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DYNAMICAL BEHAVIORS OF ATTRACTION-REPULSION CHEMOTAXIS MODEL

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Abstract. A free boundary problem for the chemotaxis model of parabolic-elliptic type is investigated in the present paper, which can be used to simulate the dynamics of cell density under the influence of the nonlinear diffusion and nonlocal attraction-repulsion forces. In particular, it is shown for supercritical case that if the initial total mass of cell density is small enough or the interaction between repulsion and attraction cancels almost each other, the strong solution for the cell density exists globally in time and converges to the self-similar Barenblatt solution at the algebraic time rate, and for subcritical case that if the initial data is a small perturbation of the steady-state solution and the attraction effect dominates the process, the strong solution for cell density exists globally in time and converges to the steady-state solution at the exponential time rate.

Key words. Chemotaxis, free boundary problem, Barenblatt solution, steady-state solution.

1. Introduction

Chemotaxis is the widespread phenomena in nature, for instance, the directional movement of biological cells, bacteria or organisms in response to chemical signals in the environment, including the positive (chemoattractive) chemotaxis and negative (chemorepulsive) chemotaxis. The first mathematical model is heuristically derived by Patlak [23] and later by Keller and Segel [10, 11] respectively to study the nonlocal aggregation process of cellular slime molds Dictyostelium Discoidium due to chemical cyclic adenosine monophosphate

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v), \quad \mathbf{x} \in \Omega,\ t \geq 0, \\ \tau v_t = \Delta v + u - av, \end{cases} \tag{1}$$

where $u = u(\mathbf{x}, t)$ and $v = v(\mathbf{x}, t)$ stand for, the density of cells and concentration of chemoattractant respectively. The non-negative parameter a denotes the mortality rate of chemical, and the parameter τ equals zero or one. Since then, this classical chemotaxis model (1) has been generalized to simulate the biological or medical phenomena [24], such as the bacteria aggregation [29], cancer invasion [2, 33] and so on.

For the sake of simulation the local repulsion of cells in biological or medical phenomena, for example, the volume exclusion or population pressure when cells are packed. Wakita et al [30] observed that the diffusive coefficient is depended on the cell density of bacterial colony through experiment. Kawasaki et al [9] introduced the porous media type bacterial diffusion by modeling spatio temporal patterns of Bacillus subtilis. Therefore, the following Patlak-Keller-Segel model with nonlinear degenerate diffusion could be taken into consideration,

$$\begin{cases} u_t = \Delta u^m - \nabla \cdot (u\nabla v), \quad m > 1, \quad \mathbf{x} \in \Omega,\ t \geq 0, \\ \tau v_t = \Delta v + u - av, \end{cases} \tag{2}$$

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where the diffusive component $m > 1$ denotes the slow diffusion. Topaz et al [27,28] and Carrillo et al [3,4] studied the model (2) to take into accounting over-crowding effects.

In order to model the aggregation of microglia in the central nervous system observed in Alzheimers disease and the quorum-sensing behaviour due to the interaction of chemoattractant and chemorepellent in the chemotaxis process, Luca et al [20] (the diffusive component $m = 1$) and Painter et al [22] (the diffusive component $m > 1$) introduced the following attraction-repulsion chemotaxis model as

$$\begin{cases} u_t = \Delta u^m - \nabla\cdot(a_1 u\nabla v) + \nabla\cdot(b_1 u\nabla w), & \mathbf{x}\in\Omega,\ t\geq 0, \\ \tau_1 v_t = \Delta v + a_2 u - a_3 v, \\ \tau_2 w_t = \Delta w + b_2 u - b_3 w, \end{cases} \tag{3}$$

where $u = u(\mathbf{x}, t)$, $v = v(\mathbf{x}, t)$, and $w = w(\mathbf{x}, t)$ stand for the density of cells, concentration of chemoattractant and chemorepellent respectively. The diffusive component $m \geq 1$, the non-negative parameters a_i and b_i $(i = 1, 2, 3)$ denote the sensitivity of cells to the chemoattractant and chemorepellent, and the growth and mortality rates of the chemicals respectively, the parameters $\tau_1, \tau_2 = 0, 1$.

The interaction between chemorepellent and chemoattractant is rather complicated which makes it difficult to analyze the mathematical properties of the solution to the system (3). Yet, there are also important progresses made recently on the well-posedness and dynamical behaviors of the solution to the system (3), refer to, for instance, [6–8, 14–18, 18, 19, 21, 25, 26, 31, 32, 34] and the references therein. In particular, the existence of classical solution or the stability of the steady-state solution had been proved in [6–8, 14–16, 18, 19, 26] for the linear diffusion case $m = 1$. In the case that the repulsion effect dominates the process (i.e., $a_1a_2 - b_1b_2 < 0$), the global existence of classical solutions was shown to the system (3) in one-dimensional [7, 16, 19] or multi-dimensional bounded domain with the Neumann boundary condition [8, 16, 18, 26], and the long-time convergence of the global classical solutions to the steady-state solution were proved in [7, 15, 18, 19, 26]. Similar results were also established for multi-dimensional Cauchy problem to the system (3), concerned with the global existence of classical solution and long-time convergence to the corresponding steady-state solution [25]. However, in the case that the attraction effect dominates the process (i.e., $a_1a_2 - b_1b_2 > 0$), there is a critical initial mass $M = \frac{8\pi}{a_1a_2-b_1b_2}$ as $\tau_1 = \tau_2 = 0$ or $M = \frac{4\pi}{a_1a_2-b_1b_2}$ as $\tau_1 = 1, \tau_2 = 0$ so that the classical solution to the system (3) in two-dimensional bounded domain with the Neumann boundary condition either existed globally in time [6,8] or blew up in finite time [8, 14, 26], depending on whether the initial total mass is larger than M or not. Similar results had also been shown for two-dimensional Cauchy problem to the system (4), related to the blow-up in finite time [25] or global existence of the classical solution [21]. For the nonlinear diffusion case $m > 1$, there are also important results shown in [17,31,32,34] to the system (3) in multi-dimensional bounded domain with the Neumann boundary condition. For instance, the global existence of weak solutions [31,34] or classical solutions [17,32] had been investigated either for $a_1a_2 - b_1b_2 < 0$ and $m \neq 2 - \frac{2}{n}$, or for $a_1a_2 - b_1b_2 > 0$ and $m > 2 - \frac{2}{n}$. Nevertheless, there existed a class of spherically symmetric weak solutions in three-dimensions which blew up in finite time [17] as it holds $m = 2 - \frac{2}{n}$.

However, although the important achievements have been obtained as above, there are few studies on the global existence and dynamical behaviors of strong solution to the congested motion problem with homogeneous nonlinear degenerate diffusion for fixed component $m > 1$. Li et al [13] have studied the free boundary value problem for Patlak-Keller-Segel model (2) with nonlinear degenerate diffusion (i.e., $\mathrm{m} > 1$) as the parameters $\tau = a = 0$, and obtained the global existence of the three-dimensional spherically symmetry strong solution and long-time behaviors with respect to the different diffusive components. Inspired by [13], we continue to consider the free boundary value problem of the following attraction-repulsion chemotaxis model (4) of parabolic-elliptic type with homogeneous nonlinear degenerate diffusion in this paper

$$
\begin{cases} u_t = \Delta u^m - \nabla \cdot (a_1 u \nabla v) + \nabla \cdot (b_1 u \nabla w), \quad \mathbf{x} \in \Omega(t),\ t \geq 0, \\ -\Delta v = a_2 u, \\ -\Delta w = b_2 u, \end{cases} \tag{4}
$$

which is a simplified version of the model (3) (i.e., $\tau_1 = \tau_2 = \mathrm{a}_3 = \mathrm{b}_3 = 0$).

Define the mixed velocity as

$$
\mathbf{V}(\mathbf{x}, t) = a_1 \nabla v - b_1 \nabla w - \frac{m}{m-1} \nabla u^{m-1}, \quad m > 1, \quad (\mathbf{x}, t) \in \Omega(t) \times (0, \infty), \tag{5}
$$

where $a_1 \nabla v - b_1 \nabla w$ is a drift velocity caused by the nonlocal attraction-repulsion, $-\frac{m}{m-1} \nabla u^{m-1}$ is a correction velocity provided by the nonlinear diffusion. Then, the first equation in system (4) can be written into the following transport form

$$
u_t + \nabla \cdot (u \mathbf{V}) = 0. \tag{6}
$$

We consider the following free boundary value problem for (4) in view of (6) as

$$
\begin{cases} u_t + \nabla \cdot (u \mathbf{V}) = 0, & \mathbf{x} \in \Omega(t),\ t > 0, \\ \mathbf{V}(\mathbf{x}, t) = a_1 \nabla v - b_1 \nabla w - \dfrac{m}{m-1} \nabla u^{m-1}, & \mathbf{x} \in \Omega(t),\ t > 0, \\ \Gamma_t(t) \triangleq \mathbf{V}(\Gamma(t), t) \cdot \mathbf{n}, & \Gamma(t) \triangleq \partial \Omega(t),\ t > 0, \\ u(\mathbf{x}, t) > 0, \quad u(\Gamma(t), t) = 0, & \mathbf{x} \in \Omega(t),\ t \geq 0, \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad u_0(\mathbf{x}) > 0, \quad u_0(\Gamma(0)) = 0, & \mathbf{x} \in \Omega(0), \end{cases} \tag{7}
$$

where $\Omega = \Omega(t)$ is a moving domain with the free boundary $\Gamma(t)$, $\mathbf{n}$ is the outward unit normal vector on the boundary $\Gamma(t)$, and the functions $v(\mathbf{x}, t)$ and $w(\mathbf{x}, t)$ satisfy

$$
\begin{cases} -\Delta v = a_2 \widetilde{u}, \quad v \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty, \\ -\Delta w = b_2 \widetilde{u}, \quad w \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty, \end{cases} \tag{8}
$$

with

$$
\widetilde{u}(\mathbf{x}, t) = \begin{cases} u(\mathbf{x}, t), & \mathbf{x} \in \overline{\Omega}(t),\ t > 0, \\ 0, & \mathbf{x} \notin \overline{\Omega}(t),\ t > 0. \end{cases} \tag{9}
$$

We shall investigate the global existence and dynamical behaviors of the strong solution to the free boundary value problem (7)-(9). To be more precise, we can show for supercritical case (i.e., $1 < m < \frac{4}{3}$) that if the initial data is a small perturbation of the self-similar Barenblatt solution, and the initial total mass M_1 of the cells density $u(\mathbf{x}, t)$ is small enough, or the interaction between the repulsion and attraction almost cancels each other (i.e., $|a_1 a_2 - b_1 b_2| \ll 1$), the strong solution for the cell density exists globally in time and converges to the self-similar Barenblatt solution at the algebraic time rate with the same total mass (refer to Theorem 2.1 for details). Moreover, we also can obtain for subcritical case (i.e., $m > \frac{4}{3}$) that if the initial data is a small perturbation of the steady-state solution and the attraction effect dominates the process (i.e., $a_1 a_2 - b_1 b_2 > 0$), the strong solution for cell density exists globally in time and converges to the steady-state solution at the exponential time rate with the same total mass (refer to Theorem 2.2 for details).

The rest of the present paper is arranged as follows. In Section 2, we reformulate the original free boundary value problem (7)-(9) in Lagrangian coordinates, and state the main results. In Section 3, we prove the main results (i.e., Theorem 2.1 and Theorem 2.2). In Section 4, we present some numerical simulations to the free boundary value problem (7)-(9), which is consistent with the main results established in Section 2.

2. Main results

We consider the spherically symmetric solution $(u(\eta,t),V(\eta,t),R(t))$ to the free boundary value problem (7)-(9) in three-dimensional spherically symmetric domain $\Omega(t)=\{\mathbf{x}\in\mathbb{R}^3 : 0\le|\mathbf{x}|\le R(t),\ R(t)>0\}$, namely, the moving region $\Omega(t)$ is a ball with the center at the origin and the radius $R(t)$

$$u(\mathbf{x},t)=u(\eta,t),\quad \mathbf{V}(\mathbf{x},t)=V(\eta,t)\frac{\mathbf{x}}{\eta},\quad \eta=|\mathbf{x}|\in[0,R(t)].$$

Therefore, the parabolic-elliptic chemotaxis system (7)-(9) can be changed to

$$\tag{10}\begin{cases}\eta^2u_t(\eta,t)+[\eta^2u(\eta,t)V(\eta,t)]_\eta=0, & \eta\in(0,R(t)),\ t>0,\\ V(\eta,t)=\dfrac{b_1b_2-a_1a_2}{\eta^2}\displaystyle\int_0^\eta 4\pi s^2u\,ds-\frac{m}{m-1}(u^{m-1})_\eta, & \eta\in(0,R(t)),\ t>0,\\ R_t(t)=V(R(t),t),\quad R(0)=R_0, & t>0,\end{cases}$$

with the following initial data and boundary conditions

$$\tag{11}\begin{cases}u(\eta,t)>0,\ u(R(t),t)=0,\ u_\eta(0,t)=0, & \eta\in[0,R(t)),\ t\ge0,\\ u(\eta,0)=u_0(\eta),\ u_0(\eta)>0,\ u_0(R_0)=0, & \eta\in[0,R_0),\\ -\infty<(u_0^{m-1})_\eta|_{\eta=R_0}<0.\end{cases}$$

Indeed, the density function $u(\eta,t)$ satisfies the conservation of mass

$$\tag{12}\int_0^{R_0}4\pi s^2u_0(s)ds=\int_0^{R(t)}4\pi s^2u(s,t)ds=M.$$

To begin with, we consider the free boundary value problem (10)-(11) *for super-critical case (i.e.,* $1<m<\frac{4}{3}$*).* If the interaction between repulsion and attraction cancels each other (i.e., $a_1a_2-b_1b_2=0$), the (10) reduces to the well-known porous media equation. Barenblatt [1] has proved that the porous media equation admits a self-similar Barenblatt solution $\widehat{u}(\mathbf{x},t)$ satisfying

$$\tag{13}\widehat{u}_t=\Delta\widehat{u}^m,$$

and

$$\tag{14}\begin{cases}\widehat{u}(\mathbf{x},t)=\widehat{u}(\eta,t)=(1+t)^{-\frac{3}{3m-1}}\left(\widehat{\beta}-\widehat{\gamma}(1+t)^{-\frac{2}{3m-1}}\eta^2\right)^{\frac{1}{m-1}}, & \eta=|\mathbf{x}|\in[0,\widehat{R}(t)],\\ \widehat{u}_0(\eta)=\widehat{u}(\eta,0)=(\widehat{\beta}-\widehat{\gamma}\eta^2)^{\frac{1}{m-1}},\quad \eta\in[0,\widehat{R}_0],\end{cases}$$

with

$$\widehat{\gamma}=\frac{m-1}{2m(3m-1)},\quad \widehat{R}(t)=\sqrt{\widehat{\beta}/\widehat{\gamma}}(1+t)^{\frac{1}{3m-1}},\quad \widehat{R}_0=\widehat{R}(0)=\sqrt{\widehat{\beta}/\widehat{\gamma}},$$

$$(m\widehat{\beta})^{\frac{3m-1}{2(m-1)}} = Mm^{\frac{1}{m-1}}(m\widehat{\gamma})^{\frac{3}{2}}\Big(\int_0^1 y^2(1-y^2)^{\frac{1}{m-1}}dy\Big)^{-1}, \quad M_1 = \int_0^{\widehat{R}_0} 4\pi s^2\widehat{u}_0 ds.$$

Similarly to (5), define the velocity by

$$\widehat{\mathbf{V}}(\mathbf{x},t) = \frac{\mathbf{x}}{\eta}\widehat{V}(\eta,t) = -\frac{m}{m-1}\nabla\widehat{u}^{m-1}(\mathbf{x},t). \tag{15}$$

The Eq. (13) can be also rewritten into a transport form as

$$\widehat{u}_t + \nabla\cdot(\widehat{u}\widehat{\mathbf{V}}) = 0, \tag{16}$$

and

$$\widehat{V}(\eta,t) = \frac{\eta}{(3m-1)(1+t)}, \quad \frac{d\widehat{R}(t)}{dt} = \widehat{V}(\widehat{R}(t),t). \tag{17}$$

Suppose that the initial total mass of the solution to the system (10)-(11) equals to that of the self-similar Barenblatt solution

$$\int_0^{R_0} 4\pi s^2 u_0(s)ds = \int_0^{\widehat{R}_0} 4\pi s^2\widehat{u}_0(s)ds = M_1 > 0. \tag{18}$$

