INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 17, Number 4, Pages 543–556

THE TIME HIGH-ORDER ENERGY-PRESERVING SCHEMES FOR THE NONLOCAL BENJAMIN-ONO EQUATION

CHUNGUANG CHEN, DONG LIANG*, AND SHUSEN XIE

Abstract. The new time high-order energy-preserving schemes are proposed for the nonlocal Benjamin-Ono equation. We get the Hamiltonian system to the nonlocal model, and it is then discretized by a Fourier pseudospectral method in space and the Hamiltonian boundary value method (HBVM) in time. This approach has high order of convergence in time and conserves the total mass and energy in discrete forms. We further develop a time second-order energy-preserving scheme and a time fourth-order energy-preserving scheme for the nonlocal Benjamin-Ono equation. Numerical experiments test the proposed schemes with a single solitary wave and the interaction of two solitary waves. Results confirm the accuracy and conservation properties of the schemes.

Key words. Nonlocal Benjamin-Ono equation, Hamiltonian boundary value method (HBVM), time high-order, energy preserving, Fourier pseudospectral method.

1. Introduction

Recently, there are more increasing interests in studying the problems of nonlocal partial differential equations in physics, mechanics, biology, materials science, and imaging science, etc. We consider the nonlocal Benjamin-Ono equation, which is a nonlocal partial differential equation arising in the study of long internal gravitation waves in deep stratified fluids and modelling the propagation of nonlinear dispersive waves ([3, 12, 14]).

The nonlocal Benjamin-Ono equation describes the remarkable properties of nonlinear dispersive wave propagation, that they permit stable, localized waveform solutions travelling at constant speeds, called solitary waves [11]. When two solitary waves overtake each other, they emerge from the interaction without any changes in shape and speed. James and Weideman [13] proposed a pseudospectral method for the Benjamin-Ono equation by the Hilbert transform, which is a convolution, reduces to a product under the spectral discretization. Boyd and Xu [9] compared three pseudospectral methods based on the Fourier, radial basis and rational orthogonal basis functions for the Benjamin-Ono equation and obtained exponential convergence in space. Thomee and Murthy [15] solved the Benjamin-Ono equation by a finite difference approximations in space and the Crank-Nicolson approximation in time. This approach has the accuracy order $O(h^2 + \Delta t^2)$. Although the spectral methods are commonly used to solve the Benjamin-Ono equation, they do not conserve the physical invariants if the system is integrated in time by non-conservative integrators such as the standard Runge-Kutta methods or multi-step methods. As a result, dissipative errors will be introduced and the shapes and speeds of solitary solutions will change in numerical simulations. Therefore, it is very important and dificult to develop time high-order energy-preserving numerical schemes to the nonlocal Benjamin-Ono equation.

Brugnano and Iavernaro et al [6, 7, 8] proposed a class of structure-conserved method, namely the Hamiltonian boundary value methods (HBVMs) that yield

Received by the editors October 1, 2019.
2000 *Mathematics Subject Classification.* 65N06, 65N12, 65M06, 65Y05, 76R05.
*Corresponding author: Dong Liang; E-mail: *dliang@mathstat.yorku.ca.*

the conservation for Hamiltonian invariants represented by polynomial functional of arbitrarily high-degree. Moreover, the methods are shown to be symmetric, precisely A-stable, and can have arbitrarily high-order accuracy. The methods have been extensively applied to simulate Hamiltonian partial differential equations, such as the semilinear wave equation [4], the nonlinear Schrödinger equation [2], the KdV equation [5] and the modified KdV equation [16]. However, to our best knowledge, the HBVMs have not been applied to approximate any *nonlocal* dispersive partial differential equations, such as the nonlocal Benjamin-Ono equation.

In this paper, we develop time high-order energy-preserving schemes for the nonlocal Benjamin-Ono equation. We first get the Hamiltonian system to the nonlocal model of Benjamin-Ono equation. We then discretize the nonlocal Benjamin-Ono equation in space by the Fourier pseudospectral method. We show that the resulting semi-discrete system can be written as a Hamiltonian system. We integrate the corresponding discrete Hamiltonian system with the HBVM approach to obtain a time second-order scheme and a time fourth-order scheme, both preserving the mass and energy indiscrete forms. Numerical experiments are given to show the preserving properties and convergence orders of the schemes and to show the physical phenomenon of the interaction of solitary waves of the nonlocal models.

This paper is organized as follows. In Section 2, we present the nonlocal model of Benjamin-Ono equation and derive out its Hamiltonian system. In Section 3, we derive the Runge-Kutta formulation of the HBVMs. In Section 4, we introduce the basic properties of the Fourier pseudospectral method and obtain the time second-order and time fourth-order energy-preserving schemes. We show numerical experiments in Section 5 and some conclusions are addressed in Section 6.

