
&\qquad + \sum_{E\in\mathcal{T}_h} h_E^{-1}\langle Q_0u - Q_bu, e_0 - e_b\rangle_{\partial E}.
\end{aligned}\tag{47}$$

Then, take the real part of the above equation, we have

$$\begin{aligned}
&\sum_{E\in\mathcal{T}_h}\|e_h\|_E^2 + \sum_{E\in\mathcal{T}_h} h_E^{-1}\|e_0 - e_b\|_{\partial E}^2 - \mathrm{Re}\langle T^N e_b, e_b\rangle_{\Gamma_R}\\
&= k^2\|e_0\|^2 + \mathrm{Re}\Big\{\sum_{E\in\mathcal{T}_h}\langle(\nabla u - R_h\nabla u)\cdot n, e_0 - e_b\rangle_{\partial E}\\
&\qquad + \sum_{E\in\mathcal{T}_h} h_E^{-1}\langle Q_0u - Q_bu, e_0 - e_b\rangle_{\partial E} + d^N(u, e_h) - d(u, e_h)\Big\}.
\end{aligned}\tag{48}$$

Since (see Lemma 3.3 in [30])

$$\mathrm{Re}\langle T^N e_b, e_b\rangle_{\Gamma_R} \leq 0,$$

then, we can deduce as follows

$$\begin{aligned}\|e_h\|^2 \leq k^2\|e_0\|^2 + \left|\sum_{E\in\mathcal{T}_h}\langle(\nabla u - R_h\nabla u)\cdot n, e_0 - e_b\rangle_{\partial E}\right| \\ + \left|\sum_{E\in\mathcal{T}_h} h_E^{-1}\langle Q_0 u - Q_b u, e_0 - e_b\rangle_{\partial E}\right| + \left|d^N(u,e_h) - d(u,e_h)\right|.\end{aligned} \tag{49}$$

Next, let us bound the terms on the right hand side of (49) one by one. From the estimates (36), (37) and Young's inequality we have

$$\left|\sum_{E\in\mathcal{T}_h} h_E^{-1}\langle Q_0 u - Q_b u, e_0 - e_b\rangle_{\partial E}\right| \leq ch\|u\|_2\|e_h\| \leq ch^2\|u\|_2^2 + \frac{1}{6}\|e_h\|^2,$$

$$\left|\sum_{E\in\mathcal{T}_h}\langle(\nabla u - R_h\nabla u)\cdot n, e_0 - e_b\rangle_{\partial E}\right| \leq ch\|u\|_2\|e_h\| \leq ch^2\|u\|_2^2 + \frac{1}{6}\|e_h\|^2.$$

In view of the trace theorem, there exist a bounded linear operator $\gamma : H^1(\Omega_R) \to H^{1/2}(\Gamma_R)$ and using the estimate (14) in Lemma 2.3, we get

$$\begin{aligned}\left|d^N(u,e_h) - d(u,e_h)\right| &= \left|\langle(T - T^N)\gamma u, e_b\rangle_{\Gamma_R}\right| \\ &\leq c\|(T - T^N)\gamma u\|_{-\frac{1}{2},\Gamma_R}\|e_b\|_{\frac{1}{2},\Gamma_R} \\ &\leq c\frac{\epsilon(N,u)}{N}\|u\|_2\|e_b\|_{\frac{1}{2},\Gamma_R} \\ &\leq c\frac{\epsilon(N,u)}{N}\|u\|_2\frac{1}{k}\|e_h\| \\ &\leq c\frac{\epsilon^2(N,u)}{k^2N^2}\|u\|_2^2 + \frac{1}{6}\|e_h\|^2,\end{aligned}$$

where $\epsilon(N,u)$ be defined in (15). Consequently,

$$\|e_h\|^2 \leq 2k^2\|e_0\|^2 + ch^2\|u\|_2^2 + c\frac{\epsilon^2(N,u)}{k^2N^2}\|u\|_2^2.$$

This completes the proof. □

Next, we analyze the error in the L^2 norm by using a duality argument as was commonly employed in the Galerkin finite element methods.