We reformulate the original free boundary value problem (10)-(11) in Lagrangian coordinates. Define the particle path $\eta(r,t)$ of the moving domain $[0,R(t)]$ by

$$\begin{cases} \eta_t(r,t) = V(\eta(r,t),t), & r\in[0,\widehat{R}_0],\ t>0,\\ \eta(r,0) = \eta_0(r), \quad \eta_0(\widehat{R}_0) = R_0, & r\in[0,\widehat{R}_0], \end{cases} \tag{19}$$

where the function $\eta_0(r)$ satisfies

$$\int_0^{\eta_0(r)} s^2u_0(s)ds = \int_0^r s^2\widehat{u}_0(s)ds, \quad r\in[0,\widehat{R}_0]. \tag{20}$$

Accordingly, set the density $f(r,t)$ and velocity $\nu(r,t)$ in Lagrangian coordinates by

$$f(r,t) = u(\eta(r,t),t), \quad \nu(r,t) = V(\eta(r,t),t), \quad r\in[0,\widehat{R}_0]. \tag{21}$$

We obtain the following equivalent system to (10) in Lagrangian coordinates as

$$\begin{cases} (\eta^2 f)_t + \eta^2 f\dfrac{\nu_r}{\eta_r} = 0, & r\in[0,\widehat{R}_0),\ t>0,\\ \nu(r,t) = \dfrac{b_1b_2-a_1a_2}{\eta^2}\displaystyle\int_0^{\eta_0(r)} 4\pi s^2u_0ds - \dfrac{m}{m-1}\dfrac{(f^{m-1})_r}{\eta_r}, & r\in[0,\widehat{R}_0),\ t>0,\\ f(r,t)>0, \quad f(\widehat{R}_0,t)=0, \quad f_r(0,t)=0, & r\in[0,\widehat{R}_0),\ t>0,\\ f(r,0) = u_0(\eta_0(r)), \quad \nu(r,0) = V(\eta_0(r)), & r\in[0,\widehat{R}_0]. \end{cases} \tag{22}$$

Define the weight function $\sigma_1(r) \triangleq \widehat{u}_0^{m-1}$ and constant $\alpha \triangleq \frac{1}{m-1}$, it is easy to verify by using (20) and the first equation of (22) that

$$\eta^2(r,t)\eta_r(r,t)f(r,t) = \eta_0^2(r)\eta_{0r}(r)u_0(\eta_0(r)) = r^2\sigma_1^\alpha, \quad r\in[0,\widehat{R}_0],\ t\geq 0. \tag{23}$$

Multiplying the second equation of (22) by σ_1^α and applying (20) and (23), we obtain the following initial boundary value problem for $\eta(r,t)$ as

$$\begin{cases} \sigma_1^\alpha\eta_t + \Big(\dfrac{\eta}{r}\Big)^2\Big[\sigma_1^{\alpha+1}\Big(\dfrac{r^2}{\eta^2\eta_r}\Big)^m\Big]_r + (a_1a_2-b_1b_2)\dfrac{\sigma_1^\alpha}{\eta^2}\displaystyle\int_0^r 4\pi s^2\widehat{u}_0ds = 0, & r\in[0,\widehat{R}_0),\ t>0,\\ \eta(r,0) = \eta_0(r), \quad \eta(0,t)=0, \quad \eta(\widehat{R}_0,t) = R(t), & r\in[0,\widehat{R}_0],\ t\geq 0. \end{cases} \tag{24}$$

In addition, it also holds that for the self-similar Barenblatt solution $(\widehat{u}(\widehat{\eta},t), \widehat{V}(\widehat{\eta},t), \widehat{R}(t))$

$$\widehat{\eta}(r,t)=r(1+t)^{\frac{1}{3m-1}},\quad \widehat{V}(\widehat{\eta},t)=\frac{\widehat{\eta}}{(3m-1)(1+t)},\quad r\in[0,\widehat{R}_0],\ t\geq 0, \tag{25}$$

and

$$\sigma_1^{\alpha}\widehat{\eta}_t+\widehat{\eta}_r^{2-3m}(\sigma_1^{\alpha+1})_r=0,\quad r\in[0,\widehat{R}_0],\ t>0. \tag{26}$$

Introduce the $\widetilde{\eta}(r,t)$ inspired by the first equation of (24) and (26) with the modification

$$\widetilde{\eta}(r,t)\triangleq\widehat{\eta}(r,t)[1+\varsigma(r,t)]=r(1+t)^{\frac{1}{3m-1}}[1+\varsigma(r,t)], \tag{27}$$

which satisfies

(28)
$$\begin{cases}\sigma_1^{\alpha}\widetilde{\eta}_t+\Big(\dfrac{\widetilde{\eta}}{r}\Big)^2\Big[\sigma_1^{\alpha+1}\Big(\dfrac{r^2}{\widetilde{\eta}^2\widetilde{\eta}_r}\Big)^m\Big]_r+(a_1a_2-b_1b_2)\dfrac{\sigma_1^{\alpha}}{\widetilde{\eta}^2}\displaystyle\int_0^r 4\pi s^2\widehat{u}_0ds=0, & r\in[0,\widehat{R}_0),\ t>0,\\ \widetilde{\eta}(r,0)=\eta_0(r),\quad \widetilde{\eta}(0,t)=0,\quad \widetilde{\eta}(\widehat{R}_0,t)=R(t), & r\in[0,\widehat{R}_0],\ t\geq 0,\end{cases}$$

and the correction term $\varsigma(r,t)$ in (27) satisfies

(29)
$$\begin{cases} r\sigma_1^{\alpha}\varsigma_t+(1+t)^{-1}\Big\{(1+\varsigma)^2\big[\sigma_1^{\alpha+1}(1+\varsigma)^{-2m}(1+\varsigma+r\varsigma_r)^{-m}\big]_r-(\sigma_1^{\alpha+1})_r+\dfrac{1}{3m-1}r\sigma_1^{\alpha}\varsigma\Big\} & \\ \quad+(a_1a_2-b_1b_2)(1+t)^{-\frac{3}{3m-1}}\dfrac{\sigma_1^{\alpha}}{r^2}(1+\varsigma)^{-2}\displaystyle\int_0^r 4\pi s^2\widehat{u}_0ds=0, & r\in[0,\widehat{R}_0),\ t>0,\\ \varsigma(0,t)=0,\quad \varsigma(r,0)=\varsigma_0(r),\quad \varsigma(\widehat{R}_0,t)=\dfrac{R(t)}{\widehat{R}(t)}-1, & r\in[0,\widehat{R}_0],\ t\geq 0.\end{cases}$$

Define the nonlinear energy functional $E_0(t)$ for the correction term $\varsigma(r,t)$ by

$$\begin{aligned}E_0(t)=&\sum_{j=0}^{3+[\frac{\alpha}{2}]}\sum_{i=1}^{5+[\alpha]-2j}(1+t)^{2j+\overline{\alpha}}\int_0^{\widehat{R}_0}\big[r^4\sigma_1^{\alpha+i+1}(\partial_t^j\partial_r^{i+1}\varsigma)^2+r^2\sigma_1^{\alpha+i-1}(\partial_t^j\partial_r^i\varsigma)^2\big](r,t)dr\\ &+\sum_{j=0}^{3+[\frac{\alpha}{2}]}(1+t)^{2j+\overline{\alpha}}\int_0^{\widehat{R}_0}\big[r^4\sigma_1^{\alpha}(\partial_t^j\varsigma)^2+r^2\sigma_1^{\alpha+1}(\partial_t^j\varsigma)^2+r^4\sigma_1^{\alpha+1}(\partial_t^j\varsigma_r)^2\big](r,t)dr.\end{aligned}$$

Applying the similar arguments as dealing with the Proposition 4.1 in [13], we can show that if there exists a constant $0<\varepsilon_0\ll 1$ such that $E_0(0)\leq\varepsilon_0^2$,

then, a unique strong solution $\varsigma(r,t)$ to the system (29) exists globally in time and satisfies for any $t>0$ that

(30)
$$(1+t)^{\frac{\overline{\alpha}}{2}}\|\varsigma\|_{H^{\frac{6+[\alpha]-\alpha}{2}}([0,\widehat{R}_0])}+\sum_{l=1}^{2}(1+t)^{l+\frac{\overline{\alpha}}{2}}\|\partial_t^l\varsigma\|_{L^\infty([0,\widehat{R}_0])}\leq C_0\varepsilon_0+M_1(b_1b_2-a_1a_2),$$

where $0<\overline{\alpha}<\frac{8-6m}{3m-1}$ and C_0 are two positive constants. The details of the proof are omitted.

Define the perturbation $\vartheta(r,t)$ of trajectory $\eta(r,t)$ in (19) around the modified term $\widetilde{\eta}(r,t)$ in (27) as

$$\vartheta(r,t)\triangleq\frac{\eta(r,t)}{r}-\frac{\widetilde{\eta}(r,t)}{r},\quad r\in[0,\widehat{R}_0],\ t>0. \tag{31}$$

Then, we can obtain the corresponding free boundary value problem for $\vartheta(r,t)$ by (24) and (28) as

(32)
$$\begin{cases} r\sigma_1^\alpha \vartheta_t - \widehat{\eta}_r^{2-3m}(1+\varsigma)^2\big[\sigma_1^{\alpha+1}(1+\varsigma)^{-2m}(1+\varsigma+r\varsigma_r)^{-m}\big]_r \\ +[\widehat{\eta}_r(1+\varsigma)+\vartheta]^2\big\{\sigma_1^{\alpha+1}[\widehat{\eta}_r(1+\varsigma)+\vartheta]^{-2m}[\widehat{\eta}_r(1+\varsigma+r\varsigma_r)+\vartheta+r\vartheta_r]^{-m}\big\}_r \\ +\sigma_1^\alpha\left\{\dfrac{a_1a_2-b_1b_2}{[\widehat{\eta}(1+\varsigma)+r\vartheta]^2}-\dfrac{a_1a_2-b_1b_2}{[\widehat{\eta}(1+\varsigma)]^2}\right\}\displaystyle\int_0^r 4\pi s^2\widehat{u}_0 ds=0, \qquad r\in[0,\widehat{R}_0),\ t>0, \\ \vartheta(r,0)=\eta_0-\widetilde{\eta}_0, \quad \vartheta(\widehat{R}_0,t)=\vartheta(0,t)=0, \qquad r\in[0,\widehat{R}_0],\ t\geq 0. \end{cases}$$

Define the nonlinear energy functional $E_1(t)$ for the perturbation $\vartheta(r,t)$ by

$$E_1(t)\triangleq\sum_{j=0}^{3+[\frac{\alpha}{2}]}\sum_{i=1}^{5+[\alpha]-2j}(1+t)^{2j}\int_0^{\widehat{R}_0}\big[r^4\sigma_1^{\alpha+i+1}(\partial_t^j\partial_r^{i+1}\vartheta)^2+r^2\sigma_1^{\alpha+i-1}(\partial_t^j\partial_r^i\vartheta)^2\big]dr$$
$$+\sum_{j=0}^{3+[\frac{\alpha}{2}]}(1+t)^{2j}\int_0^{\widehat{R}_0}\big[r^4\sigma_1^\alpha(\partial_t^j\vartheta)^2+r^2\sigma_1^{\alpha+1}(\partial_t^j\vartheta)^2+r^4\sigma_1^{\alpha+1}(\partial_t^j\vartheta_r)^2\big]dr. \tag{33}$$

Then, we have the following main results.

Theorem 2.1. *Assume that $\sigma_1=\widehat{u}_0^{m-1}$, $m\in(1,\frac{4}{3})$, $\alpha=\frac{1}{m-1}$, $0<\overline{\alpha}<\frac{8-6m}{3m-1}$, $0<M_1|a_1a_2-b_1b_2|\ll 1$, and* (18) *holds. Then, there is a small constant $\varepsilon_1>0$ such that if the initial energy $E_1(0)\leq\varepsilon_1^2$, a unique global strong solution $\vartheta(r,t)$ to the initial boundary value problem* (32) *exists and satisfies for any $t\geq 0$ that*

$$\vartheta(r,t)\in L^\infty(0,\infty;H^1([0,\widehat{R}_0]))\cap L^2(0,\infty;H^2([0,\widehat{R}_0])), \tag{34}$$

and

$$E_1(t)+\sum_{j=0}^{3+[\frac{\alpha}{2}]}\int_0^t\int_0^{\widehat{R}_0}(1+s)^{2j-1}\big\{(1+s)^2r^4\sigma_1^\alpha(\partial_s^{j+1}\vartheta)^2+r^2\sigma_1^{\alpha+1}[(\partial_s^j\vartheta)^2$$
$$+(r\partial_s^j\vartheta_r)^2]\big\}drds\leq C_1E_1(0), \tag{35}$$

where C_1 is a positive constant.

Next, we consider the free boundary value problem (10)-(11) *for subcritical case (i.e., $m>\frac{4}{3}$).* If the attraction effect dominates the process (i.e., $a_1a_2-b_1b_2>0$), there is a balance between the nonlinear diffusion and the nonlocal attraction-repulsion, Carrillo et al [5] have shown that there is a unique spherically symmetric steady-state solution $\bar{u}=\bar{u}(\eta)$ with compact support $[0,\bar{R}]$ to the system (4) as

$$(\bar{u}^m)_\eta+(a_1a_2-b_1b_2)\frac{\bar{u}}{\eta^2}\int_0^\eta 4\pi s^2\bar{u}(s)ds=0, \quad M_2=\int_0^{\bar{R}}4\pi s^2\bar{u}(s)ds, \tag{36}$$

and

$$-\infty<(\bar{u}^{m-1})_\eta|_{\eta=\bar{R}}<0, \quad \bar{u}^{m-1}(\eta)\sim\bar{R}-\eta \quad \text{as} \quad \eta\to\bar{R}. \tag{37}$$

Suppose that the initial total mass of the solution to (10) equals to that of the steady-state solution

$$\int_0^{R_0}4\pi s^2u_0(s)ds=\int_0^{\bar{R}}4\pi s^2\bar{u}(s)ds=M_2>0. \tag{38}$$

Define the particle path $\eta(r,t)$ of the moving domain $[0,R(t)]$ by

$$\begin{cases} \eta_t(r,t) = V(\eta(r,t),t), & r \in [0,\bar{R}],\ t>0, \\ \eta(r,0) = \eta_0(r), \quad \eta_0(\bar{R}) = R_0, & r \in [0,\bar{R}], \end{cases} \tag{39}$$

where the function $\eta_0(r)$ satisfies

$$\int_0^{\eta_0(r)} s^2 u_0(s)ds = \int_0^r s^2 \bar{u}(s)ds, \quad r \in [0,\bar{R}]. \tag{40}$$

Set the weight function $\sigma_2 \triangleq \bar{u}^{m-1}$ and the perturbation function

$$\varphi(r,t) \triangleq \frac{\eta}{r} - 1, \quad r \in [0,\bar{R}],\ t>0. \tag{41}$$

Applying the similar arguments to deal with the (19) and (31), we can obtain the following initial boundary value problem

$$\begin{cases} r\sigma_2^\alpha \varphi_t + (1+\varphi)^2 \Big[\sigma_2^{\alpha+1} \dfrac{(1+\varphi)^{-2m}}{(1+\varphi+r\varphi_r)^m}\Big]_r - (\sigma_2^{\alpha+1})_r (1+\varphi)^{-2} = 0, \quad r \in [0,\bar{R}),\ t>0 \\ \varphi(r,0) = \dfrac{\eta_0}{r} - 1, \quad \varphi(\bar{R},t) = \dfrac{\eta(\bar{R},t)}{\bar{R}} - 1, \quad \varphi(0,t) = 0, \quad r \in [0,\bar{R}],\ t \geq 0. \end{cases} \tag{42}$$

Define the nonlinear energy functional $E_2(t)$ for $\varphi(r,t)$ as

$$\begin{aligned} E_2(t) \triangleq & \sum_{j=0}^{3+[\frac{\alpha}{2}]} \sum_{i=1}^{5+[\alpha]-2j} \int_0^{\bar{R}} \big[r^4 \sigma_2^{\alpha+i+1} (\partial_t^j \partial_r^{i+1} \varphi)^2 + r^2 \sigma_2^{\alpha+i-1} (\partial_t^j \partial_r^i \varphi)^2\big] dr \\ & + \sum_{j=0}^{3+[\frac{\alpha}{2}]} \int_0^{\bar{R}} \big[r^4 \sigma_2^\alpha (\partial_t^j \varphi)^2 + r^2 \sigma_2^{\alpha+1} (\partial_t^j \varphi)^2 + r^4 \sigma_2^{\alpha+1} (\partial_t^j \varphi_r)^2\big] dr. \end{aligned} \tag{43}$$

Using the similar arguments as proving the Proposition 2.1 in [13], we can show that the spherically symmetric solution to the free boundary value problem (10)-(11) exists globally in time and converges exponentially to the steady-state solution (36) as follows, the details in the proof are omitted.