2. Nonlocal model of Benjamin-Ono equation and its Hamiltonian system

Consider the nonlocal model of the Benjamin-Ono equation [15]

$$\begin{cases} u_t + uu_x - Hu_{xx} = 0, \quad x \in [-L, L], t \in [0, +\infty), \\ u(x, 0) = u_0(x), \quad x \in [-L, L], \end{cases} \tag{1}$$

with $u(x + 2L, t) = u(x, t)$ and H is the Hilbert transform defined by

$$\begin{aligned} Hu(x) &= \frac{1}{\pi} P.V. \int_{-\infty}^{+\infty} \frac{u(x-y)}{y} dy \\ &= \frac{1}{2L} P.V. \int_{-L}^{L} \cot(\frac{\pi}{2L} y) u(x-y) dy \end{aligned} \tag{2}$$

for the periodic function $u(x)$. For details of the periodic Hilbert transform and theoretical analysis of (1), we refer to [15] and the references therein. It can be shown that the periodic problem (1) has many invariants, such as

$$\begin{aligned} \mathcal{M} &= \int_{-L}^{L} u dx, \\ \mathcal{I} &= \frac{1}{2} \int_{-L}^{L} u^2 dx, \\ \mathcal{H} &= \frac{1}{6} \int_{-L}^{L} \left[u^3 + 3u_x H(u) \right] dx. \end{aligned} \tag{3}$$

These invariants are usually referred as mass, momentum and energy, respectively.

Let us define the Hamiltonian functional as

$$\mathcal{H}(u)(t) = \frac{1}{6}\int_{-L}^{L} \left[u^3 + 3u_x H(u)\right] dx. \tag{4}$$

Then, we can derive the infinite-dimensional Hamiltonian system which is equivalent to the nonlocal model of Benjamin-Ono equation (1) as

$$\frac{du}{dt} = -\frac{\partial}{\partial x}\frac{\delta \mathcal{H}}{\delta u}. \tag{5}$$

In the following sections, we will develop the time high-order energy preserving schemes to the infinite-dimensional Hamiltonian system (5) with the aid of HBVMs, for solving the nonlocal Benjamin-Ono equation.

3. Preliminaries

In this section, we introduce the frame work of Hamiltonian boundary value methods (HBVMs). Let

$$\frac{dy}{dt} = J\nabla \mathcal{H}(y), \quad y(0) = y_0 \in R^{2m} \tag{6}$$

be a Hamiltonian problem in canonical form, where $J^T = -J$ is a skew-symmetric matrix. The scalar functional $\mathcal{H}(y)$ is the Hamiltonian whose value maintains constant during the motion, namely

$$\mathcal{H}(y(t)) \equiv \mathcal{H}(y(0)), \qquad \forall t \geq 0,$$

for the solution of (6) . Indeed, one has

$$\frac{d\mathcal{H}}{dt} = \nabla \mathcal{H}(y(t))^T y'(t) = \nabla \mathcal{H}(y)^T J \nabla \mathcal{H}(y) = 0, \qquad \forall t \geq 0. \tag{7}$$

In a mechanical system, $\mathcal{H}(y)$ is usually the total energy. So it is crucial to preserve $\mathcal{H}(y)$ in simulations of these problems. Assume that the Hamiltonian in (6) is a polynomial of degree ν. Starting from the initial condition y_0, we want to approximate the solution at $t = \Delta t$, say $y_1 \approx y(\Delta t)$, through a suitable path σ such that

$$\sigma(0) = y_0, \quad \sigma(\Delta t) = y_1, \quad \mathcal{H}(y_1) = \mathcal{H}(y_0).$$

We consider a polynomial path σ of degree $s \geq 1$ and denote by Π_{s-1} the set of polynomials of degree $s-1$, whose base is $\{P_0, P_1, \cdots, P_{s-1}\}$. One can expand the derivative of σ as

$$\sigma'(c\Delta t) = \sum_{l=0}^{s-1} \gamma_l P_l(c), \qquad c \in [0,1]. \tag{8}$$

By imposing the initial condition $\sigma(0) = y_0$, one can formally obtain that

$$\sigma(c\Delta t) = y_0 + \Delta t \sum_{l=0}^{s-1} \int_0^c P_l(x)dx\gamma_l, \qquad c \in [0,1]. \tag{9}$$

And $y_1 = \sigma(\Delta t)$ is an approximation of $y(\Delta t)$. Then energy conservation may be obtained by the following computation, namely,

$$\begin{aligned}\mathcal{H}(y_1) - \mathcal{H}(y_0) &= \mathcal{H}(\sigma(\Delta t)) - \mathcal{H}(\sigma(0)) \\ &= \int_0^{\Delta t} \nabla\mathcal{H}(\sigma(t))^T \sigma'(t) dt \\ &= \Delta t \int_0^1 \nabla\mathcal{H}(\sigma(c\Delta t))^T \sigma'(c\Delta t) dc \\ &= \Delta t \int_0^1 \nabla\mathcal{H}(\sigma(c\Delta t))^T \sum_{l=0}^{s-1} \gamma_l P_l(c) dc \\ &= \Delta t \sum_{l=0}^{s-1} \left[\int_0^1 \nabla\mathcal{H}(\sigma(c\Delta t))^T P_l(c) dc\right]^T \gamma_l \\ &= 0,\end{aligned}$$

provided that

$$\gamma_l = \eta_l J \int_0^1 \nabla\mathcal{H}(\sigma(c\Delta t)) P_l(c) dc \tag{10}$$

for a suitable set of nonzero scalars $\eta_0, \eta_1, \cdots, \eta_{s-1}$. If we take the shifted and scaled Legendre polynomials as the base of Π_{s-1}, all the coefficients η_l are equal to 1. In such a case, we have

$$P_0(x) \equiv 1, \quad \int_0^1 P_l(x) dx = 0, l = 1, 2, \cdots, s-1.$$

Hence, (10) can be written as

$$\gamma_l = J \int_0^1 \nabla\mathcal{H}(\sigma(c\Delta t)) P_l(c) dc. \tag{11}$$

