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ERROR ESTIMATES IN BALANCED NORMS OF FINITE ELEMENT METHODS FOR HIGHER ORDER REACTION-DIFFUSION PROBLEMS

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Abstract. Error estimates of finite element methods for reaction-diffusion problems are often realised in the related energy norm. In the singularly perturbed case, however, this norm is not adequate. A different scaling of the H^m seminorm for $2m$-th order problems leads to a balanced norm which reflects the layer behaviour correctly. We prove error estimates in such balanced norms and improve thereby existing estimates known in literature.

Key words. Balanced norms, reaction-diffusion problems, finite element methods.

1. Introduction

We shall examine the finite element method for the numerical solution of a singularly perturbed linear elliptic $2m-$th order boundary value problem in two dimensions. In the weak form it is given by

$$\varepsilon^{2k}(\nabla^m u, \nabla^m v) + \tilde{a}(u,v) = (f,v) \quad \forall v \in H_0^m(\Omega), \tag{1}$$

where $\Omega = (0,1)^2$, $0 < \varepsilon \ll 1$ is a small positive parameter, $1 \leq k \leq m$ and f is sufficiently smooth. We assume that the bilinear form $\tilde{a}(\cdot,\cdot)$ is related to a $2(m-k)-$th order operator and $\tilde{a}(u,u)$ is equivalent to $\|u\|^2_{H^{m-k}}$.

The Lax-Milgram theorem tells us that the problem has a unique solution $u \in H_0^m(\Omega)$ which is sufficiently smooth for smooth data and satisfies in the energy norm

$$|||u|||_\varepsilon := \varepsilon^k |u|_{H^m} + \|u\|_{H^{m-k}} \lesssim \|f\|_{L^2}. \tag{2}$$

Here and in the following we use the following notation: if $A \lesssim B$ then there exists a (generic) constant C independent of ε (and later also of the mesh used) such that $A \leq C\,B$.

The error of a finite element approximation $u^N \in V^N$ satisfies

$$\left|\left|\left|u - u^N\right|\right|\right|_\varepsilon \lesssim \min_{v^N \in V^N} \left|\left|\left|u - v^N\right|\right|\right|_\varepsilon \tag{3}$$

for any finite dimensional space $V^N \subset H_0^m(\Omega)$.

If we use C^{m-1}-splines, piecewise polynomial of degree $2m-1$, on a properly defined Shishkin mesh with N cells in each direction, then one can prove for the interpolation error of the Hermite interpolant $u^I \in V^N$

$$\left|\left|\left|u - u^I\right|\right|\right|_\varepsilon \lesssim \left(\varepsilon^{1/2}(N^{-1}\ln N)^m + N^{-(m+1)}\right). \tag{4}$$

It follows that the error $u - u^N$ also satisfies such an estimate. Some special one-dimensional cases are discussed, for instance, in [4, 14, 15].

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However, a typical boundary layer function $\varepsilon^{m-k}\exp(-x/\varepsilon)$ of our given problem measured in the norm $|||\cdot|||_\varepsilon$ is of order $\mathcal{O}\left(\varepsilon^{1/2}\right)$. Consequently, error estimates in this norm are less valuable as for convection diffusion equations. Therefore, we ask the fundamental question:
Is it possible to prove error estimates in the balanced norm

$$|||v|||_b := \varepsilon^{k-1/2}|v|_{H^m} + \|v\|_{H^{m-k}} \quad ? \tag{5}$$

As this norm has a different weighting of the H^m-seminorm, the layer function is measured of order $\mathcal{O}(1)$ as well as the non-layer components of the solution – the norm is balanced.

For higher order equations ($m \geq 2$), even in 1d nothing is known concerning estimates in the balanced norm for the Galerkin finite element method. The only exception is [2], where a fourth-order problem is discretised with a mixed finite element method.

The outline of this paper is as follows. In Section 2 we present a new idea to derive balanced error estimates for second order problems, improving the result in [11]. In Section 3 we generalise the idea from Section 2 to higher order problems in detail for the 1d case and give guiding principles for the (very technical) 2d case.

Notation: We denote by $(\cdot,\cdot)_D$ the L^2-scalar product on D and by $\|\cdot\|_{L^2(D)}$ the associated L^2-norm over D. Furthermore by $|\cdot|_{H^k(D)}$, $\|\cdot\|_{H^k(D)}$ and $\|\cdot\|_{W^{k,\infty}(D)}$ we denote the Sobolev-seminorm and norms in $H^k(D) = W^{k,2}(D)$ and $W^{k,\infty}(D)$. In the case of $D = \Omega$ we may skip the reference to the domain.

2. An improved estimate in a balanced norm for second order problems

Let us consider the case $m = k = 1$ and the discretization of

$$\varepsilon^2(\nabla u, \nabla v) + (cu, v) = (f, v) \quad \forall v \in V = H_0^1(\Omega), \tag{6}$$

where $c \geq \gamma > 0$ by linear finite elements on S-type meshes [10]. In [11] it was proved (on a Shishkin mesh)

$$\left|\left|\left|u - u^N\right|\right|\right|_b \lesssim N^{-1}(\ln N)^{3/2} + N^{-2}. \tag{7}$$

It was an open question to remove the factor $(\ln N)^{1/2}$ from (6). Here we modify the technique from [11] to realise that goal and use the same technique in Section 3 for higher order problems.