Theorem 2.2. *Assume that $\sigma_2 = \bar{u}^{m-1}$, $\alpha = \frac{1}{m-1}$, $m \in (\frac{4}{3},\infty)$, $a_1a_2 - b_1b_2 > 0$, and (38) holds. Then, there is a constant $0 < \varepsilon_2 \ll 1$ such that if the initial energy $E_2(0) \leq \varepsilon_2^2$, a unique global strong solution $\varphi(r,t)$ to the initial boundary value problem (42) exists and satisfies for any $t \geq 0$ that*

$$\varphi(r,t) \in L^\infty(0,\infty; H^1([0,\bar{R}])) \cap L^2(0,\infty; H^2([0,\bar{R}])), \tag{44}$$

and

$$E_2(t) + \sum_{j=0}^{3+[\frac{\alpha}{2}]} \int_0^t \int_0^{\bar{R}} \big[r^4 \sigma_2^\alpha (\partial_s^{j+1} \varphi)^2 + r^2 \sigma_2^{\alpha+1} (\partial_s^j \varphi)^2 + r^4 \sigma_2^{\alpha+1} (\partial_s^j \varphi_r)^2\big] drds \leq C_2 E_2(0), \tag{45}$$

along with the following long-time decay

$$E_2(t) \leq C_2 e^{-C_3 t} E_2(0), \tag{46}$$

where C_2 and C_3 are two positive constants.

Remark 1 : For supercritical case (i.e., $1 < m < \frac{4}{3}$), we can show by applying Theorem 2.1 and the Lagrangian variable $r = r(\eta,t)$ with $\int_0^r y^2 \widehat{u}_0 dy = \int_0^\eta s^2 u ds$, that there is a unique global spherically symmetric strong solution $(u(\eta,t), V(\eta,t),$

$R(t)$) in Eulerian coordinates to the free boundary value problem (10)-(11) so that it holds

(47)
$$\begin{cases} \left|u(\eta,t)-\widehat{u}\big(r(\eta,t)(1+t)^{\frac{1}{3m-1}},t\big)\right| \le C_4\varepsilon_1\widehat{u}_0(1+t)^{-\frac{4}{3m-1}}, & \eta\in[0,R(t)),\\ \left|V(\eta,t)-\widehat{V}\big(r(\eta,t)(1+t)^{\frac{1}{3m-1}},t\big)\right| \le C_4\varepsilon_1(1+t)^{-1+\frac{1}{3m-1}-\frac{\overline{\alpha}}{2}}, & \eta\in[0,R(t)),\\ C_4^{-1}(1+t)^{\frac{1}{3m-1}} \le R(t) \le C_4(1+t)^{\frac{1}{3m-1}},\ \left|\dfrac{d^kR(t)}{dt^k}\right| \le C_4(1+t)^{\frac{1}{3m-1}-k}, & k=1,2,\\ C_4^{-1}(1+t)^{\frac{1}{3m-1}-1} \le \left|(u^{m-1})_\eta(\eta,t)\right| \le C_4(1+t)^{\frac{1}{3m-1}-1}, & \eta\in\Big[\dfrac12R(t),R(t)\Big], \end{cases}$$

where C_4 is a positive constant.

For subcritical case (i.e., $m>\frac43$), we can obtain based on Theorem 2.2 and the Lagrangian variable $r=r(\eta,t)$ with $\int_0^r y^2\bar{u}dy=\int_0^\eta s^2uds$, that there is a unique global spherically symmetric strong solution $(u(\eta,t),V(\eta,t),R(t))$ in Eulerian coordinates to the free boundary value problem (10)-(11) satisfying

(48)
$$\begin{cases} \left|u(\eta,t)-\bar{u}(r(\eta,t))\right|+\left|V(\eta,t)\right|+\left|R(t)-\bar{R}\right|+\left|\dfrac{d^kR(t)}{dt^k}\right| \le C_5\varepsilon_2e^{-C_3t}, k=1,2,\ \eta\in[0,R(t)),\\ C_5^{-1}\varepsilon_2e^{-C_3t} \le \left|(u^{m-1})_\eta(\eta,t)-(\bar{u}^{m-1})_\eta(r,t)\right| \le C_5\varepsilon_2e^{-C_3t}, \qquad \eta\in\Big[\dfrac12R(t),R(t)\Big], \end{cases}$$

where C_3,C_5 are two positive constants.

Remark 2 : At present, we don't have an effective way to investigate the free boundary value problem (10)-(11) for critical case (i.e., $m=\frac43$). To be more precise, on one hand, if the initial data is a small perturbation of the self-similar Barenblatt solution, we expect the strong solution for the cell density exists globally in time and converges to the self-similar Barenblatt solution. Thus, the correction term $\varsigma(r,t)$ should be a strong solution of (29) and satisfy the regularity estimates (30), whereas, it is essential that the constant

$$0<\overline{\alpha}<\frac{8-6m}{3m-1}\quad\text{for}\quad m>1,$$

which means the diffusive component $1<m<\frac43$.

On the other hand, if the initial data is a small perturbation of the steady-state solution, we expect the strong solution for the cell density exists globally in time and converges to the steady-state solution, in the process of proof, for example, it is essential that the principal terms satisfy the relationship as follow

$$(9m-12)\varphi^2+(6m-8)\varphi\cdot r\varphi_r+m(r\varphi_r)^2>C(m)[\varphi^2+(r\varphi_r)^2],\quad C(m)>0,$$

which implies the diffusive component $m>\frac43$.

3. Proof of main results

3.1. Preliminaries. In this subsection, we introduce some weighted Sobolev embedding inequalities [12] which can be used to prove the main results in the Section 3.2.

Lemma 3.1. ([12]) *Assume that Ω is a bounded interval with distance function $\delta(r)\triangleq dist(r,\partial\Omega)$ near boundary. Set the weighted Sobolev spaces $H^{a,b}(\Omega)$ as*

$$H^{a,b}(\Omega)\triangleq\Big\{\delta^{\frac{a}{2}}f\in L^2(\Omega):\ \int_\Omega\delta^a|\partial_r^kf|^2dr<\infty,\ 0\le k\le b\Big\},$$

with the norm

$$\|f\|^2_{H^{a,b}(\Omega)} \triangleq \sum_{k=0}^{b} \int_\Omega \delta^a |\partial_r^k f|^2 dr, \quad b \geq \frac{a}{2} > 0.$$

Then, it holds

$$H^{a,b}(\Omega) \hookrightarrow H^{b-\frac{a}{2}}(\Omega), \quad \|f\|_{H^{b-\frac{a}{2}}(\Omega)} \leq C(a,b,\Omega)\|f\|_{H^{a,b}(\Omega)}, \tag{49}$$

where b is a positive integer and $C(a,b,\Omega)$ is a positive constant.

Lemma 3.2. ([12]) *Assume that $r_1 \in \Omega$ is a positive constant, and the function $f(r,t)$ defines on bounded interval Ω and satisfies*

$$\int_0^{r_1} r^k (f^2 + f_r^2) dr < \infty, \quad k > 1,$$

then, it holds

$$\int_0^{r_1} r^{k-2} f^2 dr \leq C(\delta,k) \int_0^{r_1} r^k (f^2 + f_r^2) dr, \tag{50}$$

where $C(\delta,k)$ is a positive constant.

Remark 3 : If $\Omega = [0, \widehat{R}_0]$ and $\widehat{R}_0 = \sqrt{\widehat{\beta}/\widehat{\gamma}}$ in Lemma 3.2, it also holds that

$$\int_{\sqrt{\widehat{\beta}/2\widehat{\gamma}}}^{\sqrt{\widehat{\beta}/\widehat{\gamma}}} \sigma_1^{k-2} f^2 dr \leq C(\widehat{\beta},\widehat{\gamma},k) \int_{\sqrt{\widehat{\beta}/2\widehat{\gamma}}}^{\sqrt{\widehat{\beta}/\widehat{\gamma}}} \sigma_1^k (f^2 + f_r^2) dr < \infty, \quad C(\widehat{\beta},\widehat{\gamma},k) > 0. \tag{51}$$

3.2. The a-priori estimates. In this subsection, we shall prove Theorem 2.1 in two steps. To begin with, we show the following basic weighted energy estimates.

Lemma 3.3. *Let $T > 0$ and the assumptions of Theorem 2.1 holds, and there is a constant $0 < \varepsilon_1 \ll 1$ such that the strong solution $\vartheta(r,t)$ to the initial boundary value problem (32) for $t \in (0,T]$ satisfies*

$$\|\vartheta\|^2_{L^\infty([0,\widehat{R}_0])} + \|\vartheta_r\|^2_{L^\infty([0,\widehat{R}_0])} + (1+t)^2 \big(\|\vartheta_t\|^2_{L^\infty([0,\widehat{R}_0])} + \|\vartheta_{rt}\|^2_{L^\infty([0,\widehat{R}_0])}\big) \leq \varepsilon_1^2 \ll 1. \tag{52}$$

Then, it holds for any $t \in [0,T]$ that

$$\begin{aligned}
&\int_0^{\widehat{R}_0} \big(r^4\sigma_1^\alpha \vartheta^2 + r^2\sigma_1^{\alpha+1}\vartheta^2 + r^4\sigma_1^{\alpha+1}\vartheta_r^2 + r^4\sigma_1^{\alpha+2}\vartheta_{rr}^2 + r^2\sigma_1^\alpha\vartheta_r^2\big)(r,t)dr \\
&+ (1+t)^2 \int_0^{\widehat{R}_0} \big(r^4\sigma_1^\alpha\vartheta_t^2 + r^2\sigma_1^{\alpha+1}\vartheta_t^2 + r^4\sigma_1^{\alpha+1}\vartheta_{rt}^2\big)(r,t)dr \\
&+ \int_0^t \int_0^{\widehat{R}_0} \big[(1+s)r^4\sigma_1^\alpha\vartheta_s^2 + (1+s)^{-1}r^2\sigma_1^{\alpha+1}(\vartheta^2 + r^2\vartheta_r^2)\big](r,s)drds \\
&+ \int_0^t \int_0^{\widehat{R}_0} \big[(1+s)^3 r^4\sigma_1^\alpha\vartheta_{ss}^2 + (1+s)r^2\sigma_1^{\alpha+1}(\vartheta_s^2 + r^2\vartheta_{rs}^2)\big](r,s)drds \\
&\leq C_6 \sum_{j=0}^{1} \int_0^{\widehat{R}_0} \big[r^4\sigma_1^\alpha(\partial_t^l\vartheta)^2 + r^2\sigma_1^{\alpha+1}(\partial_t^l\vartheta)^2 + r^4\sigma_1^{\alpha+1}(\partial_t^l\vartheta_r)^2\big](r,0)dr,
\end{aligned} \tag{53}$$

where C_6 is a positive constant.

Proof. First, multiplying the first equation of (32) by $r^3\vartheta$ and $r^3\vartheta_t$ respectively, integrating the resulted equations by parts over $[0,\widehat{R}_0]$, and applying the a-priori assumption (52), we can obtain after a complicated computation that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^{\widehat{R}_0} r^4\sigma_1^\alpha\vartheta^2dr+(1+t)^{-1}\int_0^{\widehat{R}_0} r^2\sigma_1^{\alpha+1}[\vartheta^2+(r\vartheta_r)^2]dr\\ &\le(b_1b_2-a_1a_2)\int_0^{\widehat{R}_0} r\sigma_1^\alpha\vartheta\big\{[\widehat{\eta}_r(1+\varsigma)+\vartheta]^{-2}-[\widehat{\eta}_r(1+\varsigma)]^{-2}\big\}\int_0^r 4\pi s^2\widehat{u}_0dsdr,\end{aligned}\tag{54}$$

and

$$\begin{aligned}&\frac{d}{dt}\int_0^{\widehat{R}_0} r^2\sigma_1^{\alpha+1}\mathfrak{F}_1(r,t)dr+\int_0^{\widehat{R}_0}\big\{r^4\sigma_1^\alpha\vartheta_t^2+(1+t)^{-2}r^2\sigma_1^{\alpha+1}[\vartheta^2+(r\vartheta_r)^2]\big\}dr\\ &\le(b_1b_2-a_1a_2)\int_0^{\widehat{R}_0} r\sigma_1^\alpha\vartheta_t\big\{[\widehat{\eta}_r(1+\varsigma)+\vartheta]^{-2}-[\widehat{\eta}_r(1+\varsigma)]^{-2}\big\}\int_0^r 4\pi s^2\widehat{u}_0dsdr,\end{aligned}\tag{55}$$

where the nonlinear function $\mathfrak{F}_1(r,t)$ is expressed by

$$\begin{aligned}&\mathfrak{F}_1(r,t)\\ &\triangleq\frac{\big[\widehat{\eta}_r(1+\varsigma)+\vartheta\big]^{2-2m}}{(m-1)\big[\widehat{\eta}_r(1+\varsigma+r\varsigma_r)+\vartheta+r\vartheta_r\big]^{m-1}}-(1+t)^{-1+\frac{2}{3m-1}}\frac{(1+\varsigma)^{2-2m}}{(m-1)(1+\varsigma+r\varsigma_r)^{m-1}}\\ &\quad+(1+t)^{-1+\frac{1}{3m-1}}\frac{(1+\varsigma)^{2-2m}}{(1+\varsigma+r\varsigma_r)^m}(\vartheta+r\vartheta_r)+2(1+t)^{-1+\frac{1}{3m-1}}\frac{(1+\varsigma)^{1-2m}}{(1+\varsigma+r\varsigma_r)^{m-1}}\vartheta,\end{aligned}$$

and satisfies

$$C^{-1}(1+t)^{-1}[\vartheta^2+(r\vartheta_r)^2]\le\mathfrak{F}_1(r,t)\le C(1+t)^{-1}[\vartheta^2+(r\vartheta_r)^2],\tag{56}$$

for a generic positive constant C.
Moreover, we have after a straightforward computation that

$$\begin{aligned}&\Big|(b_1b_2-a_1a_2)\int_0^{\widehat{R}_0} r\sigma_1^\alpha\vartheta\big\{[\widehat{\eta}_r(1+\varsigma)+\vartheta]^{-2}-[\widehat{\eta}_r(1+\varsigma)]^{-2}\big\}\int_0^r 4\pi s^2\widehat{u}_0dsdr\Big|\\ &\le M_1|b_1b_2-a_1a_2|(1+t)^{-\frac{3}{3m-1}}\int_0^{\widehat{R}_0} r^2\sigma_1^{\alpha+1}[\vartheta^2+(r\vartheta_r)^2]dr,\end{aligned}\tag{57}$$

$$\begin{aligned}&\Big|(b_1b_2-a_1a_2)\int_0^{\widehat{R}_0} r\sigma_1^\alpha\vartheta_t\big\{[\widehat{\eta}_r(1+\varsigma)+\vartheta]^{-2}-[\widehat{\eta}_r(1+\varsigma)]^{-2}\big\}\int_0^r 4\pi s^2\widehat{u}_0dsdr\Big|\\ &\le b\int_0^{\widehat{R}_0} r^4\sigma_1^\alpha\vartheta_t^2dr+C_bM_1(b_1b_2-a_1a_2)^2(1+t)^{-\frac{6}{3m-1}}\int_0^{\widehat{R}_0} r^2\sigma_1^{\alpha+1}[\vartheta^2+(r\vartheta_r)^2]dr,\end{aligned}\tag{58}$$

for the positive constants $0<b\ll 1$ and C_b.