By setting, hereafter,

$$f(\cdot) = J\nabla\mathcal{H}(\cdot),$$

the new approximation is then given by plugging (11) into (9):

$$\sigma(c\Delta t) = y_0 + \Delta t \sum_{l=0}^{s-1} \int_0^c P_l(x) dx \int_0^1 f(\sigma(\tau\Delta t)) P_l(\tau) d\tau, \qquad c \in [0, 1]. \tag{12}$$

In particular, for $c = 1$, we have

$$\begin{aligned} y_1 = \sigma(\Delta t) &= y_0 + \Delta t \sum_{l=0}^{s-1} \int_0^1 P_l(x) dx \int_0^1 f(\sigma(\tau\Delta t)) P_l(\tau) d\tau \\ &= y_0 + \Delta t \int_0^1 P_0(x) dx \int_0^1 f(\sigma(\tau\Delta t)) P_0(\tau) d\tau \\ &= y_0 + \Delta t \int_0^1 f(\sigma(\tau\Delta t)) d\tau. \end{aligned} \tag{13}$$

The integrals in (12) and (13) have at most degree $(\nu - 1)s + s - 1 \equiv \nu s - 1$. Therefore, by fixing a suitable set of k abscissae $0 \le c_1 < \cdots < c_k \le 1$, and corresponding quadrature weights $\{b_1, b_2, \cdots, b_k\}$, such that the resulting quadrature

formula is exact for polynomials of degree $\nu s-1$, the integrals (12) and (13) may be replaced by the corresponding quadrature formula, which yields

$$\sigma(c\Delta t)=y_0+\Delta t\sum_{l=0}^{s-1}\int_0^c P_l(x)dx\sum_{j=1}^{k}b_j f(\sigma(c_j\Delta t))P_l(c_j),\qquad c\in[0,1], \tag{14}$$

and

$$y_1=y_0+\Delta t\sum_{j=1}^{k}b_j f(\sigma(c_j\Delta t)). \tag{15}$$

Hence, by setting

$$Y_i=\sigma(c_i\Delta t),\quad i=1,...,k,$$

we have by (14) that

$$\begin{aligned} Y_i &= y_0+\Delta t\sum_{j=1}^{k}\overbrace{\left[b_j\sum_{l=0}^{s-1}P_l(c_j)\int_0^{c_i}P_l(x)dx\right]}^{=a_{ij}}f(Y_j)\\ &\equiv y_0+\Delta t\sum_{j=1}^{k}a_{ij}f(Y_j),\quad i=1,...,k, \end{aligned} \tag{16}$$

and

$$y_1=y_0+\Delta t\sum_{j=1}^{k}b_j f(Y_j). \tag{17}$$

In this way, energy conservation can always be achieved provided that the quadrature has a suitable high order. For example, we can place the k abscissae $\{c_i\}$ at the k Gauss-Legendre nodes on $[0,1]$ thus the quadrature is exact for polynomials of degree $2k-1$. In such a case, energy conservation is guaranteed for polynomial Hamiltonian of degree ν satisfying

$$\nu\le\frac{2k}{s}.$$

The method (16) and (17) defines a Hamiltonian boundary value method [7] with k stages and degree s, in short, HBVM(k,s). According to [8], the HBVM has the following properties:

Theorem 1. *For all $s=1,2,\cdots$, and $k\ge s$, the HBVM(k,s)*

1. *has order of accuracy $2s$;*
2. *is energy-preserving for polynomial Hamiltonians of degree no larger than $2k/s$;*
3. *for general C^{2k+2} Hamiltonians, the energy error at each integration step is $O(\Delta t^{2k+1})$, if Δt is the step size;*
4. *is symmetric and, therefore, precisely A-stable.*

4. Energy-preserving scheme for the Benjamin-Ono equation

We propose a time second-order energy-preserving scheme and a time fourth-order energy-preserving scheme for the problem (1), which can be written as the infinite-dimensional Hamiltonian system

$$\frac{du}{dt}=-\frac{\partial}{\partial x}\frac{\delta\mathcal{H}}{\delta u}, \tag{18}$$

where the Hamiltonian functional is defined by

$$\mathcal{H}(u)(t) = \frac{1}{6}\int_{-L}^{L}\left[u^3 + 3u_x H(u)\right]dx. \tag{19}$$