Theorem 4.5. *Let $u \in H^2(\Omega_R)$ be the solution of (4)–(6) and $u_h \in V_h$ be the solution of scheme (22). If $k^2h + \frac{k\epsilon(N,u)}{N} < 1$, there holds*

$$\|Q_0 u - u_0\| \leq ckh^2\|u\|_2 + c(k + \frac{1}{k} + hN)\frac{\epsilon(N,u)}{N^2}\|u\|_2, \tag{50}$$

where $c > 0$ is a constant independent of h and N, and $\epsilon(N,u)$ is a function of the truncation order N and the function u satisfying $\epsilon(N,u) \to 0$ as $N \to \infty$.

Proof. Now, we consider the following dual problem of (4)–(6) that seeks $\omega \in C^2(\Omega_R) \cap C^1(\overline{\Omega}_R)$ satisfying

$$-\Delta\omega - k^2\omega = e_0 \qquad \text{in}\, \Omega_R, \tag{51}$$

$$\omega = 0 \qquad \text{on}\, \Gamma_0, \tag{52}$$

$$\frac{\partial \omega}{\partial n} = T\omega \quad \text{on}\, \Gamma_R, \tag{53}$$

It can be shown that there exists a constant c such that

$$\|\omega\|_2 \leq ck\|e_0\|.$$

By testing (51) with e_0, we arrive at

$$\begin{aligned} \|e_0\|^2 &= (-\Delta\omega, e_0) - k^2(\omega, e_0) \\ &= \sum_{E\in\mathcal{T}_h} (\nabla\omega, \nabla e_0)_E - \sum_{e\in\mathcal{E}_h^0} \langle \nabla\omega \cdot n, e_0\rangle_e - \sum_{e\in\mathcal{E}_h^R} \langle T\omega, e_0\rangle_e - k^2(\omega, e_0). \end{aligned} \tag{54}$$

By the definition of the discrete weak gradient and the (33), we have

$$\begin{aligned} (\nabla_d(Q_h\omega), \nabla_d e_h)_E &= (R_h\nabla\omega, \nabla_d e_h)_E = -(\nabla\cdot(R_h\nabla\omega), e_0)_E + \langle (R_h\nabla\omega)\cdot n, e_b\rangle_{\partial E} \\ &= (\nabla e_0, R_h\nabla\omega)_E - \langle (R_h\nabla\omega)\cdot n, e_0 - e_b\rangle_{\partial E} \\ &= (\nabla e_0, \nabla\omega)_E - \langle (R_h\nabla\omega)\cdot n, e_0 - e_b\rangle_{\partial E}. \end{aligned} \tag{55}$$

It follows from the error equation (38) that

$$\begin{aligned} &\sum_{E\in\mathcal{T}_h} (\nabla_d e_h, \nabla_d(Q_h\omega))_E + s(e_h, Q_h\omega) - k^2(e_0, \omega) + d^N(e_h, Q_b\omega) \\ &= \sum_{E\in\mathcal{T}_h} \langle (\nabla u - R_h\nabla u)\cdot n, Q_0\omega - Q_b\omega\rangle_{\partial E} \\ &\quad + s(Q_h u, Q_h\omega) + d^N(u, Q_b\omega) - d(u, Q_b\omega). \end{aligned} \tag{56}$$