Therefore, substituting (57) and (58) into (54) and (55) respectively, and using $0<M_1|b_1b_2-a_1a_2|\ll 1$, we have

$$\frac{d}{dt}\int_0^{\widehat{R}_0} r^4\sigma_1^\alpha\vartheta^2dr+(1+t)^{-1}\int_0^{\widehat{R}_0} r^2\sigma_1^{\alpha+1}[\vartheta^2+(r\vartheta_r)^2]dr\le 0,\tag{59}$$

and

$$\frac{d}{dt}\int_0^{\widehat{R}_0} r^2\sigma_1^{\alpha+1}\mathfrak{F}_1(r,t)dr+\int_0^{\widehat{R}_0}\left\{r^4\sigma_1^{\alpha}\vartheta_t^2+(1+t)^{-2}r^2\sigma_1^{\alpha+1}[\vartheta^2+(r\vartheta_r)^2]\right\}dr\le 0. \tag{60}$$

Integrating the summation $(59)+(1+t)\times(60)$ with respect to time on $[0,t]$ yields to

$$\begin{aligned}&\int_0^{\widehat{R}_0}\left(r^4\sigma_1^{\alpha}\vartheta^2+r^2\sigma_1^{\alpha+1}\vartheta^2+r^4\sigma_1^{\alpha+1}\vartheta_r^2\right)dr\\&+\int_0^t\int_0^{\widehat{R}_0}(1+s)^{-1}\left[(1+s)^2r^4\sigma_1^{\alpha}\vartheta_s^2+r^2\sigma_1^{\alpha+1}\vartheta^2+r^4\sigma_1^{\alpha+1}\vartheta_r^2\right]drds\\ \le& C\int_0^{\widehat{R}_0}\left(r^4\sigma_1^{\alpha}\vartheta^2+r^2\sigma_1^{\alpha+1}\vartheta^2+r^4\sigma_1^{\alpha+1}\vartheta_r^2\right)(r,0)dr.\end{aligned}\tag{61}$$

Furthermore, differentiating the first equation of (32) about the time t, we obtain

$$\begin{aligned}&r\sigma_1^{\alpha}\vartheta_{tt}-\sigma_1^{\alpha+1}\left\{2(\theta_3+\theta_4)[\widehat{\eta}_r(1+\varsigma)+\vartheta]_r\vartheta_t-(3\theta_2-\theta_1)\vartheta_{rt}\right\}\\&+[\sigma_1^{\alpha+1}(\theta_1\vartheta_t+r\theta_2\vartheta_{rt})]_r\\&=\mathfrak{F}_2(r,t)+\mathfrak{F}_3(r,t),\end{aligned}\tag{62}$$

where the nonlinear terms $\theta_i(r,t)$, $(i=1,2,3,4)$ and the low-order terms $\mathfrak{F}_j(r,t)$, $(j=2,3)$ are given by

$$\begin{aligned}\theta_1(r,t)\triangleq&(2-2m)\frac{[\widehat{\eta}_r(1+\varsigma)+\vartheta]^{1-2m}}{[\widehat{\eta}_r(1+\varsigma+r\varsigma_r)+\vartheta+r\vartheta_r]^m}\\&-m\frac{[\widehat{\eta}_r(1+\varsigma)+\vartheta]^{2-2m}}{[\widehat{\eta}_r(1+\varsigma+r\varsigma_r)+\vartheta+r\vartheta_r]^{m+1}},\\ \theta_2(r,t)\triangleq&-m[\widehat{\eta}_r(1+\varsigma)+\vartheta]^{2-2m}[\widehat{\eta}_r(1+\varsigma+r\varsigma_r)+\vartheta+r\vartheta_r]^{-m-1},\\ \theta_3(r,t)\triangleq&(1-2m)[\widehat{\eta}_r(1+\varsigma)+\vartheta]^{-2m}[\widehat{\eta}_r(1+\varsigma+r\varsigma_r)+\vartheta+r\vartheta_r]^{-m},\\ \theta_4(r,t)\triangleq&-m[\widehat{\eta}_r(1+\varsigma)+\vartheta]^{1-2m}[\widehat{\eta}_r(1+\varsigma+r\varsigma_r)+\vartheta+r\vartheta_r]^{-m-1},\end{aligned}$$

and

$$\begin{aligned}\mathfrak{F}_2(r,t)\triangleq&2\sigma_1^{\alpha+1}\Big[\theta_3[\widehat{\eta}_r(1+\varsigma)+\vartheta]_r+(2m-1)(1+t)^{-1}\frac{(1+\varsigma)^{-2m}}{(1+\varsigma+r\varsigma_r)^m}\varsigma_r\Big][\widehat{\eta}_r(1+\varsigma)]_t\\&+2\sigma_1^{\alpha+1}\Big\{\theta_4[\widehat{\eta}_r(1+\varsigma)+\vartheta]_r+m(1+t)^{-1}\frac{(1+\varsigma)^{1-2m}}{(1+\varsigma+r\varsigma_r)^{m+1}}\varsigma_r\Big\}[\widehat{\eta}_r(1+\varsigma)]_t\\&-\Big\{\sigma_1^{\alpha+1}\Big[\theta_1-\theta_2-(2-2m)(1+t)^{-1}\frac{(1+\varsigma)^{1-2m}}{(1+\varsigma+r\varsigma_r)^m}\Big][\widehat{\eta}_r(1+\varsigma)]_t\Big\}_r\\&-\Big\{\sigma_1^{\alpha+1}\Big[\theta_2+m(1+t)^{-1}\frac{(1+\varsigma)^{2-2m}}{(1+\varsigma+r\varsigma_r)^{m+1}}\Big][\widehat{\eta}_r(1+\varsigma+r\varsigma_r)]_t\Big\}_r\\&+2\sigma_1^{\alpha+1}\Big[\frac{1}{2-2m}(\theta_1-\theta_2)-(1+t)^{-1}\frac{(1+\varsigma)^{1-2m}}{(1+\varsigma+r\varsigma_r)^m}\Big](\widehat{\eta}_r\varsigma_r)_t,\end{aligned}\tag{63}$$

$$\mathfrak{F}_3(r,t)\triangleq(b_1b_2-a_1a_2)\frac{\sigma_1^{\alpha}}{r^2}\left[(\widehat{\eta}_r+\widehat{\eta}_r\varsigma+\vartheta)^{-2}-(\widehat{\eta}_r+\widehat{\eta}_r\varsigma)^{-2}\right]_t\int_0^r 4\pi s^2\widehat{u}_0ds. \tag{64}$$

We can prove by (52) that

(65)
$$
\begin{cases}
\theta_1(r,t) \leq (1+t)^{-1}\Big[(2-2m)\dfrac{(1+\varsigma)^{1-2m}}{(1+\varsigma+r\varsigma_r)^m} - m\dfrac{(1+\varsigma)^{2-2m}}{(1+\varsigma+r\varsigma_r)^{m+1}}\Big] \\
\qquad\qquad + \varepsilon_1(1+t)^{-\frac{3m}{3m-1}}\Big[1+\sum_{l=1}^{\infty}\big(|\vartheta|^l + |r\vartheta_r|^l\big)\Big], \\
\theta_2(r,t) \leq -m(1+t)^{-1}\dfrac{(1+\varsigma)^{2-2m}}{(1+\varsigma+r\varsigma_r)^{m+1}} + \varepsilon_1(1+t)^{-\frac{3m}{3m-1}}\Big[1+\sum_{l=1}^{\infty}\big(|\vartheta|^l + |r\vartheta_r|^l\big)\Big], \\
\theta_3(r,t) \leq (1-2m)(1+t)^{-1}\dfrac{(1+\varsigma)^{-2m}}{(1+\varsigma+r\varsigma_r)^m} + \varepsilon_1(1+t)^{-\frac{3m+1}{3m-1}}\Big[1+\sum_{l=1}^{\infty}\big(|\vartheta|^l + |r\vartheta_r|^l\big)\Big], \\
\theta_4(r,t) \leq -m(1+t)^{-\frac{3m}{3m-1}}\dfrac{(1+\varsigma)^{1-2m}}{(1+\varsigma+r\varsigma_r)^{m+1}} + \varepsilon_1(1+t)^{-\frac{3m+1}{3m-1}}\Big[1+\sum_{l=1}^{\infty}\big(|\vartheta|^l + |r\vartheta_r|^l\big)\Big].
\end{cases}
$$

Multiplying the Eq. (62) by $r^3\vartheta_t$ and $r^3\vartheta_{tt}$, and integrating the resulted equations by parts on $[0,\widehat{R}_0]$ respectively, we can derive the following estimates from the a-priori assumption (52) and Cauchy inequality as

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^{\widehat{R}_0} r^4\sigma_1^\alpha\vartheta_t^2dr + \int_0^{\widehat{R}_0} r^2\sigma_1^{\alpha+1}\mathfrak{F}_4(r,t)dr \\
\leq& C_b(1+t)^{-3}\Big[1+M_1(b_1b_2-a_1a_2)^2(1+t)^{-\frac{8-6m}{3m-1}}\Big]\int_0^{\widehat{R}_0} r^2\sigma_1^{\alpha+1}[\vartheta^2+(r\vartheta_r)^2]dr \\
&+ (b+M_1|b_1b_2-a_1a_2|)(1+t)^{-1}\int_0^{\widehat{R}_0} r^2\sigma_1^{\alpha+1}[\vartheta_t^2+(r\vartheta_{rt})^2]dr,
\end{aligned}
\tag{66}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^{\widehat{R}_0}\big[r^2\sigma_1^{\alpha+1}\mathfrak{F}_4(r,t) - 2r^3\vartheta_t\mathfrak{F}_2(r,t)\big]dr + \int_0^{\widehat{R}_0} r^4\sigma_1^\alpha\vartheta_{tt}^2dr \\
\leq& b\int_0^{\widehat{R}_0} r^4\sigma_1^\alpha\vartheta_{tt}^2dr + (1+t)^{-2-\frac{1}{3m-1}}\Big[\varepsilon_1+(1+t)^{\frac{2}{3m-1}}\Big]\int_0^{\widehat{R}_0} r^2\sigma_1^{\alpha+1}\big[\vartheta_t^2+(r\vartheta_{rt})^2\big]dr \\
&+ (1+t)^{-2}\Big[b+C_bM_1(b_1b_2-a_1a_2)^2(1+t)^{-\frac{8-6m}{3m-1}}\Big]\int_0^{\widehat{R}_0} r^2\sigma_1^{\alpha+1}[\vartheta_t^2+(r\vartheta_{rt})^2]dr \\
&+ C_b(1+t)^{-4}\Big[1+M_1(b_1b_2-a_1a_2)^2(1+t)^{-\frac{8-6m}{3m-1}}\Big]\int_0^{\widehat{R}_0} r^2\sigma_1^{\alpha+1}[\vartheta^2+(r\vartheta_r)^2]dr,
\end{aligned}
\tag{67}
$$

where $0 < b \ll 1$ and C_b are two positive constants, and the nonlinear function $\mathfrak{F}_4(r,t)$ is given by

$$
\mathfrak{F}_4(r,t) \triangleq -\big\{3\theta_1 + 2r(\theta_3+\theta_4)[\widehat{\eta}_r(1+\varsigma)+\vartheta]_r\big\}\vartheta_t^2 - \theta_2(r\vartheta_{rt})^2 - 2r\theta_1\vartheta_t\vartheta_{rt},
$$

and satisfies

$$
C^{-1}(1+t)^{-1}[\vartheta_t^2+(r\vartheta_{rt})^2] \leq \mathfrak{F}_4(r,t) \leq C(1+t)^{-1}[\vartheta_t^2+(r\vartheta_{rt})^2],
\tag{68}
$$

for a generic positive constant C.

Thus, based on $0 < M_1|a_1a_2-b_1b_2| \ll 1$, one has that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^{\widehat{R}_0} r^4\sigma_1^\alpha\vartheta_t^2dr + (1+t)^{-1}\int_0^{\widehat{R}_0} r^2\sigma_1^{\alpha+1}\big[\vartheta_t^2+(r\vartheta_{rt})^2\big]dr \\
\leq& C_\epsilon(1+t)^{-3}\int_0^{\widehat{R}_0} r^2\sigma_1^{\alpha+1}\big[\vartheta^2+(r\vartheta_r)^2\big]dr,
\end{aligned}
\tag{69}
$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^{\widehat{R}_0}\left[r^2\sigma_1^{\alpha+1}\mathfrak{F}_4(r,t)-2r^3\vartheta_t\mathfrak{F}_2(r,t)\right]dr+\int_0^{\widehat{R}_0}r^4\sigma_1^{\alpha}\vartheta_{tt}^2dr\\
\leq&(1+t)^{-2}\Big[\epsilon+C(1+t)^{-\frac{1}{3m-1}}\Big]\int_0^{\widehat{R}_0}r^2\sigma_1^{\alpha+1}\left[\vartheta_t^2+(r\vartheta_{rt})^2\right]dr\\
&+C_\epsilon(1+t)^{-4}\int_0^{\widehat{R}_0}r^2\sigma_1^{\alpha+1}\left[\vartheta^2+(r\vartheta_r)^2\right]dr. \qquad (70)
\end{aligned}$$

Taking the summation $(1+t)^p\times(69)$ and $(1+t)^{1+p}\times(70)$ together for $p=1,2$, and integrating the resulted inequalities with respect to the time on $[0,t]$ respectively, we can obtain by applying(68) that

$$\begin{aligned}
&(1+t)^2\int_0^{\widehat{R}_0}\left(r^4\sigma_1^{\alpha}\vartheta_t^2+r^2\sigma_1^{\alpha+1}\vartheta_t^2+r^4\sigma_1^{\alpha+1}\vartheta_{rt}^2\right)dr\\
&+\int_0^t\int_0^{\widehat{R}_0}\left\{(1+s)^3r^4\sigma_1^{\alpha}\vartheta_{ss}^2+(1+s)r^2\sigma_1^{\alpha+1}[\vartheta_s^2+(r\vartheta_{rs})^2]\right\}drds\\
\leq&C\sum_{j=0}^{1}\int_0^{\widehat{R}_0}\left[r^4\sigma_1^{\alpha}(\partial_t^l\vartheta)^2+r^2\sigma_1^{\alpha+1}(\partial_t^l\vartheta)^2+r^4\sigma_1^{\alpha+1}(\partial_t^l\vartheta_r)^2\right](r,0)dr. \qquad (71)
\end{aligned}$$

Next, we divide the first equation of system (32) by σ_1^{α} and rewrite the resulted equation to obtain

$$\begin{aligned}
&m(1+t)^{-1}\frac{(1+\varsigma)^{2-2m}}{(1+\varsigma+r\varsigma_r)^{m+1}}\left[r\sigma_1\vartheta_{rr}+4\sigma_1\vartheta_r+(1+\alpha)r\sigma_{1r}\vartheta_r\right]\\
=&r\vartheta_t+(1+\alpha)(1+t)^{-1}\frac{(1+\varsigma)^{2-2m}}{(1+\varsigma+r\varsigma_r)^{m+1}}\Big[(2-3m)+(2-2m)\frac{r\varsigma_r}{1+\varsigma}\Big]\sigma_{1r}\vartheta\\
&-2m(1+t)^{-1}\frac{(1+\varsigma)^{1-2m}}{(1+\varsigma+r\varsigma_r)^{m+1}}r\varsigma_r\sigma_1\vartheta_r+\sigma_1\mathbf{g}_1(r,t)+(1+\alpha)\sigma_{1r}\mathbf{g}_2(r,t)\\
&+\mathbf{g}_3(r,t), \qquad (72)
\end{aligned}$$

where the lower order terms $\mathbf{g}_i(r,t)$, $(i=1,2,3)$ are defined by

$$\begin{aligned}
&\mathbf{g}_1(r,t)\\
\triangleq&-m\left\{\frac{\left[\widehat{\eta}_r(1+\varsigma)+\vartheta\right]^{2-2m}}{\left[\widehat{\eta}_r(1+\varsigma+r\varsigma_r)+\vartheta+r\vartheta_r\right]^{m+1}}-(1+t)^{-1}\frac{(1+\varsigma)^{2-2m}}{(1+\varsigma+r\varsigma_r)^{m+1}}\right\}(2\vartheta_r+r\vartheta_{rr})\\
&-2m\left\{\frac{\left[\widehat{\eta}_r(1+\varsigma)+\vartheta\right]^{1-2m}}{\left[\widehat{\eta}_r(1+\varsigma+r\varsigma_r)+\vartheta+r\vartheta_r\right]^{m}}-(1+t)^{-1}\frac{(1+\varsigma)^{1-2m}}{(1+\varsigma+r\varsigma_r)^{m}}\right\}\vartheta_r\\
&-2m(1+t)^{\frac{1}{3m-1}}\left\{\frac{\left[\widehat{\eta}_r(1+\varsigma)+\vartheta\right]^{1-2m}}{\left[\widehat{\eta}_r(1+\varsigma+r\varsigma_r)+\vartheta+r\vartheta_r\right]^{m}}-(1+t)^{-1}\frac{(1+\varsigma)^{1-2m}}{(1+\varsigma+r\varsigma_r)^{m}}\right\}\varsigma_r\\
&-m(1+t)^{\frac{1}{3m-1}}\left\{\frac{\left[\widehat{\eta}_r(1+\varsigma)+\vartheta\right]^{2-2m}}{\left[\widehat{\eta}_r(1+\varsigma+r\varsigma_r)+\vartheta+r\vartheta_r\right]^{m+1}}-(1+t)^{-1}\frac{(1+\varsigma)^{2-2m}}{(1+\varsigma+r\varsigma_r)^{m+1}}\right\}\varsigma_{rr}, \qquad (73)
\end{aligned}$$

$$\begin{aligned}
\mathbf{g}_2(r,t)\triangleq&\frac{\left[\widehat{\eta}_r(1+\varsigma)+\vartheta\right]^{2-2m}}{\left[\widehat{\eta}_r(1+\varsigma+r\varsigma_r)+\vartheta+r\vartheta_r\right]^{m}}-(1+t)^{-1+\frac{1}{3m-1}}\frac{(1+\varsigma)^{2-2m}}{(1+\varsigma+r\varsigma_r)^{m}}\\
&+(1+t)^{-1}\Big[2(m-1)\frac{(1+\varsigma)^{1-2m}}{(1+\varsigma+r\varsigma_r)^{m}}+m\frac{(1+\varsigma)^{2-2m}}{(1+\varsigma+r\varsigma_r)^{m+1}}\Big]\vartheta\\
&+m(1+t)^{-1}\frac{(1+\varsigma)^{2-2m}}{(1+\varsigma+r\varsigma_r)^{m+1}}r\vartheta_r, \qquad (74)
\end{aligned}$$

$$\mathbf{g}_3(r,t) \triangleq \frac{a_1a_2 - b_1b_2}{r^2}\Big\{\frac{1}{\widehat{\eta}_r^2} - \frac{1}{[\widehat{\eta}_r(1+\varsigma)+\vartheta]^2}\Big\}\int_0^r 4\pi s^2 \widehat{u}_0 ds. \tag{75}$$