4.1. Fourier pseudospectral method for the Benjamin-Ono equation. To illustrate the main ideas of Fourier pseudospectral space discretization for the periodic Benjamin-Ono equation, we start by introducing some basic properties of the standard Fourier pseudospectral method. The interval $[-L, L]$ is divided into $2N$ equal subintervals with the length of each $h = L/N$. And we construct the following interpolation space by means of the collocation points $x_j = jh, \quad j = -N, \cdots, N-1$,

$$S_h = Span\{g_j(x) : -N \le j \le N-1\},$$

where $g_j(x)$ is a trigonometric polynomial of degree N, and $g_j(x_k) = \delta_j^k$. Actually, $g_j(x)$ can be written as

$$g_j(x) = \frac{1}{2N}\sum_{l=-N}^{N}\frac{1}{c_l}e^{il\omega(x-x_j)}, \quad j = -N, \cdots, N \tag{20}$$

with

$$c_l = \begin{cases} 1, & l = -N+1, \cdots, N-1, \\ 2, & l = -N, N, \end{cases}$$

and $\omega = \pi/L$. Let I_h be the interpolation operator

$$I_h u(x,t) = \sum_{j=-N}^{N} u_j g_j(x) \tag{21}$$

and we have $I_h u(x_k, t) = \sum_{j=-N}^{N} u_j g_j(x_k) = u_k$.

The values of the derivatives $\frac{d}{dx}I_h u(x)$ at the collocation points x_j can be obtained by the values of u_j and the differential matrices D_1, i.e.,

$$\frac{d}{dx}I_h u(x)|_{x=x_j} = (D_1 u)_j, \tag{22}$$

where D_1 represents the first-order Fourier differential matrix with the elements

$$(D_1)_{m,n} = \begin{cases} \frac{\omega}{2}(-1)^{m+n}\cot(\omega\frac{x_m - x_n}{2}), & m \ne n, \\ 0, & m = n. \end{cases} \tag{23}$$

In what follows, we shall derive the matrix of the Hilbert transform of $I_h u(x)$ at the collocation points. First of all, by (21) and (20), we have

$$HI_h u(x) = \sum_{j=-N}^{N} u_j H(g_j(x))$$

and

$$H(g_j(x)) = \frac{1}{2N}\sum_{l=-N}^{N}\frac{1}{c_l}H(e^{il\omega(x-x_j)}).$$

Noticing that $e^{il\omega(x+2L-x_j)} = e^{il\omega(x-x_j)}$, we will use indiscriminately the Hilbert transforms defined by (2) in the following discussion as they are equivalent for any $2L$-periodic functions.

Let $F(\cdot)$ and $F^{-1}(\cdot)$ represent the Fourier transform and its inverse, respectively:

$$\hat{v}(\lambda) = F[v(x)] = \int_{-\infty}^{+\infty} v(x)e^{-i\lambda x}dx,$$
$$v(x) = F^{-1}[\hat{v}(\lambda)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{v}(\lambda)e^{i\lambda x}d\lambda.$$

When $l = 0$, $H(e^{il\omega(x-x_j)}) = H(1) = 0$. When $l \neq 0$,

$$\begin{aligned} H(e^{il\omega(x-x_j)}) &= F^{-1}\left[F[H(e^{il\omega(x-x_j)})]\right] \\ &= F^{-1}\left[F[\frac{1}{\pi y}]\cdot F[(e^{il\omega(x-x_j)})]\right] \\ &= F^{-1}\left[-i\text{sign}(\lambda)\cdot F[(e^{il\omega(x-x_j)})]\right] \\ &= F^{-1}\left[-i\text{sign}(\lambda)\cdot 2\pi\delta(\lambda-l\omega)e^{-il\omega x_j}]\right] \\ &= -ie^{-il\omega x_j}\int_{-\infty}^{+\infty}\delta(\lambda-l\omega)\text{sign}(\lambda)e^{i\lambda x}d\lambda. \end{aligned}$$

By the fact that

$$\begin{aligned} &\int_{-\infty}^{+\infty}\delta(\lambda-l\omega)\text{sign}(\lambda)e^{i\lambda x}d\lambda \\ =&-\int_{-\infty}^{0}\delta(\lambda-l\omega)e^{i\lambda x}d\lambda+\int_{0}^{+\infty}\delta(\lambda-l\omega)e^{i\lambda x}d\lambda \\ =&\begin{cases} e^{il\omega x}, & l=1,\cdots,N, \\ -e^{il\omega x}, & l=-N,\cdots,-1, \end{cases} \end{aligned}$$

it can be shown that

$$\begin{aligned} H(g_j(x)) &= -\frac{i}{2N}\left[-\sum_{l=-N}^{-1}\frac{1}{c_l}e^{il\omega(x-x_j)}+\sum_{l=1}^{N}\frac{1}{c_l}e^{il\omega(x-x_j)}\right] \\ &= -\frac{i}{2N}\left[-\sum_{l=N}^{1}\frac{1}{c_{-l}}e^{-il\omega(x-x_j)}+\sum_{l=1}^{N}\frac{1}{c_l}e^{il\omega(x-x_j)}\right]. \end{aligned}$$

Noticing that $c_l = c_{-l}$, we have

$$\begin{aligned} H(g_j(x)) &= -\frac{i}{2N}\sum_{l=1}^{N}\frac{1}{c_l}\left[e^{il\omega(x-x_j)}-e^{-il\omega(x-x_j)}\right] \\ &= \frac{1}{N}\sum_{l=1}^{N}\frac{1}{c_l}\sin(l\omega(x-x_j)). \end{aligned} \tag{24}$$