Substituting (55) and (56) into (54) gives

$$\begin{aligned} \|e_0\|^2 &= \sum_{E\in\mathcal{T}_h} (\nabla\omega, \nabla e_0)_E - \sum_{e\in\mathcal{E}_h^0} \langle \nabla\omega\cdot n, e_0 - e_b\rangle_e - \sum_{e\in\mathcal{E}_h^R} \langle T\omega, e_0\rangle_e - k^2(\omega, e_0) \\ &= \sum_{E\in\mathcal{T}_h} (\nabla_d(Q_h\omega), \nabla_d e_h)_E + \sum_{E\in\mathcal{T}_h} \langle (R_h\nabla\omega)\cdot n, e_0 - e_b\rangle_{\partial E} \\ &\quad - \sum_{e\in\mathcal{E}_h^0} \langle \nabla\omega\cdot n, e_0 - e_b\rangle_e - \sum_{e\in\mathcal{E}_h^R} \langle T\omega, e_0\rangle_e - k^2(\omega, e_0) \\ &= \sum_{E\in\mathcal{T}_h} \langle (\nabla u - R_h\nabla u)\cdot n, Q_0\omega - Q_b\omega\rangle_{\partial E} + \sum_{e\in\Gamma_R} \langle (R_h\nabla\omega)\cdot n, e_0 - e_b\rangle_e \\ &\quad + \sum_{e\in\mathcal{E}_h^0} \langle (R_h\nabla\omega)\cdot n - \nabla\omega\cdot n, e_0 - e_b\rangle_e + s(Q_h u, Q_h\omega) - s(e_h, Q_h\omega) \\ &\quad + d^N(u, Q_b\omega) - d(u, Q_b\omega) - d^N(e_h, Q_b\omega) + d(\omega, e_0) \\ &= \sum_{E\in\mathcal{T}_h} \langle (\nabla u - R_h\nabla u)\cdot n, Q_0\omega - Q_b\omega\rangle_{\partial E} + s(Q_h u, Q_h\omega) - s(e_h, Q_h\omega) \\ &\quad + \sum_{E\in\mathcal{T}_h} \langle (R_h\nabla\omega)\cdot n - \nabla\omega\cdot n, e_0 - e_b\rangle_{\partial E} + d^N(u, Q_b\omega) \\ &\quad - d(u, Q_b\omega) - d^N(e_h, Q_b\omega) + d(e_h, \omega). \end{aligned} \tag{57}$$

Next, we bound each term on the right hand side of (57). It follows from the Cauchy-Schwarz inequality, the definition of Q_b ,the trace inequality and the estimate (34) in Lemma 4.1 that

$$\begin{aligned}
&\left|\sum_{E\in\mathcal{T}_h}\langle(\nabla u - R_h\nabla u)\cdot n, Q_0\omega - Q_b\omega\rangle_{\partial E}\right| \\
&\quad\le \left(\sum_{E\in\mathcal{T}_h}\|\nabla u - R_h\nabla u\|^2_{\partial E}\right)^{\frac{1}{2}}\left(\sum_{E\in\mathcal{T}_h}\|Q_0\omega - Q_b\omega\|^2_{\partial E}\right)^{\frac{1}{2}} \\
&\quad\le c\left(\sum_{E\in\mathcal{T}_h}\|\nabla u - R_h\nabla u\|^2_{\partial E}\right)^{\frac{1}{2}}\left(\sum_{E\in\mathcal{T}_h}\|Q_0\omega - \omega\|^2_{\partial E}\right)^{\frac{1}{2}} \\
&\quad\le ch^2\|u\|_2\|\omega\|_2 \le ckh^2\|u\|_2\|e_0\|. \qquad (58)
\end{aligned}$$

Using the estimates (37) in Lemma 4.2 and (34) in Lemma 4.1, we arrive at

$$\begin{aligned}
&\left|\sum_{E\in\mathcal{T}_h}\langle R_h(\nabla\omega)\cdot n - \nabla\omega\cdot n, e_0 - e_b\rangle_{\partial E}\right| \\
&\quad\le \left(h_E\sum_{E\in\mathcal{T}_h}\|\nabla\omega - R_h\nabla\omega\|^2_{\partial E}\right)^{\frac{1}{2}}\left(\sum_{E\in\mathcal{T}_h}h_E^{-1}\|e_0 - e_b\|^2_{\partial E}\right)^{\frac{1}{2}} \\
&\quad\le ch|\!|\!|e_h|\!|\!|\|\omega\|_2 \le ckh|\!|\!|e_h|\!|\!|\|e_0\|. \qquad (59)
\end{aligned}$$