In [11] the L^2-projection $\pi u \in V^N$ from u was used instead of the Lagrange interpolant. Based on

$$u - u^N = u - \pi u + \pi u - u^N$$

we estimated for constant c the discrete error $\pi u - u^N$ starting from:

$$\begin{aligned}\left|\left|\left|\pi u - u^N\right|\right|\right|_\varepsilon^2 &\lesssim \varepsilon^2\|\nabla(\pi u - u^N)\|_{L^2}^2 + c\,\|\pi u - u^N\|_{L^2}^2 \\ &= \varepsilon^2(\nabla(\pi u - u), \nabla(\pi u - u^N)) + c\,(\pi u - u, \pi u - u^N).\end{aligned} \tag{8}$$

With $(\pi u - u, \xi) = 0$ for $\xi \in V^N$, the last term vanishes and the problem was to estimate $\|\nabla(\pi u - u)\|_{L^2}$. The use of the global projection leads to difficulties, especially in 2D: it is known that the L^2 projection is not on every mesh L^p stable, and there are examples which show that for the $W^{1,p}$ stability restrictions on the mesh are necessary even in the one-dimensional case [1,7]. Fortunately, on tensor product meshes like our S-type meshes (and their triangular versions) the L^2-projection is

L^∞-stable as shown in [7] using the Gram matrix of nodal basis functions and a precise analysis of its entries.

Here we modify the definition of the projection into V^N, the space of piecewise polynomials of degree $p \geq 1$ in each coordinate direction. In order to do so we start by defining our mesh for the number N of cells in each direction divisible by 4. Let φ be a monotonically increasing function with $\varphi(0) = 0$, $\varphi(1/2) = \ln N$ – the so-called *mesh-generating function* – and $\psi := \ln(-\varphi)$ the *mesh characterising function*, see [10]. Furthermore let $\lambda := \sigma\varepsilon \ln N$ be the transition parameter, where σ is a user chosen parameter to be specified later and $\lambda \leq 1/4$ is assumed.

The idea for defining the transition parameter comes is related to the Assumption 2.1 on a solution decomposition, see [3].

Assumption 2.1. *We assume the decomposition* $u = v + w = v + \sum_{k=1}^{4} w_k + \sum_{k=1}^{4} c_k$ *into a smooth part* v *and a layer part, consisting of boundary layer parts* w_k *and corner layer parts* c_k. *To be more precise we assume for* $0 \leq i, j \leq p+1$

$$
\begin{aligned}
|\partial_x^i \partial_y^j v(x,y)| &\lesssim 1,\\
|\partial_x^i \partial_y^j w_1(x,y)| &\lesssim \varepsilon^{-i} \exp(-x/\varepsilon),\\
|\partial_x^i \partial_y^j c_1(x,y)| &\lesssim \varepsilon^{-(i+j)} \exp(-(x+y)/\varepsilon)
\end{aligned}
$$

and similarly for the remaining terms.

Now we have $|w_1(\lambda, y)| \lesssim N^{-\sigma}$ and the size of the layer components in Ω_c can be adjusted by σ.

The mesh-points are then defined by

$$
x_i = y_i = \begin{cases} \sigma\varepsilon\varphi\left(\frac{2i}{N}\right), & i \in \{0, \ldots, N/4\},\\ \lambda + \left(\frac{4i}{N} - 1\right)\left(\frac{1}{2} - \lambda\right), & i \in \{N/4, \ldots, 3N/4\},\\ 1 - \sigma\varepsilon\varphi\left(2 - \frac{2i}{N}\right), & i \in \{3N/4, \ldots, N\}. \end{cases}
$$

By drawing axis-parallel lines through the so-defined mesh points we obtain an S-Type mesh with equidistant cells in the coarse region $\Omega_c := (\lambda, 1-\lambda)^2$ and anisotropic cells in the layer region $\Omega \setminus \Omega_c$. Note that in the layer region the small mesh-sizes can be estimated by $h_i := x_{i-1} - x_i \leq h$ and $k_j = y_{j+1} - y_j \leq h$ with

$$
\varepsilon N^{-1} \ln N \lesssim h \lesssim \varepsilon, \tag{9}
$$

and similarly for the y-direction.

Assumption 2.2. *Let the mesh-generating function* φ *be convex.*

Most of the generating functions of S-type-meshes fulfil this assumption, i.e. the most prominent two

- Shishkin mesh: $\varphi(t) = 2t \ln N$,
- Bakhvalov-S-mesh: $\varphi(t) = -\ln(1 - 2t(1 - N^{-1}))$.