Therefore,

$$H(g_j(x_k)) = \frac{1}{N}\sum_{l=1}^{N-1}\sin\frac{l\pi(k-j)}{N},$$

where the term with $l = N$ in (24) vanishes. Noticing that

$$\sum_{l=1}^{N-1}\sin(l\theta) = -\frac{1}{2\sin\frac{\theta}{2}}\left[\cos(N-\frac{1}{2})\theta-\cos\frac{\theta}{2}\right]$$

and

$$\cos(N-\frac{1}{2})\frac{\pi(k-j)}{N} = \cos\left[(k-j)\pi - \frac{(k-j)\pi}{2N}\right]$$
$$= \begin{cases} \cos\frac{(k-j)\pi}{2N}, & if \quad k-j \quad is \quad even, \\ -\cos\frac{(k-j)\pi}{2N}, & if \quad k-j \quad is \quad odd, \end{cases}$$

we obtain that

$$H(g_j(x_k)) = \begin{cases} 0, & if \quad k-j \quad is \quad even, \\ \frac{1}{N}\cot\frac{(k-j)\pi}{2N}, & if \quad k-j \quad is \quad odd, \end{cases}$$

and

$$HI_h u(x)\big|_{x=x_k} = \sum_{j=-N}^{N} u_j H(g_j(x))\big|_{x=x_k}$$
$$= \begin{cases} \frac{1}{N}\sum\limits_{j=-N,odd}^{N-1} u_j \cot\frac{(k-j)\pi}{2N}, & if \quad k \quad is \quad even, \\ \frac{1}{N}\sum\limits_{j=-N,even}^{N-1} u_j \cot\frac{(k-j)\pi}{2N}, & if \quad k \quad is \quad odd, \end{cases} \tag{25}$$

where $k = -N, -N+1, \cdots, N-1$.

Let

$$a_k = \begin{cases} \frac{1}{N}\cot\frac{k\pi}{2N}, & if \quad k \quad is \quad odd, \\ 0, & if \quad k \quad is \quad even, \end{cases}$$

and define the following $2N \times 2N$ matrix

$$H = \begin{bmatrix} a_0 & a_{-1} & \cdots & a_{-N} & \cdots & a_{-2N+1} \\ a_1 & a_0 & \cdots & a_{-N+1} & \cdots & a_{-2N+2} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{2N-1} & a_{2N-2} & \cdots & a_{N-1} & \cdots & a_0 \end{bmatrix}$$
$$= \begin{bmatrix} a_0 & a_{-1} & \cdots & a_{-N} & \cdots & a_1 \\ a_1 & a_0 & \cdots & a_{-N+1} & \cdots & a_2 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{-1} & a_{-2} & \cdots & a_{-N} & \cdots & a_0 \end{bmatrix},$$

where the last equality holds since $a_k = a_{k+2N}$ and H is actually a circulant and skew-symmetric matrix. The Hilbert transform of $I_h u(x)$ at the collocation points (25) can be expressed as

$$HI_h u(x)\big|_{x=x_k} = (H \cdot \mathbf{u})_k,$$

where $k = -N, -N+1, \cdots N-1$ and $\mathbf{u} = \{u_k\}_{k=-N}^{N-1}$.

Remark 1. *This expression is exactly the same with the discrete Hilbert transform obtained by the mid-point rule quadrature in* [15].

We approximate $u(x,t)$ by $I_h u(x,t)$, which interpolate $u(x,t)$ at the collocation points $x_j, j = -N, \cdots, N-1$. The semi-discrete Fourier pseudospectral approximation is constructed as follows: find $I_h u(x,t) \in S_h$ such that

$$\left[\left(I_h u(x,t)\right)_t + \left(I_h u(x,t)\right)\left(I_h u(x,t)\right)_x - H\left(I_h u(x,t)\right)_{xx}\right]\Big|_{x=x_j} = 0, \tag{26}$$

where $j = -N, -N+1, \cdots, N-1$. Substituting the spectral differentiation matrix D_1 and the Hilbert transform matrix H into (26), we obtain the semi-discrete system

$$\frac{d\mathbf{u}}{dt} + \frac{1}{2}D_1\mathbf{u}^2 - D_1^2 H\mathbf{u} = 0 \tag{27}$$

and the approximation of the Hamiltonian (3)

$$\mathbf{H} = \frac{h}{6}\left[\sum_{j=-N}^{N-1} u_j^3 + 3\mathbf{u}^T(D_1^T H)\mathbf{u}\right], \tag{28}$$

where D_1 and H are skew-symmetric and commutable matrices.

Problem (27) is Hamiltonian as it can be written as

$$\frac{d\mathbf{u}}{dt} = J_h \nabla \mathbf{H}(\mathbf{u}), \tag{29}$$

where $J_h = -\frac{1}{h}D_1$ is skew-symmetric. Consequently, we derive the following energy-preserving schemes by applying the HBVMs to the system (29).