Analogously, it follows from the estimates (36) in Lemma 4.2 and (35) in Lemma 4.1, we obtain

$$\begin{aligned}
&|s(e_h, Q_h\omega)| \\
&\le \left|\sum_{E\in\mathcal{T}_h}h_E^{-1}\langle e_0 - e_b, Q_0\omega - Q_b\omega\rangle\right| = \left|\sum_{E\in\mathcal{T}_h}h_E^{-1}\langle e_0 - e_b, Q_0\omega - \omega\rangle\right| \\
&\le \left(\sum_{E\in\mathcal{T}_h}h_E^{-1}\|e_0 - e_b\|^2_{\partial E}\right)^{\frac{1}{2}}\left(\sum_{E\in\mathcal{T}_h}h_E^{-2}\|Q_0\omega - \omega\|^2_{\partial E} + \|\nabla(Q_0\omega - \omega)\|^2_{\partial E}\right)^{\frac{1}{2}} \\
&\le ch|\!|\!|e_h|\!|\!|\|\omega\|_2 \le ckh|\!|\!|e_h|\!|\!|\|e_0\|. \qquad (60)
\end{aligned}$$

Based on the trace inequality and the estimate (34) in Lemma 4.1, it yields

$$\begin{aligned}
|s(Q_hu, Q_h\omega)| &\le \left|\sum_{E\in\mathcal{T}_h}h_E^{-1}\langle Q_0u - Q_bu, Q_0\omega - Q_b\omega\rangle\right| \\
&\le \left(\sum_{E\in\mathcal{T}_h}h_E^{-1}\|Q_0u - u\|^2_{\partial E}\right)^{\frac{1}{2}}\left(\sum_{E\in\mathcal{T}_h}h_E^{-1}\|Q_0\omega - \omega\|^2_{\partial E}\right)^{\frac{1}{2}} \\
&\le ch^2\|u\|_2\|\omega\|_2 \le ckh^2\|u\|_2\|e_0\|. \qquad (61)
\end{aligned}$$

According to the trace theorem, there exist a bounded linear operator $\gamma : H^1(\Omega_R) \to H^{1/2}(\Gamma_R)$ and using the estimate (14) in Lemma 2.3 and the definition of Q_b, we can get

$$
\begin{aligned}
\left|d(\omega, e_h) - d^N(Q_b\omega, e_h)\right| &= \left|\langle (T - T^N)\gamma\omega, e_b\rangle\right| \\
&\le c\|(T - T^N)\gamma\omega\|_{-\frac{1}{2},\Gamma_R}\|e_b\|_{\frac{1}{2},\Gamma_R} \\
&\le c\frac{\epsilon(N,u)}{N}\|\omega\|_2\|e_b\|_{\frac{1}{2},\Gamma_R} \\
&\le c\frac{\epsilon(N,u)}{kN}\|\omega\|_2 |\!|\!| e_h |\!|\!| \\
&\le c\frac{\epsilon(N,u)}{N} |\!|\!| e_h |\!|\!| \|e_0\|, \qquad (62)
\end{aligned}
$$

where $\epsilon(N, u)$ be defined in (15). Similarly, we also have

$$
\begin{aligned}
\left|d(u, Q_b\omega) - d^N(u, Q_b\omega)\right| &\le c\frac{\epsilon(N,u)}{N^2}\|u\|_2\|Q_b\omega\|_2 \\
&\le c\frac{\epsilon(N,u)}{N^2}\|u\|_2\|\omega\|_2 \\
&\le ck\frac{\epsilon(N,u)}{N^2}\|u\|_2\|e_0\|. \qquad (63)
\end{aligned}
$$

Substituting (58)–(63) and the estimate (46) into (57), we get

$$
\|e_0\| \le c(k^2h + \frac{k\epsilon(N,u)}{N})\|e_0\| + ckh^2\|u\|_2 + c(k + \frac{1}{k} + hN)\frac{\epsilon(N,u)}{N^2}\|u\|_2. \qquad (64)
$$

Assume that $k^2h + \frac{k\epsilon(N,u)}{N} < 1$, we arrive at

$$
\|e_0\| \le ckh^2\|u\|_2 + c(k + \frac{1}{k} + hN)\frac{\epsilon(N,u)}{N^2}\|u\|_2. \qquad (65)
$$

This completes the proof. □

Combining the results of Theorem 4.4 and Theorem 4.5, we have the following error estimate.