It is easy to verify by (52) that

$$\mathbf{g}_1(r,t) \le C(1+t)^{-1}\big[\varepsilon_1(|\vartheta_r| + |r\vartheta_{rr}|) + \varepsilon_0(1+t)^{-\frac{\overline{\alpha}}{2}}(|\vartheta| + |r\vartheta_r|)\big], \tag{76}$$

$$\mathbf{g}_2(r,t) \le \varepsilon_1(1+t)^{-\frac{3m}{3m-1}}\big(|\vartheta| + |r\vartheta_r|\big), \tag{77}$$

$$\mathbf{g}_3(r,t) \le C|a_1a_2 - b_1b_2|(1+t)^{-\frac{3}{3m-1}}\frac{|\vartheta|}{r^2}\int_0^r 4\pi s^2\widehat{u}_0 ds. \tag{78}$$

Multiplying (72) by $r\sigma_1^{\frac{\alpha}{2}}(1+t)(1+\varsigma)^{2m-2}(1+\varsigma+r\varsigma_r)^{m+1}$ and making L^2-norm over $[0,\widehat{R}_0]$, it follows from (76)-(78) and the a-priori assumption (52) that

$$\begin{aligned}
&\int_0^{\widehat{R}_0}\big[r^2\sigma_1^{1+\frac{\alpha}{2}}\vartheta_{rr} + 4r\sigma_1^{1+\frac{\alpha}{2}}\vartheta_r + (1+\alpha)r^2\sigma_1^{\frac{\alpha}{2}}\sigma_{1r}\vartheta_r\big]^2 dr\\
\le& C\int_0^{\widehat{R}_0}\big[(1+t)^2r^4\sigma_1^{\alpha}\vartheta_t^2 + r^4\sigma_1^{\alpha}\vartheta^2 + r^2\sigma_1^{1+\alpha}\vartheta^2 + r^4\sigma_1^{1+\alpha}\vartheta_r^2\big]dr\\
&+ C\varepsilon_1^2\int_0^{\widehat{R}_0}\big(r^4\sigma_1^{2+\alpha}\vartheta_{rr}^2 + r^2\sigma_1^{2+\alpha}\vartheta_r^2 + r^4\sigma_1^{\alpha}\sigma_{1r}^2\vartheta_r^2\big)dr.
\end{aligned} \tag{79}$$

We can also estimate the terms on left hand side of (79) as

$$\begin{aligned}
&\int_0^{\widehat{R}_0}\big[r^2\sigma_1^{1+\frac{\alpha}{2}}\vartheta_{rr} + 4r\sigma_1^{1+\frac{\alpha}{2}}\vartheta_r + (1+\alpha)r^2\sigma_1^{\frac{\alpha}{2}}\sigma_{1r}\vartheta_r\big]^2 dr\\
=& \int_0^{\widehat{R}_0} r^4\sigma_1^{2+\alpha}\vartheta_{rr}^2 dr + 4\int_0^{\widehat{R}_0} r^2\sigma_1^{2+\alpha}\vartheta_r^2 dr - \int_0^{\widehat{R}_0} r^3\sigma_1^{1+\alpha}[4\sigma_{1r} + (1+\alpha)r\sigma_{1rr}]\vartheta_r^2 dr,
\end{aligned} \tag{80}$$

and

$$\begin{aligned}
(1+\alpha)^2\int_0^{\widehat{R}_0} r^4\sigma_1^{\alpha}\sigma_{1r}^2\vartheta_r^2 dr \le& 2\int_0^{\widehat{R}_0}\big[r^2\sigma_1^{1+\frac{\alpha}{2}}\vartheta_{rr} + 4r\sigma_1^{1+\frac{\alpha}{2}}\vartheta_r + (1+\alpha)r^2\sigma_1^{\frac{\alpha}{2}}\sigma_{1r}\vartheta_r\big]^2 dr\\
&+ 2\int_0^{\widehat{R}_0}\big(r^2\sigma_1^{1+\frac{\alpha}{2}}\vartheta_{rr} + 4r\sigma_1^{1+\frac{\alpha}{2}}\vartheta_r\big)^2 dr.
\end{aligned} \tag{81}$$

According to the inequalities (79)-(81) and the equivalence of σ_{1r} and r, it holds

$$\begin{aligned}
&\int_0^{\widehat{R}_0}\big(r^4\sigma_1^{2+\alpha}\vartheta_{rr}^2 + r^2\sigma_1^{2+\alpha}\vartheta_r^2 + r^6\sigma_1^{\alpha}\vartheta_r^2\big)dr\\
\le& C\int_0^{\widehat{R}_0}\Big[\sum_{l=0}^{1}(1+t)^{2l}r^4\sigma_1^{\alpha}(\partial_t^l\vartheta)^2 + r^2\sigma_1^{1+\alpha}\vartheta^2 + r^4\sigma_1^{1+\alpha}\vartheta_r^2\Big]dr.
\end{aligned} \tag{82}$$

In addition, one has

$$\begin{aligned}
&\int_0^{\widehat{R}_0} r^2\sigma_1^{\alpha}\vartheta_r^2 dr\\
\le& \int_0^{\frac{\widehat{R}_0}{2}} r^2\sigma_1^{2+\alpha}\vartheta_r^2 dr + \int_{\frac{\widehat{R}_0}{2}}^{\widehat{R}_0} r^6\sigma_1^{\alpha}\vartheta_r^2 dr \le C\int_0^{\widehat{R}_0}\big(r^2\sigma_1^{2+\alpha}\vartheta_r^2 + r^6\sigma_1^{\alpha}\vartheta_r^2\big)dr.
\end{aligned} \tag{83}$$

The combination of inequalities (82)-(83) and (61)-(71) leads to

$$\int_0^{\widehat{R}_0}\big(r^4\sigma_1^{\alpha+2}\vartheta_{rr}^2 + r^2\sigma_1^{\alpha}\vartheta_r^2\big)dr$$

$$\leq C\sum_{j=0}^{1}\int_0^{\widehat{R}_0}\left\{r^4\sigma_1^{\alpha}(\partial_t^l\vartheta)^2+r^2\sigma_1^{\alpha+1}[(\partial_t^l\vartheta)^2+(r\partial_t^l\vartheta_r)^2]\right\}(r,0)dr. \tag{84}$$

Finally, we can conclude (53) by (61), (71) and (84). □

Next, we establish the below higher weighted energy estimates by an inductive method.

Lemma 3.4. *Let $T>0$ and the assumptions of Theorem 2.1 holds, and there is a constant $0<\varepsilon_1\ll 1$ such that the strong solution $\vartheta(r,t)$ to the initial boundary value problem* (32) *for $t\in(0,T]$ satisfies*

$$\begin{aligned}&\sum_{j=0}^{2}(1+t)^j\|\partial_t^j\vartheta\|^2_{L^\infty([0,\widehat{R}_0])}+\sum_{j=0}^{1}(1+t)^j\|\partial_t^j\vartheta_r\|^2_{L^\infty([0,\widehat{R}_0])}\\&+\sum_{0\leq j\leq[\frac{2+\alpha}{2}],3<i+2j\leq 3+[\alpha]}(1+t)^j\|\sigma_1^{i+j-\frac{3}{2}}\partial_t^j\partial_r^i\vartheta\|^2_{L^\infty([0,\widehat{R}_0])}\\&+\sum_{0\leq j\leq[\frac{3+\alpha}{2}],i+2j=4+[\alpha]}(1+t)^j\|r\sigma_1^{i+j-\frac{3}{2}}\partial_t^j\partial_r^i\vartheta\|^2_{L^\infty([0,\widehat{R}_0])}\\&+\sum_{0\leq j\leq[\frac{4+\alpha}{2}],i+2j=5+[\alpha]}(1+t)^j\|r^2\sigma_1^{i+j-\frac{3}{2}}\partial_t^j\partial_r^i\vartheta\|^2_{L^\infty([0,\widehat{R}_0])}\leq\varepsilon_1^2\ll 1.\end{aligned}\tag{85}$$

Then, it holds for any $t\in[0,T]$ that

$$\begin{aligned}&\sum_{l=0}^{[\frac{i+2j+1}{2}]}(1+t)^{2l}\int_0^{\widehat{R}_0}\left[r^4\sigma_1^{\alpha}(\partial_t^l\vartheta)^2+r^2\sigma_1^{\alpha+1}(\partial_t^l\vartheta)^2+r^4\sigma_1^{\alpha+1}(\partial_t^l\vartheta_r)^2\right](r,t)dr\\&+(1+t)^{2j}\int_0^{\widehat{R}_0}\left[r^4\sigma_1^{\alpha+i+1}(\partial_t^j\partial_r^{i+1}\vartheta)^2+r^2\sigma_1^{\alpha+i-1}(\partial_t^j\partial_r^i\vartheta)^2\right](r,t)dr\\&+\sum_{l=0}^{[\frac{i+2j+1}{2}]}\int_0^t\int_0^{\widehat{R}_0}(1+s)^{2l-1}\left\{(1+s)^2r^4\sigma_1^{\alpha}(\partial_s^{l+1}\vartheta)^2+r^2\sigma_1^{\alpha+1}[(\partial_s^j\vartheta)^2\right.\\&\left.+(r\partial_s^l\vartheta_r)^2]\right\}(r,s)dsdr\\\leq&C_7\sum_{l=0}^{[\frac{i+2j+1}{2}]}\int_0^{\widehat{R}_0}\left[r^4\sigma_1^{\alpha}(\partial_t^l\vartheta)^2+r^2\sigma_1^{\alpha+1}(\partial_t^l\vartheta)^2+r^4\sigma_1^{\alpha+1}(\partial_t^l\vartheta_r)^2\right](r,0)dr,\end{aligned}\tag{86}$$

where $j\geq 0$, $i\geq 1$, $i+2j\leq 5+[\alpha]$, and C_7 is a positive constant.

Proof. First, we show the following higher weighted energy estimates for $j=2,3,\cdots,3+[\frac{\alpha}{2}]$ by the inductive method

$$\begin{aligned}&(1+t)^{2j}\int_0^{\widehat{R}_0}\left[r^4\sigma_1^{\alpha}(\partial_t^j\vartheta)^2+r^2\sigma_1^{\alpha+1}(\partial_t^j\vartheta)^2+r^4\sigma_1^{\alpha+1}(\partial_t^j\vartheta_r)^2\right]dr\\&+\int_0^t\int_0^{\widehat{R}_0}(1+s)^{2j-1}\left\{(1+s)^2r^4\sigma_1^{\alpha}(\partial_s^{j+1}\vartheta)^2+r^2\sigma_1^{\alpha+1}[(\partial_s^j\vartheta)^2+(r\partial_s^j\vartheta_r)^2]\right\}dsdr\\\leq&C\sum_{l=0}^{j}\int_0^{\widehat{R}_0}\left[r^4\sigma_1^{\alpha}(\partial_t^l\vartheta)^2+r^2\sigma_1^{\alpha+1}(\partial_t^l\vartheta)^2+r^4\sigma_1^{\alpha+1}(\partial_t^l\vartheta_r)^2\right](r,0)dr,\end{aligned}\tag{87}$$

where C is a generic positive constant.

Suppose that $\vartheta(r,t)$ satisfies (85) and the following inequality for $1\le j\le k-1$, $k=1,2,\cdots,3+[\frac{\alpha}{2}]$ as

$$\begin{aligned}&(1+t)^{2j}\int_0^{\widehat{R}_0}\big[r^4\sigma_1^{\alpha}(\partial_t^j\vartheta)^2+r^2\sigma_1^{\alpha+1}(\partial_t^j\vartheta)^2+r^4\sigma_1^{\alpha+1}(\partial_t^j\vartheta_r)^2\big]dr\\&+\int_0^t\int_0^{\widehat{R}_0}(1+s)^{2j-1}\big\{(1+s)^2r^4\sigma_1^{\alpha}(\partial_s^{j+1}\vartheta)^2+r^2\sigma_1^{\alpha+1}[(\partial_s^j\vartheta)^2+(r\partial_s^j\vartheta_r)^2]\big\}dsdr\\&\le C_1\sum_{l=0}^{j}\int_0^{\widehat{R}_0}\big[r^4\sigma_1^{\alpha}(\partial_t^l\vartheta)^2+r^2\sigma_1^{\alpha+1}(\partial_t^l\vartheta)^2+r^4\sigma_1^{\alpha+1}(\partial_t^l\vartheta_r)^2\big](r,0)dr.\end{aligned}\tag{88}$$

We just need to prove the following case $j=k$. Taking the time derivative ∂_t^{k-1} on Eq. (62), one has

$$\begin{aligned}&r\sigma_1^{\alpha}\partial_t^{k+1}\vartheta+[\sigma_1^{\alpha+1}(\theta_1\partial_t^k\vartheta+\theta_2r\partial_t^k\vartheta_r)]_r-2\sigma_1^{\alpha+1}(\theta_3+\theta_4)[\widehat{\eta}_r(1+\varsigma)+\vartheta]_r\partial_t^k\vartheta\\&+\sigma_1^{\alpha+1}(3\theta_2-\theta_1)\partial_t^k\vartheta_r=\partial_t^{k-1}[\mathfrak{F}_2(r,t)+\mathfrak{F}_3(r,t)]+\mathfrak{F}_5(r,t),\end{aligned}\tag{89}$$

where the nonlinear function $\mathfrak{F}_5(r,t)$ is defined by

$$\begin{aligned}\mathfrak{F}_5(r,t)\triangleq&2\sigma_1^{\alpha+1}\big\{\partial_t^{k-1}[(\theta_3+\theta_4)(\widehat{\eta}_r(1+\varsigma)+\vartheta)_r\vartheta_t]-(\theta_3+\theta_4)[\widehat{\eta}_r(1+\varsigma)+\vartheta]_r\partial_t^k\vartheta\big\}\\&-\big\{\sigma_1^{\alpha+1}[\partial_t^{k-1}(\theta_1\vartheta_t+r\theta_2\vartheta_{rt})-\theta_1\partial_t^k\vartheta-r\theta_2\partial_t^k\vartheta_r]\big\}_r\\&-\sigma_1^{\alpha+1}\partial_t^{k-1}[(3\theta_2-\theta_1)\vartheta_{rt}]+\sigma_1^{\alpha+1}(3\theta_2-\theta_1)\partial_t^k\vartheta_r.\end{aligned}\tag{90}$$

Multiplying the Eq. (89) by $r^3\partial_t^k\vartheta$, and integrating the resulted equations by parts on $[0,\widehat{R}_0]$ respectively, with the help of the a-priori assumption (52) and Cauchy inequality, we have after a tedious computation that

$$\begin{aligned}&\frac12\frac{d}{dt}\int_0^{\widehat{R}_0}r^4\sigma_1^{\alpha}(\partial_t^k\vartheta)^2dr+\int_0^{\widehat{R}_0}r^2\sigma_1^{\alpha+1}\mathfrak{F}_6(r,t)dr\\&\le(b+\varepsilon_1+M_1|b_1b_2-a_1a_2|)(1+t)^{-1}\int_0^{\widehat{R}_0}\big[r^4\sigma_1^{\alpha}(\partial_t^k\vartheta)^2+r^2\sigma_1^{\alpha+1}(\partial_t^k\vartheta)^2+r^4\sigma_1^{\alpha+1}(\partial_t^k\vartheta_r)^2\big]dr\\&+C_b\sum_{l=0}^{k-1}(1+t)^{2l-2k-1}\int_0^{\widehat{R}_0}\big[r^4\sigma_1^{\alpha}(\partial_t^l\vartheta)^2+r^2\sigma_1^{\alpha+1}(\partial_t^l\vartheta)^2+r^4\sigma_1^{\alpha+1}(\partial_t^l\vartheta_r)^2\big]dr\\&+C_bM_1(b_1b_2-a_1a_2)^2\sum_{l=0}^{k-1}(1+t)^{2l-2k+1-\frac{6}{3m-1}}\int_0^{\widehat{R}_0}r^4\sigma_1^{\alpha}(\partial_t^l\vartheta)^2dr,\end{aligned}\tag{91}$$

where $0<b\ll1$ and C_b are two positive constants, and the nonlinear function $\mathfrak{F}_6(r,t)$ is given by

$$\begin{aligned}&\mathfrak{F}_6(r,t)\\&\triangleq-\big\{3\theta_1+2r(\theta_3+\theta_4)[\widehat{\eta}_r(1+\varsigma)+\vartheta]_r\big\}(\partial_t^k\vartheta)^2-\theta_2(r\partial_t^k\vartheta_r)^2-2r\theta_1\partial_t^k\vartheta\partial_t^k\vartheta_r.\end{aligned}\tag{92}$$

It is easy to derive by the a-priori assumption (85) and Cauchy inequality that

$$C^{-1}(1+t)^{-1}\big[(\partial_t^k\vartheta)^2+(r\partial_t^k\vartheta_r)^2\big]\le\mathfrak{F}_6(r,t)\le C(1+t)^{-1}\big[(\partial_t^k\vartheta)^2+(r\partial_t^k\vartheta_r)^2\big],\tag{93}$$

for a generic positive constant C.