As a result of Assumption 2.2 the cells in the layer region adjacent to the transition line have a width of h orthogonal to the transition line. We then define another domain by enlarging Ω_c one ply of cells in each direction:

$$
\Omega_c^* := (\lambda - h, 1 - (\lambda - h))^2.
$$

Let us denote by $\mathcal{I}$ the piecewise Gauss-Lobatto interpolation operator that uses as local interpolation points the quadrature nodes $(\hat{x}_k, \hat{k}_\ell)$ for $k, \ell \in \{0, \ldots, p+1\}$

of the Gauss-Lobatto quadrature rule. Furthermore, we denote by π the weighted, Ω_c-global L^2-projection $\pi v \in V^N$ defined by

$$(c(v-\pi v), \omega)_{\Omega_c} = 0 \quad \forall \omega \in V^N,$$

where we have denoted by $(\cdot,\cdot)_{\Omega_c}$ the restriction of the L^2-scalar product to Ω_c. Additionally, we denote by $\chi_\tau \in V^N$ on each element $\tau \in \Omega_c^* \setminus \Omega_c$ the discrete function with

$$\chi_\tau(\hat{x}_k, \hat{y}_\ell) = \begin{cases} 1, & (\hat{x}_k, \hat{y}_\ell) \in \partial\Omega_c, \\ 0, & \textit{otherwise}. \end{cases}$$

Note that on $\Omega_c^* \setminus \Omega_c$ only two types of χ_τ exist: They are one in either exactly one corner or on exactly one side of τ.

Now we can finally define our new interpolation operator. Let the interpolation operator P into V^N for $u = v + w$, see Assumption 2.1, be defined by

$$Pw|_\tau := \begin{cases} 0, & \tau \subset \Omega_c, \\ \mathcal{I}w|_\tau, & \tau \subset \Omega \setminus \Omega_c^*, \\ (\mathcal{I}w - \mathcal{I}\chi_\tau w)|_\tau, & \tau \subset \Omega_c^* \setminus \Omega_c, \end{cases}$$

$$Pv|_\tau := \begin{cases} \pi v|_\tau, & \tau \subset \Omega_c, \\ \mathcal{I}v|_\tau, & \tau \subset \Omega \setminus \Omega_c^*, \\ (\mathcal{I}v - \mathcal{I}\chi_\tau(v-\pi v))|_\tau, & \tau \subset \Omega_c^* \setminus \Omega_c. \end{cases}$$

Note that $\mathcal{I}\chi_\tau$ uses for $\tau \subset \Omega_c^* \setminus \Omega_c$ only information located on $\partial\Omega_c$ and could also be written using the basis functions associated with $\partial\Omega_c$. Therefore, above application of $\mathcal{I}\chi_\tau \pi v$ for $\tau \not\subset \Omega_c$ is defined.

Lemma 2.3. *For any $v \in W^{p+1,\infty}(\Omega_c)$ it holds*

$$\|\mathcal{I}v - \pi v\|_{L^\infty(\partial\Omega_c)} \lesssim N^{-(p+1)}.$$

Proof. Using $\pi(\mathcal{I}v) = \mathcal{I}v$ due to π being a projection we have

$$\|\mathcal{I}v - \pi v\|_{L^\infty(\partial\Omega_c)} \le \|\pi(\mathcal{I}v - v)\|_{L^\infty(\Omega_c)} \lesssim \|\mathcal{I}v - v\|_{L^\infty(\Omega_c)},$$

where we have used in the last step the L^∞-stability of the L^2-projection on Ω_c, see [7]. The result follows by standard interpolation error estimation on equidistant meshes. Alternatively to the L^∞-stability an L^∞-error estimate of the L^2-projection, see [6, 13], could be used. □

We will use in the following the splitting of the error into the interpolation and discrete error given by

$$u - u^N = (u - Pu) + (Pu - u^N) =: \eta + \xi.$$

Lemma 2.4. *Let $\sigma \ge p+1$. Under the Assumption 2.1 we have*

$$|(c\eta, \xi)| \lesssim \varepsilon^{1/2}\Big(N^{-(p+1)}(\ln N)^{1/2} + \big(h + N^{-1}\max|\psi'|\big)^{p+1}\Big) |||\xi|||_\varepsilon .$$

Proof. We will prove the estimate in the coarse and remaining region separately. Let us start on Ω_c. By definition of P and the L^2-orthogonality of the L^2-error we have

$$|(c\eta, \xi)_{\Omega_c}| = |(cw, \xi)_{\Omega_c}| \lesssim \|w\|_{L^2(\Omega_c)}\|\xi\|_{L^2(\Omega_c)} \lesssim \varepsilon^{1/2} N^{-\sigma} |||\xi|||_\varepsilon .$$