Theorem 4.6. *Let $u \in H^2(\Omega_R)$ be the solution of (4)–(6) and $u_h \in V_h$ be the solution of scheme (22). If $k^2h + \frac{k\epsilon(N,u)}{N} < 1$, there holds*

$$
|\!|\!| Q_hu - u_h |\!|\!| \le c(k^2h + 1)h\|u\|_2 + c(\frac{k^2+1}{N} + \frac{1}{2} + kh)\frac{\epsilon(N,u)}{N}\|u\|_2, \qquad (66)
$$

where $c > 0$ is a constant independent of h and N, and $\epsilon(N, u)$ is a function of the truncation order N and the function u satisfying $\epsilon(N, u) \to 0$ as $N \to \infty$.

5. Numerical Experiments

Throughout this section, we consider the following two dimensional Helmholtz problem,

$$
-\Delta u - k^2 u = f \quad \text{in } \Omega_R, \qquad (67)
$$

$$
u = g \quad \text{on } \Gamma_0, \qquad (68)
$$

$$
\frac{\partial u}{\partial n} = \mathrm{i}ku + \frac{1}{2R}u \quad \text{on } \Gamma_R, \qquad (69)
$$

where (69) is the first order approximation of (6), which is proposed by Feng in [10], Engquist and Majda in [8, 9], Bayliss, Gunzburger and Turkel in [1, 2]. The boundaries Γ_0 and Γ_R are two circles with radius R_0 and R respectively, and both circles share the same center $(0, 0)$. The region Ω_R is the annular between Γ_0 and Γ_R.

In this section, we present the numerical results to validate our theoretical results. Let $u_h = \{u_0, u_b\}$ be the numerical solution of (22), and u be the solution of (4)–(6). Let $Q_h u = \{Q_0 u, Q_b u\}$ be the L^2 projection of u onto the appropriately defined spaces. The error for the weak Galerkin solution of (22) shall be measured in three norms defined as follows:

$$\|Q_h u - u_h\|_{1,h} = \left(\sum_{E\in\mathcal{T}_h} \sum_{e\in\partial E} h_E^{-1} \int_e |Q_0 u - Q_b u - u_0 + u_b|^2 ds \right.$$
$$\left. + \sum_{E\in\mathcal{T}_h} \int_E |\nabla_d (Q_h u - u_h)|^2 \, dx \right)^{1/2} \quad (\text{Discrete H}^1 \text{ norm}),$$

$$\|Q_0 u - u_0\| = \left(\sum_{E\in\mathcal{T}_h} \int_E |Q_0 u - u_0|^2 \, dx \right)^{1/2} \quad (\text{Element} - \text{based L}^2 \text{ norm}),$$

$$\|Q_b u - u_b\| = \left(\sum_{e\in\mathcal{E}_h} h_e \int_e |Q_b u - u_b|^2 ds \right)^{1/2} \quad (\text{Edge} - \text{based L}^2 \text{ norm}).$$

Example 1. We first consider the Helmholtz equation defined on Ω_R. Here, we set $f = \sin(kr)/r$, where $r = \sqrt{(x^2 + y^2)}$. The boundary data g is chosen so that the exact solution is

$$u = \frac{\cos(kr)}{r} - \frac{\cos k + \mathrm{i} \sin k}{k(J_0(k) + \mathrm{i} J_1(k))} J_0(kr),$$

in polar coordinates, where $J_\xi(z)$ are Bessel functions of the first kind and order ξ.

Uniform triangular partitions were used in the computation through successive mesh refinements. The numerical errors for the corresponding weak Galerkin solutions in H^1 and L^2 norms and the convergence rate are collected in Table 1 and Table 2. Table 1 shows the performance of the weak Galerkin finite element method with $k = 4, R_0 = 0.5, R = 1$ and Table 2 illustrates the performance of the weak Galerkin finite element method with $k = 10, R_0 = 0.05, R = 0.1$. For both cases, we can observe an optimal convergence rate, i.e., the error measured in H^1 norm is $O(h)$ and in L^2 norm is $O(h^2)$, which are in agreement with the theoretical prediction.