Therefore, it holds by using $0<M_1|a_1a_2-b_1b_2|\ll1$ that

$$\frac12\frac{d}{dt}\int_0^{\widehat{R}_0}r^4\sigma_1^{\alpha}(\partial_t^k\vartheta)^2dr+C^{-1}(1+t)^{-1}\int_0^{\widehat{R}_0}\big[r^2\sigma_1^{\alpha+1}(\partial_t^k\vartheta)^2+r^4\sigma_1^{\alpha+1}(\partial_t^k\vartheta_r)^2\big]dr$$

$$\leq C_\epsilon \sum_{l=0}^{k-1}(1+t)^{2l-2k-1}\int_0^{\widehat{R}_0}\big[r^4\sigma_1^{\alpha}(\partial_t^l\vartheta)^2+r^2\sigma_1^{\alpha+1}(\partial_t^l\vartheta)^2+r^4\sigma_1^{\alpha+1}(\partial_t^l\vartheta_r)^2\big]dr. \tag{94}$$

Multiplying the Eq. (89) by $r^3\partial_t^{k+1}\vartheta$, integrating it by parts on $[0,\widehat{R}_0]$ respectively, and applying the a-priori assumption (52) and Cauchy inequality, we show after a complicated computation that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^{\widehat{R}_0}\Big\{r^2\sigma_1^{\alpha+1}\mathfrak{F}_6(r,t)-2r^3\partial_t^k\vartheta\big[\partial_t^{k-1}\mathfrak{F}_2(r,t)+\mathfrak{F}_5(r,t)\big]\Big\}dr+\int_0^{\widehat{R}_0}r^4\sigma_1^{\alpha}(\partial_t^{k+1}\vartheta)^2dr\\
\leq& C_b(\varepsilon_1^2+1)\sum_{l=0}^{k-1}(1+t)^{2(l-k-1)}\int_0^{\widehat{R}_0}\big[r^4\sigma_1^{\alpha}(\partial_t^l\vartheta)^2+r^2\sigma_1^{\alpha+1}(\partial_t^l\vartheta)^2+r^4\sigma_1^{\alpha+1}(\partial_t^l\vartheta_r)^2\big]dr\\
&+(b+\varepsilon_1)(1+t)^{-2}\int_0^{\widehat{R}_0}\big[r^4\sigma_1^{\alpha}(\partial_t^k\vartheta)^2+r^2\sigma_1^{\alpha+1}(\partial_t^k\vartheta)^2+r^4\sigma_1^{\alpha+1}(\partial_t^k\vartheta_r)^2\big]dr\\
&+C_bM_1(b_1b_2-a_1a_2)^2\sum_{l=0}^{k}(1+t)^{2(l-k-\frac{3}{3m-1})}\int_0^{\widehat{R}_0}r^4\sigma_1^{\alpha}(\partial_t^l\vartheta)^2dr.
\end{aligned}\tag{95}$$

Summing $(1+t)^p\times(94)$ and $(1+t)^q\times(95)$ together for $p=1,2,\cdots,2k$ and $q=1,2,\cdots,2k+1$ respectively, integrating the resulted inequalities on $[0,\widehat{R}_0]$, and using $0<M_1|a_1a_2-b_1b_2|\ll 1$, we get (87) for $j=k$ by the inductive hypothesis (88).

Next, we show the following higher regularity estimates for $j\geq 0, i\geq 1, 2\leq i+2j\leq 5+[\alpha]$ by an inductive method

$$\begin{aligned}
&(1+t)^{2j}\int_0^{\widehat{R}_0}\big[r^4\sigma_1^{\alpha+i+1}(\partial_t^j\partial_r^{i+1}\vartheta)^2+r^2\sigma_1^{\alpha+i-1}(\partial_t^j\partial_r^i\vartheta)^2\big]dr\\
\leq& C\sum_{l=0}^{[\frac{i+2j+1}{2}]}\int_0^{\widehat{R}_0}\big[r^4\sigma_1^{\alpha}(\partial_t^l\vartheta)^2+r^2\sigma_1^{\alpha+1}(\partial_t^l\vartheta)^2+r^4\sigma_1^{\alpha+1}(\partial_t^l\vartheta_r)^2\big](r,0)dr,
\end{aligned}\tag{96}$$

where C is a generic positive constant.

Suppose that $\vartheta(r,t)$ satisfies (85) and the following estimates for $j\geq 0$, $i\geq 1$, $2\leq i+2j\leq k$, $k=2,3,\cdots,4+[\alpha]$ as

$$\begin{aligned}
&(1+t)^{2j}\int_0^{\widehat{R}_0}\big[r^4\sigma_1^{\alpha+i+1}(\partial_t^j\partial_r^{i+1}\vartheta)^2+r^2\sigma_1^{\alpha+i-1}(\partial_t^j\partial_r^i\vartheta)^2\big]dr\\
\leq& C\sum_{l=0}^{[\frac{i+2j+1}{2}]}\int_0^{\widehat{R}_0}\big[r^4\sigma_1^{\alpha}(\partial_t^l\vartheta)^2+r^2\sigma_1^{\alpha+1}(\partial_t^l\vartheta)^2+r^4\sigma_1^{\alpha+1}(\partial_t^l\vartheta_r)^2\big](r,0)dr.
\end{aligned}\tag{97}$$

Thus, we just need to prove the following case $i+2j=k+1$. Taking the mixed derivatives of time and space $\partial_t^j\partial_r^{i-1}$ over $(1+t)^{-1}(1+\varsigma)^{2m-2}(1+\varsigma+r\varsigma_r)^{m+1}$(72), it holds

$$\begin{aligned}
&m\big[r\sigma_1\partial_t^j\partial_r^{i+1}\vartheta+(i+3)\sigma_1\partial_t^j\partial_r^i\vartheta+(i+\alpha)r\sigma_{1r}\partial_t^j\partial_r^i\vartheta\big]\\
=&\partial_t^j\partial_r^{i-1}\big\{(1+t)^{-1}(1+\varsigma)^{2m-2}(1+\varsigma+r\varsigma_r)^{m+1}[\sigma_1\mathbf{g}_1(r,t)+(1+\alpha)\sigma_{1r}\mathbf{g}_2(r,t)\\
&+\mathbf{g}_3(r,t)]\big\}+(1+t)^{-1}(1+\varsigma)^{2m-2}(1+\varsigma+r\varsigma_r)^{m+1}r\partial_t^{j+1}\partial_r^{i-1}\vartheta+\mathbf{G}_1(r,t),
\end{aligned}\tag{98}$$

where the lower order term $\mathbf{G}_1(r,t)$ is given by

$\mathbf{G}_1(r,t)$

$$
\begin{aligned}
&\triangleq \partial_t^j\partial_r^{i-1}\Big[(1+t)^{-1}\frac{(1+\varsigma)^{2m-2}}{(1+\varsigma+r\varsigma_r)^{-m-1}}r\vartheta_t\Big]-(1+t)^{-1}\frac{(1+\varsigma)^{2m-2}}{(1+\varsigma+r\varsigma_r)^{-m-1}}r\partial_t^{j+1}\partial_r^{i-1}\vartheta\\
&\quad+\partial_t^j\partial_r^{i-1}\Big(\frac{r\varsigma_r}{1+\varsigma}\sigma_1\vartheta_r\Big)-m\partial_t^j\partial_r^{i-1}\big\{r\sigma_1\vartheta_{rr}+[(i+3)\sigma_1+(i+\alpha)r\sigma_{1r}]\vartheta_r\big\}-mr\sigma_1\partial_t^j\partial_r^{i+1}\vartheta\\
&\quad+(1+\alpha)\partial_t^j\partial_r^{i-1}\Big\{\Big[2-3m+(2-2m)\frac{r\varsigma_r}{1+\varsigma}\Big]\sigma_{1r}\vartheta\Big\}-m[(i+3)\sigma_1+(i+\alpha)r\sigma_{1r}]\partial_t^j\partial_r^i\vartheta. \qquad (99)
\end{aligned}
$$

Multiplying (98) by $r\sigma_1^{\frac{\alpha+i-1}{2}}$ and taking L^2-norm on $[0,\widehat{R}_0]$, applying the similar arguments as dealing with the estimates (82) and (83), it implies after a tedious calculation that

$$
\begin{aligned}
&\int_0^{\widehat{R}_0}\big[r^4\sigma_1^{\alpha+i+1}(\partial_t^j\partial_r^{i+1}\vartheta)^2+r^2\sigma_1^{\alpha+i-1}(\partial_t^j\partial_r^i\vartheta)^2\big]dr\\
&\le\int_0^{\widehat{R}_0}r^2\sigma_1^{\alpha+i+1}\Big\{\partial_t^j\partial_r^{i-1}\Big[(1+t)^{-1}\frac{(1+\varsigma)^{2m-2}}{(1+\varsigma+r\varsigma_r)^{-m-1}}\big(\sigma_1\mathbf{g}_1+(1+\alpha)\sigma_{1r}\mathbf{g}_2+\mathbf{g}_3\big)\Big]\Big\}^2dr\\
&\quad+\int_0^{\widehat{R}_0}r^2\sigma_1^{\alpha+i-1}\mathbf{G}_1^2dr+C(1+t)^2\int_0^{\widehat{R}_0}r^2\sigma_1^{\alpha+i-1}(\partial_t^{j+1}\partial_r^{i-2}\vartheta)^2dr\\
&\quad+C\int_0^{\widehat{R}_0}\big[r^4\sigma_1^{\alpha+i}(\partial_t^j\partial_r^i\vartheta)^2+(1+t)^2r^4\sigma_1^{\alpha+i-1}(\partial_t^{j+1}\partial_r^{i-1}\vartheta)^2\big]dr. \qquad (100)
\end{aligned}
$$

In addition, the terms on the right hand side of (100) can be estimated by

$$
\begin{aligned}
&\int_0^{\widehat{R}_0}r^2\sigma_1^{\alpha+i-1}\mathbf{G}_1^2(r,t)dr\\
&\le C\varepsilon_1^2\sum_{l=0}^{j}\sum_{l_1=0}^{i-1}(1+t)^{-2l-\overline{\alpha}}\int_0^{\widehat{R}_0}\big[r^4\sigma_1^{\alpha+i-l_1}(\partial_t^{j-l}\partial_r^{i-l_1}\vartheta)^2+r^2\sigma_1^{\alpha+i-l_1-2}(\partial_t^{j-l}\partial_r^{i-l_1-1}\vartheta)^2\big]dr\\
&\quad+C(1+t)^{2j}\int_0^{\widehat{R}_0}\big[r^4\sigma_1^{\alpha+i-1}(\partial_t^j\partial_r^{i-1}\vartheta)^2+r^2\sigma_1^{\alpha+i-3}(\partial_t^j\partial_r^{i-2}\vartheta)^2\big]dr. \qquad (101)
\end{aligned}
$$

Due to the definitions of $\mathbf{g}_i(r,t)$, $(i=1,2,3)$ in (73)-(75) and inequalities (76)-(78), it is also easy to verify that

$$
\begin{aligned}
&\int_0^{\widehat{R}_0}r^2\sigma_1^{\alpha+i+1}\Big\{\partial_t^j\partial_r^{i-1}\Big[(1+t)^{-1}\frac{(1+\varsigma)^{2m-2}}{(1+\varsigma+r\varsigma_r)^{-m-1}}\mathbf{g}_3(r,t)\Big]\Big\}^2dr\\
&\le C(b_1b_2-a_1a_2)^2\sum_{\beta=0}^{j}\sum_{\gamma=0}^{i-2}(1+t)^{-2l-\frac{8-6m}{3m-1}}\int_0^{\widehat{R}_0}r^2\sigma_1^{\alpha+i-2-\gamma}(\partial_t^{j-\beta}\partial_r^{i-1-\gamma}\vartheta)^2dr, \qquad (102)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^{\widehat{R}_0}r^2\sigma_1^{\alpha+i+1}\Big\{\partial_t^j\partial_r^{i-1}\Big[(1+t)^{-1}\frac{(1+\varsigma)^{2m-2}}{(1+\varsigma+r\varsigma_r)^{-m-1}}\big(\sigma_1\mathbf{g}_1(r,t)+(1+\alpha)\sigma_{1r}\mathbf{g}_2(r,t)\big)\Big]\Big\}^2dr\\
&\le\sum_{\beta=0}^{j}\sum_{\gamma=0}^{i-1}\sum_{\kappa=0}^{\beta}(1+t)^{-\frac{2}{3m-1}-2\kappa}\Big[\mathfrak{D}_1(t)+(1+t)^{-\frac{2}{3m-1}}\sum_{\lambda=2}^{\infty}\mathfrak{D}_2(t)\Big], \qquad (103)
\end{aligned}
$$

where the nonlinear functions $\mathfrak{D}_1(t)$ and $\mathfrak{D}_2(t)$ are defined by

$$
\begin{aligned}
\mathfrak{D}_1(t)\triangleq&\int_0^{\widehat{R}_0}r^2\sigma_1^{\alpha+i-1}\big[|r\sigma_1\partial_t^{j-\beta}\partial_r^{i+1-\gamma}\vartheta|+|(\sigma_1+r^2)\partial_t^{j-\beta}\partial_r^{i-\gamma}\vartheta|\\
&+|r\partial_t^{j-\beta}\partial_r^{i-1-\gamma}\vartheta|+|\partial_t^{j-\beta}\partial_r^{i-2-\gamma}\vartheta|\big]^2\times\big[|r\partial_t^{\beta-\kappa}\partial_r^{\gamma+1}\vartheta|+|\partial_t^{\beta-\kappa}\partial_r^{\gamma}\vartheta|\big]^2dr, \qquad (104)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathfrak{D}_2(t)\triangleq&\int_0^{\widehat{R}_0}r^2\sigma_1^{\alpha+i-1}\big[|r\sigma_1\partial_t^{j-\beta}\partial_r^{i+1-\gamma}\vartheta|+|(\sigma_1+r^2)\partial_t^{j-\beta}\partial_r^{i-\gamma}\vartheta|\\
&+|r\partial_t^{j-\beta}\partial_r^{i-1-\gamma}\vartheta|+|\partial_t^{j-\beta}\partial_r^{i-2-\gamma}\vartheta|\big]^2\times\big[|r\partial_t^{\beta-\kappa}\partial_r^{\gamma+1}\vartheta^{\lambda}|\\
&+|\partial_t^{\beta-\kappa}\partial_r^{\gamma}\vartheta^{\lambda}|\big]^2dr. \qquad (105)
\end{aligned}
$$

Then, we investigate the following inequality by two cases.