In the remaining domain we have

$$(c\eta,\xi)_{\Omega\setminus\Omega_c} = (c(u-\mathcal{I}u),\xi)_{\Omega\setminus\Omega_c} + (c(\mathcal{I}u-Pu),\xi)_{\Omega_c^*\setminus\Omega_c},$$

where we extended the application of I into the ply of elements around Ω_c. For the first term it holds with a Hlder inequality

$$\begin{aligned}|(c(u-\mathcal{I}u),\xi)_{\Omega\setminus\Omega_c}| &\lesssim \left(\operatorname{meas}^{1/2}(\Omega\setminus\Omega_c)\|v-\mathcal{I}v\|_{L^\infty(\Omega\setminus\Omega_c)} + \|w-\mathcal{I}w\|_{L^2(\Omega\setminus\Omega_c)}\right)|||\xi|||_\varepsilon\\ &\lesssim \varepsilon^{1/2}\left(N^{-(p+1)}(\ln N)^{1/2} + \left(h+N^{-1}\max|\psi'|\right)^{p+1}\right)|||\xi|||_\varepsilon\,,\end{aligned}$$

while for the second term we have using the special function $\chi\in V^N$

$$\begin{aligned}|(c(\mathcal{I}u-Pu),\xi)_{\Omega_c^*\setminus\Omega_c}| &\lesssim \left(\|\mathcal{I}v-Pv\|_{L^2(\Omega_c^*\setminus\Omega_c)} + \|\mathcal{I}w-Pw\|_{L^2(\Omega_c^*\setminus\Omega_c)}\right)|||\xi|||_\varepsilon\\ &\lesssim \left(\|\mathcal{I}v-\pi v\|_{L^\infty(\partial\Omega_c)} + \|\mathcal{I}w\|_{L^\infty(\partial\Omega_c)}\right)\|\chi\|_{L^2(\Omega_c^*\setminus\Omega_c)}|||\xi|||_\varepsilon\,.\end{aligned}$$

Applying Lemma 2.3, the boundedness of Gauss-Lobatto-basis functions and the L^∞-stability of I we obtain

$$\begin{aligned}|(c(\mathcal{I}u-Pu),\xi)_{\Omega_c^*\setminus\Omega_c}| &\lesssim \operatorname{meas}^{1/2}(\Omega_c^*\setminus\Omega_c)\left(\|\mathcal{I}v-\pi v\|_{L^\infty(\partial\Omega_c)} + \|w\|_{L^\infty(\partial\Omega_c)}\right)|||\xi|||_\varepsilon\\ &\lesssim \varepsilon^{1/2}\left(N^{-(p+1)} + N^{-\sigma}\right)|||\xi|||_\varepsilon\,,\end{aligned}$$

where $\operatorname{meas}(\Omega_c^*\setminus\Omega_c)\lesssim h\lesssim\varepsilon$ was used. With $\sigma\ge p+1$ the proof is finished. □

The final ingredient for our proof is the estimation of the interpolation error in the balanced norm.

Lemma 2.5. *Let $\sigma\ge p+1$. Under the Assumptions 2.1 and 2.2 we have*

$$|||\eta|||_b \lesssim \left(h+N^{-1}\max|\psi'|\right)^p.$$

Proof. We start by splitting the error into

$$|||\eta|||_b \lesssim |||\eta|||_{b,\Omega_c} + |||u-\mathcal{I}u|||_{b,\Omega\setminus\Omega_c} + |||\mathcal{I}u-Pu|||_{b,\Omega_c^*\setminus\Omega_c}\,.$$

By standard anisotropic interpolation error estimation we obtain

$$|||u-\mathcal{I}u|||_{b,\Omega\setminus\Omega_c} \lesssim \left(h+N^{-1}\max|\psi'|\right)^p.$$

Using the definition of P on Ω_c we have

$$\begin{aligned}|||\eta|||_{b,\Omega_c}^2 &\le \varepsilon\|\nabla(v-\pi v)\|_{L^2(\Omega_c)}^2 + \varepsilon\|\nabla w\|_{L^2(\Omega_c)}^2 + \gamma\|v-\pi v\|_{L^2(\Omega_c)}^2 + \gamma\|w\|_{L^2(\Omega_c)}^2\\ &\lesssim \varepsilon N^{-2p} + N^{-2\sigma} + N^{-2(p+1)}.\end{aligned}$$

For the remaining term we apply an inverse inequality. By Assumption 2.2 the small size of the cells in $\Omega_c^*\setminus\Omega_c$ is h and this can be bounded from below by

$$h \ge 4\sigma\varepsilon N^{-1}\ln N, \tag{10}$$

see also (9). Thus we get

$$\begin{aligned}|||\mathcal{I}u-Pu|||_{b,\Omega_c^*\setminus\Omega_c} &\lesssim \varepsilon^{1/2}\|\nabla(\mathcal{I}u-Pu)\|_{L^2(\Omega_c^*\setminus\Omega_c)} + \|\mathcal{I}u-Pu\|_{L^2(\Omega_c^*\setminus\Omega_c)}\\ &\lesssim \left(\frac{\varepsilon}{\min\{h,N^{-1}\}} + \varepsilon^{1/2}\right)\|\mathcal{I}u-Pu\|_{L^\infty(\Omega_c^*\setminus\Omega_c)}\\ &\lesssim N(N^{-\sigma}+N^{-(p+1)}),\end{aligned}$$

where Lemma 2.3 was used in the last step. Together with $\sigma\ge p+1$ the proof is complete. □

Using these Lemmas we obtain the main result for this section.