TABLE 1. Convergence rates with $k = 4, R_0 = 0.5, R = 1$.

h	$\|Q_h u - u_h\|_{1,h}$	rate	$\|Q_0 u - u_0\|$	rate	$\|Q_b u - u_b\|$	rate
0.44026	2.1074e+00	-	2.8850e−01	-	2.8634e−01	-
0.22518	5.3785e−01	2.04	3.5335e−02	3.13	3.3836e−02	3.19
0.11421	2.4113e−01	1.18	7.8663e−03	2.21	7.5125e−03	2.22
0.0576	1.1683e−01	1.06	1.9142e−03	2.06	1.8268e−03	2.07
0.028941	5.7881e−02	1.02	4.7537e−04	2.02	4.5358e−04	2.02
0.014508	2.9215e−02	0.99	1.1864e−04	2.01	1.1320e−04	2.02

Example 2. In this example, we take $f = 0$ and the exact solution is chosen as

$$u = J_\xi(k\sqrt{(x+1)^2 + y^2}) \cos\left(\xi \arctan\left(\frac{y}{x+1}\right)\right).$$

The boundary condition g are given by the exact solutions u for $\xi = 1, \xi = \frac{3}{2}, \xi = \frac{2}{3}$, respectively. Here, J_ξ denotes the Bessel function of the first kind and order ξ. It

TABLE 2. Convergence rates with $k = 10, R_0 = 0.05, R = 0.1$.

h	$\|Q_h u - u_h\|_{1,h}$	rate	$\|Q_0 u - u_0\|$	rate	$\|Q_b u - u_b\|$	rate
0.044026	5.0527e−02	-	6.9358e−04	-	7.3935e−05	-
0.022518	2.6375e−02	0.970	1.7201e−04	2.08	1.3435e−05	2.54
0.011421	1.3577e−02	0.978	4.2973e−05	2.04	2.8624e−06	2.28
0.00576	7.5221e−03	0.863	1.0743e−05	2.03	6.7314e−07	2.11
0.0028941	4.4173e−03	0.773	2.6858e−06	2.03	1.6458e−07	2.05
0.0014508	2.6887e−03	0.719	6.7146e−07	2.01	4.0790e−08	2.02

can be checked that u is smooth for $\xi \in \mathbb{N}$, while its derivative has a singularity at $(-1, 0)$ for $\xi \notin \mathbb{N}$.

With $k = 4, R_0 = 0.5, R = 1$, Tables 3–5 show the errors in H^1 norm and L^2 norm for $\xi = 1, \xi = \frac{3}{2}, \xi = \frac{2}{3}$, respectively. It can be seen that in all three cases, the numerical convergence rates in H^1 norm and L^2 norm for weak Galerkin solution are first and second orders.

TABLE 3. Convergence rates with $k = 4, R_0 = 0.5, R = 1, \xi = 1$.

h	$\|Q_h u - u_h\|_{1,h}$	rate	$\|Q_0 u - u_0\|$	rate	$\|Q_b u - u_b\|$	rate
0.44026	2.6242e+00	-	3.7813e−01	-	3.6757e−01	-
0.22518	6.9062e−01	1.99	4.7471e−02	3.10	3.8784e−02	3.35
0.11421	3.1223e−01	1.17	1.0517e−02	2.22	8.1035e−03	2.31
0.0576	1.5475e−01	1.03	2.5560e−03	2.07	1.9731e−03	2.09
0.028941	7.8362e−02	0.987	6.3457e−04	2.02	4.7883e−04	2.03
0.014508	4.0623e−02	0.951	1.5837e−04	2.01	1.1937e−04	2.01

TABLE 4. Convergence rates with $k = 4, R_0 = 0.5, R = 1, \xi = 3/2$.