$$
\begin{aligned}
&\mathfrak{D}_1(t)+(1+t)^{-\frac{2}{3m-1}}\sum_{\lambda=2}^{\infty}\mathfrak{D}_2(t)\\
\leq & C_1\varepsilon_1^2(1+t)^{2(\kappa-\beta)}\sum_{\theta=i-\gamma-2}^{i+2\beta-2k+1}\int_0^{\widehat{R}_0}\left[r^4\sigma_1^{\alpha+\theta+1}(\partial_t^{j-\beta}\partial_r^{\theta+1}\vartheta)^2+r^2\sigma_1^{\alpha+\theta-1}(\partial_t^{j-\beta}\partial_r^{\theta}\vartheta)^2\right]dr\\
&+C_1\varepsilon_1^2(1+t)^{2(\beta-j)}\sum_{\theta=\gamma}^{i+2j-2\beta+2\kappa-1}\int_0^{\widehat{R}_0}\left[r^4\sigma_1^{\alpha+\theta+1}(\partial_t^{\beta-\kappa}\partial_r^{\theta+1}\vartheta)^2+r^2\sigma_1^{\alpha+\theta-1}(\partial_t^{\beta-\kappa}\partial_r^{\theta}\vartheta)^2\right]dr.
\end{aligned}\tag{106}
$$

<u>*Case 1:*</u> One the one hand, if $2\gamma+2\beta\geq i+j+\kappa$ for $j\geq 0, i\geq 1, i+2j\leq 5+[\alpha], 0\leq\beta\leq j, 0\leq\kappa\leq\beta, 0\leq\gamma\leq i-1$, it implies

$$\alpha-i-2j+2\beta+2\gamma+2\geq 0.\tag{107}$$

Thus, we can deduce by (85) that

$$\mathfrak{D}_1(t)\leq\varepsilon_1^2(1+t)^{2(\beta-j)}\int_0^{\widehat{R}_0}r^2\sigma_1^{\alpha+2\gamma+2\beta-i-2j+2}(|r\partial_t^{\beta-\kappa}\partial_r^{\gamma+1}\vartheta|+|\partial_t^{\beta}\partial_r^{\gamma}\vartheta|)^2dr.\tag{108}$$

Obviously, it holds either for $\gamma+2\beta-i-2j+2\geq 0$ that

$$\mathfrak{D}_1(t)\leq\varepsilon_1^2(1+t)^{2(\beta-j)}\int_0^{\widehat{R}_0}\left[r^4\sigma_1^{\alpha+\gamma}(\partial_t^{\beta-\kappa}\partial_r^{\gamma+1}\vartheta)^2+r^2\sigma_1^{\alpha+\gamma-1}(\partial_t^{\beta-\kappa}\partial_r^{\gamma}\vartheta)^2\right]dr,\tag{109}$$

or for $\gamma+2\beta-i-2j+2<0$ that

$$
\begin{aligned}
\mathfrak{D}_1(t)\leq & C\varepsilon_1^2(1+t)^{2(\beta-j)}\sum_{\theta=\gamma}^{i+2j-2\beta+2\kappa-1}\int_0^{\widehat{R}_0}r^2\sigma_1^{\alpha+\theta-1}\left[(r\partial_t^{\beta-\kappa}\partial_r^{\theta+1}\vartheta)^2\right.\\
&\left.+(\partial_t^{\beta-\kappa}\partial_r^{\theta}\vartheta)^2\right]dr,
\end{aligned}\tag{110}
$$

where we have used the following inequality

$$
\begin{aligned}
&\int_{\frac{\widehat{R}_0}{2}}^{\widehat{R}_0}\sigma_1^{\alpha+2\gamma+2\beta-i-2j+2}(|\partial_t^{\beta-\kappa}\partial_r^{\gamma+1}\vartheta|+|\partial_t^{\beta-\kappa}\partial_r^{\gamma}\vartheta|)^2dr\\
\leq & C\sum_{\theta=\gamma}^{i+2j-2\beta+2\kappa}\int_{\frac{\widehat{R}_0}{2}}^{\widehat{R}_0}r^4\sigma_1^{\alpha+i+2j-2\beta+2\kappa}(\partial_t^{\beta-\kappa}\partial_r^{\theta}\vartheta)^2dr.
\end{aligned}
$$

<u>*Case 2:*</u> On the other hand, if $2\gamma+2\beta<i+j+\kappa$ for $j>0, i\geq 1, i+2j\leq 5+[\alpha], 0\leq\beta\leq j, 0\leq\kappa\leq\beta, 0\leq\gamma\leq i-1$, it also implies

$$\alpha+i-2\gamma-2\beta+2\kappa\geq 0.$$

Then, we can deduce

$$
\begin{aligned}
\mathfrak{D}_1(t)\leq & \varepsilon_1^2(1+t)^{2(\kappa-\beta)}\int_0^{\widehat{R}_0}r^2\sigma_1^{1-2\beta-\gamma+2\kappa}\left[\left|r\sigma_1^{\frac{\alpha+i+1-\gamma}{2}}\partial_t^{j-\beta}\partial_r^{i+1-\gamma}\vartheta\right|^2\right.\\
&+\left|\sigma_1^{\frac{\alpha+i-1-\gamma}{2}}\partial_t^{j-\beta}\partial_r^{i-\gamma}\vartheta\right|^2+\left|r\sigma_1^{\frac{\alpha+i-1-\gamma}{2}}\partial_t^{j-\beta}\partial_r^{i-1-\gamma}\vartheta\right|^2\\
&\left.+\left|\sigma_1^{\frac{\alpha+i-1-\gamma}{2}}\partial_t^{j-\beta}\partial_r^{i-2-\gamma}\vartheta\right|^2\right]dr,
\end{aligned}
$$

it holds naturally either for $\gamma+2\beta-2\kappa=0,1$ that

$$\begin{aligned}\mathfrak{D}_1(t)\le&\varepsilon_1^2(1+t)^{2(\kappa-\beta)}\int_0^{\widehat{R}_0}\big[r^4\sigma_1^{\alpha+i+1-\gamma}(\partial_t^{j-\beta}\partial_r^{i+1-\gamma}\vartheta)^2\\&+r^2\sigma_1^{\alpha+i-1-\gamma}(\partial_t^{j-\beta}\partial_r^{i-\gamma}\vartheta)^2+r^4\sigma_1^{\alpha+i-1-\gamma}(\partial_t^{j-\beta}\partial_r^{i-1-\gamma}\vartheta)^2\\&+r^2\sigma_1^{\alpha+i-3-\gamma}(\partial_t^{j-\beta}\partial_r^{i-2-\gamma}\vartheta)^2\big]dr,\end{aligned}\tag{111}$$

or for $\gamma+2\beta-2\kappa\ge 2$ that

$$\begin{aligned}\mathfrak{D}_1(t)\le&\varepsilon_1^2(1+t)^{2(\kappa-\beta)}\sum_{\theta=i-\gamma-2}^{i+2\beta-2k+1}\int_0^{\widehat{R}_0}\big[r^4\sigma_1^{\alpha+\theta+1}(\partial_t^{j-\beta}\partial_r^{\theta+1}\vartheta)^2\\&+r^2\sigma_1^{\alpha+\theta-1}(\partial_t^{j-\beta}\partial_r^{\theta}\vartheta)^2\big]dr,\end{aligned}\tag{112}$$

where we have used the following inequality

$$\begin{aligned}&\sum_{q=-2}^{1}\int_{\frac{\widehat{R}_0}{2}}^{\widehat{R}_0}r^4\sigma_1^{\alpha+i-2\gamma-2\beta+2\kappa}(\partial_t^{j-\beta}\partial_r^{i-\gamma+q}\vartheta)^2dr\\\le&\sum_{q=-2}^{\gamma+2\beta-2\kappa+2}\int_{\frac{\widehat{R}_0}{2}}^{\widehat{R}_0}r^4\sigma_1^{\alpha+i+2+2\beta-2\kappa}(\partial_t^{j-\beta}\partial_r^{i-\gamma+q}\vartheta)^2dr.\end{aligned}$$

The $\mathfrak{D}_2(t)$ can be estimated by the similar argument as proving the $\mathfrak{D}_1(t)$, and the details of the proof are omitted. Thus, we can obtain (106).

Substituting (101)-(103) into (100), it is natural to conclude (96) for $i+2j=k+1$ under the inequality (106) and the inductive hypothesis (97) for $2\le i+2j\le k$. □

Proof of Theorem 2.1 : Summing (86) from $j=0$ to $j=3+[\frac{\alpha}{2}]$ and from $i=1$ to $5+[\alpha]-2j$ respectively, we can prove (35) in Theorem 2.1. According to the weighted energy estimates (35) and weighted Sobolev embedding inequality (49) and Hardy inequality (51) on $[0,\widehat{R}_0]$, we can verify the a-priori assumption (85), and conclude (34) in Theorem 2.1 as follows.

First, we verify the a-priori assumption (85). Obviously, it follows from the weighted energy inequality (35) that

$$E_1(t)\le C_1E_1(0).\tag{113}$$

What left for us is to prove the a-priori assumption satisfying

$$\begin{aligned}&\sum_{j=0}^{2}(1+t)^{2j}\|\partial_t^j\vartheta\|^2_{L^\infty([0,\widehat{R}_0])}+\sum_{j=0}^{1}(1+t)^{2j}\|\partial_t^j\vartheta_r\|^2_{L^\infty([0,\widehat{R}_0])}\\&+\sum_{0\le j\le[\frac{2+\alpha}{2}],3<i+2j\le 3+[\alpha]}(1+t)^{2j}\|\sigma_1^{i+j-\frac32}\partial_t^j\partial_r^i\vartheta\|^2_{L^\infty([0,\widehat{R}_0])}\\&+\sum_{0\le j\le[\frac{3+\alpha}{2}],i+2j=4+[\alpha]}(1+t)^{2j}\|r\sigma_1^{i+j-\frac32}\partial_t^j\partial_r^i\vartheta\|^2_{L^\infty([0,\widehat{R}_0])}\\&+\sum_{0\le j\le[\frac{4+\alpha}{2}],i+2j=5+[\alpha]}(1+t)^{2j}\|r^2\sigma_1^{i+j-\frac32}\partial_t^j\partial_r^i\vartheta\|^2_{L^\infty([0,\widehat{R}_0])}\le C_8E_1(t).\end{aligned}\tag{114}$$

In the case that the space variable $r\in[0,\frac{\widehat{R}_0}{2}]$, we show the following inequality as

$$\sum_{i+2j\le 3+[\alpha]}(1+t)^{2j}\|\partial_t^j\partial_r^i\vartheta\|^2_{L^\infty([0,\frac{\widehat{R}_0}{2}])}+\sum_{i+2j=4+[\alpha]}(1+t)^{2j}\|r\partial_t^j\partial_r^i\vartheta\|^2_{L^\infty([0,\frac{\widehat{R}_0}{2}])}$$

$$+\sum_{i+2j=5+[\alpha]}(1+t)^{2j}\|r^2\partial_t^j\partial_r^i\vartheta\|^2_{L^\infty([0,\frac{\widehat{R}_0}{2}])}\leq C_8E_1(t). \tag{115}$$

Based on the weighted Sobolev embedding inequality (49) and the following embedding inequality

$$\|f(r,t)\|_{L^\infty([0,\widehat{R}_0])}\leq C(\delta)\|f(r,t)\|_{H^{\frac{1}{2}+\delta}([0,\widehat{R}_0])}<\infty,\quad C(\delta)>0, \tag{116}$$

it holds that

$$\sum_{i+2j\leq 3+[\alpha]}\|\partial_t^j\partial_r^i\vartheta\|^2_{L^\infty([0,\frac{\widehat{R}_0}{2}])}\leq C\|\partial_t^j\vartheta\|^2_{H^{2,5+[\alpha]-2j}([0,\frac{\widehat{R}_0}{2}])}\leq C_8(1+t)^{-2j}E_1(t), \tag{117}$$

$$\sum_{i+2j=4+[\alpha]}\|r\partial_t^j\partial_r^i\vartheta\|^2_{L^\infty([0,\frac{\widehat{R}_0}{2}])}\leq C\|r\partial_t^j\partial_r^{4+[\alpha]-2j}\vartheta\|^2_{H^1([0,\frac{\widehat{R}_0}{2}])}\leq C_8(1+t)^{-2j}E_1(t), \tag{118}$$

and

$$\sum_{i+2j=5+[\alpha]}\|r^2\partial_t^j\partial_r^i\vartheta\|^2_{L^\infty([0,\frac{\widehat{R}_0}{2}])}\leq C\|r\partial_t^j\partial_r^{5+[\alpha]-2j}\vartheta\|^2_{H^1([0,\frac{\widehat{R}_0}{2}])}\leq C_8(1+t)^{-2j}E_1(t). \tag{119}$$

Summing the inequalities (117), (118) and (119) together, we can obtain (115). In the case that the space variable $r\in[\frac{\widehat{R}_0}{2},\widehat{R}_0]$, we investigate the following inequality as

$$\begin{aligned}&\sum_{j=0}^{2}(1+t)^{2j}\|\partial_t^j\vartheta\|^2_{L^\infty([\frac{\widehat{R}_0}{2},\widehat{R}_0])}+\sum_{j=0}^{1}(1+t)^{2j}\|\partial_t^j\vartheta_r\|^2_{L^\infty([\frac{\widehat{R}_0}{2},\widehat{R}_0])}\\&+\sum_{4\leq i+2j\leq 3+[\alpha]}(1+t)^{2j}\|\sigma_1^{i+j-\frac{3}{2}}\partial_t^j\partial_r^i\vartheta\|^2_{L^\infty([\frac{\widehat{R}_0}{2},\widehat{R}_0])}\leq C_8E_1(t).\end{aligned} \tag{120}$$

With the help of (116) and the weighted Sobolev embedding inequality (49), we have

$$\sum_{j=0}^{2}\|\partial_t^j\vartheta\|^2_{L^\infty([\frac{\widehat{R}_0}{2},\widehat{R}_0])}\leq C\|\partial_t^j\vartheta\|^2_{H^{6+[\alpha]+\alpha-2j,6+[\alpha]-2j}([\frac{\widehat{R}_0}{2},\widehat{R}_0])}\leq C_8(1+t)^{-2j}E_1(t), \tag{121}$$

and

$$\sum_{j=0}^{1}\|\partial_t^j\vartheta_r\|^2_{L^\infty([\frac{\widehat{R}_0}{2},\widehat{R}_0])}\leq C\|\partial_t^j\vartheta\|^2_{H^{6+[\alpha]+\alpha-2j,6+[\alpha]-2j}([\frac{\widehat{R}_0}{2},\widehat{R}_0])}\leq C_8(1+t)^{-2j}E_1(t), \tag{122}$$

If the constants $i\geq 2$, $4\leq i+2j\leq 5+[\alpha]$, $0\leq l\leq 6-i-2j$, $0\leq q\leq l$, it holds by the Hardy inequality (51) that

$$\begin{aligned}&\|\sigma_1^{i+j-\frac{3}{2}}\partial_t^j\partial_r^i\vartheta\|^2_{L^\infty([\frac{\widehat{R}_0}{2},\widehat{R}_0])}\\&\leq C\|\sigma_1^{i+j-\frac{3}{2}}\partial_t^j\partial_r^i\vartheta\|^2_{H^{\alpha+[\alpha]+10-2i-4j,6-i-2j}([\frac{\widehat{R}_0}{2},\widehat{R}_0])}\leq C_8(1+t)^{-2j}E_1(t).\end{aligned} \tag{123}$$

Taking the summation of (121), (122) and (123) together, we can obtain (120). The combination of (115) and (120) leads to (114).