Theorem 2.6. *Let $\sigma \geq p+1$ and Assumptions 2.1 and 2.2 hold. Then we have for the solutions u of* (6) *and u^N of the corresponding Galerkin method*

$$\left|\left|\left|u - u^N\right|\right|\right|_b \lesssim \left(h + N^{-1} \max|\psi'|\right)^p.$$

Proof. Let us start with the discrete error ξ. Using coercivity in the energy norm and Galerkin orthogonality we have

$$|||\xi|||_\varepsilon^2 \leq \varepsilon^{1/2} \, |||\eta|||_b \, |||\xi|||_\varepsilon + |(c\eta, \xi)|.$$

With Lemma 2.4 we get

$$|||\xi|||_\varepsilon^2 \lesssim \varepsilon^{1/2}(|||\eta|||_b + \left(h + N^{-1} \max|\psi'|\right)^p) \, |||\xi|||_\varepsilon$$

and therefore

$$\varepsilon^{1/2}\|\nabla\xi\|_{L^2} \leq \varepsilon^{-1/2} \, |||\xi|||_\varepsilon \lesssim |||\eta|||_b + \left(h + N^{-1} \max|\psi'|\right)^p.$$

Together with the energy-norm result for ξ

$$\|\xi\|_{L^2} \leq |||\xi|||_\varepsilon \lesssim \left(h + N^{-1} \max|\psi'|\right)^p$$

we have

$$|||\xi|||_b \lesssim \left(h + N^{-1} \max|\psi'|\right)^p.$$

Now the triangle inequality and Lemma 2.5 yield the assertion

$$\left|\left|\left|u - u^N\right|\right|\right|_b \leq |||\eta|||_b + |||\xi|||_b \lesssim \left(h + N^{-1} \max|\psi'|\right)^p. \qquad \square$$

Remark 2.7. *In [9] we proved for linear elements on S-type meshes the estimate*

$$\|u - u^N\|_b \lesssim h + N^{-1}(\ln N)^{1/2} \max|\psi'| \tag{11}$$

under the assumption

$$N^{-1} \lesssim \varphi(1/N). \tag{12}$$

This assumption guarantees that the minimal mesh size (φ is convex and monotonically increasing) is not too small, which is guaranteed for Shishkin and Bakhvalov-Shishkin meshes, but not, for instance, for polynomial Shishkin-meshes. Our new approach improves upon the estimate (11) *by the factor* $(\ln N)^{1/2}$ *without this assumption.*

We retain from giving results of numerical simulations and instead refer to [11], where on a Shishkin mesh and polynomial degrees $p \in \{1, 2\}$ convergence of order $\mathcal{O}\left((N^{-1}\ln N)^p\right)$ in the balanced norm was obtained numerically.

3. Higher order problems

Let us consider the higher-order version of our problem in 1d, i.e.

$$\varepsilon^{2k}(u^{(m)}, v^{(m)}) + \tilde{a}(u, v) = (f, v) \quad \forall v \in H_0^m((0,1)), \tag{13}$$

where $\tilde{a}(\cdot,\cdot)$ is equivalent to $\|\cdot\|_{H^{m-k}((0,1))}$. We sketch the rather technical extension into 2d and general polynomial degrees in Remark 3.7. We assume for our analysis to work a solution decomposition of u.

Assumption 3.1. *We assume a decomposition $u = v + w$ into a smooth part v and boundary layer parts w_1, w_2, for which holds*

$$|\partial_x^i v(x,y)| \lesssim 1, \qquad |\partial_x^i w_1(x,y)| \lesssim \varepsilon^{m-k-i} \exp -x/\varepsilon,$$

where $0 \leq i \leq 2m$ and analogously for w_2.

The mesh for the problem of this section is a 1d-version of the S-type mesh from the previous section with $\Omega_c = (\lambda_x, 1-\lambda_x)$ and $\Omega_c^* = (\lambda_x - h, 1-\lambda_x + h)$.

The discrete space V^N is the H_0^m-conforming space of Hermite-polynomials of degree $p = 2m-1$. Beside the canonical Hermite-interpolation $\mathcal{I}$ we introduce a Ritz-projection π into $V^N(\Omega_c)$ by

$$\begin{aligned} \tilde{a}(v-\pi v, \chi) &= 0 \quad \text{in } \Omega_c \text{ for all } \chi \in V^N(\Omega_c), \\ \partial_x^n (v - \pi v) &= 0 \quad \text{on } \partial\Omega_c \text{ for all } n \in \{0, \dots, m-k-1\}. \end{aligned}$$

It is well known [5], that on the uniform mesh Ω_c the error bound

$$\|v - \pi v\|_{L^\infty(\Omega_c)} \lesssim N^{-(p+1)} \tag{14}$$

holds for polynomial degrees $p \geq 2$.