h	$\|Q_h u - u_h\|_{1,h}$	rate	$\|Q_0 u - u_0\|$	rate	$\|Q_b u - u_b\|$	rate
0.44026	2.2505e+00	-	3.1144e−01	-	3.137e−01	-
0.22518	5.8287e−01	2.01	3.7614e−02	3.15	3.3020e−02	3.36
0.11421	2.6463e−01	1.16	8.2983e−03	2.23	6.9901e−03	2.29
0.0576	1.3128e−01	1.02	2.0152e−03	2.07	1.6793e−03	2.08
0.028941	6.6734e−02	0.983	5.0023e−04	2.02	4.1574e−04	2.03
0.014508	3.4729e−02	0.946	1.2484e−04	2.01	1.0368e−04	2.01

TABLE 5. Convergence rates with $k = 4, R_0 = 0.5, R = 1, \xi = 2/3$.

h	$\|Q_h u - u_h\|_{1,h}$	rate	$\|Q_0 u - u_0\|$	rate	$\|Q_b u - u_b\|$	rate
0.44026	2.9608e+00	-	4.3740e−01	-	4.1104e−01	-
0.22518	8.3163e−01	1.89	5.7760e−02	3.02	4.6698e−02	3.24
0.11421	3.8679e−01	1.13	1.2932e−02	2.20	1.0126e−02	2.25
0.0576	1.9697e−01	0.986	3.1597e−03	2.06	2.5104e−03	2.04
0.028941	1.0354e−01	0.934	7.8887e−04	2.02	6.5074e−04	1.96
0.014508	5.6068e−02	0.888	1.9846e−04	2.00	1.7321e−04	1.92

Example 3. In this study, we take $f = 0$ and the exact solution is chosen as

$$u = J_\xi(k\sqrt{(x+0.1)^2+y^2})\cos\left(\xi \arctan\left(\frac{y}{x+0.1}\right)\right).$$

The boundary condition g are given by the exact solutions u for $\xi = 1, \xi = 3/2, \xi = 2/3$. It can be checked that its derivative has a singularity at $(-0.1, 0)$ for $\xi \notin \mathbb{N}$.

With $k = 10, R_0 = 0.05, R = 0.1$, Tables 6–8 show the errors in H^1 norm and L^2 norm for $\xi = 1, \xi = \frac{3}{2}, \xi = \frac{2}{3}$, respectively. It shows that for $\xi = 1$, the numerical convergence rate in H^1 norm and L^2 norm for weak Galerkin solution is first and second order. For $\xi = \frac{3}{2}$, the rates of convergence decrease in H^1 norm and keeps second order in L^2 norm. However, the convergence rates decreases in both kinds of norms for the non-smooth case $\xi = \frac{2}{3}$.

Table 6. Convergence rates with $k = 10, R_0 = 0.05, R = 0.1, \xi = 1$.

h	$\|Q_hu - u_h\|_{1,h}$	rate	$\|Q_0u - u_0\|$	rate	$\|Q_bu - u_b\|$	rate
0.044026	8.3623e−02	-	1.1667e−03	-	2.7400e−04	-
0.022518	4.2343e−02	1.02	2.9151e−04	2.07	6.5145e−05	2.14
0.011421	2.1360e−02	1.01	7.2967e−05	2.04	1.5968e−05	2.07
0.00576	1.1018e−02	0.967	1.8250e−05	2.02	3.9645e−06	2.04
0.0028941	5.8388e−03	0.923	4.5632e−06	2.01	9.8855e−07	2.02
0.0014508	3.1888e−03	0.876	1.1408e−06	2.01	2.4687e−07	2.01

Table 7. Convergence rates with $k = 10, R_0 = 0.05, R = 0.1, \xi = 3/2$.

h	$\|Q_hu - u_h\|_{1,h}$	rate	$\|Q_0u - u_0\|$	rate	$\|Q_bu - u_b\|$	rate
0.044026	6.6068e−02	-	7.6287e−04	-	3.4584e−04	-
0.022518	3.6240e−02	0.896	1.9295e−04	2.05	9.1257e−05	1.99
0.011421	1.9134e−02	0.941	4.8475e−05	2.03	2.3363e−05	2.01
0.00576	1.1000e−02	0.809	1.2139e−05	2.02	5.8936e−06	2.01
0.0028941	6.7183e−03	0.716	3.0363e−06	2.01	1.4778e−06	2.01
0.0014508	4.2287e−03	0.670	7.5922e−07	2.01	3.6974e−07	2.01