4.2. The high-order energy preserving schemes. The time second-order energy-preserving scheme for (1) takes the form

$$\begin{cases} \mathbf{K}_i = \mathbf{U}^n + \Delta t \sum_{j=1}^{2} a_{ij} f(\mathbf{K}_j), \quad i = 1, 2, \\ \mathbf{U}^{n+1} = \mathbf{U}^n + \Delta t \sum_{i=1}^{2} b_i f(\mathbf{K}_i), \end{cases} \tag{30}$$

and the time fourth-order scheme takes the form

$$\begin{cases} \mathbf{K}_i = \mathbf{U}^n + \Delta t \sum_{j=1}^{3} a_{ij} f(\mathbf{K}_j), \quad i = 1, 2, 3, \\ \mathbf{U}^{n+1} = \mathbf{U}^n + \Delta t \sum_{i=1}^{3} b_i f(\mathbf{K}_i), \end{cases} \tag{31}$$

where $f(u) = -D_1 \nabla \mathbf{H}(u)$.

In other words, we have defined a $k-$stage Runge-Kutta Method as follows:

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} = (a_{ij}) \\ \hline & \mathbf{b}^T \end{array}$$

where $\mathbf{c} = (c_1, c_2, \cdots, c_k)^T$, $\mathbf{b} = (b_1, b_2, \cdots, b_k)^T$ and $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{k\times k}$.

For example, when $s = 1,\ k = 2$,

$$\mathbf{c} = \left[\frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right), \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right)\right]^T,$$

$$\mathbf{b} = \left[\frac{1}{2}, \frac{1}{2}\right]^T,$$

and

$$A = \begin{bmatrix} \frac{1}{4}\left(1 - \frac{1}{\sqrt{3}}\right) & \frac{1}{4}\left(1 - \frac{1}{\sqrt{3}}\right) \\ \frac{1}{4}\left(1 + \frac{1}{\sqrt{3}}\right) & \frac{1}{4}\left(1 + \frac{1}{\sqrt{3}}\right) \end{bmatrix}.$$

When $s = 2,\ k = 3$,

$$\mathbf{c} = \left[\frac{1}{2}\left(1 - \sqrt{\frac{3}{5}}\right), \frac{1}{2}, \frac{1}{2}\left(1 + \sqrt{\frac{3}{5}}\right)\right]^T,$$

$$\mathbf{b} = \left[\frac{5}{18}, \frac{4}{9}, \frac{5}{18}\right]^T,$$

and

$$A = \begin{bmatrix} 0.0959 & 0.0501 & -0.0332 \\ 0.3003 & 0.2222 & -0.0225 \\ 0.3110 & 0.3944 & 0.1819 \end{bmatrix}.$$

The proposed schemes (30) and (31) correspond to $s = 1$ and $s = 2$, respectively. Thus, according to Theorem 1 the schemes (30) and (31) are second-order and fourth-order in time, respectively, and will be referred to as HBVM(2,1) and HBVM(3,2) in the following section. Both schemes exactly preserve the mass and the energy in the discrete level. Actually, the Hamiltonian (28) is a polynomial with degree $\nu = 3$, therefore, according to Theorem 1, the proposed schemes are energy-preserving when $3s \le 2k$.

5. Numerical Experiments

In this section, we approximate the motion of a solitary wave and the interaction of two solitary waves. These numerical tests show that the proposed schemes are effective in simulating the nonlocal Benjamin-Ono equation, and they conserve both the mass and energy in the discrete forms. To estimate the accuracy of the methods, we introduce the following errors in different norms:

$$\begin{aligned} L_1 &= \sum_{j=-N}^{N-1} |u_h(x_j) - u(x_j)|h, \\ L_2 &= \left(\sum_{j=-N}^{N-1} |u_h(x_j) - u(x_j)|^2 h\right)^{1/2}, \\ L_\infty &= \max_{-N \le j \le N-1} |u_h(x_j) - u(x_j)|, \end{aligned} \tag{32}$$

where u_h is the numerical solution with space step h and u is the exact solution. The discrete versions of the invariants (3) at time $t_n = n\Delta t$ are given by

$$\begin{aligned} M_d(U^n) &= h \sum_{j=-N}^{N-1} U_k^n, \\ K_d(U^n) &= \frac{h}{2} \sum_{j=-N}^{N-1} (U_k^n)^2, \\ H_d(U^n) &= \frac{h}{6}\left[\sum_{j=-N}^{N-1} (U_k^n)^3 - 3(U^n)^T (D_1 H) U^n\right]. \end{aligned} \tag{33}$$

5.1. Single solitary wave. The periodic problem (1) has soliton solutions [15], e.g., for c arbitrary,

$$u(x,t) = \frac{2c\delta^2}{1 - \sqrt{1-\delta^2}\cos(c\delta(x - ct - x_0))}, \quad \text{with} \quad \delta = \frac{\pi}{cL}. \tag{34}$$

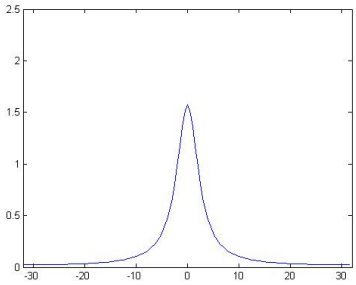

FIGURE 1. Solitary wave at $t_0 = 0$ with $c = 0.4$.