Now the second interpolation operator $Pu \in V^N$ is given for $u = v + w$ by

$$Pw\big|_\tau = \begin{cases} \mathcal{I}w\big|_\tau & \tau \subset \Omega \setminus \Omega_c^*, \\ 0 & \tau \subset \Omega_c, \end{cases} \qquad Pv\big|_\tau = \begin{cases} \mathcal{I}w\big|_\tau & \tau \subset \Omega \setminus \Omega_c^*, \\ \pi w\big|_\tau & \tau \subset \Omega_c. \end{cases}$$

Note that the definition of P is complete by $Pu \in V^N$. Before we start with the analysis we state a third assumption.

Assumption 3.2. *We assume for the bilinear form $\tilde{a}(\cdot,\cdot)$ to hold*

$$\tilde{a}(u,v)_{\Omega_c} \lesssim \|u\|_{W^{p,m-k}(\Omega_c)} \|v\|_{W^{q,m-k}(\Omega_c)}$$

for $p = q = 2$ and $p = \infty$, $q = 1$.

This assumption is fulfilled for symmetric bilinear forms $\tilde{a}(\cdot,\cdot)$ equivalent to the H^{m-k}-norm.

The analysis can now be conducted as in the previous section. We denote the error components by

$$u - u^N = (u - Pu) + (Pu - u^N) =: \eta + \xi.$$

Lemma 3.3. *Let $\sigma \geq 2m = p+1$. Under the Assumption 3.1 we have*

$$|\tilde{a}(\eta,\xi)| \lesssim \varepsilon^{1/2}\Big(N^{-(m+k)}\left((Nh)^{k-1} + (\ln N)^{1/2}\right) + \left(h + N^{-1}\max|\psi'|\right)^{m+k}\Big) \left|\left|\left|\xi\right|\right|\right|_\varepsilon.$$

Proof. The proof follows the proof of Lemma 2.4 but has some differences in the details. Therefore, we give the full proof here.

We will prove the estimate in the coarse and remaining region separately. Let us start on Ω_c. By definition of P and the orthogonality of the Ritz-error we have

$$|\tilde{a}(\eta,\xi)_{\Omega_c}| = |\tilde{a}(w,\xi)_{\Omega_c}| \lesssim \|w\|_{H^{m-k}(\Omega_c)} \left|\left|\left|\xi\right|\right|\right|_\varepsilon \lesssim \varepsilon^{1/2} N^{-\sigma} \left|\left|\left|\xi\right|\right|\right|_\varepsilon.$$

In the remaining domain we have

$$\tilde{a}(\eta,\xi)_{\Omega\setminus\Omega_c} = \tilde{a}(u - \mathcal{I}u, \xi)_{\Omega\setminus\Omega_c} + \tilde{a}(\mathcal{I}u - Pu, \xi)_{\Omega_c^*\setminus\Omega_c}.$$

For the first term it holds with Assumption 3.2

$$\begin{aligned} &|\tilde{a}(u - \mathcal{I}u, \xi)_{\Omega\setminus\Omega_c}| \\ &\lesssim \left(\operatorname{meas}^{1/2}(\Omega\setminus\Omega_c)\|v - \mathcal{I}v\|_{W^{m-k,\infty}(\Omega\setminus\Omega_c)} + \|w - \mathcal{I}w\|_{H^{m-k}(\Omega\setminus\Omega_c)}\right) \left|\left|\left|\xi\right|\right|\right|_\varepsilon \\ &\lesssim \varepsilon^{1/2}\left(N^{-(m+k)}(\ln N)^{1/2} + \left(h + N^{-1}\max|\psi'|\right)^{m+k}\right) \left|\left|\left|\xi\right|\right|\right|_\varepsilon, \end{aligned}$$

where the interpolation errors were estimated in the usual way. Local (anisotropic) interpolation error formulas can be found in [12].

For the second term we start by

$$
\begin{aligned}
&|\tilde{a}(\mathcal{I}u - Pu, \xi)_{\Omega_c^*\setminus\Omega_c}| \\
&\lesssim (\|\mathcal{I}v - Pv\|_{H^{m-k}(\Omega_c^*\setminus\Omega_c)} + \|\mathcal{I}w - Pw\|_{H^{m-k}(\Omega_c^*\setminus\Omega_c)})\, |||\xi|||_\varepsilon \\
&\lesssim \varepsilon^{1/2}(\|\mathcal{I}v - Pv\|_{W^{m-k,\infty}(\Omega_c^*\setminus\Omega_c)} + \|\mathcal{I}w - Pw\|_{W^{m-k,\infty}(\Omega_c^*\setminus\Omega_c)})\, |||\xi|||_\varepsilon\,. \qquad (15)
\end{aligned}
$$

Let us look at $\tau = (\lambda - h, \lambda) \subset \Omega_c^* \setminus \Omega_c$, the other interval follows analogously. We denote by φ_n the basis-functions that have as degrees of freedom the C^n-compatibility at $x = \lambda$ for $n \in \{0, \dots, m-1\}$. Then it holds

$$
\|\varphi_n\|_{W^{m-k,\infty}(\tau)} \lesssim h^n \left(1 + h^{-(m-k)}\right), \quad n \in \{0, \dots, m-1\}.
$$