Table 8. Convergence rates with $k = 10, R_0 = 0.05, R = 0.1, \xi = 2/3$.

h	$\|Q_hu - u_h\|_{1,h}$	rate	$\|Q_0u - u_0\|$	rate	$\|Q_bu - u_b\|$	rate
0.044026	1.4470e−01	-	1.4832e−03	-	1.0788e−03	-
0.022518	8.5593e−02	0.783	3.8020e−04	2.03	3.5780e−04	1.64
0.011421	5.1677e−02	0.743	9.9683e−05	1.97	1.1693e−04	1.65
0.00576	3.1754e−02	0.711	2.6730e−05	1.92	3.7795e−05	1.65
0.0028941	1.9740e−02	0.691	7.3538e−06	1.88	1.2112e−05	1.65
0.0014508	1.2354e−02	0.679	2.0788e−06	1.83	3.8582e−06	1.66

Example 4. In this numerical test, we will investigate the performance of weak Galerkin finite element method for the same setting as Example 1 with large wave number k. We solve the problem with $R_0 = 0.05, R = 1$ for four wave numbers $k = 5, k = 10, k = 50, k = 100$. The errors of weak Galerkin solution in H^1 norm and L^2 norm respective to k are shown in Figure 2 and Figure 3. It can be seen that the weak Galerkin finite element method are convergent for four cases.

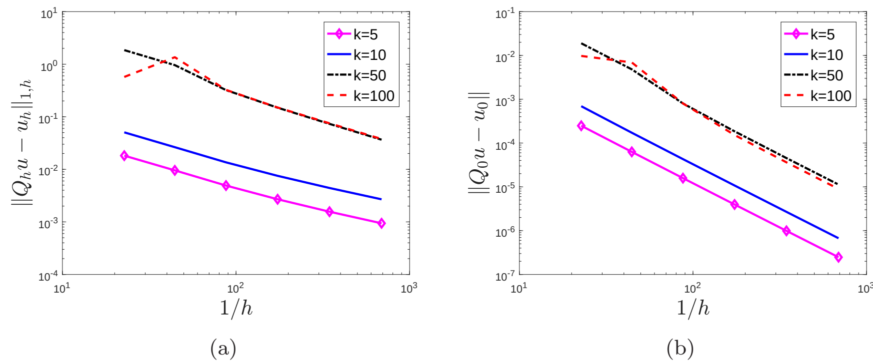

FIGURE 2. The errors for four wave numbers $k = 5, k = 10, k = 50, k = 100$. (a) Discrete H^1 norm. (b) Element-based L^2 norm.

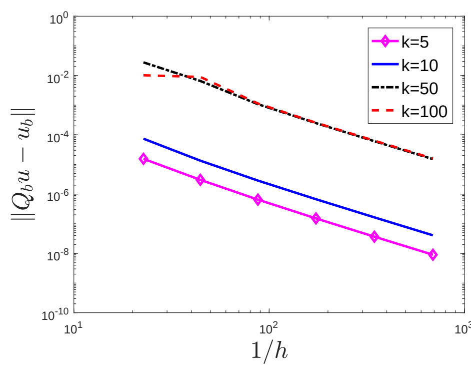

FIGURE 3. The errors in edge-based L^2 norm for four wave numbers $k = 5, k = 10, k = 50, k = 100$.

6. Conclusions

This paper introduces a weak Galerkin finite element method for the exterior Helmholtz problem. We reduce the original boundary value problem equivalently to a nonlocal boundary value problem by the exact Dirichlet to Neumann (DtN) boundary condition. In practice, one needs to truncate the infinite series of the exact DtN operator at a finite order. Thus, the error estimate include the effect of truncation of the DtN operator and the numerical discretization.

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