TABLE 1. The L_∞, L_2 and L_1 errors and convergence orders in time for the second-order energy-preserving scheme.

Δt	L_∞	Order	L_2	Order	L_1	Order
1/2	3.501050e-04	-	5.942609e-04	-	1.624535e-03	-
1/4	8.751689e-05	2.000154	1.488174e-04	1.997554	4.090576e-04	1.989651
1/8	2.187900e-05	2.000015	3.721994e-05	1.999395	1.027569e-04	1.993069
1/16	5.469744e-06	2.000001	9.305985e-06	1.999845	2.571423e-05	1.998596
1/32	1.367436e-06	2.000000	2.326559e-06	1.999961	6.429772e-06	1.999727

TABLE 2. The L_∞, L_2 and L_1 errors and convergence orders in time for the fourth-order energy-preserving scheme.

Δt	L_∞	Order	L_2	Order	L_1	Order
1/2	1.908176e-06	-	2.672955e-06	-	6.918758e-06	-
1/4	1.058457e-07	4.172160	1.512783e-07	4.143158	4.143091e-07	4.061734
1/8	6.422192e-09	4.042753	9.289610e-09	4.025444	2.625850e-08	3.979851
1/16	3.987093e-10	4.009657	5.781807e-10	4.006026	1.646343e-09	3.995448
1/32	2.487455e-11	4.002622	3.609793e-11	4.001525	1.029830e-10	3.998738

In order to estimate the errors of the schemes, we take $c = 0.4$ and compute the solution of (1) with period $2L = 64$ over $[-32, 32]$. The initial state of the solitary wave is shown in Fig 1.

Table 1 and Table 2 list the L_∞, L_2 and L_1 errors and convergence orders in time for the second-order and the fourth-order energy-preserving schemes. We take $h = 1/4$ and therefore the errors due to space discretization is negligible. The numerical results clearly show that the proposed scheme can reach second-order and fourth-order accuracy in time, respectively, that coincide with the Theorem1.

We perform a simulation with space step $h = 1/4$ and time step $\Delta t = 0.1$ for a single solitary wave over the interval $[-32, 32]$. Fig 2 (left) shows that the solitary wave moves to the right at a constant speed and unchanged amplitude as is expected. We also calculate the relative errors of the invariants evaluated along the numerical solution with respect to the conserved quantities at $t = 0$. Fig 2 (right) shows that the time second-order scheme preserves the discrete mass and energy to within the machine precision. The numerical results coincide with Theorem1.

Concerning the conservation of the invariants in long-term simulations, we approximate the single solitary wave with the time fourth-order scheme with space step $h = 1/4$ and time step $\Delta t = 0.1$ on the time interval $[0, 1000]$. Fig 3 (left) shows that the solitary wave propagates with constant speed without changing its amplitude. And Fig 3 (right) indicates that the scheme preserves the discrete mass and energy to within the machine precision.

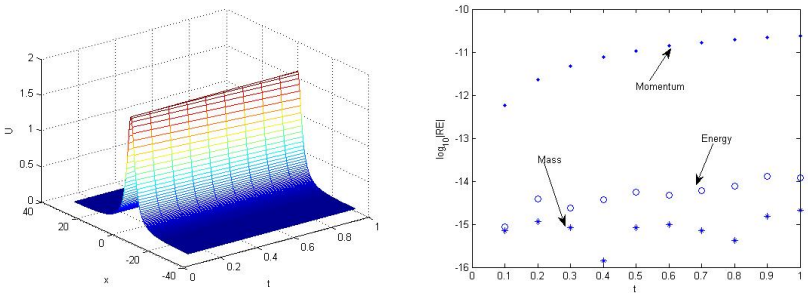

FIGURE 2. *Left*: Numerical solution of HBVM(2,1) with $c = 0.4$, $x_0 = 0$, $h = 0.25$, $\Delta t = 0.1$ and $T = 1$. *Right*: The relative errors of the invariants.

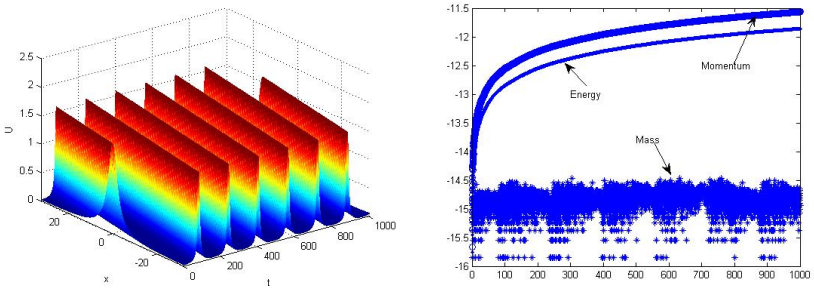

FIGURE 3. *Left*: Numerical solution of HBVM(3,2) with $c = 0.4$, $x_0 = 0$, $h = 0.25$, $\Delta t = 0.1$ and $T = 1000$. *Right*: The relative errors of the invariants.