Now we have for the boundary layers $w = w_1 + w_2$ and the smooth part v

$$
\mathcal{I}w - Pw = \sum_{n=0}^{m-1} \partial_x^n w(\lambda)\varphi_n \quad \text{and} \quad \mathcal{I}v - Pv = \sum_{n=m-k}^{m-1} \partial_x^n(\mathcal{I}v - \pi v)(\lambda)\varphi_n,
$$

where the definition of I and the boundary conditions of the Ritz-projection were used in the representations. For the first norm in (15) we use inverse inequalities and the L^∞-error estimate (14) of the Ritz-projection to obtain locally

$$
\begin{aligned}
&\|\mathcal{I}v - Pv\|_{W^{m-k,\infty}(\tau)} \\
&\lesssim \sum_{n=m-k}^{m-1} |\partial_x^n(\mathcal{I}v - \pi v)(\lambda)|\|\chi_n\|_{W^{m-k,\infty}(\tau)} \\
&\lesssim \sum_{n=m-k}^{m-1} N^n(\|\mathcal{I}v - v\|_{L^\infty(\Omega_c)} + \|v - \pi v\|_{L^\infty(\Omega_c)})h^n \left(1 + h^{-(m-k)}\right) \\
&\lesssim \sum_{n=m-k}^{m-1} N^n N^{-2m} h^n \left(1 + h^{-(m-k)}\right) \\
&\lesssim N^{-(m+k)}\left(1 + (Nh)^{k-1}\right), \qquad (16)
\end{aligned}
$$

while for the second norm we use $h \lesssim \varepsilon$ and $\frac{\varepsilon}{h} \lesssim N$, see (9), and have locally

$$
\begin{aligned}
\|\mathcal{I}w - Pw\|_{W^{m-k,\infty}(\tau)} &\lesssim \sum_{n=0}^{m-1} |\partial_x^n w(\lambda)|\|\chi_n\|_{W^{m-k,\infty}(\tau)} \\
&\lesssim \sum_{n=0}^{m-1} \varepsilon^{m-k-n} N^{-\sigma} h^n \left(1 + h^{-(m-k)}\right) \\
&\lesssim N^{-\sigma}\left(1 + \left(\frac{\varepsilon}{h}\right)^{m-k}\right) \lesssim N^{-(\sigma-(m-k))}. \qquad (17)
\end{aligned}
$$

Choosing $\sigma \ge 2m = p + 1$ the proof is done by collecting the separate bounds. $\square$

Lemma 3.4. *Let $\sigma \ge 2m$. Under the Assumptions 3.1 and 2.2 we have*

$$
|||\eta|||_b \lesssim N^{-m}(1 + (Nh)^{k-1}) + \left(h + N^{-1}\max|\psi'|\right)^m.
$$

Proof. We can follow the proof of Lemma 2.5 line by line. $\square$

Combining the results of these lemmas gives the main result for the higher-order case.

Theorem 3.5. *Let $\sigma \geq 2m = p+1$ and Assumptions 3.1 and 2.2 hold. Then we have for the solutions u of* (13) *and u^N of the corresponding Galerkin method*

$$|||u - u^N|||_b \lesssim N^{-m}(1 + (hN)^{k-1}) + \left(h + N^{-1} \max |\psi'|\right)^m .$$

Remark 3.6. *Under the additional assumption $Nh \lesssim 1$, which is equivalent to $h \lesssim N^{-1}$, Theorem 3.5 yields the shorter estimate*

$$|||u - u^N|||_b \lesssim \left(N^{-1} \max |\psi'|\right)^m = \left(N^{-1} \max |\psi'|\right)^{p+1-m} .$$

This assumption on h is true for the Shishkin mesh with

$$|||u - u^N|||_b \lesssim (N^{-1} \ln N)^{p+1-m}$$

or the Bakhvalov-S-mesh for $\varepsilon \lesssim N^{-1}$ with

$$|||u - u^N|||_b \lesssim N^{-(p+1-m)}.$$

Numerical simulations of the 1d-case on a different layer adapted mesh can be found in [15], where a fourth-order problem is discretized by Hermite-polynomials of degrees $p \in \{3, 5\}$ and optimal convergence orders in the balanced norm analogue to our theoretical ones are found numerically.

Remark 3.7. *For the 2d-case similar ideas can be used. Altogether it is a quite technical but straightforward task. We will show the idea for the case $m = 2$ and $k = 1$, thus a fourth order-problem with $\tilde{a}(\cdot, \cdot)$ a second order bilinear form like the one considered in [15] in 1d.*

We start with an assumption on a decomposition of $u = v + \sum_{i=1}^{4}(w_i + c_i)$ into a smooth part v, four boundary layer parts w_i with $|\partial_x^i \partial_y^j w_1(x, y)| \lesssim \varepsilon^{1-i} \mathrm{e}^{-x/\varepsilon}$ and four corner layer parts with $|\partial_x^i \partial_y^j c_1(x, y)| \lesssim \varepsilon^{1-i-j} \mathrm{e}^{-x/\varepsilon} \mathrm{e}^{-y/\varepsilon}$ (and analogously for the remaining parts) for $0 \leq i, j \leq 2m$. The mesh is defined as in Section 2, our discrete space V^N is the space of bicubic C^1-Hermite-splines and $\mathcal{I}$ the canonical Hermite-interpolation into V^N.