Next, we prove (34) in Theorem 2.1 as follows. For $1 < m < \frac{4}{3}$, $\alpha = \frac{1}{m-1}$, $i+2j = 5+[\alpha]$, if $j = 0$, $i = 5+[\alpha]$, $a = \alpha+i-1 = \alpha+4+[\alpha]$, $b = i = 5+[\alpha]$, it implies

$$\begin{aligned}\|\vartheta\|^2_{H^{\frac{6+[\alpha]-\alpha}{2}}([0,\widehat{R}_0])} \leq & C\Big[\|\vartheta\|^2_{H^{\alpha+4+[\alpha],5+[\alpha]}([0,\frac{\widehat{R}_0}{2}])} + \|\vartheta\|^2_{H^{\alpha+4+[\alpha],5+[\alpha]}([\frac{\widehat{R}_0}{2},\widehat{R}_0])}\Big] \\ \leq & C_4\varepsilon_1^2,\end{aligned}$$

and

$$\|\vartheta_{rr}\|^2_{L^\infty([0,\widehat{R}_0])} \leq C\|\vartheta\|^2_{H^{\frac{6+[\alpha]-\alpha}{2}}([0,\widehat{R}_0])} \leq C_4\varepsilon_1^2.$$

Similarly, if $j = 1$, $i = 3+[\alpha]$, $a = \alpha+i-1 = \alpha+2+[\alpha]$, $b = i = 3+[\alpha]$, it also holds that

$$\|\vartheta_t\|^2_{L^\infty([0,\widehat{R}_0])} \leq C\|\vartheta_t\|^2_{H^{\alpha+2+[\alpha],3+[\alpha]}([0,\widehat{R}_0])} \leq C_4\varepsilon_1^2(1+t)^{-2},$$

and if $j = 2$, $i = 1+[\alpha]$, $a = \alpha+i-1 = \alpha+[\alpha]$, $b = i = 1+[\alpha]$, that

$$\|\vartheta_{tt}\|^2_{L^\infty([0,\widehat{R}_0])} \leq C\|\vartheta_{tt}\|^2_{H^{\alpha+[\alpha],1+[\alpha]}([0,\widehat{R}_0])} \leq C_4\varepsilon_1^2(1+t)^{-4}.$$

Therefore, we can conclude (34) from the above facts.

4. Numerical simulations

In this section, we carry out numerical simulations for the free boundary value problem (10)-(11), which is consistent with the main results of Theorem 2.1 and Theorem 2.2 in Section 2.

For the supercritical case (i.e., $1 < m < \frac{4}{3}$), To begin with, we discretize the spatial domain by placing a grid over the domain $[0, \widehat{R}_0]$, and for simplicity, we use the uniform grid with the grid spacing $\triangle r = 1/N$ (N is a positive integer). Similarly, we discretize the temporal interval with the grid spacing $\triangle t$. The discretized solution at each discrete time is presented as a vector

(124)
$$\eta_i^n = \eta(r_i, t^n), \quad r_i = i\Delta r, \quad t^n = n\Delta t, \quad \text{for} \quad i = 0,1,2,\cdots,N,\ n = 0,1,2,\cdots.$$

Moreover, define the discretized mass by

$$M_{1i} = \begin{cases} \displaystyle\sum_{k=1}^{i} \pi(r_{k-1}+r_{k+1})^2\widehat{u}_0(r_k)\Delta r, & i = 1,2,\cdots N, \\ 0, & i = 0. \end{cases} \tag{125}$$

Discretize η_r and η_{rr} by applying the central difference method and η_t by using the forward Euler scheme respectively, and modify the unstable scheme by replacing η_i^n with $\frac{1}{2}(\eta_{i-1}^n + \eta_{i+1}^n)$. Thus, an explicit discrete scheme for the initial boundary value problem (24) can be written into a following form as

$$\begin{cases} \dfrac{\eta_i^{n+1} - \frac{1}{2}(\eta_{i+1}^n + \eta_{i-1}^n)}{\Delta t} = -\dfrac{1}{\sigma_1^\alpha(r_i)}P_i^n - (a_1a_2 - b_1b_2)\Big(\dfrac{1}{\eta_i^n}\Big)^2 M_{1i}, \\ \eta_i^0 = \eta_0(r_i), \quad \eta_0^n = 0, \quad \eta_N^n = R(t^n), \end{cases} \tag{126}$$

where the constants $i = 1,2,\cdots,N-1$, $n = 0,1,2,\cdots$, and $P_i^n = P(\eta_i^n)$ represents an appropriate discretization to the spatial operator $P(\eta) \triangleq \big(\frac{\eta}{r}\big)^2\Big[\sigma_1^{\alpha+1}\Big(\frac{r^2}{\eta^2\eta_r}\Big)^m\Big]_r$ of the first equation in (24), which is formed as

$$P(\eta_i^n) = m\left[\frac{r_i^2\widehat{u}_0(r_i)\Delta r}{(\eta_i^n)^2(\eta_i^n - \eta_{i-1}^n)}\right]^{m-1}\Bigg\{\frac{2r_i\widehat{u}_0(r_i)\Delta r + [\widehat{u}_0(r_{i+1}) - \widehat{u}_0(r_i)]}{(\eta_i^n)^2(\eta_i^n - \eta_{i-1}^n)}$$

$$- \frac{r_i^2 \widehat{u}_0(r_i)}{(\eta_i^n)^3} - \frac{r_i^2 \widehat{u}_0(r_i)(\eta_i^n - 2\eta_i^n + \eta_{i-1}^n)}{[\eta_i^n(\eta_i^n - \eta_{i-1}^n)]^2 \Delta r} \Bigg\}. \tag{127}$$

Consistently, for the self-similar Barenblatt solution $(\widehat{u}(\widehat{\eta},t), \widehat{V}(\widehat{\eta},t), \widehat{R}(t))$, we also obtain the following discretized equation

$$\widehat{\eta}_i^{n+1} = \frac{1}{2}(\widehat{\eta}_{i+1}^n + \widehat{\eta}_{i-1}^n) - \frac{\Delta t}{\sigma_1^\alpha(r_i)} P(\widehat{\eta}_i^n), \quad \text{for} \quad i = 1, 2, \cdots, N-1, \ n = 0, 1, 2, \cdots. \tag{128}$$

Selecting an effective total mass $M_{1N} \approx M_1$ and using least square method, we can solve the system (14) and obtain approximate value of M_1.

Next, define the initial data η_i^0 be a small perturbation of the self-similar Barenblatt solution as

$$\eta_i^0 \triangleq \widehat{\eta}(r_i, 0)[1 + \epsilon(r_i)] = r_i[1 + \epsilon(r_i)], \quad 0 < \epsilon(r_i) \ll 1, \quad i = 0, 1, 2, \cdots, N. \tag{129}$$

Then, we can define the discrete density function $u_i^n = u(\eta_i^n, t^n)$ and the velocity function $V_i^n = V(\eta_i^n, t^n)$ by discretizing (23) as

$$u_i^n = \frac{2r_i \sigma_1^\alpha(r_i)\Delta r}{(\eta_i^n)^2(\eta_{i+1}^n - \eta_{i-1}^n)} u_i^0, \quad V_i^n = (\eta_t)_i^n, \quad \text{for} \quad i = 1, 2, \cdots, N-1, \ n = 1, 2, \cdots. \tag{130}$$

For the subcritical case (i.e., $m > \frac{4}{3}$), using the similar method to deal with the mass M_2 and the partial derivatives of η on fixed region $[0, \bar{R}]$ and modifying unstable scheme by replacing η_i^n with $\frac{1}{3}(\eta_{i-1}^n + \eta_i^n + \eta_{i+1}^n)$, we can obtain the explicit difference scheme and the initial data which is similar to (126) and (129) respectively. To derive the certain discrete spherically symmetric steady-state solution $\bar{u}_i = \bar{u}(r_i)$, we define an approximate discrete scheme in the following form

$$\bar{u}(r_{i+1}) = \bar{u}(r_i) - \frac{a_1 a_2 - b_1 b_2}{m(r_i)^2 \bar{u}(r_i)^{m-2}} \sum_{k=1}^{i} \pi(r_{k-1} + r_{k+1})^2 \bar{u}(r_k) \Delta r, i = 1, 2, \cdots, N-1, \tag{131}$$

where the right boundary condition satisfies $\bar{u}_N = 0$. Dealing with the left boundary condition $\bar{u}_0$ by the one sided difference to obtain $\bar{u}_1 = \bar{u}_0$, we can express the density and velocity function by replacing u_i^0 in (130) with $\bar{u}_i$.

4.1. Simulation for the supercritical case. For the supercritical case $m \in (1, \frac{4}{3})$, we have proved that if the initial total mass of cell density is small enough or the interaction between repulsion and attraction cancels almost each other, the strong solution to the free boundary value problem (10)-(11) exists globally in time and converges to the self-similar Barenblatt solution at the algebraic time rate as shown in Theorem 10 in Section 2. This is verified numerically by Figure 1.

Indeed, Figure 1 (a) demonstrates that the maximum value of the cell density decays in time. Figure 1 (b) shows that the difference of the cell density u and the self-similar Barenblatt solution $\widehat{u}$ tends to zero as $t \to \infty$. Moreover, Figure 1 (c) and Figure 1 (d) present that the velocity decays and then levels off, the difference between the two velocities V and $\widehat{V}$ decays sharply and tends to zero as $t \to \infty$. These numerical simulations are consistent with what we have shown in Theorem 10: The cell density concentrated at the center will expand outward and decay as the time grows up, and the cell density function and velocity function converge to the self-similar Barenblatt solution as time $t \to \infty$.

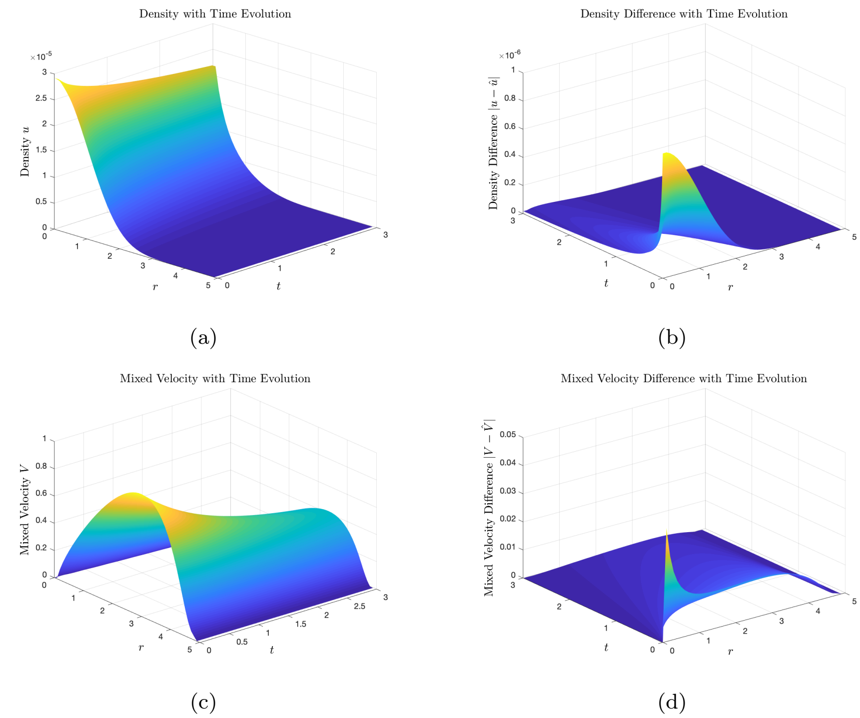

FIGURE 1. Time evolution of the density (a), the difference of the cell density u and the self-similar Barenblatt solution $\widehat{u}$ (b), the mixed velocity (c) and the mixed velocity difference (d). Spreading for the supercritical case $m = \frac{13}{12}$. Here the initial data is fixed as $\eta_i^0 = r_i \times 1.01$ and the boundary condition $R(t^n)$ is fixed as a small perturbation of $\widehat{R}(t^n)$ as $R(t^n) = \widehat{R}(t^n) \times 1.01$. The initial total mass M_1 is small ($M_1 \approx 4 \times 10^{-4}$ as $a_1a_2 - b_1b_2 = 1$) or the interaction between repulsion and attraction cancels each other ($|a_1a_2 - b_1b_2| \ll 1$).

4.2. Simulation for the subcritical case. For the subcritical case $m > \frac{4}{3}$, we have shown that if the initial data is a small perturbation of the steady-state solution and the attraction effect dominates the process, the strong solution for the cell density function exists globally in time and converges to the corresponding steady-state solution at the exponential time rate as established in Theorem 2.2 in Section 2. This is verified numerically by Figure 2 and Figure 3.

Indeed, Figure 2 (a) shows that the time evolution of the cell density on the compact support $[0, \bar{R}]$. Figure 2 (b) presents the difference of the cell density u and the steady-state solution $\bar{u}$ tends to zero as the time $t \to \infty$. Figure 2 (c) and Figure 2 (d) demonstrate the time evolution of the mixed velocity, namely, the velocity decays and then level off, and the difference between the two velocities V and $\bar{V}$ decays sharply and tends to zero as the time $t \to \infty$. These numerical simulations are consistent with what we have obtained in Theorem 2.2.

In addition, we plot the computed solution at various constant coefficient $a_1a_2 - b_1b_2$ in Figure 3, It is worth noting that faster reduction with coefficient $a_1a_2 - b_1b_2 = 10$, yet, it takes more time for the strong solution of the cell density to the free boundary value problem (10)-(11) with coefficient $a_1a_2 - b_1b_2 = 1$ to converge to the steady-state solution given by [5].

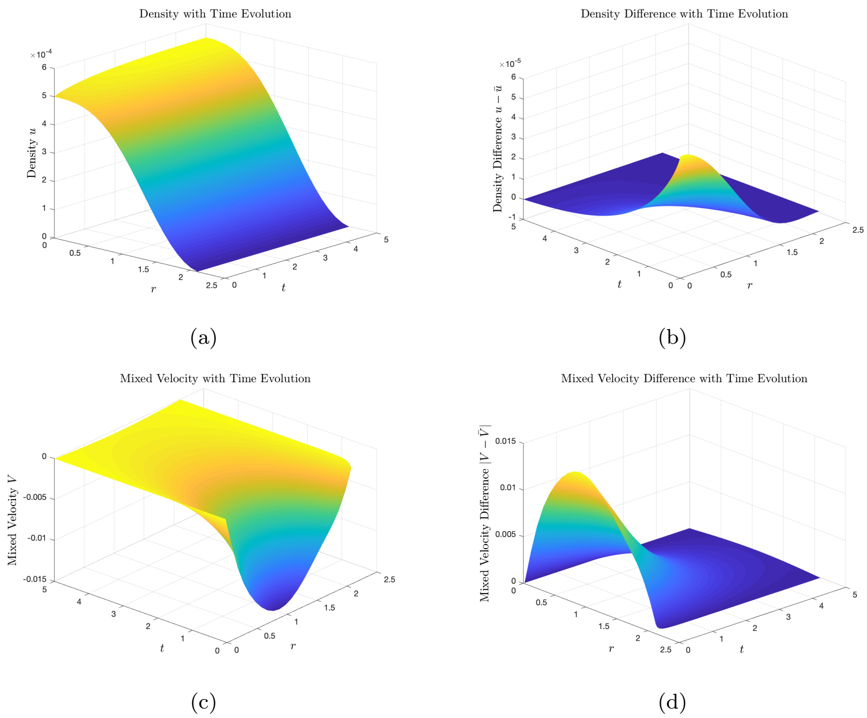

FIGURE 2. Time evolution of the density (a), the difference of the cell density u and the steady-state solution $\bar{u}$ (b), the mixed velocity (c) and the difference between the two velocities V and $\bar{V}$ (d). Convergence to the steady-state solution for the subcritical case $m = \frac{5}{3}$. Where the initial data is fixed as $\eta_i^0 = r_i \times 1.0001$ and the boundary condition $R(t^n)$ is fixed as a small perturbation of $\bar{R}(t^n)$ as $R(t^n) = \bar{R}(t^n) \times \left(1 + \frac{10^{-5}}{t^n + 0.1}\right)$. The initial total mass M_2 is fixed ($M_2 \approx 5 \times 10^{-3}$), the interaction between repulsion and attraction is nonnegative (we assume $a_1 a_2 - b_1 b_2 = 10$).

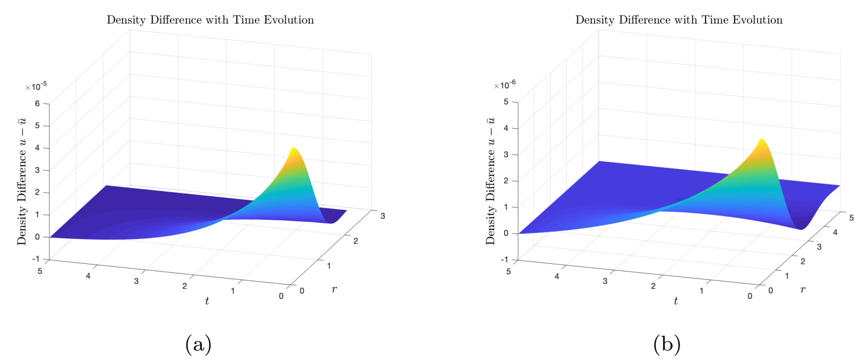

FIGURE 3. Time evolution of the density difference with $a_1 a_2 - b_1 b_2 = 10$ (a) and the density difference with $a_1 a_2 - b_1 b_2 = 1$ (b). Convergence to the steady-state solution for the subcritical case $m > \frac{4}{3}$. The initial total mass M_2 is fixed ($M_2 \approx 5 \times 10^{-3}$).

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