5.2. Interaction of two solitary waves. We consider the solution involving two solitary waves. The exact solution is given by

$$u(x,t) = \frac{2c_1\delta_1^2}{1-\sqrt{1-\delta_1^2}\cos(c\delta_1(x-c_1t-x_{01}))} + \frac{2c_1\delta_2^2}{1-\sqrt{1-\delta_2^2}\cos(c_2\delta(x-c_2t-x_{02}))},$$

where $c_1 = 0.8, c_2 = 0.25, x_{01} = -20, x_{02} = -5$ and $\delta_i = \pi/(c_i L), \quad i = 1, 2$. We simulate the problem by the fourth-order scheme (31) over the region $[-32, 32]$ up to time $T = 40$ with step size $h = 0.25$ and $\Delta t = 0.1$. Fig 4 shows that the numerical simulations at four different moments. The taller solitary wave was initially located at the left and catch up with the shorter wave at around $t = 15$. The two waves started to separate about $t = 30$ and the taller wave overtook the shorter one and travelled to the right after $t = 40$. The interaction between the two solitary waves can also be observed from Fig 5 (*left*) and the scheme preserves the mass and the energy to within the machine precision.

6. Conclusion

In this paper, we proposed the new time high-order energy-preserving schemes for the nonlocal Benjamin-Ono equation. We discretize the equation by a Fourier

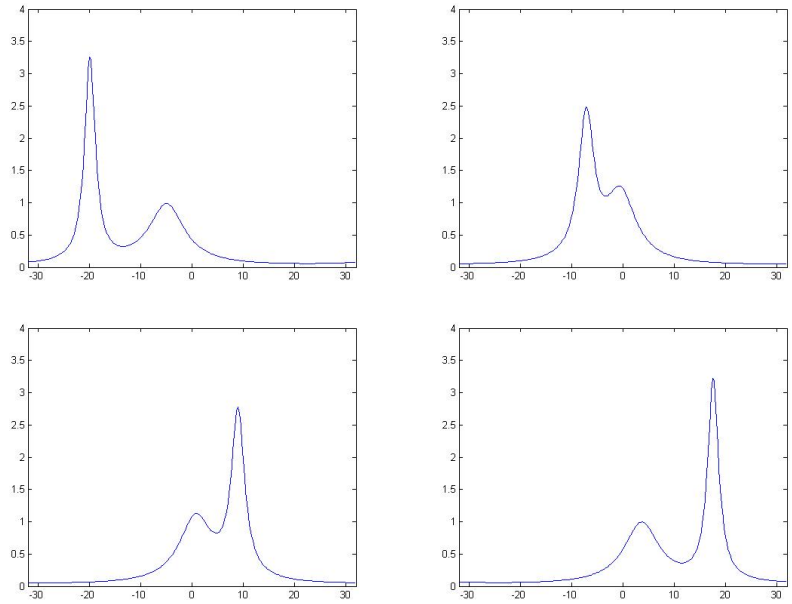

FIGURE 4. Numerical simulations of the collision of two solitary waves by HBVM(3,2) with $c_1 = 0.8, c_2 = 0.25, x_{01} = -20, x_{02} = -5, h = 0.25$ and $\Delta t = 0.1$ at $T = 0$ (*upper left*), $T = 15$ (*upper right*), $T = 30$ (*lower left*) and $T = 40$ (*lower right*).

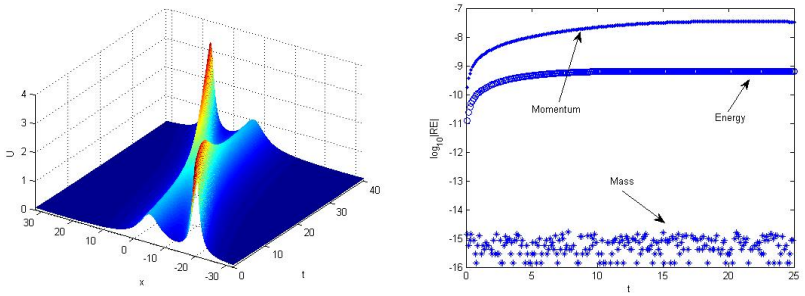

FIGURE 5. *Left*: Numerical solution of HBVM(3,2) with $c_1 = 0.8, c_2 = 0.25, x_{01} = -20, x_{02} = -5, h = 0.25$ and $\Delta t = 0.1$. *Right*: The relative errors of the invariants.

pseudospectral method in space and apply the HBVM in time to the discrete Hamiltonian system. We tested the proposed schemes to be second-order and fourth-order in time. Our numerical examples also showed that the two schemes are stable in simulations of solitary waves and conserve mass and energy in the discrete forms.

Acknowledgement

This work was supported partially by Fundamental Research Funds for the Central Universities and by Natural Sciences and Engineering Research Council of Canada.

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School of Mathematical Sciences, Ocean University of China, Qingdao, Shandong, 266100, China
E-mail: cgchen@ouc.edu.cn

Department of Mathematics and Statistics, York University, Toronto, Ontario, M3J 1P3, Canada
E-mail: dliang@mathstat.yorku.ca

School of Mathematical Sciences, Ocean University of China, Qingdao, Shandong, 266100, China
E-mail: shusenxie@ouc.edu.cn