The main task is to define the projection P into V^N. We define it separately for each part of the decomposition. Let $\Omega_1 := (0, \lambda) \times (0, 1)$ and $\Omega_1^* := (\lambda - h, \lambda) \times (0, 1)$. Then

$$Pw_1|_\tau := \begin{cases} \mathcal{I}w_1|_\tau, & \tau \subset \Omega_1 \setminus \Omega_1^*, \\ 0, & \tau \subset \Omega \setminus \Omega_1. \end{cases}$$

Again Pw_1 is completely defined by $Pw_1 \in V^N$. For the corner-component c_1 we define similarly $\widehat{\Omega}_1 := (0, \lambda) \times (0, \lambda)$, $\widehat{\Omega}_1^* := (\lambda - h, \lambda) \times (0, \lambda) \cup (0, \lambda) \times (\lambda - h, \lambda)$ and

$$Pc_1|_\tau := \begin{cases} \mathcal{I}c_1|_\tau, & \tau \subset \widehat{\Omega}_1 \setminus \widehat{\Omega}_1^*, \\ 0, & \tau \subset \Omega \setminus \widehat{\Omega}_1. \end{cases}$$

For the other layer components we proceed similarly. That leaves the smooth part. With Ω_c and Ω_c^* from Section 2 we define

$$Pv|_\tau := \begin{cases} \mathcal{I}v|_\tau, & \tau \subset \Omega \setminus \Omega_c^*, \\ \pi v|_\tau, & \tau \subset \Omega_c, \end{cases}$$

where πv is the Ritz-projection into $V_c^N := \{v \in C^1(\Omega_c) \,:\, v|_\tau \subset \mathcal{Q}_3(\tau)\}$ given by

$$\begin{aligned}\tilde{a}(v - \pi v, \chi)_{\Omega_c} &= 0 \textit{ for all } \chi \in V_c^N \cap H_0^1(\Omega_c)\\ \mathcal{I}v - \pi v &= 0 \textit{ on } \partial\Omega_c.\end{aligned}$$

Note that the boundary condition implies

$$\partial_t(\mathcal{I}v - \pi v) = 0 \textit{ on } \partial\Omega_c,$$

where ∂_t denotes the tangential derivative.

Given this interpolation operator P it is straightforward to show

$$\begin{aligned}\|\mathcal{I}w_i - Pw_i\|_{W^{1,\infty}(\Omega_i^*)} + \|\mathcal{I}c_i - Pc_i\|_{W^{1,\infty}(\widehat{\Omega}_i^*)} &\lesssim N^{-(\sigma-1)},\\ \|\mathcal{I}v - Pv\|_{W^{1,\infty}(\Omega_c^*\setminus\Omega_c)} &\lesssim (1+hN)N^{-3},\end{aligned}$$

where the additional assumption on the minimal mesh width $h_{min} \geq \varepsilon N^{-1}$ is needed for the first and an L^∞-error estimation for the Ritz projection or an L^∞-stability result for π is assumed for the second estimate (for a fourth-order problem discretised on a triangular mesh by Clough-Tocher elements see [8]).

Similarly we obtain

$$\begin{aligned}|\tilde{a}(\eta,\xi)| &\lesssim \varepsilon^{1/2}(1+hN)\left(h + N^{-1}\max|\psi'|\right)^3 |||\xi|||_\varepsilon\,,\\ |||\eta|||_b &\lesssim (1+hN)\left(h + N^{-1}\max|\psi'|\right)^2\end{aligned}$$

for $\sigma \geq 4$ by a tedious estimation. Combining above steps gives the result in the 2d-case

$$\left|\left|\left|u - u^N\right|\right|\right|_b \lesssim (1+hN)\left(h + N^{-1}\max|\psi'|\right)^2$$

for $\sigma \geq 4$. Note that $p+1-m = 2$ is the convergence order.

The extension of these ideas to the general case of $m \geq k \geq 1$ is also clear. With $p = 2m-1$ we use an C^{m-1}-Hermite-space with piecewise $\mathcal{Q}_p$-polynomials and define the projection π as Ritz-projection using higher order boundary conditions depending on $m-k$. Then for $\sigma \geq 2m = p+1$ the result from Theorem 3.5 holds also in 2d.

The final extension of above analysis is to increase the polynomial degree to $p \geq 2m$ while preserving the C^{m-1}-continuity of the discrete space. With a suitable defined operator $\mathcal{I}$ and a properly defined interpolation operator P (using above π and ideas from Section 2) the balanced norm estimate can be shown for $\sigma \geq p+1$ and $hN \lesssim 1$ to be

$$\left|\left|\left|u - u^N\right|\right|\right|_b \lesssim \left(N^{-1}\max|\psi'|\right)^{p+1-m}.$$

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